CSS 322: Scientific Computing
Project #1: QR factorization
Due Date: November 6th, 2021 @17:00

Part I:
For this part, you will implement Gram-Schmidt algorithm. There are many variations of the method. We are looking at the following two algorithms in particular.

Gram-Schmidt-A:
Input: an n x n matrix $A = [a_1\ a_2 \dots a_n]$ where $a_i \in R^n$.
Let R and Q be n x n matrices
$R = [r_1\ r_2 \dots r_n]$ where $r_i \in R^n$
$Q = [q_1\ q_2 \dots q_n]$ where $q_i \in R^n$
for i = 1 to n:
    $v_i = a_i$
    for j = 1 to i – 1:
        $r_{ji} = q_j^T a_i$
        $v_i = v_i - r_{ji} q_j$
    endfor
    $r_{ii} = ||v_i||_2$
    $q_i = v_i / r_{ii}$
endfor

return [Q, R]

Gram-Schmidt-B:
Input: an n x n matrix $A = [a_1\ a_2 \dots a_n]$ where $a_i \in R^n$.
Let R and Q be n x n matrices
$R = [r_1\ r_2 \dots r_n]$ where $r_i \in R^n$
$Q = [q_1\ q_2 \dots q_n]$ where $q_i \in R^n$
for i = 1 to n:
    $v_i = a_i$
for i = 1 to n:
    $r_{ii} = ||v_i||_2$
    $q_i = v_i / r_{ii}$
    for j = i+1 to n:
        $r_{ij} = q_i^T v_j$
        $v_j = v_j - r_{ij} q_i$
    endfor
endfor

return [Q, R]

For this part, you are to implement two Matlab functions: gsa and gsb. gsa and gsb are the Gram-Schmidt-A and Gram-Schmidt-B algorithms described above. Note that both functions accept an n x n matrix A and return its [Q, R] factorization result. You are to save your code in gsa.m and gsb.m respectively. Note that, you should end every Matlab command line in the with ';' to prevent printing the output at each line.

Part II:
For this part, you will experiment with the codes you just implemented in part A using Matlab. You are to create a matrix A using the following Matlab command.

```
[U,A] = qr(randn(80));
[V,B] = qr(randn(80));
Z=diag(2.^(-1:-1:-80));
A = U*Z*V;
```

Then, you are to perform QR factorization on A using your two methods in part A.

```
[QA,RA] = gsa(A);
[QB,RB] = gsb(A);
```

Finally, you are to create the scatter plots of the diagonal elements of RA, RB and Z in the same plot. For RA, you should use the "blue" color with "+" sign. For RA, you should use the "red" color with "x" sign. For Z, you should use the "black" color with "s" sign (square).

Save the plot in png format.

For this part, you are to write a small report to discuss the followings.

a) Attach the plot in the document, and describe the approximated results of Gram-Schmidt-A and Gram-Schmidt-B to that of Z. How accurate they are in comparison to the actual value? Which one of the two version is more accurate?
b) Explain why the different ways of implementation in Part I cause the differences in the approximations shown in a).

Part III:
For this part, we will look at a special version of the Toeplitz matrix. Specifically, we will look at an upper triangular Toeplitz matrix where the main diagonal has the value of 1, and the first upper superdiagonal has the value of 3, and the rest of the matrix has the value of 0. Hence, our version of the upper triangular Toeplitz matrix of size n x n have the following form

$$A = \begin{bmatrix} 1 & 3 & & 0 \\ & 1 & \cdots & \\ & & \cdots & 3 \\ 0 & & & 1 \end{bmatrix}$$

Answer the followings in the same report in Part II.

a) What is the rank of A?
b) What is the determinant of A?
c) What are the eigenvalues of A?
d) What is A^{-1}?
e) What is the QR factorization of A?

f) Create the 10 x 10 version of matrix A in Matlab, find the answers for a), b), c) and d) using Matlab and report them here.
g) Use your gsb to perform QR factorization of the matrix A in f) and report your results here.
h) Compare and discuss the results from Matlab in f) and g) against the analytical results from a) to e).

In summarize, you have to implement two functions in Matlab and submit them for Part I, and write a report to answer the questions in Part II and Part III.