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(All Summary)

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Date :

▸ ðTS210 : Math III

- Linear system / Vectors
  - Form : $y = ax + b \Rightarrow Ax = B$; A and B is matrix
  - Solution of system : $\det(A) \neq 0 \Rightarrow$ unique solution
    $\det(A) = 0 \Rightarrow$ infinite number or no solution
    ↳ might be check by random the number.
  - homogenous system : $ax + by = 0$ * all right hand side of equal sign is 0
    → can be either trivial solution $(x=y=0)$ or infinite number of solution
  - Vectors : $\bar{x} \Rightarrow$ vector x , $\bar{\theta} \Rightarrow$ zero vector
    norm $\Rightarrow \|\bar{x}\| \Rightarrow \sqrt{\sum_{i=1}^{n} x_i^2}$
    dot-product $\Rightarrow \bar{x}\cdot\bar{y} = \sum_{i=1}^{n} x_i y_i$ → can be denoted as $(\bar{x},\bar{y})$
    $= \|\bar{x}\|\|\bar{y}\|\cos\theta$ → where θ is angle between $\bar{x},\bar{y}$
    cross-product $\Rightarrow \|\bar{x}\times\bar{y}\| = \|\bar{x}\|\|\bar{y}\|\sin\theta$
    - If vectors are perpendicular / orthogonal , then $\bar{x}\cdot\bar{y} = 0$

- Matrix
  - Guassion Elimination / GE and Guass-Jordan Elimination / GJE
  - use row-reduction operations ① kR_1 ② $R_1 \leftrightarrow R_2$ ③ $kR_1 + R_2$
  - how do you know number of free variables ?
    ↳ do GE first → then n(free variable) = n(unknown) - n(equations)
  - Gives 3 outcomes

(GE)
- Triangular Matrix
  ↳ unique solution
- Non-Triangular Matrix (* mostly contained free variables)
  ↳ infinite number of solution
  ↳ answer in parametic/vector form
- Inconsistency
  ↳ no solution

(GJE)
- Diagonal Matrix
  ↳ unique solution
- Non-Diagonal/Non-Square
  ↳ infinite number of solution
  ↳ answer in parametic/vector form
- Inconsistency
  ↳ no solution

$$\left[\begin{array}{ccc|c} x&x&x&c\\ x&x&x&c\\ x&x&x&c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1&x&x&c\\ x&x&x&c\\ x&x&x&c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1&x&x&c\\ 0&x&x&c\\ 0&x&x&c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1&x&x&c\\ 0&1&x&c\\ 0&x&x&c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1&x&x&c\\ 0&1&x&c\\ 0&0&x&c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1&x&x&c\\ 0&1&x&c\\ 0&0&1&c \end{array}\right]$$

* GE stop here

$$\rightarrow \left[\begin{array}{ccc|c} 1&x&0&c\\ 0&x&0&c\\ 0&0&1&c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1&x&0&c\\ 0&1&0&c\\ 0&0&1&c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1&0&0&c\\ 0&1&0&c\\ 0&0&1&c \end{array}\right]$$

* GJE stop here

* during calculation encounter 0 0 0 | 0 → remove it
0 0 0 | c → stop → no solution

- Determinant of Matrix / Symmetric Matrix / Inverse Matrix
  - Transpose : from A with $n \times m$ dimension
    to A^T with $m \times n$ dimension and $A_{ij} = A^T_{ji} \; \forall i,j$
  - Symmetric Matrix : $A = A^T$
  - Properties of transpose
    1) $(A^T)^T = A$
    2) $(A \pm B)^T = A^T \pm B^T$
    3) $(kA)^T = k(A^T)$
    4) $(AB)^T = B^T A^T$
    - If A is an invertible matrix, A^T also invertible
    - $(A^{-1})^T = (A^T)^{-1}$
  - Diagonal Matrix $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$
    - only diagonal line contain non-zero element
    - if A is diagonal Matrix, $A^k = \begin{bmatrix} a^k & 0 & 0 & 0 \\ 0 & b^k & 0 & 0 \\ 0 & 0 & c^k & 0 \\ 0 & 0 & 0 & d^k \end{bmatrix}$, same as invert.
  - Triangular matrix $\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$ call UT upper-triangle $\begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{bmatrix}$ call LT lower-triangle
    - If A is invertible symmetric matrix, A^{-1} also symmetric
    - $(A^{-1})^T = A^{-1}$
    - A^TA and AA^T are symmetric
    - If A is invertible, A^TA and AA^T is invertible
  - Invertible Matrix $\rightarrow \det(A) \neq 0$
    - ↳ $A^{-1} = \text{adj}(A) / \det(A)$
      $= [C_{ij}(A) \; \forall i,j]^T / \det(A)$
      $= [(-1)^{i+j} M_{ij}(A) \forall i,j]^T / \det(A)$
    - $\text{adj}(A) = [C_{ij}(A) \; \forall i,j]^T$
    - $C_{ij}(A) = (-1)^{i+j} M_{ij}(A)$
    - $M_{ij}(A) = \det(A)$ without row i and column j
    - ↳ find by GJE → $[A \mid I] = \ldots\ldots = [I \mid A^{-1}]$
    - $(AB)^{-1} = B^{-1}A^{-1}$
    - $(A_1 A_2 A_3 \ldots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \ldots A_2^{-1} A_1^{-1}$
    - $(A^{-1})^{-1} = A$
    - $A^0 = 1$
    - $A^n = \underbrace{AAA\ldots A}_{n}$
    - $A^{-n} = \underbrace{A^{-1}A^{-1}\ldots A^{-1}}_{n}$
  - Find determinant ($\det(A)$)
    - diagonal multiplication ([illegible]) $\searrow \oplus \; \nearrow \ominus$; only works max at 3x3
    - by Cofactor ; $\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \ldots + a_{1n} \times C_{1n}$ - choose column or row
      $= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + \ldots + a_{n1}C_{n1}$ - choose w/ less amount of work
    - by GJE/GE ; do GE/GJE then use determinant properties to evaluate

2D lines $ax+by+c=0 \rightarrow$ normal vector $\begin{pmatrix} a \\ b \end{pmatrix}, -\begin{pmatrix} a \\ b \end{pmatrix}$

directional vector $\begin{pmatrix} b \\ -a \end{pmatrix}, -\begin{pmatrix} b \\ -a \end{pmatrix}$

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- determinant properties — det(A)
  - entire row / column contain 0 $\rightarrow 0$
  - two propotional / equal row/column $\rightarrow 0$
  - single row/col multiply by constant k $\rightarrow k\det(A)$
  - entire matrix multiply by constant $\rightarrow k^n \det(A)$
  - matrix w/ 2 exchanging row/column $\rightarrow -\det(A)$
  - multiply 1 row and add to another row $(kR_1+R_2) \rightarrow \det(A)$
  - transpose matrix $\rightarrow \det(A)$; $\det(A^T)=\det(A)$
  - upper-triangular / lower triangular (LT/UT) $\rightarrow$ all elements in diagonal multiply together.
  - $\det(AB)=\det(A)\det(B)$ • $\det(A^{-1}) = 1/\det(A)$

- Matrix Properties

1) $A+B=B+A$ 2) $A+(B+C)=(A+B)+C$ 3) $A(BC)=(AB)C$

4) $A(B\pm C)=AB\pm AC$ 5) $(B\pm C)A = BA\pm BC$ 6) $a(A+B)=aA+aB$

7) $(a\pm b)A = aA\pm bA$ 8) $(ab)A = a(bA) = b(aA)$ 9) $a(AB)=A(aB)=(aA)B$

10) $AB \neq BA$ 11) $AB=AC \rightarrow B\neq C$

- Cramer's Rule

↳ write equation in form of $Ax=B$ and $\det(A)\neq 0$

↳ Thus $x_i = \dfrac{\det(A_i)}{\det(A)}$; A_i is matrix A that replaced column i with B

- Lines and Plane.
  - 2D Line : perpendicular to line $\rightarrow$ normal vector / parallel to line $\rightarrow$ directional vector

    normalized vector $\Rightarrow \dfrac{1}{\sqrt{a^2+b^2}}(a,b)$ | parametic equation (vector) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

    - line through 2 points : $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1} = t$ ← use in parametic form
  - 3D Lines & Plane : general equation of plane : $Ax+By+Cz+D=0$

    normal vector to plane : $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ normalized $= \dfrac{\pm 1}{\sqrt{A^2+B^2+C^2}}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$

    vector form $(N, r-r_0)=0$ where $r=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- Lines in 3D can be defined as intersection of 2 nonparallel plane

$$\begin{cases} A_1x + B_1y + C_1z = D_1 \\ A_2x + B_2y + C_2z = D_2 \end{cases} \rightarrow \text{parametric form}: \underbrace{\begin{cases} x = a_xt + b_x \\ y = a_yt + b_y \\ z = a_zt + b_z \end{cases}}_{\text{parametric}} \Rightarrow \underbrace{\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}}_{\text{directional vector}} t + \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \underbrace{at + b}_{\text{vector form}}$$

- line through two points in 3D defined by two planes given by

use in formation of parametic equ. $\rightarrow$ $t = \dfrac{z - z_1}{z_2 - z_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ $\Rightarrow$ gives directional vector $\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$

- Relative Position of Lines and Planes in 3D
  - Planes in 3D
    - Parallel: $\hookrightarrow N_{1,unit} = N_{2,unit}$ or $-N_{2,unit}$ and no solution
    - Identical (a.k.a. coincide): $\hookrightarrow N_{1,unit} = N_{2,unit}$ or $N_{2,unit}$ and infinite number of solution.
    - Intersect: $\hookrightarrow N_{1,unit} \neq N_{2,unit}$ and $N_{2,unit}$
    - Perpendicular: $\hookrightarrow (N_1, N_2) = 0$
  - Planes and Lines in 3D
    - Parallel: $\hookrightarrow (a, N) = 0$ and no solution
    - Coincide: $\hookrightarrow (a, N) = 0$ and infinite number of solution
    - Intersect: $\hookrightarrow (a, N) \neq 0$
    - Perpendicular: $\hookrightarrow a_{unit} = N_{unit}$ or $a_{unit} = -N_{unit}$
  - Lines in 3D
    - Parallel: $\hookrightarrow a_{1,unit} = a_{2\,unit}$ or $= -a_{2\,unit}$ and no solutions
    - Coincide: $\hookrightarrow a_{1\,unit} = a_{2\,unit}$ or $= -a_{2\,unit}$ and infinite number of solution
    - Intersect: $\hookrightarrow a_{1\,unit} \neq a_{2\,unit}$ and $\neq -a_{2\,unit}$ and unique solution
    - Skew: $\hookrightarrow a_{1\,unit} \neq a_{2\,unit}$ and $\neq -a_{2\,unit}$ $\hookrightarrow$ has no solution
  - use GJE / GE at its full potential
  - If given a parametic equation, just substitute it first.
- Find a plane through 3 points $\rightarrow Ax + By + Cz - D = 0$

$r_1 = (x_1, y_1, z_1)$, $r_2 = (x_2, y_2, z_2)$, $r_3 = (x_3, y_3, z_3)$ // points r_1, r_2, r_3

where $A = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}$, $B = \begin{vmatrix} x_1 & 1 & z_1 \\ x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \end{vmatrix}$, $C = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$, $D = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$