Financial Time Series and Their Characteristics

Egon Zakrajšek
Division of Monetary Affairs
Federal Reserve Board

Summer School in Financial Mathematics
Faculty of Mathematics & Physics
University of Ljubljana
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Basics of Asset Returns

Most financial studies involve returns—instead of prices—of assets:

- Asset returns is a complete and scale-free summary of the investment opportunity for an average investors.
- Return series have more attractive statistical properties than price series.
- Several definitinos of asset returns.

Define,

$P_t =$ price of an asset in period t (assume no dividends)

One-Period Simple Return

Holding the asset from one period from date $t-1$ to date t would result in a simple gross return:

$$1 + R_t = \frac{P_t}{P_{t-1}} \quad \text{or} \quad P_t = P_{t-1}(1 + R_t)$$

Corresponding one-period simple net return or simple return:

$$R_t = \frac{P_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}}$$

Multiperiod Simple Return

Holding the asset for k periods between dates $t-k$ and t gives a k-period simple gross return:

$$\begin{aligned} 1+R_t[k]=\frac{P_t}{P_{t-k}} &= \frac{P_t}{P_{t-1}}\times\frac{P_{t-1}}{P_{t-2}}\times\cdots\times\frac{P_{t-k+1}}{P_{t-k}} \\ &= (1+R_t)(1+R_{t-1})\cdots(1+R_{t-k}) \\ &= \prod_{j=0}^{k-1}(1+R_{t-j}) \end{aligned}$$

- k-period simple gross return is just the product of the k one-period simple gross returns involved—compound return
- k-period simple net return:

$$R_t[k]=\frac{P_t-P_{t-k}}{P_{t-k}}$$

Time Interval

Actual time interval is important in discussing and comparing returns (e.g., monthly, annual).

- If the time interval is not given, it is implicitly assumed to be one year.
- If the asset was held for k years, then the annualized (average) returns is defined as

$$\begin{aligned}
\text{Annualized}\{R_t[k]\} &= \left[\prod_{j=0}^{k-1}(1+R_{t-j})\right]^{1/k} - 1 \\
&= \exp\left[\frac{1}{k}\sum_{j=0}^{k-1}\ln(1+R_{t-j})\right] - 1
\end{aligned}$$

- Arithmetic averages are easier to compute than geometric ones!

Continuously Compounded Returns

The natural log of the simple gross return of an asset is called the continuously compounded return or log return:

$$r_t = \ln(1 + R_t) = \ln \frac{P_t}{P_{t-1}} = p_t - p_{t-1} \quad \text{where} \quad p_t = \ln P_t$$

Advantages of log returns:

- Easy to compute multiperiod returns:

$$\begin{aligned} r_t[k] &= \ln(1 + R_t[k]) = \ln\left[(1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1})\right] \\ &= r_t + r_{t-1} + \cdots + r_{t-k} \end{aligned}$$

- More tractable statistical properties.

Portfolio Return

The simple net return of a portfolio consisting of N assets is a weighted average of the simple net returns of the assets involved, where the weight on each asset is the fraction of the portfolio's value investment in that asset:

$$R_{p,t} = \sum_{i=1}^{N} w_i R_{it}$$

The continuously compounded returns of a portfolio, however, do not have this convenient property!
Useful approximation:

$$r_{p,t} \approx \sum_{i=1}^{N} w_i r_{it} \quad \text{if } R_{it} \text{ "small"}$$

Dividend Payments

If an asset pays dividends periodically, the definition of asset returns must be modified:

- $D_t =$ dividend payment of an asset between periods $t-1$ and t
- $P_t =$ price of the asset at the end of period t
- Total returns:

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1 \quad \text{and} \quad r_t = \ln(P_t + D_t) - \ln P_{t-1}$$

Most reference indexes include dividend payments:

- German DAX index exception.
- CRSP and MSCI indexes include reference indexes without ("price index") and with dividends ("total return index").

Excess Return

Excess return of an asset in period t is the difference between the asset's return and the return on some reference asset.

- Reference asset is often taken to be riskless (e.g., short-term U.S. Treasury bill return).
- Excess returns:

$$Z_t = R_t - R_{0t} \quad \text{and} \quad z_t = r_t - r_{0t}$$

- Excess return can be thought of as the payoff on an arbitrage portfolio that goes long in an asset and short in the reference asset with no net initial investment.

Motivation

Early work in finance imposed strong assumptions on the statistical properties of asset returns:

- Normality of log-returns:
  - Convenient assumption for many applications (e.g., Black-Scholes model for option pricing)
  - Consistent with the Law of Large Numbers for stock-index returns
- Time independency of returns:
  - To some extent, an implications of the Efficient Market Hypothesis
  - EMH only imposes unpredictability of returns

Returns as Random Variable

Assume that the random variable X (i.e., log-return) has the following cumulative distribution function (CDF):

$$F_X(x) = \Pr[X \leq x] = \int_{-\infty}^{x} f_X(u)du$$

- f_X = probability distribution function (PDF) of X

Moments of a Random Variable

- The mean (expected value) of X:

$$\mu = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- The variance of X:

$$\sigma^2 = V[X] = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) dx$$

- The k-th noncentral moment:

$$m_k = E[X^k] = \int_{-\infty}^{\infty} x^k f_X(x) dx$$

- The k-th central moment:

$$\mu_k = E[(X-m_1)^k] = \int_{-\infty}^{\infty} (x-m_1)^k f_X(x) dx$$

Skewness

The third central moment measures the skewness of the distribution:

$$\mu_3 = E[(X - m_1)^3]$$

Standardized skewness coefficient:

$$S[X] = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3}{\sigma^3}$$

- When $S[X]$ is negative, large realizations of X are more often negative than positive (i.e., crashes are more likely than booms)
- For normal distribution $S[X] = 0$

Kurtosis

The fourth central moment measures the tail heaviness/peakedness of the distribution:

$$\mu_4 = E[(X - m_1)^4]$$

Standardized kurtosis coefficient:

$$K[X] = E\left[\left(\frac{X - \mu}{\sigma}\right)^4\right] = \frac{\mu_4}{\sigma^4}$$

- Large $K[X]$ implies that large realizations (positive or negative) are more likely to occur
- For normal distribution $K[X] = 3$
- Define excess kurtosis as $K[X] - 3$

Descriptive Statistics of Returns

Let $\{r_t : t = 1, 2, \ldots, T\}$ denote a time-series of log-returns that we assume to be the realizations of a random variable.

- Measures of location:
  - Sample mean (or average) is the simplest estimate of location:

$$\bar{r} = \hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} r_t$$

  - Mean is very sensitive to “outliers”
  - Median (MED) is robust to outliers:

$$\text{MED} = \Pr[r_t \leq Q(0.5)] = \Pr[r_t > Q(0.5)] = 0.5$$

  - Other robust measures of location: α-trimmed means and α-winsorized means

Descriptive Statistics of Returns (cont.)

- Measures of dispersion:
  - Sample standard deviation (square root of variance) is the simplest estimate of dispersion:

$$s = \hat{\sigma} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2}$$

  - Std. deviation is very sensitive to "outliers"
  - Median Absolute Deviation (MAD) is robust to outliers:

$$\text{MAD} = med(|r_t - \text{MED}|)$$

    - Under normality $s = 1.4826 \times \text{MAD}$
  - Inter Quartile Range (IQR) is robust to outliers:

$$\text{IQR} = Q(0.75) - Q(0.25)$$

    - Under normality $s = \text{IQR}/1.34898$

Descriptive Statistics of Returns (cont.)

- Skewness:
  - Sample skewness coefficient is the simplest estimate of asymmetry:

$$\hat{S} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{r_t - \bar{r}}{s}\right]^3}$$

    - If $\hat{S} < 0$, the distribution is skewed to the left
    - If $\hat{S} > 0$, the distribution is skewed to the right
  - Octile Skewness (OS) is robust to outliers:

$$\text{OS} = \frac{[Q(0.875) - Q(0.5)] - [Q(0.5) - Q(0.125)]}{Q(0.875) - Q(0.125)}$$

    - If distribution is symmetric then $\text{OS} = 0$
    - $-1 \leq \text{OS} \leq 1$

Descriptive Statistics of Returns (cont.)

- Kurtosis:
  - Sample kurtosis coefficient is the simplest estimate of asymmetry:

$$\hat{K} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{r_t - \bar{r}}{s}\right]^4}$$

    - If $\hat{K} < 3$, the distribution has thinner tails than normal
    - If $\hat{K} > 3$, the distribution has thicker tails than normal
  - Left/Right Quantile Weights (LQW/RQW) are robust to outliers:

$$\mathrm{LQW} = \frac{\left[Q\left(\frac{0.875}{2}\right) + Q\left(\frac{0.125}{2}\right)\right] - Q(0.25)}{Q\left(\frac{0.875}{2}\right) - Q\left(\frac{0.125}{2}\right)}$$

$$\mathrm{RQW} = \frac{\left[Q\left(\frac{1+0.875}{2}\right) + Q\left(1 - \frac{0.875}{2}\right)\right] - Q(0.75)}{Q\left(\frac{1+0.875}{2}\right) - Q\left(1 - \frac{0.875}{2}\right)}$$

    - Distinguishes left and right tail heaviness
    - $-1 < \mathrm{LQW}, \mathrm{RQW} < 1$

Distribution of Sample Moments

Under normality, the following results hold as $T \to \infty$:

- $\sqrt{T}(\hat{\mu} - \mu) \sim N(0, \sigma^2)$
- $\sqrt{T}(\hat{\sigma}^2 - \sigma^2) \sim N(0, 2\sigma^4)$
- $\sqrt{T}(\hat{S} - 0) \sim N(0, 6)$
- $\sqrt{T}(\hat{K} - 3) \sim N(0, 24)$

These asymptotic results for the sample moments can be used to perform statistical tests about the distribution of returns.

Tests of Normality

We consider unconditional normality of the return series $\{r_t : t = 1, 2, \ldots, T\}$.

Three broad classes of tests for the null hypothesis of normality:

- Moments of the distribution (Jarque-Bera; Doornik & Hansen)
- Properties of the empirical distribution function (Kolmogorov-Smirnov; Anderson-Darling; Cramer-von Mises)
- Properties of the ranked series (Shapiro-Wilk)

Jarque-Bera (1987) Test

Based on the idea that under the null hypothesis, skewness and excess kurtosis are jointly equal to zero.

- Jarque-Bera test statistic:

$$JB = T\left[\frac{\hat{S}^2}{6} + \frac{(\hat{K}-3)^2}{24}\right]$$

- Under the null hypothesis $JB \sim \chi^2(2)$
- Doornik & Hansen (2008) test is based on transformations of S and K that are much closer to normality

Kolmogorov-Smirnov (1933) Test

Compares the empirical distribution function (EDF) with with an assumed theoretical CDF $F^*(x;\theta)$ (i.e., normal distribution)

- The return series $\{r_t : t = 1, 2, \ldots, T\}$ is drawn from an unknown CDF $F_r(\cdot)$
- Approximate F_r by its EDF G_r:

$$G_r(x) = \frac{1}{T}\sum_{t=1}^{T} I(r_t \leq x)$$

- Compare the EDF with $F^*(x;\theta)$ to see if they are "close:"

$$\begin{aligned} H_0 : G_r(x) &= F^*(x;\theta) \quad \forall x \\ H_A : G_r(x) &\neq F^*(x;\theta) \quad \text{for at least one value of } x \end{aligned}$$

Kolmogorov-Smirnov (1933) Test (cont.)

- Kolmogorov-Smirnov test statistic:

$$KS = \sup_x |F^*(x;\theta) - G_r(x)|$$

- Critical values have been tabulated for known μ and σ^2
- Lilliefors modification of the Kolmogorov-Smirnov test when testing against $N(\hat{\mu}, \hat{\sigma}^2)$