1 κ precondition test

Consider a discretised incompressible Magnetohydrodynamics problem of the following form

$$\mathcal{K}_{\rm MH} = \begin{pmatrix} A+O & B^T & C^T & 0 \\ B & 0 & 0 & 0 \\ -C & 0 & M & D^T \\ 0 & 0 & D & 0 \end{pmatrix}. \tag{1}$$

Using the inner-outer precondition

$$\mathcal{K}_{\rm MH} = \begin{pmatrix} F & B^T & C^T & 0 \\ 0 & -S & 0 & 0 \\ -C & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{pmatrix}, \tag{2}$$

from [Dan's thesis] and [Michael's thesis] as a starting point we re-oder the variables to (u, b, p, r) to yield

$$\mathcal{M}_{\rm MH} = \begin{pmatrix} F & C^T & B^T & 0 \\ -C & M+X & 0 & 0 \\ 0 & 0 & -S & 0 \\ 0 & 0 & 0 & L \end{pmatrix}. \tag{3}$$

Using the preconditioner $\mathcal{M}_{\rm MHD}$ depends on an efficient way to implement

$$\mathcal{M}_{\rm outer} = \begin{pmatrix} F & C^T \\ -C & M+X \end{pmatrix}^{-1}. \tag{4}$$

The approach used in [Dan's thesis] and [Michael's thesis] was to approximate $\mathcal{M}_{\rm outer}$ with an "inner" GMRES solve taking the block diagonal matrices as the preconditioner. However, this inner solver limits the efficiency of the preconditioning approach, therefore consider different possible approximations of $\mathcal{M}_{\text{outer}}$ may increase the performance of this approach. Performing block LDL decomposition gives two possible preconditions of the following form

$$\mathcal{M}_1 = \begin{pmatrix} F & 0 \\ 0 & M+X+CF^{-1}C^T \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} F+C^T(M+X)^{-1}C & 0 \\ 0 & M+X \end{pmatrix}. \tag{5}$$

It has been show that using $\mathcal{M}_1$ or $\mathcal{M}_2$ as a preconditioner for a Kyrlov subspace method yields exactly 3 eigenvalues [Murphy, Golub, Wathen]. Therefore, $\mathcal{M}_1^{-1}$ or $\mathcal{M}_2^{-1}$ may yield possible alternative and better approximations to $\mathcal{M}_{\text{outer}}$.

1.1 Approximation of $C^TA^{-1}C$ and $C\tilde{A}^{-1}C^T$ where A and $\tilde{A}$ are Laplacians

The operator $\tilde{\mathcal{F}}u$ is given by

$$\begin{aligned} C\tilde{A}^{-1}C^T(u) &= -b \times \{\nabla \times \Delta^{-1}[\nabla \times (u \times b)]\}, \\ &= -b \times \nabla \times \Delta^{-1}\nabla \times (u \times b). \end{aligned} \tag{6}$$

The second operator $\mathcal{F}b$ is given by

$$\begin{aligned} C^tA^{-1}C(b) &= \nabla \times ([\Delta^{-1}\{-b \times (\nabla \times b)\}] \times b), \\ &= -\nabla \times ([\tfrac{1}{2}\Delta^{-1}\nabla(b \cdot b)] \times b - [\Delta^{-1}(b \cdot \nabla)b] \times b), \\ &= -\tfrac{1}{2}\nabla \times \Delta^{-1}\nabla(b \cdot b) \times b - \nabla \times \Delta^{-1}(b \cdot \nabla)b) \times b, \\ &= \nabla \times \Delta^{-1}(-\tfrac{1}{2}\nabla(b \cdot b) \times b - (b \cdot \nabla)b) \times b), \\ &= [\nabla \times \Delta^{-1}(\nabla \times b) \times b] \times b. \end{aligned} \tag{7}$$

The efficiency of the two approaches $\mathcal{F}b$ and $\tilde{\mathcal{F}}u$ rely on a good approximation of $\nabla \times \Delta^{-1}\nabla$. In the resent technical report from [Phillips et. al] they showed that

$$\nabla \times \Delta^{-1} \nabla c = -c.$$

Using this identity the two operators simplify to:

$$\tilde{\mathcal{F}} u = b \times (u \times b), \quad \mathcal{F} b = (b \times b) \times b.$$

Here we notice that for the simplification of $\mathcal{F} b$ has an $b \times b$ term which is identically zero. However, since the equations are linearised then

$$\mathcal{F} b = (b^k \times b) \times b^k.$$

In 2-dimensions, let $b^k = (b_1^k, b_2^k, 0)$ and $b = (b_1, b_2, 0)$ then

$$(b^k \times b) \times b^k = \begin{pmatrix} (b_2^k)^2 & -b_1^k b_2^k \\ -b_1^k b_2^k & (b_1^k)^2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Using $u = (u_1, u_2, 0)$ and linearising around the two magnetic terms in $\tilde{\mathcal{F}} u$ yields

$$b_k \times (u \times b_k) = \begin{pmatrix} (b_2^k)^2 & -b_1^k b_2^k \\ -b_1^k b_2^k & (b_1^k)^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

1.2 Numerical testing

Construction of the precondition

$$\begin{pmatrix} F & C^T & B^T & 0 \\ 0 & M + X + C F^{-1} C^T & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & L \end{pmatrix}, \tag{8}$$

produces the following results

l	DoF	AV solve Time	Total picard time	picard iterations	Av Outer its
1	51	0.012178	0.455623	5	11.6
2	163	0.034960	0.543420	5	25.8
3	579	0.134840	1.109627	5	33.0
4	2179	1.488171	8.177399	5	35.2
5	8451	99.375478	498.770225	5	36.6

Table 1: $\kappa = 1$, $\nu_m = 1$ and $\nu = 1$

l	DoF	AV solve Time	Total picard time	picard iterations	Av Outer its
1	51	0.011994	0.536929	6	11.5
2	163	0.034794	0.649704	6	26.3
3	579	0.133624	1.359779	6	34.0
4	2179	1.463671	9.618563	6	36.7
5	8451	83.979013	505.978769	6	37.3

Table 2: $\kappa = 10$, $\nu_m = 1$ and $\nu = 1$

l	DoF	AV solve Time	Total picard time	picard iterations	Av Outer its
1	51	0.011866	0.616299	7	10.9
2	163	0.034967	0.756720	7	26.7
3	579	0.134432	1.582807	7	34.1
4	2179	1.464692	11.227711	7	36.4
5	8451	87.282682	613.476128	7	37.6

Table 3: $\kappa = 100$, $\nu_m = 1$ and $\nu = 1$

Now consider the following preconditioner

$$\begin{pmatrix} F - C^T(M+X)^(-1)C & C^T & B^T & 0 \\ 0 & M+X & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & L \end{pmatrix}, \tag{9}$$

Using this preconditioner yields:

l	DoF	AV solve Time	Total picard time	picard iterations	Av Outer its
1	51	0.012170	0.454375	5	11.6
2	163	0.034755	0.541899	5	25.8
3	579	0.134306	1.105293	5	33.0
4	2179	1.484157	8.151595	5	35.2
5	8451	117.987729	591.895518	5	36.6

Table 4: $\kappa = 1$, $\nu_m = 1$ and $\nu = 1$

l	DoF	AV solve Time	Total picard time	picard iterations	Av Outer its
1	51	0.013092	3.153271	6	11.5
2	163	0.036800	0.730741	6	26.3
3	579	0.140951	1.440613	6	34.0
4	2179	1.817301	11.879250	6	36.7
5	8451	114.076549	686.811843	6	37.3

Table 5: $\kappa = 10$, $\nu_m = 1$ and $\nu = 1$

l	DoF	AV solve Time	Total picard time	picard iterations	Av Outer its
1	51	0.013156	3.384350	7	10.9
2	163	0.038609	0.911713	7	26.7
3	579	0.152467	1.862579	7	34.1
4	2179	2.115897	15.969938	7	36.4
5	8451	112.773827	792.175312	7	37.6

Table 6: $\kappa = 100$, $\nu_m = 1$ and $\nu = 1$