Block Preconditioners for an Incompressible Magnetohydrodynamics Problem

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  - Movement of the conductive material that induces and modifies any existing electromagnetic field
  - Magnetic and electric fields generate a mechanical force on the fluid

MHD model: coupled Navier-Stokes and Maxwell's equations

$$
\begin{aligned}
-\nu\,\Delta u + (u\cdot\nabla)u + \nabla p - \kappa\,(\nabla\times b)\times b &= f \qquad \text{in } \Omega,\\
\nabla\cdot u &= 0 \qquad \text{in } \Omega,\\
\kappa\nu_m\,\nabla\times(\nabla\times b) + \nabla r - \kappa\,\nabla\times(u\times b) &= g \qquad \text{in } \Omega,\\
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with appropriate boundary conditions

- $(\nabla\times b)\times b$: Lorentz force accelerates the fluid particles in the direction normal to the electric and magnetic fields
- $\nabla\times(u\times b)$: electromotive force modifying the magnetic field

Steady-state Navier-Stokes equations

Incompressible Navier-Stokes Equations:

$$\begin{aligned} -\nu\,\Delta u + (u\cdot\nabla)u + \nabla p &= f && \text{in } \Omega, \\ \nabla\cdot u &= 0 && \text{in } \Omega, \\ u &= u_D && \text{on } \partial\Omega \end{aligned}$$

where u is the fluids velocity; p is the fluids pressure; f is the body force acting on the fluid and ν the kinematic viscosity

Steady-state Navier-Stokes equations

Corresponding linear system:

$$\mathcal{K}_{\mathrm{NS}} = \begin{pmatrix} A+0 & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

where $A \in \mathbb{R}^{n_u \times n_u}$ is the discrete Laplacian; $O \in \mathbb{R}^{n_u \times n_u}$ is the discrete convection operator and $B \in \mathbb{R}^{m_u \times n_u}$ is a discrete divergence operator with full row rank.

Note: due to convection term the linear system is non-symmetric

For an extensive discussion of preconditioners we refer to Elman, Silvester and Wathen 2014.

Time-Harmonic Maxwell in mixed form

Maxwell operator in mixed form:

$$\begin{aligned} \nabla \times (\nabla \times b) - k^2 b + \nabla r = g \qquad & \text{in } \Omega, \\ \nabla \cdot b = 0 \qquad & \text{in } \Omega, \\ b \times n = b_D \qquad & \text{on } \partial\Omega, \\ r = 0 \qquad & \text{on } \partial\Omega, \end{aligned}$$

where b is the magnetic vector field, r is a scalar multiplier and k is the wave number

Note: for the MHD system $k = 0$

Time-Harmonic Maxwell in mixed form

Corresponding linear system:

$$\begin{pmatrix} M - k^2 X & D^T \\ D & 0 \end{pmatrix} \begin{pmatrix} b \\ r \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix},$$

where $M \in \mathbb{R}^{n_b \times n_b}$ is the discrete curl-curl operator; $X \in \mathbb{R}^{n_b \times n_b}$ the discrete mass matrix; $D \in \mathbb{R}^{m_b \times n_b}$ the discrete divergence operator

Note: M is semidefinite with nullity m_b

A significant amount of literature on time-harmonic Maxwell: notably for us, work of Hiptmair & Xu 2007 for solving shifted curl-curl equations in a fully scalable fashion

Discretisation

- Finite element discretisation based on the formulation in Schötzau 2004
- Fluid variables: Mini element (P1+B3)/P1
- Magnetic variables: mixed Nédélec element approximation N1/P1
- Nédélec elements capture solutions correctly on non-convex domains

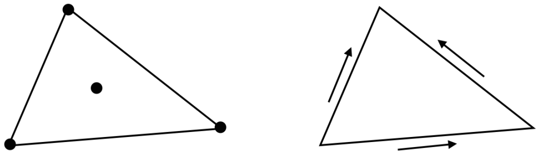

Figure: Bubble element and Nédélec element

Discretised and linearised MHD model:

$$\begin{pmatrix} A+O(u) & B^T & C(b)^T & 0 \\ B & 0 & 0 & 0 \\ -C(b) & 0 & M & D^T \\ 0 & 0 & D & 0 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta p \\ \delta b \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_p \\ r_b \\ r_r \end{pmatrix},$$

with

$$\begin{aligned} r_u &= f - Au - O(u)u - C(b)^T b - B^T p, \\ r_p &= -Bu, \\ r_b &= g - Mu + C(b)b - D^T r, \\ r_r &= -Db. \end{aligned}$$

C: coupling terms

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C: coupling terms

In some cases, it is possible to decouple

- Magnetic Decoupling (MD): if the coupling term is not large, just dump it and solve a block diagonal system
- Complete Decoupling (CD): if convection is small, then dump convective term and obtain symmetry

A Few Comments

- Little has been done with respect to a preconditioned iterative solution method
- Phillips, Elman, Cyr, Shadid, and Pawlowski 2014: block preconditioners for an exact penalty formulation, using nodal elements; resulting system is block 3-by-3
- Results show good scalability with respect to the mesh
- Our formulation: Nédélec (edge) elements, giving rise to a richer finite element space, 4-by-4 system, but with a different set of challenges

Ideal preconditioning

Non-singular $(1,1)$ block (as in Navier-Stokes)

$$\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F & B^T \\ 0 & BF^{-1}B^T \end{pmatrix}$$

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F singular with nullity m (as in time-harmonic Maxwell)

$$\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F + B^T W^{-1} B & 0 \\ 0 & W \end{pmatrix}, \text{ where } W \text{ is SPD}$$

Greif & Schötzau 2006 showed exactly two eigenvalues: ± 1

Navier-Stokes subproblem

$$\mathcal{K}_{\mathrm{NS}} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}$$

where $F = A + O$ is the discrete convection diffusion operator. Shown in Elman, Silvester & Wathen 2014 that

$$\mathcal{P}_{\mathrm{NS}} = \begin{pmatrix} F & B^T \\ 0 & S \end{pmatrix}, \quad S = A_p F_p^{-1} Q_p$$

is a good approximation to the Schur complement preconditioner. A_p: pressure space Laplacian, F_p: pressure space convection-diffusion operator, Q_p: pressure space mass matrix

Maxwell subproblem

$$\mathcal{K}_{\mathrm{NS}} = \begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix}$$

Note: M is highly rank deficient.
Greif & Schötzau 2007 show that L (scalar Laplacian) is the appropriate choice (from an inf-sup stability point of view)

$$\mathcal{P}_{\mathrm{iM}} = \begin{pmatrix} M + B^T L^{-1} B & 0 \\ 0 & L \end{pmatrix}$$

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Practical preconditioner:

$$\mathcal{P}_{\mathrm{M}} = \begin{pmatrix} M + X & 0 \\ 0 & L \end{pmatrix}$$

where X vector mass matrix is spectrally equivalent to $B^T L^{-1} B$

MHD problem

Combining the Navier-Stokes and Maxwell preconditioners

$$\mathcal{P}_{\mathrm{MH}} = \begin{pmatrix} F & B^T & C^T & 0 \\ 0 & -S & 0 & 0 \\ -C & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

Note: $\mathcal{P}_{\mathrm{MH}}$ remains challenging to solve due to coupling terms.
Schur complement approximation for velocity-magnetic unknowns

$$\mathcal{P}_{\mathrm{schurMH}} = \begin{pmatrix} F + M_C & B^T & C^T & 0 \\ 0 & -S & 0 & 0 \\ 0 & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

where $M_C = C^T(M+X)^{-1}C$

Spectral analysis (ideal preconditioner)

Note: using $X = B^T L^{-1} B$ for eigenvalue analysis

Theorem

The matrix $\mathcal{P}_{\mathrm{MH}}^{-1}\mathcal{K}_{\mathrm{MH}}$ *has an eigenvalue* $\lambda = 1$ *with algebraic multiplicity of (at least)* $n_u + n_b$ *and an eigenvalue* $\lambda = -1$ *with algebraic multiplicity of (at least)* m_b.

Theorem

The matrix $\mathcal{P}_{\mathrm{schurMH}}^{-1}\mathcal{K}_{\mathrm{MH}}$ *has an eigenvalue* $\lambda = 1$ *with algebraic multiplicity of (at least)* $n_b + n_c$ *where* n_c *is the dimension of the nullspace of* C *and an eigenvalue* $\lambda = -1$ *with algebraic multiplicity of (at least)* m_b.

Eigenvalue distribution

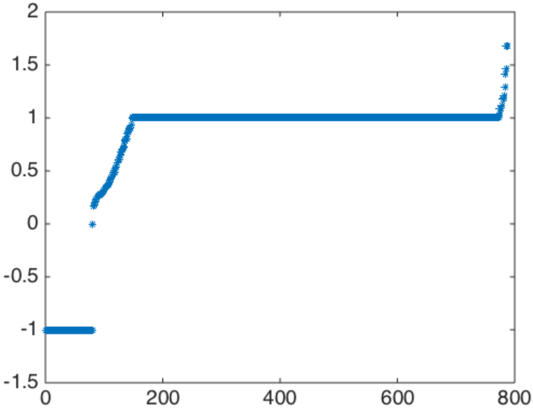

Figure: Real part of eigenvalues of preconditioned matrix $\mathcal{P}_{\mathrm{schurMH}}^{-1}\mathcal{K}_{\mathrm{MH}}$

Approximation for M_C

Inspired by Phillips, Elman, Cyr, Shadid & Pawlowski 2014, we approximate M_C by

$$M_C \approx Q_s = \kappa {\nu_m}^{-1} b \times (u \times b)$$

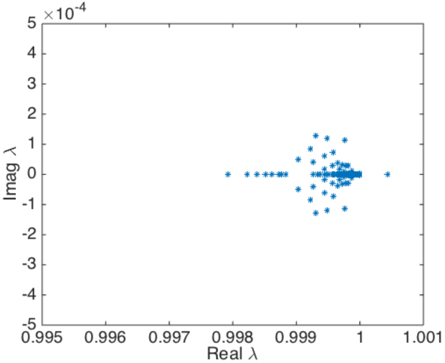

Figure: Eigenvalues of preconditioned matrix $(F + Q_S)^{-1}(F + M_C)$

Numerical software

- Finite element software FEniCS:
    - DOLFIN problem-solving interface
    - FFC compiler for finite element variational forms
    - FIAT finite element tabulator
    - Instant just-in-time compiler
    - UFC the code generator
    - UFL form language
- Linear algebra software:
    - PETSc linear algebra wrapper features
    - HYPRE as a multigrid solver
    - MUMPS sparse direct solver

2D: smooth solution

l	DoF	Average solve Time	Total time	Non-linear its	Average its
3	788	0.03	0.3	5	16.0
4	2,980	0.08	1.0	6	17.5
5	11,588	0.28	3.4	6	19.3
6	45,700	1.06	11.0	5	20.6
7	181,508	4.70	46.5	5	21.0
8	723,460	19.95	192.1	5	21.8
9	2,888,708	98.91	868.9	5	22.4
10	11,544,580	482.86	5794.1	5	24.2

Table: Number of nonlinear iterations and number of iterations to solve the MHD system with Tol = 1e-4, $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$

Large scale 3D

- Current goal: construct a high order (Taylor-Hood/Nédélec) FE discretisation
- This provides an accurate scheme and avoids stabilisation issues that are prevalent in lower order formulations.
- Work in progress: implement with direct solvers, working on implementing the Hiptmair-Xu AMG preconditioning scheme.

l	DoF	Non-linear its	Average its
1	963	3	19.7
2	5,977	3	25.3
3	41,805	3	26.7
4	312,085	3	27.0

Table: Number of nonlinear iterations and number of iterations to solve the MHD system with Tol = 1e-4, $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$

Future work

- Further develop code and release on a public repository
- Robustness with respect to kinematic viscosity
- Other non-linear solvers
- Different mixed finite element discretisations

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