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Mixed Finite Elements in $\mathbb{R}^3$

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Summary. We present here some new families of non conforming finite elements in $\mathbb{R}^3$. These two families of finite elements, built on tetrahedrons or on cubes are respectively conforming in the spaces $H(\mathrm{curl})$ and $H(\mathrm{div})$. We give some applications of these elements for the approximation of Maxwell's equations and equations of elasticity.

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First, we introduce some notations:

K	is a tetrahedron or a cube, the *volume* of which is $\int_K dx$;
∂K	is its boundary;
f	is a face of K, the *surface* of which is $\int_f d\gamma$;
a	is an edge, the length of which is $\int_a ds$;
$L^2(K)$	is the usual Hilbert space of square integrable functions defined on K;
$H^m(K)$	$=\{\phi\in L^2(K);\ \partial^\alpha\phi\in L^2(K);\ \lvert\alpha\rvert\leqq m\}$, where $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index; $\lvert\alpha\rvert=\alpha_1+\alpha_2+\alpha_3$;
$\mathrm{curl}\,u$	$=\nabla\wedge u$, (defined by using the distributional derivative) for $u=(u_1,u_2,u_3)$; $u_i\in L^2(K)$;
$H(\mathrm{curl})$	$=\{u\in(L^2(K))^3;\ \mathrm{curl}\,u\in(L^2(K))^3\}$;
$\mathrm{div}\,u$	$=\nabla\cdot u$;
$H(\mathrm{div})$	$=\{u\in(L^2(K))^3;\ \mathrm{div}\,u\in L^2(K)\}$;
$D^k u$	is the k-th differential operator associated to u, which is a $(k+1)$-multilinear operator acting on $\mathbb{R}^3$;
k	is an index;
$\mathbb{P}_k$	is the linear space of polynomials, the degree of which is less or equal to k;
σ_k	is the group of all permutations of the set $\{1,2,\dots,k\}$;
c or c_ε	will stand for any constant depending possibly on ε.

1. Study of the Finite Elements Built on Tetrahedrons

1.1 Some Space of Polynomials on $\mathbb{R}^n$

We introduce here some linear spaces of polynomials which will be used later to build the conforming finite elements in $H(\mathrm{curl})$ or in $H(\mathrm{div})$. We consider here the case of $\mathbb{R}^n$ though hereafter we will only use $n=2$ and $n=3$.

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Definition 1. For $u\in(C^k(\mathbb{R}^n))^n$, $\varepsilon^k u$ is the $k+1$ multi-linear form obtained from $D^k u$ by the following symmetrization:

$$\forall\,\xi_1,\xi_2,\dots,\xi_k,\xi_{k+1}\in\mathbb{R}^n,$$

$$\varepsilon^k u(\xi_1,\xi_2,\dots,\xi_k,\xi_{k+1})=\sum_{\sigma\in\sigma_{k+1}}\frac{1}{(k+1)!}D^k u(\xi_{\sigma(1)},\dots,\xi_{\sigma(k)},\xi_{\sigma(k+1)}). \tag{1}$$

Remark. When $k=1$, εu is the following bilinear symmetric form:

$$(\varepsilon u)_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right).$$

εu is the strain tensor of elasticity when $n=3$. ■

Let us prove a property of the operator $\varepsilon^k u$. Let $e_1,\dots,e_n$ be the basis in $\mathbb{R}^n$. We have

Lemma 1. *For $\alpha=(\alpha_1,\dots,\alpha_n)$, a multi-index, $|\alpha|=k+1$, we have*

$$\begin{aligned}\frac{\partial^{k+1}u_j}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}=\sum_{\substack{i=1\\ i\neq j}}\Bigg[\alpha_i\frac{\partial}{\partial x_i}\varepsilon^k u(\overbrace{e_1,\dots,e_1}^{\alpha_1\text{-times}};\dots;\overbrace{e_i,\dots,e_i}^{\alpha_i-1\text{-times}};\dots;\overbrace{e_j,\dots,e_j}^{\alpha_j+1\text{-times}};\dots;\overbrace{e_n,\dots,e_n}^{\alpha_n\text{-times}})\Bigg]\\ -(k-\alpha_j)\frac{\partial}{\partial x_j}\varepsilon^k u(\overbrace{e_1,\dots,e_1}^{\alpha_1\text{-times}};\dots;\overbrace{e_n,\dots,e_n}^{\alpha_n\text{-times}}).\end{aligned}\tag{2}$$

Proof. Using definition (1), we obtain

$$\begin{aligned}*\quad&\frac{\partial}{\partial x_i}\varepsilon^k u(\overbrace{e_1,\dots,e_1}^{\alpha_1\text{-times}};\dots;\overbrace{e_i,\dots,e_i}^{\alpha_i-1\text{-times}};\dots;\overbrace{e_j,\dots,e_j}^{\alpha_j+1\text{-times}};\dots;\overbrace{e_n,\dots,e_n}^{\alpha_n\text{-times}})\\ &=\frac{1}{(k+1)}\Bigg(\sum_{\substack{l=1\\ l\neq i,j}}\alpha_l\frac{\partial^{k+1}u_l}{\partial x_1^{\alpha_1}\dots\partial x_l^{\alpha_l-1}\dots\partial x_i^{\alpha_i}\dots\partial x_j^{\alpha_j+1}\dots\partial x_n^{\alpha_n}}\\ &\quad+(\alpha_{i-1})\frac{\partial^{k+1}u_i}{\partial x_1^{\alpha_1}\dots\partial x_i^{\alpha_i-1}\dots\partial x_j^{\alpha_j+1}\dots\partial x_n^{\alpha_n}}\\ &\quad+(\alpha_{j+1})\frac{\partial^{k+1}u_j}{\partial x_1^{\alpha_1}\dots\partial x_i^{\alpha_i}\dots\partial x_j^{\alpha_j}\dots\partial x_n^{\alpha_n}}\Bigg)\end{aligned}$$

$$\begin{aligned}*\quad&\frac{\partial}{\partial x_j}\varepsilon^k u(\overbrace{e_1,\dots,e_1}^{\alpha_1\text{-times}};\dots;\overbrace{e_j,\dots,e_j}^{\alpha_j\text{-times}};\dots;\overbrace{e_n,\dots,e_n}^{\alpha_n\text{-times}})\\ &=\frac{1}{(k+1)}\Bigg(\sum_{\substack{l=1\\ l\neq j}}\alpha_l\frac{\partial^{k+1}u_l}{\partial x_1^{\alpha_1}\dots\partial x_l^{\alpha_l-1}\dots\partial x_j^{\alpha_j+1}\dots\partial x_n^{\alpha_n}}\\ &\quad+\alpha_j\frac{\partial^{k+1}u_j}{\partial x_1^{\alpha_1}\dots\partial x_j^{\alpha_j}\dots\partial x_n^{\alpha_n}}\Bigg).\end{aligned}$$

Just adding, we obtain equality (2). ■

Now, we introduce the following linear space

$$\mathscr{R}^k = \{u \in (C^{k+1}(\mathbb{R}^n))^n;\ \varepsilon^k u = 0\}.$$

Lemma 1 shows that $\mathscr{R}^k \subset (\mathbb{P}_k)^n$. It is clear that $(\mathbb{P}_{k-1})^n \subset \mathscr{R}^k$.

Definition 2. We define

$$\mathscr{R}^k = \{u \in (\mathbb{P}_k)^n;\ \varepsilon^k u = 0\}.$$

S^k is the space of homogeneous polynomials of degree k contained in $\mathscr{R}^k$.

Examples. 1) $n=2$

a) $k=1,\ \varepsilon^1 u = 0 \Leftrightarrow \dfrac{\partial u_1}{\partial x_1} = 0,\ \dfrac{\partial u_2}{\partial x_2} = 0,\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} = 0,$

$$\mathscr{R}^1 = \mathscr{R} = \left\{ u = \alpha \oplus \beta \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix};\ \alpha \in (\mathbb{P}_0)^2,\ \beta \in \mathbb{P}_0 \right\}.$$

b) $k=2,\ \varepsilon^2 u = 0 \Leftrightarrow \dfrac{\partial^2 u_1}{\partial x_1^2} = 0,\ \dfrac{\partial^2 u_2}{\partial x_2^2} = 0,\ \dfrac{\partial^2 u_1}{\partial x_2^2} + 2\dfrac{\partial^2 u_2}{\partial x_1 \partial x_2} = 0,$

$$\frac{\partial^2 u_2}{\partial x_1^2} + 2\frac{\partial^2 u_1}{\partial x_1 \partial x_2} = 0,$$

$$\mathscr{R}^2 = (\mathbb{P}_1)^2 \oplus S^2;$$

S^2 is a two-dimensional space generated by the two vectors

$$\begin{bmatrix} x_2^2 \\ -x_1 x_2 \end{bmatrix} \text{ and } \begin{bmatrix} -x_1 x_2 \\ x_1^2 \end{bmatrix}.$$

Therefore, the dimension of $\mathscr{R}^2$ is 8.

2) $n=3$

a) $k=1,\ \varepsilon^1 u = 0 \Leftrightarrow \dfrac{\partial u_i}{\partial x_i} = 0 \quad (i=1,2,3);$

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = 0 \quad (i \neq j,\ i,j=1,2,3),$$

$$\mathscr{R}^1 = \mathscr{R} = \left\{ u = \alpha + \beta \wedge r;\ \alpha \in (\mathbb{P}_0)^3;\ \beta \in (\mathbb{P}_0)^3;\ r = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\}, \qquad \dim \mathscr{R} = 6.$$

b) $k=2,\ \varepsilon^2 u = 0 \Leftrightarrow \dfrac{\partial^2 u_i}{\partial x_i^2} = 0 \quad (i=1,2,3); \qquad \dfrac{\partial^2 u_i}{\partial x_j^2} + 2\dfrac{\partial^2 u_j}{\partial x_i \partial x_j} = 0 \quad (i \neq j);$

$$\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + \frac{\partial^2 u_2}{\partial x_1 \partial x_3} + \frac{\partial^2 u_3}{\partial x_1 \partial x_2} = 0,$$

which are 10 relations,

$$\mathscr{R}^2 = (\mathbb{P}_1)^3 \oplus S^2;$$

S^2 is a space of dimension 8 generated by the following basis:

$$\begin{bmatrix} x_2^2 \\ -x_1x_2 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ -x_2x_3 \\ x_2^2 \end{bmatrix}; \begin{bmatrix} -x_1x_2 \\ x_1^2 \\ 0 \end{bmatrix}; \begin{bmatrix} -x_1x_3 \\ 0 \\ x_1^2 \end{bmatrix}; \begin{bmatrix} x_3^2 \\ 0 \\ -x_1x_3 \end{bmatrix}; \begin{bmatrix} 0 \\ x_3^2 \\ -x_2x_3 \end{bmatrix}; \begin{bmatrix} x_2x_3 \\ -x_1x_3 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ x_1x_3 \\ -x_1x_2 \end{bmatrix}.$$

$\dim \mathscr{R}^2 = 20$.

Remark. Let Ω be a bounded regular domain of $\mathbb{R}^n$. In the case $k=1$, it results from Lemma 1 that

$$\|\varepsilon u\|_{(L^2(\Omega))/(n(n+1))/2} = \left(\sum_{i,j=1}^{n} \int_\Omega (\varepsilon(u))^2 \, dx \right)^{\frac{1}{2}}$$

is a norm on $(H^1(\Omega))^n/\mathscr{R}$, where $\mathscr{R}$ stand for $\mathscr{R}^1$ and are the so-called rigid body displacements. It can be shown that in fact this norm is equivalent to the usual one and this is called Korn's inequality. Using the proof of Korn's inequality given in Duvault-Lions [4], we obtain an equivalent result for $k \geqq 1$ (N is the dimension of the space of $k+1$ symmetric linear form):

$$\|\varepsilon^k u\|_{(L2(\Omega))^N} \geqq \alpha \, \|u\|_{(H^k(\Omega))^n/\mathscr{R}^k}; \qquad \alpha > 0. \quad \blacksquare \tag{3}$$

We shall now give some properties of the operator $\varepsilon^k u$ and of the space $\mathscr{R}^k$. The following lemmas will be useful for that purpose. Let us define $\operatorname{curl} u$ by

$$\operatorname{curl} u_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}; \qquad i \neq j; \; 1 \leqq i, j \leqq n.$$

Lemma 2. *We have*

$$\operatorname{curl} u \in (\mathbb{P}_{k-2})^{(n(n-1))/2}$$

implies

$$\mathbb{D}^k u = \varepsilon^k u.$$

Proof. Let α_i be a family of $k+1$ index with $1 \leqq \alpha_i \leqq$ n. We have

$$\begin{aligned}
&(D^k u - \varepsilon^k u)(e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_k}, e_{\alpha_{k+1}}) \\
&= \frac{1}{k+1} \sum_{i=1}^{k} [D^k u(e_{\alpha_1}, \ldots, e_{\alpha_i}, \ldots, e_{\alpha_{k+1}}) \\
&\qquad - D^k u(e_{\alpha_1}, \ldots, e_{\alpha_{i-1}}, e_{\alpha_{k+1}}, \ldots, e_{\alpha_n}, e_{\alpha_i})] \\
&= \frac{1}{k+1} \sum_{i=1}^{k} \frac{\partial^{k-1}}{\partial x_{\alpha_1} \ldots \partial x_{\alpha_{i-1}} \partial x_{\alpha_{i+1}} \ldots \partial x_{\alpha_k}} \left(\frac{\partial u_{\alpha_{k+1}}}{\partial x_{\alpha_i}} - \frac{\partial u_{\alpha_i}}{\partial x_{\alpha_{k+1}}} \right). \quad \blacksquare
\end{aligned}$$

Lemma 3. *We have*

$$\operatorname{curl} u = 0 \quad \textit{and} \quad \varepsilon^k u = 0$$

that are equivalent to

$$u = \operatorname{grad} \phi \quad \text{and} \quad \phi \in \mathbb{P}_k.$$

Proof. From Lemma 2, we know that $\operatorname{curl} u = 0$ implies $D^k u = \varepsilon^k u = 0$, so that $u \in (\mathbb{P}_{k-1})^n$. But, $\operatorname{curl} u = 0$ implies $u = \operatorname{grad} \phi$. ■

Now, we give a characterization of S^k.

Proposition 1. *We have $u \in S^k$ if and only if u is homogeneous, of degree k and*

$$r \cdot u = \sum_{i=1}^{n} x_i u_i \equiv 0.$$

Proof. Since u is homogeneous, we have, by using Euler's identity,

$$D^k u(\underbrace{r, \ldots, r}_{k\text{-times}}) = k!\, u,$$

and multiplying by r in the scalar product, we obtain

$$\varepsilon^k u(\underbrace{r, \ldots, r, r}_{(k+1)\text{-times}}) = k!\, u \cdot r.$$

Now, $\varepsilon^k u = 0$ implies $u \cdot r \equiv 0$. Conversely, $u \cdot r \equiv 0$ implies

$$\varepsilon^k u(r, \ldots, r) \equiv 0.$$

But

$$r = \sum_{i=1}^{n} x_i e_i$$

and by developing, we obtain a homogeneous polynomial of degree $k+1$ which is zero if and only if $\varepsilon^k u = 0$. ■

Suppose that we have an affine transformation on $\mathbb{R}^n$ defined by

$$x = B\hat{x} + b; \qquad B \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n);\ b \in \mathbb{R}^n. \tag{4}$$

The corresponding vectors are respectively denoted by u and $\hat{u}$. We will transform vectors like gradients, i.e.

$$u = (\mathbb{B}^*)^{-1} \hat{u}. \tag{5}$$

We have

Proposition 2. *$\mathscr{R}^k$ is invariant under an affine transformation if we transform u by using formula* (5).

Proof. It is easy to check that

$$D^k \hat{u}(\hat{\xi}_1, \ldots, \hat{\xi}_k, \hat{\xi}_{k+1}) = D^k u(B\hat{\xi}_1, B\hat{\xi}_2, \ldots, B\hat{\xi}_k, B\hat{\xi}_{k+1}),$$

and this implies

$$\varepsilon^k \hat{u}(\hat{\xi}_1, \ldots, \hat{\xi}_k, \hat{\xi}_{k+1}) = \varepsilon^k u(B\hat{\xi}_1, B\hat{\xi}_2, \ldots, B\hat{\xi}_k, B\hat{\xi}_{k+1}),$$

so that

$$\varepsilon^k \hat{u} = 0 \Leftrightarrow \varepsilon^k u = 0. \quad ■$$

Lemma 4. *$\mathscr{R}^k$ is a linear space of dimension*

$$N_k = \frac{(k+n)(k+n-1)\dots(k+2)k}{(n-1)!}$$

and, in the case $n=3$,

$$N_k = \frac{(k+3)(k+2)k}{2}.$$

Proof. The number of independent relations imposed by $\varepsilon^k u = 0$ is the dimension of the spaces of polynomials of degree $(k-1)$ in $(n-1)$ variables, that is $\binom{n+k}{n-1}$. So that we have

$$N_k = n\binom{n+k}{n} - \binom{n+k}{n-1}. \quad \blacksquare$$

We now introduce another space of polynomials which will be used to build finite elements for the approximation of $H(\mathrm{div})$.

Definition 3. We define

$$\mathbb{D}^k = (\mathbb{P}_{k-1})^n \oplus \tilde{\mathbb{P}}_{k-1} \cdot r$$

where $\tilde{\mathbb{P}}_{k-1}$ is the space of homogeneous polynomials of degree $k-1$.

Let us consider an affine transformation on $\mathbb{R}^n$. It is clear that $\mathbb{D}^k$ is not invariant if we use the transformation (5) on the vectors. We will transform vectors by

$$u = B\hat{u}. \tag{6}$$

Lemma 5. *$\mathbb{D}^k$ is invariant in an affine transformation if we use formula* (6).

Proof. The proof is easy and does not need to be detailed. $\blacksquare$

The dimension of $\mathbb{D}^k$ is

$$n\binom{n+k-1}{n} + \binom{n+k-2}{n-1};$$

$$M_k = \frac{(n+k)(n+k-2)(n+k-3)\dots(k+1)k}{(n-1)!},$$

and in the case $n=3$

$$M_k = \frac{(3+k)(k+1)k}{2}; \qquad \begin{cases} k=1 : M_1 = 4; \\ k=2 : M_2 = 15. \end{cases}$$

Remark. $\tilde{\mathbb{D}}^k$ will be the homogeneous polynomials of degree k of $\mathbb{D}^k$. We can respectively characterize the spaces $\mathbb{D}^k$ and $\tilde{\mathbb{D}}^k$:

$$\tilde{\mathbb{D}}^k = \{u \in (\tilde{\mathbb{P}}_k)^3;\ u \cdot \xi = 0;\ \forall \xi \in \mathbb{R}^n,\ \xi \cdot r = 0\},$$

and in the case $n=3$,

$$\tilde{\mathbb{D}}^k=\{u\in(\tilde{\mathbb{P}}_k)^3;\ u\wedge r=0\}.$$

We have also

$$\mathbb{D}^k=\{u\in(C^\infty)^3;\ \mathbb{D}^k u(\xi_1,\xi_2,\ldots,\xi_k,\xi_{k+1})=0,\ \forall\,\xi_i\in\mathbb{R}^n \text{ such that } \xi_{k+1}\cdot\xi_i=0,\ \forall\, i=1,\ldots,k\}.$$

We now prove a proposition which holds only in the case $n=3$. We call $\mathbb{D}_0^k$ the subspace of polynomials of $\mathbb{D}^k$ such that $\operatorname{div} u=0$. For $\tilde{p}\in\tilde{\mathbb{P}}_{k-1}$, we have by Euler's identity,

$$\operatorname{div}(\tilde{p}\,r)=(k+2)\,\tilde{p},$$

so that

$$\operatorname{div} u=0 \Rightarrow \tilde{p}=0 \Rightarrow u\in(\mathbb{P}_{k-1})^n.$$

That is

$$\mathbb{D}_0^k=\{u\in(\mathbb{P}_{k-1})^n,\ \operatorname{div} u=0\}.$$

Let us define $\tilde{\mathbb{D}}_0^k$:

$$\tilde{\mathbb{D}}_0^k=\{u\in(\tilde{\mathbb{P}}_{k-1})^n,\ \operatorname{div} u=0.$$

We have

Proposition 3. *When $n=3$, the operator* curl *is an isomorphism of S^k onto $\tilde{\mathbb{D}}_0^k$.*

Proof. The operator curl is a linear operator of S^k into $\tilde{\mathbb{D}}_0^k$. By Lemma 3, $u\in S^k$, $\operatorname{curl} u=0$, imply $u=0$. So that this operator is injective. To conclude, it is enough to prove that both spaces S^k and $\tilde{\mathbb{D}}_0^k$ have the same dimension:

$$\dim S^k=3\binom{k+2}{2}-\binom{k+3}{2}=k(k+2);$$

$$\dim\tilde{\mathbb{D}}_0^k=3\binom{k+1}{2}-\binom{k}{2}\qquad=k(k+2).\quad\blacksquare$$

Proposition 4. *When $n=3$, the operator* curl *is surjective from $\mathscr{R}^k$ onto $\mathbb{D}_0^k$ and maps $\mathbb{D}^k$ into $\mathscr{R}^{k-1}$.*

Proof. The first part of the proposition results from Proposition 2. For the second part, it is enough to take the homogeneous term

$$\operatorname{curl}(\tilde{p}_{k-1}r)=\operatorname{grad}\tilde{p}_{k-1}\wedge r,$$

so that

$$r\cdot(\operatorname{curl}(\tilde{p}_{k-1}r))=0$$

which is a characterization of elements in S^{k-1}. $\blacksquare$

Remarks. The operator div is bijective from $\tilde{\mathbb{D}}^k$ onto $\tilde{\mathbb{P}}_{k-1}$ since

$$\operatorname{div}(\tilde{p}\,r)=(k+n-1)\,\tilde{p};\qquad\forall\,\tilde{p}\in\tilde{\mathbb{P}}_{k-1}.$$

Let G^k be the homogeneous polynomials of degree k that are the gradients of a homogeneous polynomial of degree $k+1$. Euler's identity proves that

$$\phi \in \tilde{\mathbb{P}}_{k+1} \Rightarrow \phi = (k+1)(r \cdot \operatorname{grad} \phi),$$

so that, it results from Proposition 1

$$\mathsf{G}^k \cap S^k = 0;$$

$$\dim \mathsf{G}^k + \dim S^k = \frac{(k+2)(k+3)}{2} + k(k+2) = \frac{3(k+2)(k+1)}{2} = \dim (\tilde{\mathbb{P}}_k)^3.$$

This proves that

$$\mathsf{G}^k \oplus S^k = (\tilde{\mathbb{P}}_k)^3.$$

1.2. Finite Elements in $H(\text{curl})$

A finite element is defined by the following:

K a domain which will be in our case a tetrahedron;
P a space of polynomials on K of dimension N;
A a set of N degrees of freedom which are linear functionals acting on P.

This finite element is said to be unisolvent if

$$\forall p \in \mathsf{P}, \quad \alpha_i(p) = 0; \quad \forall \alpha_i \in \mathsf{A} \Rightarrow p \equiv 0.$$

We work here with vectorial finite elements on $\mathbb{R}^3$ so that P is a subspace of $(C^\infty(\bar{K}))^3$. Then, for any $u \in (C^\infty(\bar{K}))^3$, we can define a unique interpolate Πu such that

$$\alpha_i(u - \Pi u) = 0, \quad \forall \alpha_i \in \mathsf{A}; \ \Pi u \in \mathsf{P}.$$

A finite element is said to be conforming in $H(\text{curl})$ if the following property is satisfied

[Let K_1 and K_2 be two elements with a common face f and let $u \in (C^\infty(\overline{K_1 \cup K_2}))^3$. Then, the functions v defined by $v = \Pi_1 u$ on K_1, $v = \Pi_2 u$ on K_2, belong to $H(\text{curl}, \mathrm{K}_1 \cup \mathrm{K}_2)$.

Using the theory of distributions and Stokes' formula, we can verify

Lemma 6. *A finite element is conforming in* $H(\text{curl})$ *if and only if*

- *the tangential components of* $\Pi_1 u$ *and* $\Pi_2 u$ *are the same on* f, *or equivalently,*
- *for any* $\alpha_i \in \mathsf{A}$ *defined only on* f, $\alpha_i(p) = 0$, $p \in \mathsf{P}$, *implies* $n \wedge p = 0$ *on* f *where* n *is the normal to the face* f.

Let us now introduce a family of conforming finite elements in $H(\text{curl})$:

K is a tetrahedron;
P $= \mathscr{R}^k$ (see Definition 2).

Definition 4. We have the following *degrees of freedom:*

1) $\int_a u \cdot t\, q\, ds; \quad \forall q \in \mathbb{P}_{k-1}$,

where t is the unit vector directed along the edge a. These are $6k$ degrees of freedom;

2) $\int_f (u \wedge n \cdot q)\, d\gamma; \quad \forall q \in (\mathbb{P}_{k-2})^2$,

these are $4k(k-1)$ degrees of freedom;

3) $\int_K u \cdot q\, dx; \quad \forall q \in (\mathbb{P}_{k-3})^3$,

these are $\frac{k(k-1)(k-2)}{2}$ degrees of freedom. ■

We check that the total number of degrees is N_k:

$$N_k = \frac{k(k+2)(k+3)}{2}.$$

Before stating that this finite element is conforming in $H(\text{curl})$ and unisolvent, we prove

Lemma 7. *If* $u \in (\mathbb{P}_k)^3$ *and*

$$\alpha^i(u) = 0, \qquad \forall \alpha_i \in \mathsf{A},$$

then

$$\text{curl}\, u = 0.$$

Proof. On each face f of the tetrahedron K, we can define operators, whose expressions are given in local orthonormal coordinates ξ_1, ξ_2:

$$\overrightarrow{\text{curl}}_f\, q = \begin{cases} -\dfrac{\partial q}{\partial \xi_2} \\ \dfrac{\partial q}{\partial \xi_1} \end{cases} \tag{7}$$

and for a vector tangential to this face $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, we have

$$\text{curl}_f\, u = \frac{\partial u_1}{\partial \xi_2} - \frac{\partial u_2}{\partial \xi_1}. \tag{8}$$

Stokes' formula gives

$$\int_f (\overrightarrow{\text{curl}}_f\, q \cdot u - q\, \text{curl}_f\, u)\, d\gamma = \int_{\partial f} u \cdot t\, q\, ds. \tag{9}$$

We apply this equality with q defined on f, $q \in \mathbb{P}_{k-1}$ and the tangential component of u, u_f. It is easy to check that (with a correct orientation of the faces)

$$\operatorname{curl}_f u_f = \operatorname{curl} u \cdot n,$$

so that equality (9) and the use of degrees of freedom of types (1) and (2) yields to

$$\int_f q \operatorname{curl} u \cdot n \, d\gamma = 0,$$

and so (since $\operatorname{curl} u \cdot n \in \mathbb{P}_{k-1}$),

$$\operatorname{curl} u \cdot n \equiv 0 \quad \text{on each face.} \tag{10}$$

Using Stokes' formula in K, we have

$$\int_K (u \cdot \operatorname{curl} q - q \operatorname{curl} u) \, dx = \int_{\partial K} u \wedge n \cdot q \, d\gamma. \tag{11}$$

Using then the degrees of freedom of types (2) and (3), we have, for $q \in (\mathbb{P}_{k-2})^3$,

$$\int_K \operatorname{curl} u \cdot q \, dx = 0, \qquad \forall q \in (\mathbb{P}_{k-2})^3. \tag{12}$$

Using an affine transformation and formula (5), we can associate to K the tetrahedron $\hat{K}$ the vertices of which are

$$(0,0,0); \quad (1,0,0); \quad (0,1,0); \quad (0,0,1),$$

and by (10) and (12), we obtain

$$\begin{aligned} &\operatorname{curl} \hat{u} \cdot \hat{n} = 0 \qquad \text{on each face} \\ &\int_{\hat{K}} \operatorname{curl} \hat{u} \cdot \hat{q} \, dx = 0, \quad \forall \hat{q} \in (\mathbb{P}_{k-2})^3. \end{aligned} \tag{13}$$

This implies

$$\begin{aligned} &\hat{v}_1 = (\operatorname{curl} \hat{u})_1 = \hat{x}_1 \hat{\psi}_1, &&\hat{\psi}_1 \in \mathbb{P}_{k-2}, \\ &\hat{v}_2 = \hat{x}_2 \hat{\psi}_2, &&\hat{\psi}_2 \in \mathbb{P}_{k-2}, \\ &\hat{v}_3 = \hat{x}_3 \hat{\psi}_3, &&\hat{\psi}_3 \in \mathbb{P}_{k-2}, \end{aligned}$$

and setting $\hat{q} = (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3)$ in (13), we finally obtain

$$\operatorname{curl} \hat{u} = 0$$

which implies

$$\operatorname{curl} u = 0. \quad \blacksquare$$

Proposition 5. *If $u \in \mathscr{R}^k$ and for one face f of K, we have*

$$\int_f u \wedge n \cdot q \, d\gamma = 0, \qquad \forall q \in (\mathbb{P}_{k-2})^2 \tag{14}$$

$$\int_a u \cdot t \cdot q \, ds = 0, \qquad \forall q \in \mathbb{P}_{k-1} \tag{15}$$

for the three edges contained in f. Then

$$u_f \equiv 0 \quad on\ f,$$

where u_f is the tangential part of u.

Remark. This proposition proves that our finite element is conforming in $H(\mathrm{curl})$, but first we need prove that it is unisolvent.

Proof. We remark that $u \in \mathscr{R}^k$ if and only if

$$\varepsilon^k u(\overbrace{r, \ldots, r}^{(k+1)\text{-times}}) = 0$$

which is also equivalent to

$$D^k u(\overbrace{t, \ldots, t}^{(k+1)\text{-times}}) = 0$$

for any unit vector t, i.e. also $u \cdot t$ *is of degree* $k-1$ *on each line directed along* t. So that (15) implies

$$u \cdot t = 0 \quad \text{on each edge belonging to } f. \tag{16}$$

Now, using again (9), we obtain

$$\mathrm{curl}_f\, u_f = 0 \quad \text{on } f. \tag{17}$$

Using vectors which are tangential to the face f in the definition of $\mathscr{R}^k$ and ε^k, it can be easily checked that $u_f \varepsilon \mathscr{R}^k(f)$, where $\mathscr{R}^k(f)$ is the corresponding space of polynomials with two variables defined on the face f. Now, Lemma 3 implies

$$u_f = \mathrm{grad}_f\, \phi, \qquad \phi \in \mathbb{P}_k(f).$$

The function ϕ is defined up to an additive constant and using (16), we see that we can choose this constant such that

$$\phi|_{\partial f} = 0.$$

Then, if $\lambda_1, \lambda_2, \lambda_3$, are the barycentric coordinates in the triangle f, we have

$$\phi = \lambda_1 \lambda_2 \lambda_3 \psi; \qquad \psi \in \mathbb{P}_{k-3}, \tag{18}$$

and finally

$$\int_f u_f \cdot q\, d\gamma = -\int_f \phi\, \mathrm{div}\, q\, d\gamma = 0; \qquad \forall\, q \in (\mathbb{P}_{k-2})^2, \tag{19}$$

But the operator div is surjective from $(\mathbb{P}_{k-2})^2$ onto $\mathbb{P}_{k-3}$, so that we can choose

$$\mathrm{div}\, q = \psi$$

and then, (18)–(19) imply $\psi = 0$, i.e.

$$u_f \equiv 0 \quad \text{on } f. \quad \blacksquare$$

Theorem 1. *The finite element defined on the tetrahedron K by $\mathsf{P}=\mathscr{R}^k$ (Definition 2) and A (Definition 4) is unisolvent and conforming in the Hilbert space $H(\text{curl})$.*

Proof. It is sufficient to prove that

$$\alpha_i(u)=0; \qquad \alpha_i\in \mathsf{A} \Rightarrow u=0.$$

But, from Lemma 7, we know that this implies

$$\operatorname{curl} u=0.$$

Using then Lemma 3, we obtain

$$u=\operatorname{grad}\phi; \qquad \phi\in \mathbb{P}_k.$$

Proposition 5 then proves that $u_f=0$ on each face which is also equivalent to (choosing the constant in ϕ)

$$\phi|_f=0.$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, be the barycentric coordinates in K. We have

$$\phi=\lambda_1\lambda_2\lambda_3\lambda_4\psi; \qquad \psi\in \mathbb{P}_{k-4}. \tag{20}$$

Using Green's formula, we have

$$\int_K \phi \operatorname{div} q\, dx = -\int_K q\cdot u\, dx + \int_{\partial K} \phi q\cdot n\, d\gamma = 0, \qquad \forall q\in(\mathbb{P}_{k-3})^3. \tag{21}$$

We can choose $q\in(\mathbb{P}_{k-3})^3$ such that

$$\operatorname{div} q=\psi\in \mathbb{P}_{k-4}$$

so that (20) and (21) imply $\psi=0$ and $u=0$. ∎

Thus we can give a new version of Lemma 7:

Corollary 1. *If Πu is the interpolate of u, then we have*

$$\operatorname{curl}(u-\Pi u)=0 \qquad \text{if } u\in(\mathbb{P}_k)^3.$$

Remark. This family of finite elements is invariant under affine transformations provided that we use formula (5) to transform the vectors u. It is easy to check that we have

$$\widehat{\Pi u}=\hat{\Pi}\hat{u},$$

where

$$\hat{u}=B^*u, \qquad \widehat{\Pi u}=B^*(\Pi u).$$

However, we must be careful we choose degrees of freedom that are not completely invariant since the images of the unit vectors $\hat{t}$ and $\hat{n}$ are not unit vectors. It can be convenient to use invariant degrees of freedom. For instance, when $k=1$, we can use the six moments $\dfrac{1}{\text{length}(a)}\int_a u\cdot a\, ds$, where a is the vector with ends the two vertices defining a. Or equivalently, $u \cdot a(M_a)$, where M_a is the middle point of a. ■

In what concerns the error of interpolation, we have the following result. Let h be the diameter of K and ρ the radius of the largest inscribed sphere. Suppose that we have a family of K such that

$$\frac{h}{\rho} \leqq c,$$

where c is a fixed constant.

Theorem 2. *Let Π be the interpolation operator associated to the finite element $(K, \mathscr{R}^k, \mathsf{A}_k)$; we have*

$$\|u - \Pi u\|_{H(\mathrm{curl}, K)} \leqq c\, h^k |u|_{(H^{k+1}(K))^3}. \tag{22}$$

Proof. Classically, we use a reference finite element $(\hat{K}, \mathscr{R}^k, \hat{\mathsf{A}}_k)$ and the transformation

$$x = B\hat{x} + b,$$
$$u = (B^*)^{-1}\hat{u}.$$

In order to obtain convenient formulas of transformation on curl u, it is better to write it as the following skew-matrix

$$\operatorname{curl} u = \begin{pmatrix} 0 & \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_3}{\partial x_1} - \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_1}{\partial x_2} - \dfrac{\partial u_2}{\partial x_1} & 0 & \dfrac{\partial u_3}{\partial x_2} - \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_2}{\partial x_3} - \dfrac{\partial u_3}{\partial x_2} & 0 \end{pmatrix}. \tag{23}$$

We have now

$$\operatorname{curl} u = (B^*)^{-1} \circ \operatorname{curl} \hat{u} \circ (B)^{-1} \tag{24}$$

and the norm of the matrix is equivalent to the usual norm. We have

$$u - \Pi u = (B^*)^{-1}(\hat{u} - \hat{\Pi}\hat{u}),$$

so that

$$\|u - \Pi u\|_{(L^2(K))^3} \leqq \frac{c\,|\det B|}{h} \|\hat{u} - \hat{\Pi}\hat{u}\|_{(L^2(\hat{K}))^3}.$$

Using the fact that $\hat{\Pi}\hat{u} = \hat{u}$ when $\hat{u} \in (\mathbb{P}_{k-1})^3$, we obtain (see P.G. Ciarlet [3] for instance),

$$\|u - \Pi u\|_{(L^2(K))^3} \leqq \frac{c\,|\det B|}{h} |\hat{u}|_{k,\hat{K}}, \quad (\text{and } |\hat{u}|_{2,\hat{K}} \text{ if } k=1),$$

where $|\ |_{k,\hat{K}}$ is the semi-norm of $(H^k(\hat{K}))^3$ which corresponds to the derivatives of order k. Using now

$$D^k \hat{u}(\hat{\xi}_1, \dots, \hat{\xi}_k) = D^k u(B\hat{\xi}_1, \dots, B\hat{\xi}_k),$$

we obtain

$$\|u - \Pi u\|_{(L^2(K))^3} \leqq c h^k |u|_{(H^k(K))^3}.$$

Using (23) and (24), we obtain in the same way

$$\|\operatorname{curl}(u - \Pi u)\|_{L^2(K))^3} \leqq \frac{c\,|\det B|}{h} \|\operatorname{curl}(\hat{u} - \hat{\Pi}\hat{u})\|_{(L^2(\hat{K}))^3}.$$

$\int_{\hat{K}} \operatorname{curl}(\hat{u} - \hat{\Pi}\hat{u}) \cdot \phi\, dx$ is a linear form in $\hat{u}$ when $\phi \in (L^2(\hat{K}))^3$ and we have

$$\left|\int_{\hat{K}} \operatorname{curl}(\hat{u} - \hat{\Pi}\hat{u}) \cdot \phi\, dx\right| \leqq c\, \|\phi\|_{(L^2(K))^3} \|\hat{u}\|_{(H^{k+1}(\hat{K}))^3}.$$

But when $\hat{u} \in (\mathbb{P}_k)^3$, this linear form is zero. So that, it is continuous on the quotient space $(H^{k+1}(\hat{K})/\mathbb{P}_k)^3$. Thus we have

$$\left|\int_{\hat{K}} \operatorname{curl}(\hat{u} - \hat{\Pi}\hat{u}) \cdot \phi\, dx\right| \leqq c\, \|\phi\|_{(L^2(\hat{K}))^3} |\hat{u}|_{(H^{k+1}(\hat{K}))^3},$$

which is equivalent to

$$\|\operatorname{curl}(\hat{u} - \hat{\Pi}\hat{u})\|_{(L^2(\hat{K}))^3} \leqq c\, |\hat{u}|_{(H^{k+1}(\hat{K}))^3},$$

and using

$$\|\hat{u}\|_{(H^{k+1}(\hat{K}))^3} \leqq \frac{c\, h^{k+2}}{|\det B|} |u|_{(H^{k+1}(K))^3},$$

we obtain the final result. ■

1.3. Finite Elements in $H(\operatorname{div})$

We introduce here a family of vectorial elements that are conforming in $H(\operatorname{div})$. We have

Lemma 8. *A finite element* $(K, \mathsf{P}, \mathsf{A})$ *is conforming in* $H(\operatorname{div})$ *if it satisfies for any* $\alpha_i \in \mathsf{A}$, *defined only on the face* f *with normal* n: $u \in P$, $\alpha_i(u) = 0$, *implies*

$$u \cdot n = 0 \quad \textit{on this face.}$$

Definition 5. For any index k, we define a finite element in the following way;

K is a tetrahedron;

P $= \mathbb{D}^k$ (Cf. Definition 3);

A are the following momenta:

(1) $\int_f u \cdot n\, q\, d\gamma$, $\forall q \in \mathbb{P}_{k-1}$

(2) $\int_K u \cdot q\, dx$, $\forall q \in (\mathbb{P}_{k-2})^3$. ■

The total number of degrees of freedom is

$$4\frac{k(k+1)}{2}+3\frac{(k+1)k(k-1)}{6}=\frac{(k+3)(k+1)k}{2},$$

equal to M_k, the dimension of $\mathbb{D}^k$. We give a lemma before proving the unisolvence.

Lemma 9. *Let* A *be given as in Definition 5. If* $u\in(\mathbb{P}_k)^3$ *and* $\alpha_i(u)=0$, $\forall\,\alpha_i\in\mathsf{A}$, *then*

$$\operatorname{div} u=0.$$

Proof. Using Green's formula, we have

$$\int_K \operatorname{div} u\, q\, dx=-\int_K u\cdot \operatorname{grad} q\, dx+\int_{\partial K} u\cdot n\, q\, d\gamma=0; \qquad \forall\, q\in\mathbb{P}_{k-1}.$$

But $\operatorname{div} u\in\mathbb{P}_{k-1}$, and this implies $\operatorname{div} u=0$. ∎

Theorem 3. *The finite element given by Definition 5 is unisolvent and conforming in* $H(\operatorname{div})$.

Proof. We shall prove that

$$u\in\mathbb{D}^k, \quad \alpha_i(u)=0; \quad \forall\,\alpha_i\in\mathsf{A} \text{ implies } u=0.$$

First, let us notice that the polynomials of $\mathbb{D}^k$ are such that the normal component to a face $u\cdot n$ is of degree $k-1$ (this is a characterization of $\mathbb{D}^k$). Thus, the degree of freedom of type (1) implies

$$u\cdot n=0 \quad \text{on each face.}$$

This is also the property of conforming finite element in $H(\operatorname{div})$. Now, using Lemma 9, we have

$$\operatorname{div} u=0.$$

But

$$u=v+pr, \quad v\in(\mathbb{P}_{k-1})^3; \quad p\in\tilde{\mathbb{P}}_{k-1}.$$

$$\operatorname{div} u=\operatorname{div} v+(k+2)p.$$

Thus, we obtain

$$p=0.$$

Using finally an affine transformation of K onto $\hat{K}$, the vertex of which are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, we have

$$\hat{u}_1=\hat{x}_1\psi_1,$$
$$\hat{u}_2=\hat{x}_2\psi_2,$$
$$\hat{u}_3=\hat{x}_3\psi_3.$$

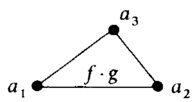
Now, choosing $q=(\psi_1,\psi_2,\psi_3)$ with degrees of freedom of type (2), we obtain

$$\psi=0 \quad\text{and}\quad u=0. \blacksquare$$

Remark. These finite elements are invariant by an affine transformation if we change the vectors u by formula (6). We can choose also invariant degrees of freedom. For instance, when $k=1$, it is more convenient to use

$$\frac{1}{\operatorname{mes}(f)}\int u\cdot v\,d\gamma=u\cdot v(g),$$

where g is the center of gravity of the face f and v is the normal (not unit) vector which is the exterior product of two edges of this face divided by $\det(B)$.



$$v=\frac{1}{\det(B)}\overrightarrow{a_1a_2}\wedge\overrightarrow{a_1a_3}$$

In what concerns the error of interpolation, we have

Theorem 4. *The finite element given by Definition 5 is such that*

$$\|u-\Pi u\|_{(L^2(K))^3}\leqq c\,h^k\,|u|_{(H^k(\Omega))^3};\tag{25}$$

$$|\operatorname{div}(u-\Pi u)|_{L^2(K)}\leqq c\,h^k\,|u|_{(H^{k+1}(\Omega))^3}.\tag{26}$$

Proof. It is very similar to the one of Theorem 2 for the relation (25) by using the fact that, if we define $\hat{u}$ by (6), i.e.

$$\hat{u}=B^{-1}u,$$

we have

$$\widehat{\Pi u}=\hat{\Pi}\,\hat{u}.$$

For (26), it results from Lemma 9 that

$$\operatorname{div}(u-\Pi u)=0,\qquad \forall\, u\in(\mathbb{P}_k)^3.$$

It is easy to verify that

$$\operatorname{div}\hat{u}=\sum_{i=1}^{3}\frac{\partial\hat{u}_i}{\partial x_i}=\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}=\operatorname{div}u.$$

Then, the proof of similar to the one of Theorem 3. ■

2. Study of Finite Elements Built on Cubes

Let us first give some notations:

$Q_{l,m,n}$ will be the polynomials in three variables (x_1,x_2,x_3) the maximum degree of which are respectively l in x_1, m in x_2, n in x_3;

$Q_{l,m}$ will be the polynomials in two variables (x_1,x_2) the maximum degree of which are respectively l in x_1, m in x_2.

Let C be the following cube:

$$\mathsf{C}=\{x\in\mathbb{R}^3;\,0\leqq x_i\leqq 1;\,i=1,2,3\}.$$

We will define our finite element on this reference cube. Using an affine transformation, it corresponds to finite elements on a non degenerate parallelipiped. First, let us define *conforming* finite elements in $H(\mathrm{curl})$.

Definition 6. For any index k, we define a finite element $(\mathsf{C}, \mathsf{P}, \mathsf{A})$ with

$$\mathsf{P}=\{u; u_1\in Q_{k-1,k,k}; u_2\in Q_{k,k-1,k}; u_3\in Q_{k,k,k-1}\}$$

A representing the following momenta:

(1) $\int_a u\cdot t\,q\,ds$; $q\in\mathbb{P}_{k-1}$, for each edge a with a tangential vector t; these are $12k$ degrees;

(2) $\int_f u\wedge n\,q\,d\gamma$; $q=(q_1,q_2)$; $q_1\in Q_{k-2,k-1}$; $q_2\in Q_{k-1,k-2}$; for each face; these are $6\times 2k(k-1)$ degrees;

(3) $\int_{\mathsf{C}} u\cdot q\,d\gamma$; $q=(q_1,q_2,q_3)$; $q_1\in Q_{k-1,k-2,k-2}$; $q_2\in Q_{k-2,k-1,k-2}$; $q_3\in Q_{k-2,k-2,k-1}$; these are $3k(k-1)^2$ degrees. ■

The total number of degrees is

$$12k+12k(k-1)+3k(k-1)^2=3k(k+1)^2$$

which is also the dimension of the space P.

Theorem 5. *The finite element given by Definition 6 is unisolvent and conforming in* $H(\mathrm{curl})$.

Proof. We first remark that on a face, for instance $x_3=0$, the tangential components of u, let them be $u_f=(u_1,u_2)$, are such that

$$u_1\in Q_{k-1,k};\qquad u_2\in Q_{k,k-1}.$$

On the restriction of an edge, the tangential component is of degree $k-1$. We have to prove that

$$u\in\mathsf{P},\quad \alpha_i(u)=0;\qquad \forall\,\alpha_i\in\mathsf{A},$$

implies

$$u=0.$$

The degrees of type (1) implies

$$u\cdot t=0\quad\text{on each edge.}$$

Now, this implies that on a face ($x_3=0$ for instance), we have

$$u_1=x_2(1-x_2)v_1;\qquad v_1\in Q_{k-1,k-2};$$

$$u_2=x_1(1-x_1)v_1;\qquad v_2\in Q_{k-2,k-1}.$$

We can then choose $q_1=v_1$, $q_2=v_2$ in degrees of type (2) and that implies $u_f=0$. This proves the conformity. Thus we know that u has the expression:

$$u_1 = x_2(1-x_2)\,x_3(1-x_3)\,w_1; \qquad w_1 \in Q_{k-1,k-2,k-2};$$
$$u_2 = x_1(1-x_1)\,x_3(1-x_3)\,w_2; \qquad w_2 \in Q_{k-2,k-1,k-2};$$
$$u_3 = x_1(1-x_1)\,x_2(1-x_2)\,w_3; \qquad w_3 \in Q_{k-2,k-2,k-1}.$$

Finally, using $q=(w_1, w_2, w_3)$ in degrees of type (3), we can deduce $u=0$. ■

In order to prove the error estimates of interpolation, we need

Lemma 10. *When* $u \in (\mathbb{P}_k)^3$,

$$\alpha_i(u)=0; \quad \forall \alpha_i \in \mathsf{A},$$

implies

$$\operatorname{curl} u = 0.$$

Proof. Using Stoke's formula, we have

$$\int_f \operatorname{curl} u \cdot n\, q\, d\gamma = -\int_f (u \wedge n) \cdot \operatorname{grad}_f q\, d\gamma + \int_{\partial f} u \cdot t\, q\, d\gamma, \qquad \forall q \in Q_{k-1,k-1}$$

and this implies (since $\operatorname{curl} u \cdot n \in \mathbb{P}_{k-1}$)).

$$\operatorname{curl} u \cdot n = 0.$$

Thus, we have

$$v = \operatorname{curl} u,$$
$$v_1 = x_1(1-x_1)\,w_1; \qquad w_1 \in \mathbb{P}_{k-3};$$
$$v_2 = x_2(1-x_2)\,w_2; \qquad w_2 \in \mathbb{P}_{k-3};$$
$$v_3 = x_3(1-x_3)\,w_3; \qquad w_3 \in \mathbb{P}_{k-3};$$

Using now

$$\int_{\mathsf{C}} \operatorname{curl} u \cdot q\, dx + \int_{\mathsf{C}} u \operatorname{curl} q\, dx = \int_{\partial \mathsf{C}} u \wedge n \cdot q\, d\gamma, \qquad \forall q \in (\mathbb{P}_{k-3})^3,$$

we obtain

$$\int_{\mathsf{C}} \operatorname{curl} u \cdot q\, dx = 0$$

and this implies

$$\operatorname{curl} u = 0. \quad ■$$

When C_h is a cube the edge of which has the length h, we obtain the following error estimate.

Theorem 6. *The finite element introduced in Definition 6 is such that*

$$\|u - \Pi u\|_{(L^2(K))^3} \leqq c\, h^k\, |u|_{(H^{k+1}(K))^3}$$
$$|\operatorname{curl}(u - \Pi u)|_{(L^2(K))^3} \leqq c\, h^k\, |u|_{(H^{k+1}(K))^3}. \quad ■$$

We now introduce the family of conforming finite elements in $H(\operatorname{div})$.

Definition 7. We define a finite element $(\mathsf{C}, \mathsf{P}, \mathsf{A})$ by

$$\mathsf{P} = \{u;\, u_1 \in Q_{k,k-1,k-1};\, u_2 \in Q_{k-1,k,k-1};\, u_3 \in Q_{k-1,k-1,k}\}$$

A representing

(1) $\int_f u \cdot n q \, d\gamma$; $\forall q \in Q_{k-1,k-1}$;

(2) $\int_C u \cdot q \, dx$; $\forall q = (q_1, q_2, q_3)$; $q_1 \in Q_{k-2,k-1,k-1}$; $q_2 \in Q_{k-1,k-2,k-1}$; $q_3 \in Q_{k-1,k-1,k-2}$. ■

The dimensions of P is $3k(k-1)^2$ which is also the number of degrees of freedom.

Theorem 7. *The finite element given by Definition 7 is unisolvent and conforming in $H(\mathrm{div})$.*

Proof. Since $u \cdot n \in Q_{k-1,k-1}$, it is clear that

$$\int_f u \cdot n q \, d\gamma = 0 \Rightarrow u \cdot n = 0 \quad \text{on } f$$

and this proves the conformity. Then, $u \cdot n = 0$ implies

$$\begin{aligned} u_1 &= x_1(1-x_1)v_1; & v_1 &\in Q_{k-2,k-1,k-1}; \\ u_2 &= x_2(1-x_2)v_2; & v_2 &\in Q_{k-1,k-2,k-1}; \\ u_3 &= x_3(1-x_3)v_3; & v_3 &\in Q_{k-1,k-1,k-2}. \end{aligned}$$

Thus, we can take $q = v$ in degrees of type (2) and this implies $u = 0$. ■

We have also the following error estimate on C_h.

Theorem 8. *The finite element given by Definition 7 is such that*

$$|u - \Pi u|_{(L^2(K))^3} \leqq c h^k |u|_{(H^k(K))^3}$$
$$|\mathrm{div}(u - \Pi u)|_{L^2(K)} \leqq c h^k |u|_{(H^{k+1}(K))^3}.$$

Proof. We only need prove the equivalent of Lemma 9, i.e.

$$u \in (\mathbb{P}_k)^3 \quad \text{and} \quad \alpha_i(u) = 0, \quad \forall \alpha_i \in A, \quad \text{implies } \mathrm{div}\, u = 0.$$

For all q in the space $\mathbb{P}_{k-1}$, we have

$$\int_C \mathrm{div}\, u q \, dx = -\int_C u \cdot \mathrm{grad}\, q \, dx + \int_{\partial C} q u \cdot n \, d\gamma = 0$$

and this implies $\mathrm{div}\, u = 0$. ■

3. Some Applications

We shall briefly indicate here some of the applications which are at the origin of this study.

3.1. Maxwell's Equations in Dielectric Medias

Let Ω be a bounded domain in $\mathbb{R}^3$. We are looking for two fields B and E depending on x and t (and H related to B), which satisfy, in Ω,

$$\frac{1}{c}\frac{\partial B}{\partial t}+\operatorname{curl} E=0; \qquad \operatorname{div} B=0; \tag{27}$$

$$\frac{1}{c}\frac{\partial E}{\partial t}-\operatorname{curl} H=-\frac{1}{c}j; \qquad \operatorname{div} E=\rho \ \ (\rho \text{ and } j \text{ are given}); \tag{28}$$

$$B=\mu H \qquad (\mu \text{ depending only on the media}); \tag{29}$$

and some boundary conditions on the boundary Γ of Ω. These can be, for example (n is the normal to Γ):

$$B \cdot n|_\Gamma=0;$$

$$E \wedge n|_\Gamma=0.$$

The fields B and E must also satisfy some initial conditions at the time $t=0$:

$$B(x,0)=B_0(x); \qquad \operatorname{div} B_0=0;$$

$$E(x,0)=E_0(x).$$

The coefficient μ depends on the dielectric media. It can be discontinuous along a surface, and then it is known that $B \cdot n$ and $E \wedge n$ are continuous across this surface, while $B \wedge n$ and $E \cdot n$ are discontinuous. In such a case, it is natural to approximate E in $H(\text{curl})$ and B in $H(\text{div})$.

We have the following associate variational formulation:

$$\frac{1}{c}\int_\Omega \frac{\partial E}{\partial t}\cdot E^*\,dx-\int_\Omega \frac{B}{\mu}\cdot \operatorname{curl} E^*\,dx=-\frac{1}{c}\int_\Omega jE^*\,dx; \qquad \forall E^*\in\mathscr{E}, \tag{30}$$

$$\frac{1}{c}\frac{\partial B}{\partial t}+\operatorname{curl} E=0; \qquad B\in\mathsf{B} \quad (\text{and thus, } \operatorname{div} B=\operatorname{div} B_0=0); \tag{31}$$

$$\mathscr{E}=\{E^*\in H(\text{curl},\Omega);\, E^*\wedge n|_\Gamma=0\},$$

$$\mathsf{B}=\{B\in H(\text{div});\, B\cdot n|_\Gamma=0\}.$$

Let there be given a "triangulation" of Ω using tetrahedrons; then $\mathscr{E}_h$ and B_h are

$$\mathscr{E}_h=\{E_h^*\in H(\text{curl});\, E_h^*|_K\in\mathscr{R}^k;\, E_h^*\wedge n|_\Gamma=0\};$$

$$\mathsf{B}_h=\{B_h^*\in H(\text{div});\, B_h^*|_K\in\mathbb{D}^k;\, B_h^*\cdot n|_\Gamma=0\},$$

and the approximate problem is

$$\frac{1}{c}\int_\Omega \frac{\partial E_h}{\partial t}E_h^*\,dx-\int_\Omega \frac{B_h}{\mu}\cdot \operatorname{curl} E_h^*\,dx=-\frac{1}{c}\int_\Omega jE_h^*\,dx; \qquad \forall E_h^*\in\mathscr{E}_h; \tag{32}$$

$$\frac{1}{c}\frac{\partial B_h}{\partial t}+\operatorname{curl} E_h=0; \qquad B_h\in\mathsf{B}_h. \tag{33}$$

Proposition 3 shows that Eq. (33) is compatible with the structure of $\mathscr{E}_h$ and B_h since $E_h \in \mathscr{E}_h \Rightarrow \operatorname{curl} E_h \in \mathsf{B}_h$. Equation (33) implies

$$\begin{aligned} &\operatorname{div} B_h = 0, \\ &B_h \cdot n|_\Gamma = 0. \end{aligned} \tag{34}$$

If we use explicit schemes in time, then B_h^n can be eliminated at each time step by using Eq. (33).

Remark. As B_h verifies (34), we have

$$B_h = \operatorname{curl} A.$$

But it is not clear that this A (which anyway is not unique) can be associated to a finite element approximation using potential vectors.

When the media is ferromagnetic, the coefficient μ depends non linearly on $|H|$. Our approximation is still available in this case.

We refer to the work of J.C. Adam, A. Gourdin-Serveniere, J.C. Nedelec [1] for the use of similar finite elements for the solution of Maxwell's bidimensional equations. This work studies also the discretization in time. See also Petravic [7] for three-dimensional Maxwell's equations.

3.2. *Eddy Currents Equations in a Conductor*

In an electric conductor, Maxwell's equations take a different form. The unknowns are now B and the current j. The equations can be written

$$\operatorname{curl} H = \frac{j}{c} \qquad (\text{and thus } \operatorname{div} j = 0); \tag{35}$$

$$\frac{\partial B}{\partial t} + \operatorname{curl}\left(\frac{j}{\sigma}\right) = 0 \qquad (\text{and thus } \operatorname{div} B = 0); \tag{36}$$

$$B = \mu H; \tag{37}$$

σ is a constant depending on the conductor, μ is a constant or a monotonic function of the norm of H in a ferromagnetic media. We must add a boundary condition on the boundary Γ of the conductor Ω. It can be, for example,

$$\begin{aligned} &j \cdot n|_\Gamma = I; \qquad I \text{ given}; \\ &H \wedge n|_\Gamma = 0. \end{aligned} \tag{38}$$

We need also the initial condition B_0.

An approximate problem will be associated to the following variational formulation:

$$\operatorname{curl} H = \frac{j}{c};$$

$$\int_\Omega \frac{\partial B}{\partial t} \cdot H^* \, dx + \int_\Omega \frac{j}{c} \cdot \operatorname{curl} H^* \, dx = 0; \qquad \forall H^* \in \mathscr{H}$$

$$B = \mu H;$$

$$\mathscr{H} = \{H^* \in H(\operatorname{curl}); H^* \wedge n|_\Gamma = 0\}.$$

Then, the approximate problem will be

$$\operatorname{curl} H_h = \frac{j_h}{c}; \qquad j_h \in \mathsf{J}_h;$$

$$\int_\Omega \frac{\partial B_h}{\partial t} \cdot H_h^* \, dx + \int_\Omega \frac{j_h}{\mu} \cdot \operatorname{curl} H_h^* \, dx = 0; \qquad \forall H_h^* \in \mathscr{H}_h;$$

$$\mathsf{J}_h = \{j_h \in H(\operatorname{div}, \Omega); j_h|_K \in \mathbb{D}^k; j_h \cdot n|_\Gamma = I\};$$

$$\mathscr{H}_h = \{H_h^* \in H(\operatorname{curl}, \Omega); H_h^*|_K \in \mathscr{R}^k; H^* \wedge n|_\Gamma = 0\}.$$

We refer to R. Glowinski – A. Marroco [6] for an approximation of this problem by finite elements using potential vectors.

3.3. Elasticity and Stokes' Equations

The classical linear elasticity equations can be written in the following form

$$\begin{aligned} &\text{find } u = (u_1, u_2, u_3) \text{ defined in a bounded domain } \Omega \text{ such that} \\ &(\mu > 0; \lambda \geqq 0) \\ &2\mu \operatorname{curl} \operatorname{curl} u - (3\mu + \lambda) \operatorname{grad} \operatorname{div} u = f \quad \text{in } \Omega; \\ &\text{with some boundary conditions.} \end{aligned} \tag{39}$$

It is interesting here to consider boundary conditions of the non-usual type

$$u \wedge n|_\Gamma = 0,$$

$$\operatorname{div} u|_\Gamma = 0,$$

and this can be seen to be equivalent to

$$\begin{aligned} &u \wedge n|_\Gamma = 0 \quad \text{(non slipping condition)} \\ &\sigma n \cdot n|_\Gamma = 0. \end{aligned} \tag{40}$$

Let us set

$$p = \operatorname{div} u.$$

Equation (39) admits the following variational formulation:

$$\begin{aligned} &2\mu \int_\Omega \operatorname{curl} u \cdot \operatorname{curl} v \, dx - (3\mu + \lambda) \int_\Omega \operatorname{grad} p \cdot v \, dx = \int_\Omega f \cdot v \, dx, \\ &\int_\Omega p q \, dx + \int_\Omega \operatorname{grad} q \cdot u \, dx = 0; \quad \forall v \in \mathsf{U}; \ \forall q \in H_0^1(\Omega). \end{aligned} \tag{41}$$

If we consider the same boundary conditions, the Stokes' equations admit a similar formulation which is

$$\begin{aligned}
&\nu\int_\Omega \operatorname{curl} u\cdot\operatorname{curl} v\,dx+\int_\Omega \operatorname{grad} p\cdot v\,dx=\int_\Omega f\cdot v\,dx;\\
&\int_\Omega u\cdot\operatorname{grad} q\,dx=0;\qquad \forall\, v\in \mathsf{U};\ \forall\, q\in H_0^1(\Omega);\\
&\mathsf{U}(=\mathscr{H}=\mathscr{E})=\{u\in H(\operatorname{curl},\Omega);\, u\wedge n|_\Gamma=0\}.
\end{aligned}\tag{42}$$

Equation (41) will be approximated by

$$\begin{aligned}
&2\mu\int_\Omega \operatorname{curl} u_h\cdot\operatorname{curl} v_h\,dx-(3\mu+\lambda)\int_\Omega \operatorname{grad} p_h\cdot v_h\,dx=\int_\Omega f\cdot v_h\,dx,\\
&\int_\Omega p_h\cdot q_h\,dx+\int_\Omega \operatorname{grad} q_h\cdot u_h\,dx=0;\qquad \forall\, v_h\in \mathsf{U}_h;\ \forall\, q_h\in Q_h^0;\\
&\mathsf{U}_h=\{u_h\in\mathsf{U};\, u_h|_K\in\mathscr{R}^k\}\qquad (=\mathscr{H}_h=\mathscr{E}_h);\\
&Q_h^0=\{q_h\in H^1(\Omega);\, q_h|_K\in\mathbb{P}_k;\, q_h|_\Gamma=0\},
\end{aligned}\tag{43}$$

while Eq. (42) will be approximated by

$$\begin{aligned}
&\nu\int_\Omega \operatorname{curl} u_h\cdot\operatorname{curl} v_h\,dx+\int_\Omega \operatorname{grad} p_h\cdot v_h\,dx=\int_\Omega f\cdot v_h\,dx;\\
&\int_\Omega u_h\cdot\operatorname{grad} q_h\,dx=0;\qquad \forall\, v_h\in \mathsf{U}_h;\ \forall\, q_h\in Q_h^0.
\end{aligned}\tag{44}$$

The error estimates for this type of systems have been studied by F. Brezzi [2] and M. Fortin [5].

Using these results, we can prove the following statement.

Theorem 9. *Suppose that the triangulation T_h is such that*

$$\max_{K\in\mathsf{T}_h}\operatorname{diam}(K)\leqq c\min_{K\in\mathsf{T}_h}\operatorname{diam}(K).$$

Let (u,p) be the solution of Eq. (41) (resp. (42)), and (u_h,p_h) be the solution of Eq. (43) (resp. (44)); then we have

$$\|u-u_h\|_{H(\operatorname{curl},\Omega)}+\|p-p_h\|_{H_0^1(\Omega)}\leqq c\,h^k(\|u\|_{(H^{k+1}(\Omega))}+\|p\|_{H^{k+1}(\Omega)}).\tag{45}$$

Proof. From F. Brezzi [2] we know that it is sufficient to verify two hypotheses which are

$$\sup_{\substack{v_h\in\mathsf{U}_h\\ \|v_h\|_{H(\operatorname{curl},\Omega)}\leqq 1}}\int_\Omega v_h\cdot\operatorname{grad} q_h\,dx\geqq\beta\,\|q_h\|_{H_0^1(\Omega)};\qquad \forall\, q_h\in Q_h^0;\ \beta>0,\tag{46}$$

and

$$\forall\, v_h\in\mathsf{U}_h;\quad \int_\Omega v_h\cdot\operatorname{grad} q_h\,dx=0;\ \ \forall\, q_h\in Q_h^0;\ \text{imply}$$

$$\int_\Omega |\operatorname{curl} v_h|^2\,dx\geqq\gamma\int_\Omega |v_h|^2\,dx;\qquad \gamma>0.\tag{47}$$

Hypotheses (46) can be easily verified choosing

$$v_h = \operatorname{grad} q_h$$

which is possible since $q_h \in Q_h^0 \Rightarrow \operatorname{grad} q_h \in \mathrm{U}_h$. Before proving (47), we give

Lemma 11. *Let $\Pi_h u$ be the interpolate of u defined on each tetrahedron as Πu. Let u be such that*

$$u = \operatorname{grad} p; \qquad p \in H_0^1(\Omega).$$

Then, if u is regular enough to ensure the existence of $\Pi_h u$, we have

$$\Pi_h u = \operatorname{grad} p_h; \qquad p_h \in Q_h^0.$$

Proof. On each tetrahedron, we have

$$\operatorname{curl} u = 0$$

and for any face f, we have

$$\int_f \operatorname{curl} u \cdot n\, q\, d\gamma = \int_{\partial f} u \cdot t\, q\, ds - \int_f u \cdot \overrightarrow{\operatorname{curl}}_f\, q\, d\gamma; \qquad \forall q \in \mathbb{P}_{k-1}.$$

This equality implies (using the definition of Πu):

$$\operatorname{curl} \Pi u \cdot n = 0 \quad \text{on each face } f. \tag{48}$$

We also have, for any tetrahedron,

$$\int_K \operatorname{curl} u \cdot q\, dx = \int_{\partial K} u \wedge n \cdot q\, d\gamma - \int_K u \cdot \operatorname{curl} q\, d\gamma; \qquad \forall q \in (\mathbb{P}_{k-2})^3.$$

This equality and the definition of Πu imply

$$\int_K \operatorname{curl} \Pi u \cdot q\, dx = 0, \qquad \forall q \in (\mathbb{P}_{k-2})^3. \tag{49}$$

Now, identities (48) and (49) imply

$$\operatorname{curl} \Pi u = 0 \quad \text{on each tetrahedron } K.$$

Since $\Pi_h u$ belongs to the space $H(\mathrm{rot})$, this implies also

$$\operatorname{curl} \Pi_h u = 0.$$

Thus

$$\Pi_h u = \operatorname{grad} p_h, \qquad p_h \in H^1(\Omega).$$

From Lemma 3, it results that p_h belongs to the space Q_h. Finally, the boundary condition

$$\Pi_h u \wedge n|_\Gamma = 0$$

yields to

$$p_h|_\Gamma = 0$$

so that p_h belongs to Q_h^0. ■

We can now prove property (47). Let us first solve the following equation

$$\begin{aligned} \Delta p &= \operatorname{div} v_h \\ p|_\Gamma &= 0. \end{aligned} \tag{50}$$

Then, p belongs to the space $H_0^1(\Omega)$. We consider

$$w = v_h - \operatorname{grad} p.$$

We have

$$\begin{aligned} \operatorname{curl} w &= \operatorname{curl} v_h \\ \operatorname{div} w &= 0, \\ w \wedge n|_\Gamma &= 0. \end{aligned} \tag{51}$$

Thus, we can verify that

$$\begin{aligned} \Delta w &= \operatorname{curl}\operatorname{curl} v_h \\ \frac{\partial w_n}{\partial n} &= 0 \\ w \wedge n|_\Gamma &= 0. \end{aligned} \tag{52}$$

For some $q>2$, the regularity properties of Eq. (52) ensure us that

$$\|w\|_{(W^{1,q}(\Omega))^3} \leqq c\,\|\operatorname{curl} v_h\|_{(L^q(\Omega))^3}. \tag{53}$$

We also have

$$\|w\|_{(H^1(\Omega))^3} \leqq c\,\|\operatorname{curl} v_h\|_{(L^2(\Omega))^3}. \tag{54}$$

In each tetrahedron K, the function Πw is defined if $w \in W^{1,q}(\Omega)$ for any $q>2$ (by the trace theorem). The study of error leads to

$$\|\Pi w\|_{(L^2(K))^3} \leqq \|w\|_{(L^2(K))^3} + c\,h\,\|Dw\|_{(L^q(K))^9}.$$

This yields to

$$\begin{aligned} \|\Pi_h w\|^2_{(L^2(\Omega))^3} &\leqq c\Big[\|w\|^2_{(L^2(\Omega))^3} + h^2 \sum_{K\in T_h} \Big(\int_K |Dw|^q\,dx\Big)^{2/q}\Big] \\ &\leqq c\Big[\|w\|^2_{(L^2(\Omega))^3} + h^2 \Big(\sum_{K\in T_h} 1\Big)^{(q-2)/q} \Big(\int_\Omega |Dw|^q\,dx\Big)^{2/q}\Big]. \end{aligned}$$

The number of tetrahedra is bounded by

$$\sum_{K\in T_h} 1 \leqq \frac{c}{h^3}.$$

Thus, we obtain

$$\|\Pi_h w\|^2_{(L^2(\Omega))^3} \leqq c\,\|w\|^2_{(L^2(\Omega))^3} + c\,\frac{h^2}{h^{(3(q-2))/q}}\,\|Dw\|^2_{(L^q(\Omega))^9}. \tag{55}$$

Using now inequalities (53) and (54), we obtain

$$\|\Pi_h w\|^2_{(L^2(\Omega))^3} \leqq c\,\|\mathrm{curl}\, v_h\|^2_{(L^2(\Omega))} + c\,\frac{h^2}{h^{(3(q-2))/q}}\,\|\mathrm{curl}\, v_h\|^2_{(L^q(\Omega))^3}.$$

Using the reference finite element, it is classical to prove

$$\|\mathrm{curl}\, v_h\|^2_{(L^q(\Omega))^3} \leqq \frac{c}{h^{(3(q-2))/q}}\,\|\mathrm{curl}\, v_h\|_{(L^2(\Omega))^3},$$

so that, finally,

$$\|\Pi_h w\|^2_{(L^2(\Omega))^3} \leqq c\left(1+\frac{h^2}{h^{(6(q-2))/q}}\right)\|\mathrm{curl}\, v_h\|^2_{(L^2(\Omega))^3}.$$

Now, choosing q such that $2<q\leqq 3$, we obtain

$$\|\Pi_h w\|_{(L^2(\Omega))^3} \leqq c\,\|\mathrm{curl}\, v_h\|_{(L^2(\Omega))^3}. \tag{56}$$

From Lemma 11, we have

$$v_h = \Pi_h w + \mathrm{grad}\, p_h$$

so that, using (48), we obtain

$$\int_\Omega |v_h|^2\,dx \leqq \int_\Omega v_h\cdot \Pi_h w\,dx,$$

and with (56), we get inequality (47). ■

It is easy to check that, in both systems (43) and (44), the equations giving p_h can be written independently

$$\int_\Omega \mathrm{grad}\, p_h\cdot \mathrm{grad}\, q_h\,dx = \int_\Omega f\,\mathrm{grad}\, q_h\,dx; \qquad \forall\, q_h\in Q_h^0. \tag{57}$$

And using Eq. (56), we can also obtain the error estimate on p. This property is related to the type of boundary condition that we have considered here and is not true for other boundary conditions.

Remark. We can easily modify the variational formulation of these problems in order to consider different boundary conditions (such as, for example, $u|_\Gamma=0$). We must notice that, in this case, it is more difficult to prove hypothesis (46).

Conclusion

Our finite elements are a natural generalization of the mixed finite element introduced by P.A. Raviart and J.M. Thomas [8] in two dimensions. It appears that in the bi-dimensional case, the conforming finite elements in $H(\mathrm{curl})$ are easily related to finite elements conforming in $H(\mathrm{div})$ by a twisting of $\frac{\pi}{2}$. In three dimensions, these finite elements are completely different.

The main advantage of these finite elements is the possibility of approximating Maxwell's equations while exactly verifying one of the physical law.

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