Iterative Solution of a Mixed Finite Element Discretisation of an Incompressible Magnetohydrodynamics Problem

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- MHD models electrically conductive fluids (such as liquid metals, plasma, salt water, etc) in an elective field
- Applications: electromagnetic pumping, aluminium electrolysis, the Earth's molten core and solar flares
- MHD models couple electromagnetism (governed by Maxwell's equations) and fluid dynamics (governed by the Navier-Stokes equations)
- Movement of the conductive material that induces a magnetic field which then modifies any existing electromagnetic field
- Magnetic and electric fields generate a mechanical force on the fluid

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Incompressible Navier-Stokes Equations:

$$
\begin{aligned}
-\nu\,\Delta u + (u\cdot\nabla)u + \nabla p &= f && \text{in } \Omega,\\
\nabla\cdot u &= 0 && \text{in } \Omega,\\
u &= u_D && \text{on } \partial\Omega
\end{aligned}
$$

- u: fluid velocity
- p: fluid pressure
- f: body force acting on the fluid
- ν: kinematic viscosity

Time-harmonic Maxwell's equations:

$$\begin{aligned} \nabla \times \nabla \times b - k^2 b + \nabla r &= g && \text{in } \Omega, \\ \nabla \cdot b &= 0 && \text{in } \Omega, \\ b \times n &= b_D && \text{on } \partial\Omega, \\ r &= 0 && \text{on } \partial\Omega \end{aligned}$$

,

- b: electric field
- r: scalar multiplier
- k: wave number
- n: unit outward normal

MHD model: coupled Navier-Stokes and Maxwell's equations

$$\begin{aligned} -\nu\,\Delta u + (u\cdot\nabla)u + \nabla p - \kappa\,(\nabla\times b)\times b &= f && \text{in } \Omega,\\ \nabla\cdot u &= 0 && \text{in } \Omega,\\ \kappa\nu_m\,\nabla\times(\nabla\times b) + \nabla r - \kappa\,\nabla\times(u\times b) &= g && \text{in } \Omega,\\ \nabla\cdot b &= 0 && \text{in } \Omega, \end{aligned}$$

with appropriate boundary conditions.

- $(\nabla\times b)\times b$: Lorentz force accelerates the fluid particles in the direction normal to the electric and magnetic fields.
- $\nabla\times(u\times b)$: electromotive force modifying the magnetic field

- Finite element discretisation base on Schötzau 2004
- Fluid variables: lowest order Taylor-Hood ($\mathcal{P}_2/\mathcal{P}_1$)
- Magnetic variables: mixed element approximation

- MHD model is non-linear: $\mathcal{K}(x)x = b$
- Standard Oseem iteration: $\mathcal{K}(x_k)x_{k+1} = b$
- Re-arrange to solve for updates
- $\mathcal{K}(x_k)\delta x = b - \mathcal{K}(x_k)x_k$ where $x_{k+1} = x_k + \delta x$

Discretised and linearised MHD model:

$$\begin{pmatrix} A+O(u) & B^T & C(b)^T & 0 \\ B & 0 & 0 & 0 \\ -C(b) & 0 & M & D^T \\ 0 & 0 & D & 0 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta p \\ \delta b \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_p \\ r_b \\ r_r \end{pmatrix},$$

with

$$\begin{aligned} r_u &= f - Au - O(u)u - C(b)^T b - B^T p, \\ r_p &= -Bu, \\ r_b &= g - Mu + C(b)b - D^T r, \\ r_r &= -Db. \end{aligned}$$

A: discrete Laplacian operator, O: discrete convection-diffusion operator, B: discrete divergence operator, M: discrete curl-curl operator, X: discrete mass matrix and D: discrete divergence operator.

Decoupling schemes

Magnetic Decoupling (MD):

$$\mathcal{K}_{\mathrm{MD}} = \left(\begin{array}{cc|cc} A + O(u) & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array} \right)$$

Complete Decoupling (CD):

$$\mathcal{K}_{\mathrm{CD}} = \left(\begin{array}{cc|cc} A & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array} \right)$$

Preconditioning

Non-singular $(1,1)$ block

$$\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F & B^T \\ 0 & BF^{-1}B^T \end{pmatrix}$$

Murphy, Golub & Wathen 2000 showed exactly three eigenvalues: ± 1

Singular $(1,1)$ block

$$\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F + B^T W^{-1} B & 0 \\ 0 & W \end{pmatrix},$$ where W is SPD

Greif & Schötzau 2006 showed exactly two eigenvalues: ± 1

Navier-Stokes subproblem

$$\mathcal{K}_{\mathrm{NS}} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}$$

where $F = A + O$ is the discrete convection diffusion operator. Shown in Elman, Silvester Wathen 2005/2014 that

$$\mathcal{P}_{\mathrm{NS}} = \begin{pmatrix} F & B^T \\ 0 & S \end{pmatrix}, \quad S = A_p F_p^{-1} Q_p$$

is a good approximation to the Schur complement preconditioner. A_p: pressure space Laplacian, F_p: pressure space convection-diffusion operator, Q_p: pressure space mass matrix

Maxwell subproblem

$$\mathcal{K}_{\mathrm{NS}} = \begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix}$$

*N*ote: M is highly singular.

Greif & Schötzau 2007 shows that L (scalar Laplacian) is the appropriate choice for W

$$\mathcal{P}_{\mathrm{iM}} = \begin{pmatrix} M + B^T L^{-1} B & 0 \\ 0 & L \end{pmatrix}$$

Practical preconditioner:

$$\mathcal{P}_{\mathrm{M}} = \begin{pmatrix} M + X & 0 \\ 0 & L \end{pmatrix}$$

where X vector mass matrix

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MHD problem

Combing the Navier-Stokes and Maxwell preconditioners

$$\mathcal{M}_{\mathrm{MH}} = \begin{pmatrix} F & B^T & C^T & 0 \\ 0 & -S & 0 & 0 \\ -C & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

with the inner preconditioner

$$\mathcal{M}_{\mathrm{innerMH}} = \begin{pmatrix} F & B^T & 0 & 0 \\ 0 & -S & 0 & 0 \\ 0 & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

Iteration scheme	Coefficient matrix	Preconditioner
(MD)	$\left(\begin{array}{cc\|cc} F & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right)$	$\left(\begin{array}{cc\|cc} F & B^T & 0 & 0 \\ 0 & -S & 0 & 0 \\ \hline 0 & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{array}\right)$
(CD)	$\left(\begin{array}{cc\|cc} A & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right)$	$\left(\begin{array}{cc\|cc} A & 0 & 0 & 0 \\ 0 & \frac{1}{\nu}Q_p & 0 & 0 \\ \hline 0 & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{array}\right)$

TABLE : Summary of coefficient matrices and corresponding preconditioners for each the decoupling scheme

Numerical software used

- Finite element software FEniCS: core libraries are the problem-solving interface DOLFIN, the compiler for finite element variational forms FFC, the finite element tabulator FIAT for creating finite element function spaces, the just-in-time compiler Instant, the code generator UFC and the form language UFL.
- Linear algebra software: HYPRE as a multigrid solver and the sparse direct solvers UMFPACK, PASTIX, SuperLU and MUMPS

Navier-Stokes subproblem in isolation

ℓ	Dofs	Average iterations			
	u_h/p_h	$\nu = 10$	$\nu = 1$	$\nu = 0.1$	$\nu = 0.01$
5	8,450/1,089	17	17	21	58
6	33,282/4,225	17	17	22	30
7	132,098/16,641	18	17	22	21
8	526,338/66,049	18	18	22	20
9	2,101,250/263,169	18	19	22	21

TABLE : Iteration table for a PCD preconditioned 2D example for various values of ν

Maxwell subproblem in isolation

2D results:

ℓ	Dofs b_h/r_h	Number of iterations			
		$\nu_m = 10$	$\nu_m = 100$	$\nu_m = 1000$	$\nu_m = 10000$
5	10,368/4,225	5	4	6	6
6	41,216/16,641	5	6	6	6
7	164,352/66,049	5	6	6	6
8	656,384/263,169	5	6	6	8
9	2,623,488/1,050,625	4	6	6	8
10	10,489,856/4,198,401	4	6	8	10

TABLE : Iteration count for Maxwell preconditioner for 2D example

3D results:

ℓ	Dofs b_h/r_h	Number of iterations			
		$\nu_m = 10$	$\nu_m = 100$	$\nu_m = 1000$	$\nu_m = 10000$
2	2,936/729	4	4	6	5
3	21,424/49,13	4	4	6	5
4	163,424/35,937	4	6	6	5
5	1,276,096/274,625	4	6	6	5

TABLE : Iteration table for Maxwell preconditioner for 3D example

MHD: inner-outer preconditioner

MHD: (MD) scheme

2D example

ℓ	Dofs	Av solve time	Total time	Its_{NL}	Its_{NS}	Its_{M}
5	24,132	0.5	7.3	5	22.0	3.4
6	95,364	2.5	30.4	5	21.2	3.4
7	379,140	13.1	134.7	5	21.2	3.4
8	1,511,940	69.8	627.3	5	21.4	3.4
9	6,038,532	407.7	3159.7	5	21.6	3.2
10	24,135,684	3022.1	19668.3	5	21.6	3.4

TABLE : Number of non-linear iterations and average number of iterations to solve the Navier-Stokes and Maxwell's subproblem for the MD scheme with $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$.

3D example

l	Dofs	Av solve time	Total time	Its_{NL}	Its_{NS}	Its_{M}
1	963	0.04	2.7	5	23.2	3.4
2	5977	0.20	17.2	5	34.2	3.0
3	41805	4.86	151.1	5	34.2	3.4
4	312,085	242.1	2222.8	5	32.4	3.4
5	2,410,533	30222.3	159032.8	5	30.8	3.2

TABLE : Number of non-linear iterations and average number of iterations to solve the Navier-Stokes and Maxwell's subproblem for the MD scheme with $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$ in 3D.

MHD: (CD) scheme

2D example

ℓ	Dofs	Av solve time	Total time	Its$_{\mathrm{NL}}$	Its$_{\mathrm{S}}$	Its$_{\mathrm{M}}$
5	24,132	0.4	7.7	11	29.0	3.3
6	95,364	2.6	38.8	11	28.5	3.4
7	379,140	13.0	181.7	11	27.5	3.4
8	1,511,940	66.5	888.0	11	28.3	3.5
9	6,038,532	358.0	4565.4	11	28.3	3.4
10	24,135,684	2335.7	28337.4	11	27.4	3.5

TABLE : Number of non-linear iterations and average number of iterations to solve the Stokes and Maxwell's subproblem for the CD scheme with $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$.

3D example

ℓ	Dofs	Av solve time	Total time	Its_{NL}	Its_{S}	Its_{M}
1	963	0.03	1.8	6	30.0	3.5
2	5,977	0.20	9.9	6	45.7	3.2
3	41,805	4.32	89.2	6	43.0	2.8
4	312,085	214.3	1786.5	6	42.3	2.8
5	2,410,533	26954.4	165671.1	6	41.3	2.8

TABLE : Number of non-linear iterations and average number of iterations to solve the Stokes and Maxwell's subproblem for the CD scheme with $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$ in 3D.

Future work:

- Scalable inner solvers
- Release code on a public repository
- Parallelisation of the code
- Robustness with respect to kinematic viscosity
- Other non-linear solvers
- Different mixed finite element discretisations