MHD - Neumann bilinear form

The variational for the MHD model with inhomogeneous Neumann conditions is

$$\begin{aligned}
A(\boldsymbol{u}_h,\boldsymbol{v}) + O(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}) + C(\boldsymbol{b}_h;\boldsymbol{v},\boldsymbol{b}_h) + B(\boldsymbol{v},p_h) &= (\boldsymbol{f},\boldsymbol{v})_\Omega - (\boldsymbol{p}_N, v)_{\Omega_N} \\
B(\boldsymbol{u}_h,q) &= 0, \\
M(\boldsymbol{b}_h,\boldsymbol{c}) - C(\boldsymbol{b}_h;\boldsymbol{u}_h,\boldsymbol{c}) + D(\boldsymbol{c},r_h) &= (\boldsymbol{g},\boldsymbol{c})_\Omega, \\
D(\boldsymbol{b}_h,s) &= 0,
\end{aligned} \tag{1}$$

where $\boldsymbol{p}_N$ is the Neumann condition. Then the Picard iteration is given by:

$$\begin{aligned}
A(\delta\boldsymbol{u}_h,\boldsymbol{v}) + O(\boldsymbol{u}_h;\delta\boldsymbol{u}_h,\boldsymbol{v}) + C(\boldsymbol{b}_h;\boldsymbol{v},\delta\boldsymbol{u}_h) + B(\boldsymbol{v},\delta p_h) &= R_u(\boldsymbol{u}_h,\boldsymbol{b}_h,p_h;\boldsymbol{v}), \\
B(\delta\boldsymbol{u}_h,q) &= R_p(\boldsymbol{u}_h;q), \\
M(\delta\boldsymbol{b}_h,\boldsymbol{c}) + D(\boldsymbol{c},\delta r_h) - C(\boldsymbol{b}_h;\delta\boldsymbol{u}_h,\boldsymbol{v}) &= R_b(\boldsymbol{u}_h,\boldsymbol{b}_h,r_h;\boldsymbol{c}), \\
D(\delta\boldsymbol{b}_h,s) &= R_r(\boldsymbol{b}_h;s),
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
R_u(\boldsymbol{u}_h,\boldsymbol{b}_h,p_h;\boldsymbol{v}) =& (\boldsymbol{f},\boldsymbol{v})_\Omega - (\boldsymbol{p}_N,v)_{\Omega_N} - A(\boldsymbol{u}_h,\boldsymbol{v}) - O(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}) \\
& - C(\boldsymbol{b}_h;\boldsymbol{v},\boldsymbol{b}_h) - B(\boldsymbol{v},p_h), \\
R_p(\boldsymbol{u}_h;q) =& - B(\boldsymbol{u}_h,q), \\
R_b(\boldsymbol{u}_h,\boldsymbol{b}_h,r_h;\boldsymbol{c}) =& (\boldsymbol{g},c)_\Omega - M(\boldsymbol{b}_h,\boldsymbol{c}) + C(\boldsymbol{b}_h;\boldsymbol{u}_h,\boldsymbol{c}) - D(\boldsymbol{c},r_h), \\
R_r(\boldsymbol{b}_h;s) =& - D(\boldsymbol{b}_h,s),
\end{aligned} \tag{3}$$

Therefore you need to enforce the inhomogeneous Neumann conditions at each non-linear iteration???? Also, if you enforce homogeneous boundary conditions within the non-linear iteration doesn't this stop model from capturing the pressure driven flow?

A final thought, the expression for $\boldsymbol{p}_N$ is linear and hence would arise in its original form throughout the non-linear iterations.

MHD - smooth Neumann conditions

Consider the exact solution:

```
u = ( x*y*exp(x + y) + x*exp(x + y) ,  -x*y*exp(x + y) - y*exp(x + y) )

p = ( exp(y)*sin(x) )

b = ( x*cos(x) ,  x*y*sin(x) - y*cos(x) )

r = ( x*sin(2*pi*x)*sin(2*pi*y) )
```

The error tables are given below. Here I used a unit square domain with Neumann conditions on the left and right boundaries and Dirichlet on the top and bottom ones.

ℓ	Dofs $\boldsymbol{u}_h/p_h$	$\|\boldsymbol{e}_u\|_{L^2(\Omega)}$	r	$\|\boldsymbol{e}_u\|_{H^1(\Omega)}$	r	$\|e_p\|_{L^2(\Omega)}$	r
1	50/9	9.0942e-02	-	1.2457e+00	-	5.6315e-01	-
2	162/25	1.1441e-02	3.53	3.2338e-01	2.29	7.8755e-02	3.85
3	578/81	1.3656e-03	3.34	8.1946e-02	2.16	9.2087e-03	3.65
4	2,178/289	1.6719e-04	3.17	2.0637e-02	2.08	1.0929e-03	3.35
5	8,450/1,089	2.0858e-05	3.07	5.1826e-03	2.04	1.6142e-04	2.88
6	33,282/4,225	2.6265e-06	3.02	1.2990e-03	2.02	3.1921e-05	2.39
7	132,098/16,641	2.7450e-07	3.28	3.2522e-04	2.01	7.4632e-06	2.12

Table 1: Convergence for 2D MHD - fluid variables

ℓ	Dofs $\boldsymbol{b}_h/r_h$	$\|\boldsymbol{e}_b\|_{L^2(\Omega)}$	l	$\|\boldsymbol{e}_b\|_{H(\mathrm{curl},\Omega)}$	l
1	16/9	1.8060e-01	-	2.6788e-01	-
2	56/25	9.1265e-02	1.09	1.3398e-01	1.11
3	208/81	4.5753e-02	1.05	6.7003e-02	1.06
4	800/289	2.2892e-02	1.03	3.3503e-02	1.03
5	3,136/1,089	1.1448e-02	1.01	1.6752e-02	1.01
6	12,416/4,225	5.7241e-03	1.01	8.3759e-03	1.01
7	49,408/16,641	2.8621e-03	1.00	4.1879e-03	1.00

Table 2: Convergence for 2D MHD - magnetic variable

ℓ	Dofs $\boldsymbol{b}_h/r_h$	$\|\boldsymbol{e}_r\|_{L^2(\Omega)}$	l	$\|\boldsymbol{e}_r\|_{H^1(\Omega)}$	l
1	16/9	2.7524e-01	-	2.4780e+00	-
2	56/25	1.4850e-01	1.21	1.7787e+00	0.65
3	208/81	4.8879e-02	1.89	1.0042e+00	0.97
4	800/289	1.3198e-02	2.06	5.1942e-01	1.04
5	3,136/1,089	3.3659e-03	2.06	2.6198e-01	1.03
6	12,416/4,225	8.4572e-04	2.04	1.3128e-01	1.02
7	49,408/16,641	2.1170e-04	2.02	6.5676e-02	1.01

Table 3: Convergence for 2D MHD - multiplier variable

MHD - Hartmann Neumann conditions

Consider the exact solution: I have considered exactly the same parameter setup

```
u = ( -99998.3333527776*cosh(0.01*y) + 100003.333311111 ,  0 )

p = ( -10.0*x - 50.0*(-y + 99.9983333527776*sinh(0.01*y))**2 )

b = ( -10.0*y + 999.983333527776*sinh(0.01*y) ,  1 )

r = ( 0 )
```

as in your CNAME paper.

Small domain - $(0,1) \times (0,1)$

ℓ	Dofs $\boldsymbol{u}_h/p_h$	$\|\boldsymbol{e}_u\|_{L^2(\Omega)}$	r	$\|\boldsymbol{e}_u\|_{H^1(\Omega)}$	r	$\|e_p\|_{L^2(\Omega)}$	r
1	50/9	1.0907e-06	-	5.9809e-06	-	6.8112e-07	-
2	162/25	2.4709e-07	2.53	1.5978e-06	2.25	1.1813e-07	3.43
3	578/81	1.2943e-07	1.02	4.0806e-07	2.15	9.4741e-08	0.38
4	2,178/289	1.6700e-07	0.38	1.0278e-07	2.08	1.0221e-07	0.12
5	8,450/1,089	1.5944e-07	0.07	2.5669e-08	2.05	1.0260e-07	0.01
6	33,282/4,225	1.6102e-07	0.01	7.1451e-09	1.87	1.0227e-07	0.00
7	132,098/16,641	1.6184e-07	0.01	5.4389e-09	0.40	1.0223e-07	0.00

Table 4: Convergence for 2D MHD - fluid variables - small domain

ℓ	Dofs $\boldsymbol{b}_h/r_h$	$\|\boldsymbol{e}_b\|_{L^2(\Omega)}$	l	$\|\boldsymbol{e}_b\|_{H(\mathrm{curl},\Omega)}$	l	$\|\boldsymbol{e}_r\|_{L^2(\Omega)}$
1	16/9	2.0609e-05	-	6.6536e-05	-	7.9542e-18
2	56/25	1.0642e-05	1.06	3.3834e-05	1.08	1.0057e-11
3	208/81	5.3645e-06	1.04	1.6987e-05	1.05	7.6667e-11
4	800/289	2.6877e-06	1.03	8.5022e-06	1.03	4.6064e-10
5	3,136/1,089	1.3445e-06	1.01	4.2522e-06	1.01	5.5048e-09
6	12,416/4,225	6.7236e-07	1.01	2.1262e-06	1.01	6.7659e-08
7	49,408/16,641	3.3619e-07	1.00	1.0631e-06	1.00	4.2016e-07

Table 5: Convergence for 2D MHD - magnetic variable - small domain

Original domain - $(0,10)\times(1,1)$

ℓ	Dofs $\boldsymbol{u}_h/p_h$	$\Vert\boldsymbol{e}_u\Vert_{L^2(\Omega)}$	r	$\Vert\boldsymbol{e}_u\Vert_{H^1(\Omega)}$	r	$\Vert e_p\Vert_{L^2(\Omega)}$	r
1	210/33	8.4039e-05	-	1.5327e-04	-	1.2410e-05	-
2	738/105	2.2328e-05	2.11	4.2124e-05	2.06	3.1605e-06	2.36
3	2,754/369	5.6093e-06	2.10	1.0752e-05	2.07	4.5085e-06	0.57
4	10,626/1,377	1.2471e-06	2.23	2.7021e-06	2.05	4.5568e-06	0.02
5	41,730/5,313	6.6421e-07	0.92	6.7647e-07	2.02	4.5738e-06	0.01
6	165,378/20,865	7.1821e-07	0.11	1.6886e-07	2.02	4.5791e-06	0.00

Table 6: Convergence for 2D MHD - fluid variables - original domain

ℓ	Dofs $\boldsymbol{b}_h/r_h$	$\Vert\boldsymbol{e}_b\Vert_{L^2(\Omega)}$	l	$\Vert\boldsymbol{e}_b\Vert_{H(\mathrm{curl},\Omega)}$	l	$\Vert\boldsymbol{e}_r\Vert_{L^2(\Omega)}$
1	72/33	1.5144e-04	-	5.5277e-04	-	8.9585e-16
2	264/105	9.0833e-05	0.79	2.9756e-04	0.95	2.0906e-15
3	1,008/369	4.7431e-05	0.97	1.5131e-04	1.01	2.3104e-15
4	3,936/1,377	2.3971e-05	1.00	7.5968e-05	1.01	3.1749e-15
5	15,552/5,313	1.2017e-05	1.01	3.8023e-05	1.01	4.8842e-15
6	61,824/20,865	6.0127e-06	1.00	1.9016e-05	1.00	5.7900e-15

Table 7: Convergence for 2D MHD - magnetic variable - original domain

Large domain - $(0, 80) \times (1, 1)$

ℓ	Dofs $\boldsymbol{u}_h/p_h$	$\|\boldsymbol{e}_u\|_{L^2(\Omega)}$	r	$\|\boldsymbol{e}_u\|_{H^1(\Omega)}$	r	$\|e_p\|_{L^2(\Omega)}$	r
1	490/75	4.2854e-03	-	3.9356e-03	-	3.3003e-04	-
2	1746/245	1.1348e-03	2.09	1.0919e-03	2.02	9.2882e-05	2.14
3	6,562/873	2.8863e-04	2.07	2.7938e-04	2.06	6.7975e-05	0.49
4	25,410/3,281	7.1669e-05	2.06	6.8712e-05	2.07	7.2061e-05	0.09
5	99,970/12,705	1.7437e-05	2.06	1.7137e-05	2.03	7.2392e-05	0.01

Table 8: Convergence for 2D MHD - fluid variables - large domain

ℓ	Dofs $\boldsymbol{b}_h/r_h$	$\|\boldsymbol{e}_b\|_{L^2(\Omega)}$	l	$\|\boldsymbol{e}_b\|_{H(\mathrm{curl},\Omega)}$	l	$\|\boldsymbol{e}_r\|_{L^2(\Omega)}$
1	170/75	5.6497e-03	-	6.9921e-03	-	3.0388e-14
2	628/245	3.1297e-03	0.90	3.7639e-03	0.95	3.5901e-14
3	2,408/873	1.6074e-03	0.99	1.9140e-03	1.01	5.2011e-14
4	9,424/3,281	8.0912e-04	1.01	9.6096e-04	1.01	6.6039e-14
5	37,280/12,705	4.0524e-04	1.01	4.8097e-04	1.01	1.0988e-13

Table 9: Convergence for 2D MHD - magnetic variable - large domain

Discussion

- I have a working smooth solution model with Neumann boundary conditions - see "MHD - smooth Neumann conditions" section. The convergence rates are what we would expect for a smooth solution here.
- For the Hartmann flow example if have produced tables for three separate domains (small, original and large). In each case, the errors are small for the first mesh level, ℓ.
- I have decreased the Picard tolerance to 1e-10 but I still get the same error convergence results.
- For the original and large meshes it is possible to see that the fluid variables start to exhibit the correct convergence results. However, $\|\boldsymbol{e}_u\|_{L^2(\Omega)}$ appears to be an order less than we would expect.
- Recall that we are using TH P2/P1 for the fluid variables and N1/P1 mixed Nédélec approximation for the magnetic variables. Could the loss in order in $\|\boldsymbol{e}_u\|_{L^2(\Omega)}$ be a result of the miss match in the orders of the velocity and magnetic fields??