# Iterative Solution of a Mixed Finite Element Discretisation of an Incompressible Magnetohydrodynamics Problem

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# Problem background

- MHD models electrically conductive fluids (such as liquid metals, plasma, salt water, etc) in an electic field
- Applications: electromagnetic pumping, aluminium electrolysis, the Earth's molten core and solar flares
- MHD models couple electromagnetism (governed by Maxwell's equations) and fluid dynamics (governed by the Navier-Stokes equations)
- Movement of the conductive material that induces and modifies any existing electromagnetic field
- Magnetic and electric fields generate a mechanical force on the fluid

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Incompressible Navier-Stokes Equations:

$$
\begin{aligned}
-\nu\,\Delta u + (u\cdot\nabla)u + \nabla p &= f && \text{in } \Omega,\\
\nabla\cdot u &= 0 && \text{in } \Omega,\\
u &= u_D && \text{on } \partial\Omega
\end{aligned}
$$

- $u$: fluid velocity
- $p$: fluid pressure
- $f$: body force acting on the fluid
- $\nu$: kinematic viscosity

Maxwell operator in mixed form:

$$
\begin{aligned}
\nabla \times (\nabla \times b) + \nabla r = g \qquad & \text{in } \Omega, \\
\nabla \cdot b = 0 \qquad & \text{in } \Omega, \\
b \times n = b_D \qquad & \text{on } \partial\Omega, \\
r = 0 \qquad & \text{on } \partial\Omega,
\end{aligned}
$$

- $b$: magnetic field
- $r$: scalar multiplier
- $n$: unit outward normal

# MHD model: coupled Navier-Stokes and Maxwell's equations

$$
\begin{aligned}
-\nu\,\Delta u + (u\cdot\nabla)u + \nabla p - \kappa\,(\nabla\times b)\times b &= f && \text{in } \Omega,\\
\nabla\cdot u &= 0 && \text{in } \Omega,\\
\kappa\nu_m\,\nabla\times(\nabla\times b) + \nabla r - \kappa\,\nabla\times(u\times b) &= g && \text{in } \Omega,\\
\nabla\cdot b &= 0 && \text{in } \Omega,
\end{aligned}
$$

with appropriate boundary conditions.

- $(\nabla\times b)\times b$: Lorentz force accelerates the fluid particles in the direction normal to the electric and magnetic fields.
- $\nabla\times(u\times b)$: electromotive force modifying the magnetic field

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# Discretisation

- Finite element discretisation based on the formulation in Schötzau 2004
- Fluid variables: lowest order Taylor-Hood ($\mathcal{P}_2/\mathcal{P}_1$)
- Magnetic variables: mixed Nédélec element approximation
- Nédélec elements capture solutions correctly on non-convex domains

# Non-linear solver

- MHD model is non-linear: $\mathcal{K}(x)x = b$
- Standard Oseen iteration: $\mathcal{K}(x_k)x_{k+1} = b$
- Re-arrange to solve for updates
- $\mathcal{K}(x_k)\delta x = b - \mathcal{K}(x_k)x_k$ where $x_{k+1} = x_k + \delta x$

Discretised and linearised MHD model:

$$\begin{pmatrix} A+O(u) & B^T & C(b)^T & 0 \\ B & 0 & 0 & 0 \\ -C(b) & 0 & M & D^T \\ 0 & 0 & D & 0 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta p \\ \delta b \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_p \\ r_b \\ r_r \end{pmatrix},$$

with

$$\begin{aligned} r_u &= f - Au - O(u)u - C(b)^T b - B^T p, \\ r_p &= -Bu, \\ r_b &= g - Mu + C(b)b - D^T r, \\ r_r &= -Db. \end{aligned}$$

$A$: discrete Laplacian operator, $O$: discrete convection operator,
$B$: discrete divergence operator, $M$: discrete curl-curl operator,
$C$: coupling terms, $D$: discrete divergence operator.

# Decoupling schemes

Magnetic Decoupling (MD):

$$\mathcal{K}_{\mathrm{MD}} = \left(\begin{array}{cc|cc} A + O(u) & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right)$$

Complete Decoupling (CD):

$$\mathcal{K}_{\mathrm{CD}} = \left(\begin{array}{cc|cc} A & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right)$$

# Linear solver and Preconditioning

Consider

$$Ax = b,$$

to iteratively solve:

$$\text{find } x_k \in x_0 + \text{span}\{r_0, Ar_0, \ldots, A^{k-1}r_0\}$$

where $r_0 = b - Ax_0$ and $x_0$ is the initial guess.

Key for success: preconditioning

1. the preconditioner $P$ approximates $A$
2. $P$ easy to solve for than $A$

Want eigenvalues of $P^{-1}A$ to be clusters

# Ideal preconditioning

Non-singular $(1,1)$ block

$$\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F & B^T \\ 0 & BF^{-1}B^T \end{pmatrix}$$

Murphy, Golub & Wathen 2000 showed exactly two eigenvalues: $\pm 1$

Singular $(1,1)$ block

$$\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F + B^T W^{-1} B & 0 \\ 0 & W \end{pmatrix},$$ where $W$ is SPD

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# Navier-Stokes subproblem

$$\mathcal{K}_{\mathrm{NS}} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}$$

where $F = A + O$ is the discrete convection diffusion operator. Shown in Elman, Silvester & Wathen 2005/2014 that

$$\mathcal{P}_{\mathrm{NS}} = \begin{pmatrix} F & B^T \\ 0 & S \end{pmatrix}, \quad S = A_p F_p^{-1} Q_p$$

is a good approximation to the Schur complement preconditioner.
$A_p$: pressure space Laplacian, $F_p$: pressure space convection-diffusion operator, $Q_p$: pressure space mass matrix

# Maxwell subproblem

$$\mathcal{K}_{\rm NS} = \begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix}$$

*N*ote: M is highly rank defficient.

Greif & Schötzau 2007 shows that $L$ (scalar Laplacian) is the appropriate choice for $W$

$$\mathcal{P}_{\rm iM} = \begin{pmatrix} M + B^T L^{-1} B & 0 \\ 0 & L \end{pmatrix}$$

Practical preconditioner:

$$\mathcal{P}_{\rm M} = \begin{pmatrix} M + X & 0 \\ 0 & L \end{pmatrix}$$

where $X$ vector mass matrix is spectrally equivalent to $B^T L^{-1} B$

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# MHD problem

Combining the Navier-Stokes and Maxwell preconditioners

$$\mathcal{P}_{\text{MH}} = \begin{pmatrix} F & B^T & C^T & 0 \\ 0 & -S & 0 & 0 \\ -C & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

with the inner preconditioner

$$\mathcal{P}_{\text{innerMH}} = \begin{pmatrix} F & B^T & 0 & 0 \\ 0 & -S & 0 & 0 \\ 0 & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

# Summary of decoupling scheme preconditioners

| Iteration scheme | Coefficient matrix | Preconditioner |
|---|---|---|
| (MD) | $\left(\begin{array}{cc\|cc} F & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right)$ | $\left(\begin{array}{cc\|cc} F & B^T & 0 & 0 \\ 0 & -S & 0 & 0 \\ \hline 0 & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{array}\right)$ |
| (CD) | $\left(\begin{array}{cc\|cc} A & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right)$ | $\left(\begin{array}{cc\|cc} A & 0 & 0 & 0 \\ 0 & \frac{1}{\nu}Q_p & 0 & 0 \\ \hline 0 & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{array}\right)$ |

TABLE : Summary of coefficient matrices and corresponding preconditioners for each the decoupling scheme

# Numerical software used

- Finite element software FEniCS: core libraries are the problem-solving interface DOLFIN, the compiler for finite element variational forms FFC, the finite element tabulator FIAT for creating finite element function spaces, the just-in-time compiler Instant, the code generator UFC and the form language UFL.
- Linear algebra software: HYPRE as a multigrid solver and the sparse direct solvers UMFPACK, PASTIX, SuperLU and MUMPS

# Navier-Stokes subproblem in isolation: 2D

| $\ell$ | Dofs | Average iterations | | | |
|---|---|---|---|---|---|
| | $u_h/p_h$ | $\nu = 10$ | $\nu = 1$ | $\nu = 0.1$ | $\nu = 0.01$ |
| 5 | 8,450/1,089 | 17 | 17 | 21 | 58 |
| 6 | 33,282/4,225 | 17 | 17 | 22 | 30 |
| 7 | 132,098/16,641 | 18 | 17 | 22 | 21 |
| 8 | 526,338/66,049 | 18 | 18 | 22 | 20 |
| 9 | 2,101,250/263,169 | 18 | 19 | 22 | 21 |

TABLE : Iteration table for a PCD preconditioned for various values of $\nu$

# Maxwell subproblem in isolation: 2D

| $\ell$ | Dofs $b_h/r_h$ | Number of iterations | | | |
|---|---|---|---|---|---|
| | | $\nu_m = 10$ | $\nu_m = 100$ | $\nu_m = 1000$ | $\nu_m = 10000$ |
| 5 | 10,368/4,225 | 5 | 4 | 6 | 6 |
| 6 | 41,216/16,641 | 5 | 6 | 6 | 6 |
| 7 | 164,352/66,049 | 5 | 6 | 6 | 6 |
| 8 | 656,384/263,169 | 5 | 6 | 6 | 8 |
| 9 | 2,623,488/1,050,625 | 4 | 6 | 6 | 8 |
| 10 | 10,489,856/4,198,401 | 4 | 6 | 8 | 10 |

TABLE : Iteration count for Maxwell preconditioner

# MHD: why inner-outer?

| $\ell$ | Dofs | $\kappa = 0.1$ Its$_{\mathrm{NL}}$ | Its$_{\mathrm{O}}$ | Its$_{\mathrm{I}}$ | $\kappa = 1$ Its$_{\mathrm{NL}}$ | Its$_{\mathrm{O}}$ | Its$_{\mathrm{I}}$ | $\kappa = 10$ Its$_{\mathrm{NL}}$ | Its$_{\mathrm{O}}$ | Its$_{\mathrm{I}}$ | $\kappa = 100$ Its$_{\mathrm{NL}}$ | Its$_{\mathrm{O}}$ | Its$_{\mathrm{I}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 6,180 | 5 | 22.2 | 15.4 | 6 | 18.8 | 14.7 | 7 | 30.7 | 14.1 | 9 | 61.4 | 24.4 |
| 5 | 24,132 | 5 | 23.8 | 11.4 | 6 | 27.3 | 15.2 | 8 | 43.8 | 24.0 | 10 | 80.3 | 37.9 |
| 6 | 95,364 | 5 | 28.0 | 17.4 | 6 | 20.2 | 15.3 | 7 | 41.1 | 15.4 | 9 | 74.6 | 31.1 |
| 7 | 379,140 | 5 | 18.6 | 14.6 | 7 | 16.3 | 14.2 | 7 | 37.4 | 16.4 | 14 | 73.9 | 34.7 |
| 8 | 1,511,940 | 5 | 20.4 | 15.2 | 8 | 24.3 | 14.9 | 7 | 39.6 | 18.4 | 11 | 75.4 | 33.3 |

TABLE : Number of non-linear and average number of preconditioning iterations for various values of $\kappa$ with $\nu = 1$ and $\nu_m = 10$.

| $\ell$ | Dofs | $\kappa = 0.1$ Its$_{\mathrm{NL}}$ | Its$_{\mathrm{NS}}$ | Its$_{\mathrm{M}}$ | $\kappa = 1$ Its$_{\mathrm{NL}}$ | Its$_{\mathrm{NS}}$ | Its$_{\mathrm{M}}$ | $\kappa = 10$ Its$_{\mathrm{NL}}$ | Its$_{\mathrm{NS}}$ | Its$_{\mathrm{M}}$ | $\kappa = 100$ Its$_{\mathrm{NL}}$ | Its$_{\mathrm{NS}}$ | Its$_{\mathrm{M}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 6,180 | 4 | 22.5 | 4.5 | 5 | 23.0 | 3.4 | 10 | 21.9 | 2.3 | - | 22.2 | 2.2 |
| 5 | 24,132 | 4 | 22.1 | 4.5 | 5 | 22.0 | 3.4 | 10 | 21.4 | 2.3 | - | 21.7 | 2.2 |
| 6 | 95,364 | 4 | 21.5 | 4.5 | 5 | 21.2 | 3.4 | 10 | 21.1 | 2.3 | - | 21.5 | 2.3 |
| 7 | 379,140 | 4 | 21.5 | 4.8 | 5 | 21.2 | 3.4 | 10 | 21.1 | 2.4 | - | 21.6 | 2.2 |
| 8 | 1,511,940 | 4 | 21.5 | 4.8 | 5 | 21.4 | 3.4 | 10 | 21.1 | 2.4 | - | 21.7 | 2.2 |

TABLE : Number of non-linear iterations and average number of iterations to solve the Navier-Stokes and Maxwell's subproblem for the MD scheme with $\nu = 1$ and $\nu_m = 10$.

# MHD: (MD) scheme 2D

| $\ell$ | Dofs | Av solve time | Total time | $\text{Its}_{\text{NL}}$ | $\text{Its}_{\text{NS}}$ | $\text{Its}_{\text{M}}$ |
|---|---|---|---|---|---|---|
| 5 | 24,132 | 0.5 | 7.3 | 5 | 22.0 | 3.4 |
| 6 | 95,364 | 2.5 | 30.4 | 5 | 21.2 | 3.4 |
| 7 | 379,140 | 13.1 | 134.7 | 5 | 21.2 | 3.4 |
| 8 | 1,511,940 | 69.8 | 627.3 | 5 | 21.4 | 3.4 |
| 9 | 6,038,532 | 407.7 | 3159.7 | 5 | 21.6 | 3.2 |
| 10 | 24,135,684 | 3022.1 | 19668.3 | 5 | 21.6 | 3.4 |

TABLE : Number of non-linear iterations and average number of iterations to solve the Navier-Stokes and Maxwell's subproblem for the MD scheme with $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$.

# MHD: (MD) scheme 3D

| l | Dofs | Av solve time | Total time | $\text{Its}_{\text{NL}}$ | $\text{Its}_{\text{NS}}$ | $\text{Its}_{\text{M}}$ |
|---|---|---|---|---|---|---|
| 1 | 963 | 0.04 | 2.7 | 5 | 23.2 | 3.4 |
| 2 | 5977 | 0.20 | 17.2 | 5 | 34.2 | 3.0 |
| 3 | 41805 | 4.86 | 151.1 | 5 | 34.2 | 3.4 |
| 4 | 312,085 | 242.1 | 2222.8 | 5 | 32.4 | 3.4 |
| 5 | 2,410,533 | 30222.3 | 159032.8 | 5 | 30.8 | 3.2 |

TABLE : Number of non-linear iterations and average number of iterations to solve the Navier-Stokes and Maxwell's subproblem for the MD scheme with $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$ in 3D.

# MHD: (CD) scheme 2D

| $\ell$ | Dofs | Av solve time | Total time | $\text{Its}_{\text{NL}}$ | $\text{Its}_{\text{S}}$ | $\text{Its}_{\text{M}}$ |
|---|---|---|---|---|---|---|
| 5 | 24,132 | 0.4 | 7.7 | 11 | 29.0 | 3.3 |
| 6 | 95,364 | 2.6 | 38.8 | 11 | 28.5 | 3.4 |
| 7 | 379,140 | 13.0 | 181.7 | 11 | 27.5 | 3.4 |
| 8 | 1,511,940 | 66.5 | 888.0 | 11 | 28.3 | 3.5 |
| 9 | 6,038,532 | 358.0 | 4565.4 | 11 | 28.3 | 3.4 |
| 10 | 24,135,684 | 2335.7 | 28337.4 | 11 | 27.4 | 3.5 |

TABLE : Number of non-linear iterations and average number of iterations to solve the Stokes and Maxwell's subproblem for the CD scheme with $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$.

# MHD: (CD) scheme 3D

| $\ell$ | Dofs | Av solve time | Total time | $\text{Its}_{\text{NL}}$ | $\text{Its}_{\text{S}}$ | $\text{Its}_{\text{M}}$ |
|---|---|---|---|---|---|---|
| 1 | 963 | 0.03 | 1.8 | 6 | 30.0 | 3.5 |
| 2 | 5,977 | 0.20 | 9.9 | 6 | 45.7 | 3.2 |
| 3 | 41,805 | 4.32 | 89.2 | 6 | 43.0 | 2.8 |
| 4 | 312,085 | 214.3 | 1786.5 | 6 | 42.3 | 2.8 |
| 5 | 2,410,533 | 26954.4 | 165671.1 | 6 | 41.3 | 2.8 |

TABLE : Number of non-linear iterations and average number of iterations to solve the Stokes and Maxwell's subproblem for the CD scheme with $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$ in 3D.

Future work:

- Scalable inner solvers
- Release code on a public repository
- Parallelisation of the code
- Robustness with respect to kinematic viscosity
- Other non-linear solvers
- Different mixed finite element discretisations