MHD - Neumann bilinear form

The variational for the MHD model with inhomogeneous Neumann conditions is

$$
\begin{aligned}
A(\boldsymbol{u}_h, \boldsymbol{v}) + O(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}) + C(\boldsymbol{b}_h; \boldsymbol{v}, \boldsymbol{b}_h) + B(\boldsymbol{v}, p_h) &= (\boldsymbol{f}, \boldsymbol{v})_\Omega - (\boldsymbol{p}_N, v)_{\Omega_N} \\
B(\boldsymbol{u}_h, q) &= 0, \\
M(\boldsymbol{b}_h, \boldsymbol{c}) - C(\boldsymbol{b}_h; \boldsymbol{u}_h, \boldsymbol{c}) + D(\boldsymbol{c}, r_h) &= (\boldsymbol{g}, \boldsymbol{c})_\Omega, \\
D(\boldsymbol{b}_h, s) &= 0,
\end{aligned}
\tag{1}
$$

where $\boldsymbol{p}_N$ is the Neumann condition. Then the Picard iteration is given by:

$$
\begin{aligned}
A(\delta\boldsymbol{u}_h, \boldsymbol{v}) + O(\boldsymbol{u}_h; \delta\boldsymbol{u}_h, \boldsymbol{v}) + C(\boldsymbol{b}_h; \boldsymbol{v}, \delta\boldsymbol{u}_h) + B(\boldsymbol{v}, \delta p_h) &= R_u(\boldsymbol{u}_h, \boldsymbol{b}_h, p_h; \boldsymbol{v}), \\
B(\delta\boldsymbol{u}_h, q) &= R_p(\boldsymbol{u}_h; q), \\
M(\delta\boldsymbol{b}_h, \boldsymbol{c}) + D(\boldsymbol{c}, \delta r_h) - C(\boldsymbol{b}_h; \delta\boldsymbol{u}_h, \boldsymbol{v}) &= R_b(\boldsymbol{u}_h, \boldsymbol{b}_h, r_h; \boldsymbol{c}), \\
D(\delta\boldsymbol{b}_h, s) &= R_r(\boldsymbol{b}_h; s),
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
R_u(\boldsymbol{u}_h, \boldsymbol{b}_h, p_h; \boldsymbol{v}) =& (\boldsymbol{f}, \boldsymbol{v})_\Omega - (\boldsymbol{p}_N, v)_{\Omega_N} - A(\boldsymbol{u}_h, \boldsymbol{v}) - O(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}) \\
& - C(\boldsymbol{b}_h; \boldsymbol{v}, \boldsymbol{b}_h) - B(\boldsymbol{v}, p_h), \\
R_p(\boldsymbol{u}_h; q) =& - B(\boldsymbol{u}_h, q), \\
R_b(\boldsymbol{u}_h, \boldsymbol{b}_h, r_h; \boldsymbol{c}) =& (\boldsymbol{g}, c)_\Omega - M(\boldsymbol{b}_h, \boldsymbol{c}) + C(\boldsymbol{b}_h; \boldsymbol{u}_h, \boldsymbol{c}) - D(\boldsymbol{c}, r_h), \\
R_r(\boldsymbol{b}_h; s) =& - D(\boldsymbol{b}_h, s),
\end{aligned}
\tag{3}
$$

Therefore you need to enforce the inhomogeneous Neumann conditions at each non-linear iteration???? Also, if you enforce homogeneous boundary conditions within the non-linear iteration doesn't this stop model from capturing the pressure driven flow?

MHD - smooth Neumann conditions

Consider the exact solution:

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u = ( x*y*exp(x + y) + x*exp(x + y) ,  -x*y*exp(x + y) - y*exp(x + y) )

p = ( exp(y)*sin(x) )

b = ( x*cos(x) ,  x*y*sin(x) - y*cos(x) )

r = ( x*sin(2*pi*x)*sin(2*pi*y) )
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The error tables are given below. Here I used a unit square domain with Neumann conditions on the left and right boundaries and Dirichlet on the top and bottom ones.

ℓ	Dofs $\boldsymbol{u}_h/p_h$	$\|\boldsymbol{e}_u\|_{L^2(\Omega)}$	r	$\|\boldsymbol{e}_u\|_{H^1(\Omega)}$	r	$\|e_p\|_{L^2(\Omega)}$	r
1	50/9	9.0942e-02	-	1.2457e+00	-	5.6315e-01	-
2	162/25	1.1441e-02	3.53	3.2338e-01	2.29	7.8755e-02	3.85
3	578/81	1.3656e-03	3.34	8.1946e-02	2.16	9.2087e-03	3.65
4	2,178/289	1.6719e-04	3.17	2.0637e-02	2.08	1.0929e-03	3.35
5	8,450/1,089	2.0858e-05	3.07	5.1826e-03	2.04	1.6142e-04	2.88
6	33,282/4,225	2.6265e-06	3.02	1.2990e-03	2.02	3.1921e-05	2.39
7	132,098/16,641	2.7450e-07	3.28	3.2522e-04	2.01	7.4632e-06	2.12

Table 1: Convergence for 2D MHD - fluid variables

ℓ	Dofs $\boldsymbol{b}_h/r_h$	$\|\boldsymbol{e}_b\|_{L^2(\Omega)}$	l	$\|\boldsymbol{e}_b\|_{H(\text{curl},\Omega)}$	l
1	16/9	1.8060e-01	-	2.6788e-01	-
2	56/25	9.1265e-02	1.09	1.3398e-01	1.11
3	208/81	4.5753e-02	1.05	6.7003e-02	1.06
4	800/289	2.2892e-02	1.03	3.3503e-02	1.03
5	3,136/1,089	1.1448e-02	1.01	1.6752e-02	1.01
6	12,416/4,225	5.7241e-03	1.01	8.3759e-03	1.01
7	49,408/16,641	2.8621e-03	1.00	4.1879e-03	1.00

Table 2: Convergence for 2D MHD - magnetic variable

ℓ	Dofs $\boldsymbol{b}_h/r_h$	$\|\boldsymbol{e}_r\|_{L^2(\Omega)}$	l	$\|\boldsymbol{e}_r\|_{H^1(\Omega)}$	l
1	16/9	2.7524e-01	-	2.4780e+00	-
2	56/25	1.4850e-01	1.21	1.7787e+00	0.65
3	208/81	4.8879e-02	1.89	1.0042e+00	0.97
4	800/289	1.3198e-02	2.06	5.1942e-01	1.04
5	3,136/1,089	3.3659e-03	2.06	2.6198e-01	1.03
6	12,416/4,225	8.4572e-04	2.04	1.3128e-01	1.02
7	49,408/16,641	2.1170e-04	2.02	6.5676e-02	1.01

Table 3: Convergence for 2D MHD - multiplier variable

MHD - Hartmann Neumann conditions

Small domain - $(0, 10) \times (1, 1)$

Large domain - $(0, 50) \times (1, 1)$