Chapter 6
A Posteriori Error Estimation

In this chapter, we present some basic techniques for developing a posteriori error estimation for solving Maxwell's equations. It is known that the a posteriori error estimation plays a very important role in adaptive finite element method. In Sect. 6.1, we provide a brief overview of a posteriori error estimation. Then in Sect. 6.2, through time-harmonic Maxwell's equations, we demonstrate the fundamental ideas on how to obtain the upper and lower posteriori error estimates. In Sect. 6.3, we present a posteriori error estimator obtained for a cold plasma model described by integro-differential Maxwell's equations.

6.1 A Brief Overview of A Posteriori Error Analysis

How to use the computational solution to guide where to refine or coarsen the local mesh grid and/or how to choose the proper orders of the basis function in different regions becomes an essential problem in the adaptive finite element method. Since the pioneering work of Babuska and Rheinboldt in the late 1970s [16], the adaptive finite element method has been well developed as evidenced by the vast literature in this area. If an error estimate for the unknown exact solution is totally based on the available computational result, then this error estimate is called **a posteriori error estimator**. How to develop a robust a posteriori error estimator plays an important role in developing an effective adaptive finite element method. Due to the intelligent work of many researchers over the past three decades, the study of a posteriori error estimator for standard elliptic, parabolic and second order hyperbolic problems seems mature (e.g., review papers [32,64,111,126,227], books [4,20,21,252,287,297], and references cited therein).

On the other hand, works on a posteriori error estimators for Maxwell's equations are quite limited. The analysis of a posteriori error estimators for the edge elements was initiated by Monk in 1998 [216] and Beck et al. in 2000 [31]. So far, there are only about two dozens of papers devoted to Maxwell's equations in free space

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[48, 72, 82, 139, 147, 148, 159, 253, 305, 306]. For example, Monk [216] obtained a posteriori error estimator for a scattering problem interacting with a bounded inhomogeneous and anisotropic scatterer. Beck et al. [31] developed a residual-based a posteriori error estimator for the model problem (6.1) and (6.2) shown below. In this seminar paper, they obtained both the lower and upper bounds. In 2003, Nicaise et al. [225] considered residual-based a posteriori error estimator for the same model. Then in 2005, Nicaise [224] developed a posteriori Zienkiewicz-Zhu type error estimators for the same problem. Recently, Houston et al. [147] developed a posteriori error estimator for a mixed discontinuous Galerkin approximation of time-harmonic Maxwell's equations.

In the following two sections, we present details on those basic techniques of how to derive a posteriori error estimator for Maxwell's equations in free space and cold plasma, respectively. The rest of this chapter is mainly based on papers [72, 182].

6.2 A Posteriori Error Estimator for Free Space Model

6.2.1 Preliminaries

When discretizing the time-domain free space Maxwell's equations in time, we end up solving the following problem at each time step [31, 72]:

$$\nabla \times (\alpha(\mathbf{x})\nabla \times \mathbf{u}) + \beta(\mathbf{x})\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \tag{6.1}$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{6.2}$$

where $\mathbf{u}$ is the time approximation of either the electric field $\mathbf{E}$ or the magnetic field $\mathbf{H}$, $\mathbf{f} \in H(\text{div}; \Omega)$ is a source function, while $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are the underlying medium parameters. Here for simplicity, we only consider the perfect conductor boundary condition (6.2). Throughout this chapter, Ω is assumed to be a bounded, simply-connected domain in R^3 with connected Lipschitz polyhedral boundary, whose unit outward normal vector is denoted as $\mathbf{n}$.

For simplicity, we assume that $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are piecewise positive constant functions on Ω, and Ω is composed of two disjoint subdomains Ω_1 and Ω_2. More specifically,

$$\alpha = \alpha_i \qquad \text{and } \beta = \beta_i \quad \text{in } \Omega_i,$$

where both Ω_1 and Ω_2 are simply-connected Lipschitz polyhedra.

It is easy to obtain a weak formulation of (6.1) and (6.2): Find $\mathbf{u} \in H_0(\text{curl}; \Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall\, \mathbf{v} \in H_0(\text{curl}; \Omega), \tag{6.3}$$

where the bilinear form $a(\mathbf{u}, \mathbf{v})$ is given by

$$a(\mathbf{u}, \mathbf{v}) = (\alpha\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + (\beta\mathbf{u}, \mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v} \in H_0(\text{curl}; \Omega).$$

Recall that the space $H_0(\text{curl}; \Omega)$ is defined as

$$H_0(\text{curl}; \Omega) = \left\{ \mathbf{u} \in (L^2(\Omega))^3 : \nabla \times \mathbf{u} \in (L^2(\Omega))^3 \text{ and } \mathbf{u} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega \right\}.$$

The well-posedness of (6.3) is guaranteed by the Lax-Milgram lemma.

To develop a finite element method for solving (6.3), we consider a shape-regular mesh T_h that partitions the domain Ω into disjoint tetrahedral elements $\{K\}$, such that $\overline{\Omega} = \bigcup_{K \in T_h} K$. Following the same notations defined in Sect. 4.2.2, we denote the diameter of K by h_K, the mesh size by $h = \max_{K \in T_h} h_K$, the set of all interior faces by F_h^I, the set of all boundary faces by F_h^B, and the set of all faces by $F_h = F_h^I \cup F_h^B$. Furthermore, we denote ω_K for the union of all elements K having a common face with K, and ω_F for the union of two elements sharing F.

With the above preparation, we can develop the finite element approximation of (6.3): Find $\mathbf{u} \in \mathbf{V}_h^0$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h^0, \tag{6.4}$$

where we use the lowest order edge element space (cf. Example 3.8):

$$\mathbf{V}_h^0 = \left\{ \mathbf{v} \in H_0(\text{curl}; \Omega) : \mathbf{v}|_K = \mathbf{a}_K \times \mathbf{x} + \mathbf{b}_K, \ \ \mathbf{a}_K, \mathbf{b}_K \in R^3, \ \forall\, K \in T_h \right\}.$$

Below are some fundamental results (cf. [145, 217]) needed for deriving a posterior error estimator.

Lemma 6.1. *The space $H_0(curl; \Omega)$ admits the following β-orthogonal decomposition*

$$H_0(curl; \Omega) = H_0^0(curl; \Omega) \oplus H_0^{\perp}(curl; \Omega), \tag{6.5}$$

where

$$H_0^0(curl; \Omega) \equiv \{ \mathbf{v} \in H_0(curl; \Omega) : \nabla \times \mathbf{v} = \mathbf{0} \}$$

and

$$H_0^{\perp}(curl; \Omega) \equiv \left\{ \mathbf{v} \in H_0(curl; \Omega) : (\beta \mathbf{v}, \mathbf{v}^0) = 0, \ \mathbf{v}^0 \in H_0^0(curl; \Omega) \right\}.$$

Lemma 6.2. *If the domain Ω is simply connected with connected boundary, we have*

$$H_0^0(curl; \Omega) = \nabla H_0^1(\Omega) \tag{6.6}$$

and

$$||\mathbf{v}||_0 \leq C ||\nabla \times \mathbf{v}||_0 \quad \forall\, \mathbf{v} \in H_0^{\perp}(curl; \Omega), \tag{6.7}$$

where the constant $C > 0$ depends on Ω only.

Lemma 6.3. *[145, Lemma 2.4] Assume that Ω is a bounded Lipschitz domain, then for any $\mathbf{v} \in H_0(curl;\Omega)$, there exists the regular decomposition*

$$\mathbf{v} = \mathbf{w} + \nabla\phi, \tag{6.8}$$

where $\mathbf{w} \in H_0(curl;\Omega) \cap (H^1(\Omega))^3$ and $\phi \in H_0^1(\Omega)$. Moreover, there is a positive constant C_{hip} depending only on Ω such that

$$||\mathbf{w}||_1 \le C_{hip}||\mathbf{v}||_{curl}, \quad ||\phi||_1 \le C_{hip}||\mathbf{v}||_{curl}, \tag{6.9}$$

here and below we define the norm $||\mathbf{v}||_{curl} = (||\mathbf{v}||_0^2 + ||\nabla \times \mathbf{v}||_0^2)^{1/2}$.

Lemma 6.4. *Let D_K (resp. D_F) denote the union of elements in T_h with non-empty intersection with K (resp. F). Furthermore, we denote a generic constant $C > 0$, which depends only on the shape regularity of the mesh.*

(i) [31] for any $\mathbf{w} \in H_0(curl;\Omega) \cap (H^1(\Omega))^3$, there exists the quasi-interpolation $\Pi_h\mathbf{w} \in \mathbf{V}_h^0$ such that

$$||\mathbf{w} - \Pi_h\mathbf{w}||_{0,K} \le Ch_K|\mathbf{w}|_{1,D_K} \quad \forall\, K \in T_h, \tag{6.10}$$

$$||\mathbf{w} - \Pi_h\mathbf{w}||_{0,F} \le Ch_F^{\frac{1}{2}}|\mathbf{w}|_{1,D_F}, \quad \forall\, F \in F_h. \tag{6.11}$$

(ii) [127, Sect. I.A.3] let S_0^h be the continuous piecewise linear finite element subspace of $H_0^1(\Omega)$. Then for any $\phi \in H_0^1(\Omega)$, there exists a continuous piecewise linear approximation $I_h\phi \in S_0^h$ such that

$$||\phi - I_h\phi||_{0,K} \le Ch_K|\phi|_{1,D_K} \quad \forall\, K \in T_h, \tag{6.12}$$

$$\beta_K^{\frac{1}{2}}||\phi - I_h\phi||_{0,K} \le Ch_K||\beta^{\frac{1}{2}}\nabla\phi||_{0,D_K} \quad \forall\, K \in T_h, \tag{6.13}$$

$$||\phi - I_h\phi||_{0,F} \le Ch_F^{\frac{1}{2}}|\phi|_{1,D_F} \quad \forall\, F \in F_h, \tag{6.14}$$

$$\beta_F^{\frac{1}{2}}||\phi - I_h\phi||_{0,F} \le Ch_F^{\frac{1}{2}}||\beta^{\frac{1}{2}}\nabla\phi||_{0,D_F} \quad \forall\, F \in F_h. \tag{6.15}$$

6.2.2 An Upper Bound of A Posterior Error Estimator

Before presenting the error estimate, we need to introduce some notations:

$$\Lambda_K^{\alpha} \equiv \frac{\alpha_K}{\alpha_m}, \;\; \Lambda_F^{\alpha} \equiv \frac{\alpha_F}{\alpha_m}, \;\; \Lambda_K^{\beta\alpha} \equiv \frac{\beta_K}{\alpha_m}, \;\; \Lambda_F^{\beta\alpha} \equiv \frac{\beta_F}{\alpha_m}, \;\; \forall\, K \in T_h, \;\; F \in F_h,$$

where $\alpha_m = \min\{\alpha_1, \alpha_2\}$. Furthermore, we define the element residuals

$$\mathbf{R}_K(\mathbf{u}_h) = \mathbf{f} - \beta\mathbf{u}_h \quad \text{in } K \in T_h,$$

and the jump residuals: for any $F \in F_h$,

$$\mathbf{J}_{F1}(\mathbf{u}_h) \equiv -[\alpha(\nabla \times \mathbf{u}_h) \times \mathbf{n}]_F \ \text{ and } \ \mathbf{J}_{F2}(\mathbf{u}_h) \equiv [(\mathbf{f} - \beta \mathbf{u}_h) \cdot \mathbf{n}]_F.$$

For simplicity, in the rest of this section, we write

$$[g(\mathbf{n})]_F = g(\mathbf{n}_F)|_{K_1} + g(\mathbf{n}_F)|_{K_2}$$

with $g(\mathbf{n})$ being either $\alpha(\nabla \times \mathbf{u}_h) \times \mathbf{n}$ or $(\mathbf{f} - \beta \mathbf{u}_h) \cdot \mathbf{n}$, and K_1 and K_2 are the two neighboring elements sharing the face F with unit outward normal vector $\mathbf{n}_F$. Furthermore, without confusion, we often use the short notation

$$\mathbf{R}_K = \mathbf{R}_K(\mathbf{u}_h), \ \ \mathbf{J}_{F1} = \mathbf{J}_{F1}(\mathbf{u}_h) \ \text{ and } \ \mathbf{J}_{F2} = \mathbf{J}_{F2}(\mathbf{u}_h).$$

Denote the solution error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$. Below we will bound the energy norm $||\mathbf{e}||_a = \sqrt{a(\mathbf{e}, \mathbf{e})}$ from above and below by the local error indicators

$$\eta_h^2(K) = \eta_{1,K}^2 + \sum_{F \subset \partial K, F \in F_h} \eta_{1,F}^2 + \eta_{2,K}^2 + \sum_{F \subset \partial K, F \in F_h} \eta_{2,F}^2, \tag{6.16}$$

where

$$\eta_{1,K}^2 = \Lambda_K^{\alpha} ||h_K \alpha_K^{-1/2} \mathbf{R}_K||_{0,K}^2, \quad \eta_{1,F}^2 = \Lambda_F^{\alpha} ||h_F^{1/2} \alpha_F^{-1/2} \mathbf{J}_{F1}||_{0,F}^2,$$
$$\eta_{2,K}^2 = \max(1, \Lambda_K^{\beta\alpha}) ||h_K \beta_K^{-1/2} \mathrm{div} \mathbf{f}||_{0,K}^2, \ \ \eta_{2,F}^2 = \max(1, \Lambda_F^{\beta\alpha}) ||h_F^{1/2} \beta_F^{-1/2} \mathbf{J}_{F2}||_{0,F}^2.$$

The upper bound of the error $\mathbf{u} - \mathbf{u}_h$ is given below.

Theorem 6.1.

$$||\mathbf{u} - \mathbf{u}_h||_a^2 \le C_{up} \sum_{K \in T_h} \eta_h^2(K), \tag{6.17}$$

where the constant $C_{up} > 0$ depends only on the shape regularity of the mesh.

Proof. It is easy to see that the error $\mathbf{e} \in H_0(\mathrm{curl}; \Omega)$ satisfies the error equation

$$a(\mathbf{e}, \mathbf{v}) = r(\mathbf{v}) \quad \forall\ \mathbf{v} \in H_0(\mathrm{curl}; \Omega), \tag{6.18}$$

where the residual

$$r(\mathbf{v}) = (\mathbf{f} - \beta \mathbf{u}_h, \mathbf{v}) - (\alpha\ \nabla \times \mathbf{u}_h, \nabla \times \mathbf{v}) \quad \forall\ \mathbf{v} \in H_0(\mathrm{curl}; \Omega).$$

Using (6.4), we have the Galerkin orthogonality relation

$$a(\mathbf{e}, \mathbf{v}_h) = r(\mathbf{v}_h) = 0 \quad \forall\ \mathbf{v}_h \in \mathbf{V}_h^0. \tag{6.19}$$

Using the orthogonal decomposition (6.5) in the error equation (6.18), we have

$$||\mathbf{e}||_a^2 = r(\mathbf{e}) = r(\mathbf{e}^{\perp}) + r(\mathbf{e}^0), \tag{6.20}$$

where $\mathbf{e}^{\perp} \in H_0^{\perp}(\text{curl}; \Omega)$ and $\mathbf{e}^0 \in H_0^0(\text{curl}; \Omega)$. Then by the decomposition (6.8), we have

$$r(\mathbf{e}^{\perp}) = r(\mathbf{w}) + r(\nabla\phi), \tag{6.21}$$

where $\mathbf{w} \in H_0(\text{curl}; \Omega) \cap (H^1(\Omega))^3$ and $\phi \in H_0^1(\Omega)$.

The proof is completed by combining the estimates of $r(\mathbf{w}), r(\nabla\phi)$ and $r(\mathbf{e}^0)$ proved in Lemmas 6.5–6.7 shown below. □

Lemma 6.5.

$$r(\mathbf{w}) \le C(\sum_{K\in T_h} \eta_{1,K}^2 + \sum_{F\in F_h} \eta_{1,F}^2)^{1/2}||\mathbf{e}^{\perp}||_a.$$

Proof. By the Galerkin orthogonality (6.19), we have

$$r(\mathbf{w}) = r(\mathbf{w} - \Pi_h\mathbf{w}) = (\mathbf{f} - \beta\mathbf{u}_h, \mathbf{w} - \Pi_h\mathbf{w}) - (\alpha\nabla\times\mathbf{u}_h, \nabla\times(\mathbf{w} - \Pi_h\mathbf{w})).$$

Then using integration by parts and the property $\nabla\times(\alpha\,\nabla\times\mathbf{u}_h) = \mathbf{0}$, we obtain

$$\begin{aligned}
r(\mathbf{w}) &= \sum_{K\in T_h} (\mathbf{f} - \beta\mathbf{u}_h - \nabla\times(\alpha\nabla\times\mathbf{u}_h), \mathbf{w} - \Pi_h\mathbf{w}) - \sum_{F\in F_h} ([\alpha\nabla\times\mathbf{u}_h\times\mathbf{n}]_F, \mathbf{w} - \Pi_h\mathbf{w})_F \\
&= \sum_{K\in T_h} (\mathbf{R}_K, \mathbf{w} - \Pi_h\mathbf{w}) + \sum_{F\in F_h} (\mathbf{J}_{F1}, \mathbf{w} - \Pi_h\mathbf{w})_F \\
&\le \sum_{K\in T_h} ||\mathbf{R}_K||_{0,K}||\mathbf{w} - \Pi_h\mathbf{w}||_{0,K} + \sum_{F\in F_h} ||\mathbf{J}_{F1}||_{0,F}||\mathbf{w} - \Pi_h\mathbf{w}||_{0,F} \\
&\le C[(\sum_{K\in T_h} h_K^2||\mathbf{R}_K||_{0,K}^2)^{1/2}|\mathbf{w}|_1 + (\sum_{F\in F_h} h_F||\mathbf{J}_{F1}||_{0,F}^2)^{1/2}|\mathbf{w}|_1] \\
&\le C(\sum_{K\in T_h} \eta_{1,K}^2 + \sum_{F\in F_h} \eta_{1,F}^2)^{1/2}||\alpha^{1/2}\nabla\times\mathbf{e}^{\perp}||_0,
\end{aligned}$$

which concludes the proof. In the above, we used the approximation property (6.10) and (6.11), and the estimate (6.9). □

Lemma 6.6.

$$r(\nabla\phi) \le C(\sum_{K\in T_h} \eta_{2,K}^2 + \sum_{F\in F_h} \eta_{2,F}^2)^{1/2}||\mathbf{e}^{\perp}||_a.$$

Proof. It is known that (cf. (2.6) of [72]):

$$\mathbf{V}_h^0 \cap H_0^0(\text{curl}; \Omega) = \nabla S_0^h, \tag{6.22}$$

which implies that $\nabla\phi_h$ belongs to $\mathbf{V}_h^0$ for any $\phi_h \in S_0^h$.

By the Galerkin orthogonality (6.19), integration by parts and the fact that $\mathrm{div}(\beta\mathbf{u}_h) = 0$ on each element K, we have

$$
\begin{aligned}
r(\nabla\phi) &= r(\nabla(\phi - \Pi_h\phi)) = (\mathbf{f} - \beta\mathbf{u}_h, \nabla(\phi - \Pi_h\phi)) \\
&= \sum_{K\in T_h} (\mathrm{div}(\beta\mathbf{u}_h) - \mathrm{div}\mathbf{f}, \phi - \Pi_h\phi)_K + \sum_{F\in F_h} ([\mathbf{f} - \beta\mathbf{u}_h \cdot \mathbf{n}]_F, \phi - \Pi_h\phi)_F \\
&= \sum_{K\in T_h} (-\mathrm{div}\mathbf{f}, \phi - \Pi_h\phi)_K + \sum_{F\in F_h} (\mathbf{J}_{F_2}, \phi - \Pi_h\phi)_F \\
&\le \sum_{K\in T_h} ||\mathrm{div}\mathbf{f}||_{0,K}||\phi - \Pi_h\phi||_{0,K} + \sum_{F\in F_h} ||\mathbf{J}_{F_2}||_{0,F}||\phi - \Pi_h\phi||_{0,F} \\
&\le C[(\sum_{K\in T_h} h_K^2||\mathrm{div}\mathbf{f}||_{0,K}^2)^{\frac{1}{2}}|\phi|_1 + (\sum_{F\in F_h} h_F||\mathbf{J}_{F_2}||_{0,F}^2)^{\frac{1}{2}}|\phi|_1] \\
&\le C[(\sum_{K\in T_h} h_K^2||\mathrm{div}\mathbf{f}||_{0,K}^2)^{\frac{1}{2}}||\mathbf{e}^{\perp}||_0 + (\sum_{F\in F_h} h_F||\mathbf{J}_{F_2}||_{0,F}^2)^{\frac{1}{2}}||\mathbf{e}^{\perp}||_0] \\
&\le C[(\sum_{K\in T_h} \Lambda_K^{\beta\alpha}||h_K\beta_K^{-\frac{1}{2}}\mathrm{div}\mathbf{f}||_{0,K}^2 + \sum_{F\in F_h} \Lambda_F^{\beta\alpha}||h_F^{\frac{1}{2}}\beta_F^{-\frac{1}{2}}\mathbf{J}_{F2}||_{0,F}^2)^{\frac{1}{2}}||\alpha^{\frac{1}{2}}\nabla\times\mathbf{e}^{\perp}||_0 \\
&\le C(\sum_{K\in T_h} \eta_{2,K}^2 + \sum_{F\in F_h} \eta_{2,F}^2)^{\frac{1}{2}}||\alpha^{\frac{1}{2}}\nabla\times\mathbf{e}^{\perp}||_0,
\end{aligned}
$$

which concludes the proof. Here we used the approximation property (6.12)–(6.14), and the estimates (6.9) and (6.7). □

Lemma 6.7.

$$r(\mathbf{e}^0) \le C(\sum_{K\in T_h} \eta_{2,K}^2 + \sum_{F\in F_h} \eta_{2,F}^2)^{1/2}||\beta^{1/2}\mathbf{e}^0||_0.$$

Proof. By (6.6), we know that there exists some $\psi \in H_0^1(\Omega)$ such that $\mathbf{e}^0 = \nabla\psi$. Hence similar to the proof of Lemma 6.6, we have

$$
\begin{aligned}
r(\mathbf{e}^0) &= r(\nabla(\psi - \Pi_h\psi)) \\
&\le C(\sum_{K\in T_h} ||h_K\beta_K^{-\frac{1}{2}}\mathrm{div}\mathbf{f}||_{0,K}^2 + \sum_{F\in F_h} ||h_F^{\frac{1}{2}}\beta_F^{-\frac{1}{2}}\mathbf{J}_{F2}||_{0,F}^2)^{\frac{1}{2}}||\beta^{\frac{1}{2}}\nabla\psi||_0 \\
&\le C(\sum_{K\in T_h} \eta_{2,K}^2 + \sum_{F\in F_h} \eta_{2,F}^2)^{\frac{1}{2}}||\beta^{\frac{1}{2}}\mathbf{e}^0||_0,
\end{aligned}
$$

which completes the proof. □

6.2.3 A Lower Bound of A Posterior Error Estimator

To obtain a lower bound of the error, we need to use the bubble function technique originally introduced by Verfurth [286] for the elliptic problem. We denote b_K for the standard polynomial bubble function on an element K, and b_F for the standard polynomial bubble function on an interior element face F, shared by two elements K and K'. For simplicity, in the following we denote $UF = \{K, K'\}$ for the union of elements K and K'. For a tetrahedron K, an exemplary element bubble function $b_K = 256\Pi_{i=1}^4\lambda_i$, and face bubble function $b_F = 27\Pi_{i=1}^3\lambda_i$, where λ_i is the standard basis function in S_0^h at vertex x_i.

With these notation, we have the following classical estimates.

Lemma 6.8. *For any polynomial function v on K, there exists a constant $C > 0$ independent of v and h_K such that*

$$||b_K v||_{0,K} \le C||v||_{0,K}, \quad ||v||_{0,K} \le C||b_K^{\frac{1}{2}} v||_{0,K}, \tag{6.23}$$

$$||\nabla(b_K v)||_{0,K} \le Ch_K^{-1}||v||_{0,K}. \tag{6.24}$$

On the other hand, for any polynomial function w on F, there exists a constant $C > 0$ independent of w and h_F such that

$$||w||_{0,F} \le C||b_F^{\frac{1}{2}} w||_{0,F}, \tag{6.25}$$

$$||Ex(b_F w)||_{0,K} \le Ch_F^{\frac{1}{2}}||w||_{0,F} \quad \forall\ K \in UF, \tag{6.26}$$

$$||\nabla Ex(b_F w)||_{0,K} \le Ch_F^{-\frac{1}{2}}||w||_{0,F} \quad \forall\ K \in UF, \tag{6.27}$$

where $Ex(b_F w) \in H_0^1((\overline{K} \cup \overline{K}')^\circ)$ is an extension of $b_F w$ such that $Ex(b_F w)|_F = b_F w$.

The same estimates as (6.23)–(6.27) hold true for vector functions. Moreover, for a vector ploynomial function $\mathbf{v}$ on K, there exists a constant $C > 0$ independent of $\mathbf{v}$ and h_K such that

$$||\nabla \times (b_K \mathbf{v})||_{0,K} \le Ch_K^{-1}||\mathbf{v}||_{0,K}. \tag{6.28}$$

Similarly, for any vector polynomial function $\mathbf{w}$ on F, there exists a constant $C > 0$ independent of $\mathbf{w}$ and h_F such that

$$||\nabla \times Ex(b_F \mathbf{w})||_{0,K} \le Ch_F^{-\frac{1}{2}}||\mathbf{w}||_{0,F} \quad \forall\ K \in UF, \tag{6.29}$$

where $Ex(b_F \mathbf{w}) \in H_0^1((\overline{K} \cup \overline{K}')^\circ)^3$ is an extension of $b_F \mathbf{w}$ such that $Ex(b_F \mathbf{w})|_F = b_F \mathbf{w}$.

Proof. The proof of (6.23), (6.25), and (6.26) can be found in [286, Lemma 4.1]. The proof of (6.24) and (6.27) can be obtained from Eqs. (2.35) and (2.39) of [4], respectively. The proof of (6.28) and (6.29) can be obtained by similar arguments as the proof of (6.24) and (6.27). □

Before deriving the lower bound of the error, let us introduce a few more notations. Let $\overline{\mathbf{R}}_K$ be the integral mean of $\mathbf{R}_K$ over element K, and $\overline{\mathrm{div}\mathbf{f}_K}$ be the integral mean of $\mathrm{div}\mathbf{f}_K$ over element K. Let $\Lambda^{\alpha}_{\omega_K} = \max_{K' \in \omega_K}(\Lambda^{\alpha}_{K'})$ and $\Lambda^{\beta\alpha}_{\omega_K} = \max_{K' \in \omega_K}(\Lambda^{\beta\alpha}_{K'}, 1)$.

First, we have the following lower bound for the local error estimator $\eta^2_{1,K}$.

Lemma 6.9.

$$\eta^2_{1,K} \le C\Lambda^{\alpha}_K[||\alpha^{\frac{1}{2}}\,\nabla \times (\mathbf{u} - \mathbf{u}_h)||^2_{0,K} + h^2_K\beta_K\alpha^{-1}_K||\beta^{\frac{1}{2}}(\mathbf{u} - \mathbf{u}_h)||^2_{0,K}$$
$$+h^2_K\alpha^{-1}_K||\overline{\mathbf{R}}_K - \mathbf{R}_K||^2_{0,K}]. \tag{6.30}$$

Proof. Using (6.23), the facts that $b_K\overline{\mathbf{R}}_K \in H^1_0(\Omega)^3$ and $\nabla \times (\alpha\,\nabla \times \mathbf{u}_h) = 0$, and integration by parts, we have

$$\begin{aligned}
C||\overline{\mathbf{R}}_K||^2_{0,K} &\le (\overline{\mathbf{R}}_K, b_K\overline{\mathbf{R}}_K)_K = (\mathbf{R}_K, b_K\overline{\mathbf{R}}_K)_K + (\overline{\mathbf{R}}_K - \mathbf{R}_K, b_K\overline{\mathbf{R}}_K)_K \\
&= (\mathbf{f} - \beta\mathbf{u}_h - \nabla \times (\alpha\,\nabla \times \mathbf{u}_h), b_K\overline{\mathbf{R}}_K)_K + (\overline{\mathbf{R}}_K - \mathbf{R}_K, b_K\overline{\mathbf{R}}_K)_K \\
&= (\alpha\,\nabla \times (\mathbf{u} - \mathbf{u}_h), \nabla \times (b_K\overline{\mathbf{R}}_K))_K + (\beta(\mathbf{u} - \mathbf{u}_h), b_K\overline{\mathbf{R}}_K)_K \\
&\quad +(\overline{\mathbf{R}}_K - \mathbf{R}_K, b_K\overline{\mathbf{R}}_K)_K \\
&\le C[\alpha^{\frac{1}{2}}h^{-1}_K||\alpha^{\frac{1}{2}}\,\nabla \times (\mathbf{u} - \mathbf{u}_h)||_{0,K} + \beta^{\frac{1}{2}}_K||\beta^{\frac{1}{2}}(\mathbf{u} - \mathbf{u}_h)||_{0,K} \\
&\quad +||\overline{\mathbf{R}}_K - \mathbf{R}_K||_{0,K}]||\overline{\mathbf{R}}_K||_{0,K},
\end{aligned}$$

where in the last step we used the standard inverse estimate and (6.23).

Combining the above estimate with the triangle inequality, we obtain

$$\begin{aligned}
||\mathbf{R}_K||_{0,K} &\le ||\overline{\mathbf{R}}_K||_{0,K} + ||\mathbf{R}_K - \overline{\mathbf{R}}_K||_{0,K} \\
&\le C[\alpha^{\frac{1}{2}}h^{-1}_K||\alpha^{\frac{1}{2}}\,\nabla \times (\mathbf{u} - \mathbf{u}_h)||_{0,K} + \beta^{\frac{1}{2}}_K||\beta^{\frac{1}{2}}(\mathbf{u} - \mathbf{u}_h)||_{0,K} \\
&\quad +||\overline{\mathbf{R}}_K - \mathbf{R}_K||_{0,K}].
\end{aligned} \tag{6.31}$$

Recall the definition of $\eta_{1,K}$, we have

$$\begin{aligned}
\eta^2_{1,K} &= \Lambda^{\alpha}_K h^2_K\alpha^{-1}_K||\mathbf{R}_K||^2_{0,K} \\
&\le C\Lambda^{\alpha}_K[||\alpha^{\frac{1}{2}}\,\nabla \times (\mathbf{u} - \mathbf{u}_h)||^2_{0,K} + h^2_K\beta_K\alpha^{-1}_K||\beta^{\frac{1}{2}}(\mathbf{u} - \mathbf{u}_h)||^2_{0,K} \\
&\quad +h^2_K\alpha^{-1}_K||\overline{\mathbf{R}}_K - \mathbf{R}_K||^2_{0,K}],
\end{aligned}$$

which completes the proof. □

For the local error estimator $\eta_{1,F}^2$, we have the following lower bound.

Lemma 6.10.

$$\eta_{1,F}^2 \le C\Lambda_F^\alpha[\sum_{K\in\omega_F} ||\alpha^{\frac{1}{2}}\nabla\times(\mathbf{u}-\mathbf{u}_h)||_{0,K}^2 + \sum_{K\in\omega_F} h_K^2\beta_K\alpha_K^{-1}||\beta^{\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)||_{0,K}^2 + \sum_{K\in\omega_F} h_K^2\alpha_K^{-1}||\mathbf{R}_K-\overline{\mathbf{R}}_K||_{0,K}^2]. \tag{6.32}$$

Proof. Using (6.25), the fact $\nabla\times(\alpha\ \nabla\times\mathbf{u}_h)=0$, and integration by parts, we have

$$\begin{aligned}
&C||\mathbf{J}_{F1}||_{0,F}^2 \le (\mathbf{J}_{F1}, b_F\mathbf{J}_{F1})_F = -([\alpha\ \nabla\times\mathbf{u}_h\times\mathbf{n}]_F, b_F\mathbf{J}_{F1})_F\\
&= \sum_{K\in\omega_F}(\nabla\times(\alpha\ \nabla\times\mathbf{u}_h), b_F\mathbf{J}_{F1})_K - \sum_{K\in\omega_F}(\alpha\ \nabla\times\mathbf{u}_h, \nabla\times(b_F\mathbf{J}_{F1}))_K\\
&= r(b_F\mathbf{J}_{F1}) - \sum_{K\in\omega_F}(\mathbf{R}_K, b_F\mathbf{J}_{F1})_K\\
&= \sum_{K\in\omega_F}[(\alpha\ \nabla\times(\mathbf{u}-\mathbf{u}_h), \nabla\times(b_F\mathbf{J}_{F1}))_K - (\beta(\mathbf{u}-\mathbf{u}_h), b_F\mathbf{J}_{F1})_K]\\
&\quad - \sum_{K\in\omega_F}(\mathbf{R}_K, b_F\mathbf{J}_{F1})_K\\
&\le C[\sum_{K\in\omega_F}\alpha_K^{\frac{1}{2}}h_K^{-1}||\alpha^{\frac{1}{2}}\nabla\times(\mathbf{u}-\mathbf{u}_h)||_{0,K}||b_F\mathbf{J}_{F1}||_{0,K}\\
&\quad + \sum_{K\in\omega_F}\beta_K^{\frac{1}{2}}||\beta^{\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)||_{0,K}||b_F\mathbf{J}_{F1}||_{0,K} + \sum_{K\in\omega_F}||\mathbf{R}_K||_{0,K}||b_F\mathbf{J}_{F1}||_{0,K}]\\
&\le C[\sum_{K\in\omega_F}\alpha_K^{\frac{1}{2}}h_K^{-\frac{1}{2}}||\alpha^{\frac{1}{2}}\nabla\times(\mathbf{u}-\mathbf{u}_h)||_{0,K} + \sum_{K\in\omega_F}h_K^{\frac{1}{2}}\beta_K^{\frac{1}{2}}||\beta^{\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)||_{0,K}\\
&\quad + \sum_{K\in\omega_F}h_K^{\frac{1}{2}}||\mathbf{R}_K-\overline{\mathbf{R}}_K||_{0,K}]||\mathbf{J}_{F1}||_{0,F}],
\end{aligned}$$

where in the above derivation we used the standard inverse estimate, estimates (6.31) and (6.27).

Hence by the definion of $\eta_{1,F}$, we obtain

$$\begin{aligned}
\eta_{1,F}^2 &= \Lambda_F^\alpha h_F\alpha_F^{-1}||\mathbf{J}_{F1}||_{0,F}^2\\
&\le C\Lambda_F^\alpha[\sum_{K\in\omega_F}||\alpha^{\frac{1}{2}}\nabla\times(\mathbf{u}-\mathbf{u}_h)||_{0,K}^2 + \sum_{K\in\omega_F}h_K^2\beta_K\alpha_K^{-1}||\beta^{\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)||_{0,K}^2\\
&\quad + \sum_{K\in\omega_F}h_K^2\alpha_K^{-1}||\mathbf{R}_K-\overline{\mathbf{R}}_K||_{0,K}^2],
\end{aligned}$$

which concludes the proof. □

For the local error estimator $\eta^2_{2,K}$, we have the following lower bound.

Lemma 6.11.

$$\eta^2_{2,K} \le C \max(\Lambda_K^{\beta\alpha}, 1)[||\beta^{\frac{1}{2}}(\mathbf{u} - \mathbf{u}_h)||^2_{0,K} + h_K^2 \beta_K^{-1} ||\overline{div\mathbf{f}}_K - div\mathbf{f}||^2_{0,K}].$$

Proof. Similar to the proof of (6.31), by using the facts that

$$\mathrm{div}(\beta \mathbf{u}_h) = 0 \text{ and } \mathrm{div}\mathbf{f} = \mathrm{div}(\beta \mathbf{u}),$$

we easily have

$$\begin{aligned} C||\overline{\mathrm{div}\mathbf{f}}_K||^2_{0,K} &\le (\overline{\mathrm{div}\mathbf{f}}_K, b_K \overline{\mathrm{div}\mathbf{f}}_K)_K \\ &= (\mathrm{div}(\mathbf{f} - \beta \mathbf{u}_h), b_K \overline{\mathrm{div}\mathbf{f}}_K)_K + (\overline{\mathrm{div}\mathbf{f}}_K - \mathrm{div}\mathbf{f}, b_K \overline{\mathrm{div}\mathbf{f}}_K)_K \\ &= -(\beta(\mathbf{u} - \mathbf{u}_h), \nabla(b_K \overline{\mathrm{div}\mathbf{f}}_K))_K + (\overline{\mathrm{div}\mathbf{f}}_K - \mathrm{div}\mathbf{f}, b_K \overline{\mathrm{div}\mathbf{f}}_K)_K \\ &\le C[\beta_K^{\frac{1}{2}} h_K^{-1} ||\beta^{\frac{1}{2}}(\mathbf{u} - \mathbf{u}_h)||_{0,K} + ||\overline{\mathrm{div}\mathbf{f}}_K - \mathrm{div}\mathbf{f}||_{0,K}]||\overline{\mathrm{div}\mathbf{f}}_K||_{0,K}, \end{aligned}$$

which, along with the triangle inequality, leads to

$$||\mathrm{div}\mathbf{f}||_{0,K} \le C[\beta_K^{\frac{1}{2}} h_K^{-1} ||\beta^{\frac{1}{2}}(\mathbf{u} - \mathbf{u}_h)||_{0,K} + ||\overline{\mathrm{div}\mathbf{f}}_K - \mathrm{div}\mathbf{f}||_{0,K}].$$

Recall the definition of $\eta^2_{2,K}$, we obtain

$$\eta^2_{2,K} \le C \max(\Lambda_K^{\beta\alpha}, 1)[||\beta^{\frac{1}{2}}(\mathbf{u} - \mathbf{u}_h)||^2_{0,K} + h_K^2 \beta_K^{-1} ||\overline{\mathrm{div}\mathbf{f}}_K - \mathrm{div}\mathbf{f}||^2_{0,K}],$$

which concludes the proof. □

Finally, we can prove the following lower bound for the local error estimator $\eta^2_{2,F}$.

Lemma 6.12.

$$\eta^2_{2,F} \le C \max(\Lambda_F^{\beta\alpha}, 1)[\sum_{K \in \omega_F} ||\beta^{\frac{1}{2}}(\mathbf{u} - \mathbf{u}_h)||^2_{0,K} + \sum_{K \in \omega_F} h_K^2 \beta_K^{-1} ||div\mathbf{f} - \overline{div\mathbf{f}}_K||^2_{0,K}]. \tag{6.33}$$

Proof. Applying the extension operator Ex to the jump $\mathbf{J}_{F_2}$, we obtain

$$\begin{aligned} C||\mathbf{J}_{F2}||^2_{0,F} &\le (\mathbf{J}_{F2}, b_F Ex(\mathbf{J}_{F2}))_F = ([(\mathbf{f} - \beta \mathbf{u}_h) \cdot \mathbf{n}]_F, b_F Ex(\mathbf{J}_{F2}))_F \\ &= (\mathbf{f} - \beta \mathbf{u}_h, \nabla(b_F Ex(\mathbf{J}_{F2})))_{\omega_F} + (\mathrm{div}(\mathbf{f} - \beta \mathbf{u}_h), b_F Ex(\mathbf{J}_{F2}))_{\omega_F} \\ &= (\beta(\mathbf{f} - \mathbf{u}_h), \nabla(b_F Ex(\mathbf{J}_{F2})))_{\omega_F} + (\mathrm{div}\mathbf{f}, b_F Ex(\mathbf{J}_{F2}))_{\omega_F} \\ &\le C[\sum_{K \in \omega_F} \beta_K^{\frac{1}{2}} h_K^{-\frac{1}{2}} ||\beta^{\frac{1}{2}}(\mathbf{f} - \mathbf{u}_h)||_{0,K} + \sum_{K \in \omega_F} h_K^{\frac{1}{2}} ||\mathrm{div}\mathbf{f}||_{0,K}]||\mathbf{J}_{F2}||_{0,F}, \end{aligned}$$

where we used (6.26). Using the estimate of div$\mathbf{f}$, we have

$$||\mathbf{J}_{F2}||_{0,F} \leq C[\sum_{K\in\omega_F} \beta_K^{\frac{1}{2}} h_K^{-\frac{1}{2}} ||\beta^{\frac{1}{2}}(\mathbf{f}-\mathbf{u}_h)||_{0,K} + \sum_{K\in\omega_F} h_K^{\frac{1}{2}} ||\mathrm{div}\mathbf{f} - \overline{\mathrm{div}\mathbf{f}}_K||_{0,K}],$$

from which and the definition of $\eta_{2,F}^2$ we obtain

$$\begin{aligned}\eta_{2,F}^2 &= \max(\Lambda_F^{\beta\alpha},1) h_F^{-1}\beta_F^{-1}||\mathbf{J}_{F2}||_{0,F}^2 \\ &\leq C\max(\Lambda_F^{\beta\alpha},1)[\sum_{K\in\omega_F}||\beta^{\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)||_{0,K}^2 + \sum_{K\in\omega_F} h_K^2\beta_K^{-1}||\mathrm{div}\mathbf{f} - \overline{\mathrm{div}\mathbf{f}}_K||_{0,K}^2],\end{aligned}$$

which completes the proof. □

Combining Lemmas 6.9–6.12, we obtain the following lower bound of a posterior error estimator.

Theorem 6.2.

$$\begin{aligned}\sum_{K\in T_h}\eta_h^2(K) \leq C_{low}\sum_{K\in T_h}[&\Lambda_{\omega_K}^{\alpha}||\alpha^{\frac{1}{2}}\nabla\times(\mathbf{u}-\mathbf{u}_h)||_{0,\omega_K}^2 \\ &+\Lambda_{\omega_K}^{\alpha}\sum_{K'\in\omega_K} h_{K'}^2\beta_{K'}\alpha_{K'}^{-1}||\beta^{\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)||_{0,K'}^2 \\ &+\Lambda_{\omega_K}^{\beta\alpha}\sum_{K'\in\omega_K}||\beta^{\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)||_{0,K'}^2 + \Lambda_{\omega_K}^{\alpha}\sum_{K'\in\omega_K} h_{K'}^2\alpha_{K'}^{-1}||\mathbf{R}_{K'} - \overline{\mathbf{R}}_{K'}||_{0,K'}^2 \\ &+\Lambda_{\omega_K}^{\beta\alpha}\sum_{K'\in\omega_K} h_{K'}^2\beta_{K'}^{-1}||div\mathbf{f} - \overline{div\mathbf{f}}_{K'}||_{0,K'}^2].\end{aligned}$$

6.2.4 Zienkiewicz-Zhu Error Estimator

Another simple and effective posterior error estimator is the so-called Zienkiewicz-Zhu (ZZ) estimator introduced by Zienkiewicz and Zhu [309] and later improved by many researchers (cf. [172, 296, 298] and references cited therein). The basic idea is to use some post-processing procedure to compute an improved gradient of the numerical solution first, then use the difference between this recovered gradient and the original gradient for the error estimator. In practical implementation, a gradient (or flux) is often computed, hence it is cheap to implement the ZZ error estimator. Moreover, the estimator has been proved to be very robust for a variety of problems, and has been quite popular. In this section, we present a nice Zienkiewicz-Zhu error estimator obtained by Nicaise [224] for the Maxwell's equations (6.1) and (6.2).

We denote $\mathscr{N}$ the set of all (interior or boundary) nodes of T_h, $\omega_{\mathbf{x}}$ the union of all elements sharing node $\mathbf{x}$, and the jump of a function $\mathbf{v}$ across a face F as:

$$[|\mathbf{v}(\mathbf{y})|] = \lim_{\epsilon \to +0} (\mathbf{v}(\mathbf{y} + \epsilon \mathbf{n}_F) - \mathbf{v}(\mathbf{y} - \epsilon \mathbf{n}_F)), \quad \mathbf{y} \in F,$$

where $\mathbf{n}_F$ is the unit outward vector to F.

Before we define a ZZ type recovered operator, let us first recall the barycentric coordinate $\lambda_{\mathbf{x}}$ at any node $\mathbf{x}$ defined in Chap. 2, i.e., $\lambda_{\mathbf{x}}$ is a continuous piecewise linear function on T_h such that

$$\lambda_{\mathbf{x}}(\mathbf{y}) = \delta_{\mathbf{x},\mathbf{y}}, \quad \forall\, \mathbf{y} \in \mathscr{N},$$

where $\delta_{\mathbf{x},\mathbf{y}} = 1$ if $\mathbf{x} = \mathbf{y}$, and 0 otherwise. Moreover, let us denote W_h the space of piecewise linear vector fields on T_h, and $V_h = W_h \cap C(\Omega, \mathscr{R}^3)$.

With the above notation, a ZZ type recovered operator $R_{ZZ} : W_h \to V_h$ can be defined by [224]: $\mathbf{v}_h \to \sum_{\mathbf{x} \in \mathscr{N}} (R_{ZZ}\mathbf{v}_h)(\mathbf{x})\lambda_{\mathbf{x}}$, where

$$(R_{ZZ}\mathbf{v}_h)(\mathbf{x}) = \sum_{K \in \omega_{\mathbf{x}}} \mu_{K,\mathbf{x}} \mathbf{v}_h|_K(\mathbf{x}), \quad \mathbf{x} \in \mathscr{N}, \tag{6.34}$$

where $\mu_{K,\mathbf{x}} \geq 0$ are the weights, which can be freely chosen such that $\sum_{K \in \omega_{\mathbf{x}}} \mu_{K,\mathbf{x}} = 1$. Furthermore, the local and global ZZ estimators are defined as:

$$\begin{aligned} \eta_{Z,K}^2 &= ||R_{ZZ}\mathbf{u}_h - \mathbf{u}_h||_{0,K}^2 + ||R_{ZZ}(\mathrm{curl}_h \mathbf{u}_h) - \mathrm{curl}_h \mathbf{u}_h||_{0,K}^2, \\ \eta_Z^2 &= \sum_{K \in T_h} \eta_{Z,K}^2, \end{aligned}$$

where curl_h is calculated elementwisely.

Nicaise [224] proved that the above defined ZZ estimator is equivalent to a residual type error estimator. Furthermore, both lower and upper bounds for the ZZ estimators are obtained.

Theorem 6.3. *For problem (6.1) and (6.2), the error* $\mathbf{u} - \mathbf{u}_h$ *is bounded locally from below and globally from above:*

$$\begin{aligned} \eta_{Z,K} &\leq C[||\mathbf{u} - \mathbf{u}_h||_{H(curl;\omega_K)} + \sum_{K' \subset \omega_K} \xi_{K'}], \\ ||\mathbf{u} - \mathbf{u}_h||_{H(curl;\Omega)} &\leq C[\eta_Z + \eta_{el} + \xi], \end{aligned}$$

where

$$\xi_K^2 = h_K^2 ||r_K - R_K||_{0,K}^2, \quad \xi^2 = \sum_{K \in T_h} \xi_K^2, \quad \eta_{el}^2 = \sum_{K \in T_h} h_K^2 ||r_K||_{0,K}^2.$$

Here R_K is the exact residual defined by

$$R_K = f - (\nabla \times (\alpha \nabla \times \mathbf{u}_h) + \beta \mathbf{u}_h) \quad \forall\ K \in T_h,$$

and r_K is the corresponding approximated residual.

The proof of Theorem 6.3 is quite technical, interested readers can consult the original paper [224, Theorem 3.9].

6.3 A Posteriori Error Estimator for Cold Plasma Model

In this section, we develop a posteriori error estimator for a semi-discrete DG scheme used to solve the cold plasma model discussed in Sect. 4.2.2. For simplicity, we assume that Ω is partitioned into disjoint tetrahedral elements $\{K\}$ such that $\overline{\Omega} = \bigcup_{K \in T_h} K$. Hence the according finite element space is given by

$$\mathbf{V}_h = \{\mathbf{v} \in (L^2(\Omega))^3 : \ \mathbf{v}|_K \in (P_l(K))^3, \ K \in T_h\}, \ \ l \geq 1, \tag{6.35}$$

Note that all results below hold true for a mesh of affine hexahedral elements, in which case on each element K, $\mathbf{v}|_K$ is a polynomial of degree at most l in each variable.

To simplify the presentation, we assume that all physical parameters in the governing equation (4.5) are one (i.e., $C_v = \nu = \omega_p = 1$) and adding a source term $\mathbf{f}$ to the right hand side of (4.5), in which case the governing equation is simplified as:

$$\mathbf{E}_{tt} + \nabla \times \nabla \times \mathbf{E} + \mathbf{E} - \mathbf{J}(\mathbf{E}) = \mathbf{f}, \tag{6.36}$$

where the polarization current density $\mathbf{J}$ is

$$\mathbf{J}(\mathbf{E}) \equiv \mathbf{J}(\mathbf{x}, t; \mathbf{E}) = \int_0^t e^{-(t-s)} \mathbf{E}(\mathbf{x}, s) ds. \tag{6.37}$$

We can form a semi-discrete DG scheme for (6.36): For any $t \in (0, T)$, find $\mathbf{E}^h(\cdot, t) \in \mathbf{V}_h$ such that

$$(\mathbf{E}^h_{tt}, \phi) + a_h(\mathbf{E}^h, \phi) - (\mathbf{J}(\mathbf{E}^h), \phi) = (\mathbf{f}, \phi), \ \ \forall\ \phi \in \mathbf{V}_h, \tag{6.38}$$

subject to the initial conditions

$$\mathbf{E}^h|_{t=0} = \Pi_2 \mathbf{E}_0, \quad \mathbf{E}^h_t|_{t=0} = \Pi_2 \mathbf{E}_1, \tag{6.39}$$

where Π_2 denotes the standard L_2-projection onto $\mathbf{V}_h$. Moreover, the bilinear form a_h is defined on $\mathbf{V}_h \times \mathbf{V}_h$ as

$$a_h(\mathbf{u},\mathbf{v}) = \sum_{K\in T_h}\int_K (\nabla\times\mathbf{u}\cdot\nabla\times\mathbf{v}+\mathbf{u}\cdot\mathbf{v})d\mathbf{x} - \sum_{F\in F_h}\int_F [[\mathbf{u}]]_T\cdot\{\{\nabla\times\mathbf{v}\}\}dA$$
$$-\sum_{F\in F_h}\int_F [[\mathbf{v}]]_T\cdot\{\{\nabla\times\mathbf{u}\}\}dA + \sum_{F\in F_h}\int_F a[[\mathbf{u}]]_T\cdot[[\mathbf{v}]]_T dA.$$

Here $[[\mathbf{v}]]$ and $\{\{\mathbf{v}\}\}$ are the standard notation for the tangential jumps and averages of $\mathbf{v}$ across interior faces defined in Sect. 4.2.2. Finally, a is a penalty function, which is defined on each face $F \in F_h$ as:

$$a|_F = \gamma\hbar^{-1},$$

where $\hbar|_F = \min\{h_{K^+}, h_{K^-}\}$ for an interior face $F = \partial K^+ \cap \partial K^-$, and $\hbar|_F = h_K$ for a boundary face $F = \partial K \cap \partial\Omega$. The penalty parameter γ is a positive constant.

Following Sect. 4.2.2, we denote the space $\mathbf{V}(h) = H_0(\text{curl};\Omega) + \mathbf{V}_h$ and define the DG energy norm by

$$||\mathbf{v}||_h^2 = ||\mathbf{v}||_{0,\Omega}^2 + \sum_{K\in T_h}||\nabla\times\mathbf{v}||_{0,K}^2 + \sum_{F\in F_h}||a^{1/2}[[\mathbf{v}]]_T||_{0,F}^2.$$

In order to carry out the posteriori analysis, we introduce an auxiliary bilinear form $\tilde{a}_h$ on $\mathbf{V}(h)\times\mathbf{V}(h)$ defined as

$$\tilde{a}_h(\mathbf{u},\mathbf{v}) = \sum_{K\in T_h}\int_K (\nabla\times\mathbf{u}\cdot\nabla\times\mathbf{v}+\mathbf{u}\cdot\mathbf{v})d\mathbf{x} - \sum_{K\in T_h}\int_K \mathscr{L}(\mathbf{u})\cdot(\nabla\times\mathbf{v})dx$$
$$-\sum_{K\in T_h}\int_K \mathscr{L}(\mathbf{v})\cdot(\nabla\times\mathbf{u})dx + \sum_{F\in F_h}\int_F a[[\mathbf{u}]]_T\cdot[[\mathbf{v}]]_T dA,$$

where the lifting operator $\mathscr{L}(\mathbf{v}) \in \mathbf{V}_h$ for any $\mathbf{v}\in\mathbf{V}(h)$ is defined by

$$\int_\Omega \mathscr{L}(\mathbf{v})\cdot\mathbf{w}dx = \sum_{F\in F_h}\int_F [[\mathbf{v}]]_T\cdot\{\{\mathbf{w}\}\}dA \quad \forall\, \mathbf{w}\in\mathbf{V}_h, \tag{6.40}$$

from which it is easy to see that the lifting operator $\mathscr{L}(\mathbf{v})$ can be bounded as follows [148]:

$$||\mathscr{L}(\mathbf{v})||_{0,\Omega}^2 \le \alpha^{-1}C_{lift}\sum_{F\in F_h}||a^{\frac{1}{2}}[[\mathbf{v}]]_T||_{0,F}^2. \tag{6.41}$$

In the rest two subsections, we present detailed derivation of upper and lower bounds of the posteriori error estimator.

6.3.1 Upper Bound of the Posteriori Error Estimator

One of the main tools used in the posteriori error estimate for DG methods is to find a conforming finite element function close to the discontinuous one. For this purpose, we define the conforming finite element space

$$\mathbf{V}_h^c = \mathbf{V}_h \cap H_0(\text{curl}; \Omega), \tag{6.42}$$

i.e., $\mathbf{V}_h^c$ is the second family of Nédélec element [223]. Moreover, we have the following approximation property [148].

Lemma 6.13. *For any* $\mathbf{v}^h \in \mathbf{V}_h$*, there exists a conforming approximation* $\mathbf{v}_c^h \in \mathbf{V}_h^c$ *such that*

$$\sum_{K \in T_h} ||\nabla \times (\mathbf{v}^h - \mathbf{v}_c^h)||_{0,K}^2 \leq C_{app} \sum_{F \in F_h} h_F^{-1} ||[[\mathbf{v}^h]]_T||_{0,F}^2,$$

$$||\mathbf{v}^h - \mathbf{v}_c^h||_{0,\Omega}^2 \leq C_{app} \sum_{F \in F_h} h_F ||[[\mathbf{v}^h]]_T||_{0,F}^2,$$

and

$$||\mathbf{v}^h - \mathbf{v}_c^h||_h^2 \leq (2\alpha^{-1} C_{app} + 1) \sum_{F \in F_h} ||a^{\frac{1}{2}} [[\mathbf{v}^h]]_T||_{0,F}^2,$$

where the constant $C_{app} > 0$ *depends only on the shape regularity of the mesh and the approximation order* l *in space* $\mathbf{V}_h$*.*

Before we state the posteriori error estimator, let us introduce some local error indicators. Let $\mathbf{f}_h \in \mathbf{V}_h$ be some approximation of $\mathbf{f}$, and

$$\eta_{R_K}^2 = h_K^2 ||\mathbf{f}_h - \mathbf{E}_{tt}^h - \nabla \times \nabla \times \mathbf{E}^h - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h)||_{0,K}^2,$$

which measures the residual of the approximated governing Maxwell's equations (6.36).

We denote

$$\eta_{T_K}^2 = \frac{1}{2} \sum_{F \in \partial K \setminus \Gamma} h_F ||[[\nabla \times \mathbf{E}^h]]_T||_{0,F}^2$$

for the face residual about the jump of $\nabla \times \mathbf{E}^h$.

To measure the tangential jumps of the approximate solution $\mathbf{E}^h$, we denote

$$\eta_{J_K}^2 = \frac{1}{2} \sum_{F \in \partial K \setminus \Gamma} ||a^{\frac{1}{2}} [[\mathbf{E}^h]]_T||_{0,F}^2.$$

Noting that $\nabla \cdot \nabla \times (\nabla \times \mathbf{E}^h) = 0$, hence

$$\eta_{D_K}^2 = h_K^2 ||\nabla \cdot (\mathbf{f}_h - \mathbf{E}_{tt}^h - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h))||_{0,K}^2$$

measures the error in the divergence of the governing Maxwell's equations (6.36).

Furthermore, we denote

$$\eta_{N_K}^2 = \frac{1}{2} \sum_{F \in \partial K \backslash \Gamma} h_K ||[[\mathbf{f}_h - \mathbf{E}_{tt}^h - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h)]]_N||_{0,F}^2$$

for measuring the normal jump of $\mathbf{f}_h - \mathbf{E}_{tt}^h - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h)$ over the interior faces.

Similarly, we can define the following local estimators:

$$\begin{aligned}
\eta_{R_K^t}^2 &= h_K^2 ||(\mathbf{f}_h - \mathbf{E}_{tt}^h - \nabla \times \nabla \times \mathbf{E}^h - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h))_t||_{0,K}^2, \\
\eta_{T_K^t}^2 &= \frac{1}{2} \sum_{F \in \partial K \backslash \Gamma} h_F ||[[\nabla \times \mathbf{E}_t^h]]_T||_{0,F}^2, \\
\eta_{J_K^t}^2 &= \frac{1}{2} \sum_{F \in \partial K \backslash \Gamma} ||a^{\frac{1}{2}} [[\mathbf{E}_t^h]]_T||_{0,F}^2, \\
\eta_{D_K^t}^2 &= h_K^2 ||\nabla \cdot (\mathbf{f}_h - \mathbf{E}_{tt}^h - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h))_t||_{0,K}^2, \\
\eta_{N_K^t}^2 &= \frac{1}{2} \sum_{F \in \partial K \backslash \Gamma} h_K ||[[(\mathbf{f}_h - \mathbf{E}_{tt}^h - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h))_t]]_N||_{0,F}^2.
\end{aligned}$$

Theorem 6.4. *Let* $\mathbf{E}$ *be the solution of (6.36) and* $\mathbf{E}^h$ *be the DG solution of (6.38) with* $\gamma \geq \gamma_{min}$. *Then the following estimation holds:*

$$\begin{aligned}
& ||\mathbf{E} - \mathbf{E}^h||_h^2(t) + ||(\mathbf{E} - \mathbf{E}^h)_t||_h^2(t) \\
\leq\ & C[||\mathbf{E} - \mathbf{E}^h||_h^2(0) + ||(\mathbf{E} - \mathbf{E}^h)_t||_h^2(0)] \\
& + C \int_0^t \sum_{F \in F_h} h_F (||[[\mathbf{E}_{tt}^h]]_T||_{0,F}^2 + ||[[\mathbf{E}_t^h]]_T||_{0,F}^2 + ||[[\mathbf{E}^h]]_T||_{0,F}^2) dt \\
& + C \sum_{F \in F_h} [||a^{\frac{1}{2}} [[\mathbf{E}^h]]_T||_{0,F}^2(t) + ||a^{\frac{1}{2}} [[\mathbf{E}_t^h]]_T||_{0,F}^2(t) \\
& + ||a^{\frac{1}{2}} [[\mathbf{E}^h]]_T||_{0,F}^2(0) + ||a^{\frac{1}{2}} [[\mathbf{E}_t^h]]_T||_{0,F}^2(0)] \\
& + C[||\mathbf{f} - \mathbf{f}_h||_{0,\Omega}^2(t) + \sum_{K \in T_h} (\eta_{R_K}^2 + \eta_{T_K}^2 + \eta_{J_K}^2 + \eta_{D_K}^2 + \eta_{N_K}^2)(t)] \\
& + C[||\mathbf{f} - \mathbf{f}_h||_{0,\Omega}^2(0) + \sum_{K \in T_h} (\eta_{R_K}^2 + \eta_{T_K}^2 + \eta_{J_K}^2 + \eta_{D_K}^2 + \eta_{N_K}^2)(0)] \\
& + C \int_0^t [\sum_{K \in T_h} (\eta_{R_K^t}^2 + \eta_{T_K^t}^2 + \eta_{J_K^t}^2 + \eta_{D_K^t}^2 + \eta_{N_K^t}^2) + ||(\mathbf{f} - \mathbf{f}_h)_t||_0^2] dt.
\end{aligned}$$

Proof. Denote $\mathbf{w} = \mathbf{E} - \mathbf{E}_c^h \in H_0(\text{curl};\Omega)$, where $\mathbf{E}_c^h$ is the conforming approximation of $\mathbf{E}^h$. Then for any $\phi \in H_0(\text{curl};\Omega)$, we have

$$
\begin{aligned}
(\mathbf{w}_{tt},\phi) + \tilde{a}_h(\mathbf{w},\phi) &= ((\mathbf{E} - \mathbf{E}^h + \mathbf{E}^h - \mathbf{E}_c^h)_{tt},\phi) + \tilde{a}_h(\mathbf{E} - \mathbf{E}^h + \mathbf{E}^h - \mathbf{E}_c^h,\phi) \\
&= (\mathbf{E}_{tt} - \mathbf{E}_{tt}^h,\phi) + \tilde{a}_h(\mathbf{E},\phi) - \tilde{a}_h(\mathbf{E}^h,\phi) \\
&\quad + ((\mathbf{E}^h - \mathbf{E}_c^h)_{tt},\phi) + \tilde{a}_h(\mathbf{E}^h - \mathbf{E}_c^h,\phi).
\end{aligned} \tag{6.43}
$$

Using the fact that $\tilde{a}_h(\mathbf{u},\mathbf{v}) = \int_\Omega (\nabla\times\mathbf{u}\cdot\nabla\times\mathbf{v} + \mathbf{u}\cdot\mathbf{v})dx$ on $H_0(\text{curl};\Omega)\times H_0(\text{curl};\Omega)$, we can write the weak formulation of (6.36) as: Find $\mathbf{E}\in H_0(\text{curl};\Omega)$ such that

$$
(\mathbf{E}_{tt},\phi) + \tilde{a}_h(\mathbf{E},\phi) - (\mathbf{J}(\mathbf{E}),\phi) = (\mathbf{f},\phi) \quad \forall\, \phi \in H_0(\text{curl};\Omega). \tag{6.44}
$$

Using the fact that $\tilde{a}_h = a_h$ on $\mathbf{V}_h \times \mathbf{V}_h$, we can rewrite the semi-discrete scheme (6.38) as

$$
(\mathbf{E}_{tt}^h,\phi_h) + \tilde{a}_h(\mathbf{E}^h,\phi_h) - (\mathbf{J}(\mathbf{E}^h),\phi_h) = (\mathbf{f},\phi_h), \quad \forall\, \phi_h \in \mathbf{V}_h. \tag{6.45}
$$

From (6.44) and (6.45), we have

$$
\begin{aligned}
&(\mathbf{E}_{tt} - \mathbf{E}_{tt}^h,\phi) + \tilde{a}_h(\mathbf{E},\phi) - \tilde{a}_h(\mathbf{E}^h,\phi) \\
={}& (\mathbf{f} + \mathbf{J}(\mathbf{E}) - \mathbf{E}_{tt}^h,\phi) - \tilde{a}_h(\mathbf{E}^h,\phi_h) - \tilde{a}_h(\mathbf{E}^h,\phi - \phi_h) \\
={}& (\mathbf{f} + \mathbf{J}(\mathbf{E}^h) - \mathbf{E}_{tt}^h,\phi - \phi_h) + (\mathbf{J}(\mathbf{E} - \mathbf{E}^h),\phi) - \tilde{a}_h(\mathbf{E}^h,\phi - \phi_h),
\end{aligned}
$$

substituting which into (6.43), we obtain

$$
\begin{aligned}
(\mathbf{w}_{tt},\phi) + \tilde{a}_h(\mathbf{w},\phi) &= (\mathbf{f} + \mathbf{J}(\mathbf{E}^h) - \mathbf{E}_{tt}^h,\phi - \phi_h) + (\mathbf{J}(\mathbf{w} + \mathbf{E}_c^h - \mathbf{E}^h),\phi) \\
&\quad - \tilde{a}_h(\mathbf{E}^h,\phi - \phi_h) + ((\mathbf{E}^h - \mathbf{E}_c^h)_{tt},\phi) + \tilde{a}_h(\mathbf{E}^h - \mathbf{E}_c^h,\phi).
\end{aligned} \tag{6.46}
$$

Choosing $\phi = \mathbf{w}_t$ in (6.46), then integrating both sides from 0 to t, and multiplying both sides by 2, we obtain

$$
||\mathbf{w}(t)||_h^2 + ||\mathbf{w}_t(t)||_0^2 \le ||\mathbf{w}(0)||_h^2 + ||\mathbf{w}_t(0)||_0^2 + \sum_{i=1}^{5} Err_i. \tag{6.47}
$$

With careful estimates of all $Err_i, i = 1,\cdots,5$ (cf. [182]), we have

$$
\begin{aligned}
&||\mathbf{w}(t)||_h^2 + ||\mathbf{w}_t(t)||_h^2 \\
&\le C(||\mathbf{w}(0)||_h^2 + ||\mathbf{w}_t(0)||_h^2) + C\int_0^t (||\mathbf{w}(t)||_h^2 + ||\mathbf{w}_t(t)||_h^2)dt \\
&+ C\int_0^t \sum_{F\in F_h} h_F(||[[\mathbf{E}_{tt}^h]]_T||_{0,F}^2 + ||[[\mathbf{E}_t^h]]_T||_{0,F}^2 + ||[[\mathbf{E}^h]]_T||_{0,F}^2)dt \\
&+ \delta_1||\mathbf{w}(t)||_h^2 + \frac{C}{\delta_1}\sum_{F\in F_h} ||a^{\frac{1}{2}}[[\mathbf{E}^h]]_T||_{0,F}^2 + C\sum_{F\in F_h} ||a^{\frac{1}{2}}[[\mathbf{E}^h(0)]]_T||_{0,F}^2 \\
&+ \delta_2||\mathbf{w}(t)||_{curl}^2 + \frac{C}{\delta_2}[||\mathbf{f}-\mathbf{f}_h||_{0,\Omega}^2(t) + \sum_{K\in T_h}(\eta_{R_K}^2 + \eta_{T_K}^2 + \eta_{J_K}^2 + \eta_{D_K}^2 + \eta_{N_K}^2)(t)] \\
&+ \delta_3||\mathbf{w}(0)||_{curl}^2 + \frac{C}{\delta_3}[||\mathbf{f}-\mathbf{f}_h||_{0,\Omega}^2(0) + \sum_{K\in T_h}(\eta_{R_K}^2 + \eta_{T_K}^2 + \eta_{J_K}^2 + \eta_{D_K}^2 + \eta_{N_K}^2)(0)] \\
&+ C\int_0^t [\sum_{K\in T_h}(\eta_{R_K^t}^2 + \eta_{T_K^t}^2 + \eta_{J_K^t}^2 + \eta_{D_K^t}^2 + \eta_{N_K^t}^2) + ||(\mathbf{f}-\mathbf{f}_h)_t||_0^2]dt. \qquad (6.48)
\end{aligned}
$$

By the definition of $||\cdot||_h$ and Lemma 6.13, we easily have

$$
||(\mathbf{E}^h - \mathbf{E}_c^h)(t)||_h^2 \le C\sum_{F\in F_h} ||a^{\frac{1}{2}}[[\mathbf{E}^h]]_T||_{0,F}^2,
$$

and

$$
||(\mathbf{E}^h - \mathbf{E}_c^h)_t(t)||_h^2 \le C\sum_{F\in F_h} ||a^{\frac{1}{2}}[[\mathbf{E}_t^h]]_T||_{0,F}^2,
$$

which, along with (6.48), the triangle inequality, and the Gronwall inequality (choosing δ_1 and δ_2 small enough), concludes the proof. □

6.3.2 Lower Bound of the Local Error Estimator

Theorem 6.5. *Let* $\mathbf{E}$ *be the solution of (6.36) and* $\mathbf{E}^h$ *be the DG solution of (6.38) with* $\gamma \ge \gamma_{min}$. *Then the following local bounds hold:*

$$
\begin{aligned}
(i)\quad \eta_{R_K} &\le C[h_K||(\mathbf{E}-\mathbf{E}^h)_{tt}||_{0,K} + h_K||\mathbf{E}-\mathbf{E}^h||_{0,K} + h_K\int_0^t ||\mathbf{E}-\mathbf{E}^h||_{0,K}(s)ds \\
&\quad + h_K||\mathbf{f}_h - \mathbf{f}||_{0,K} + ||\nabla\times(\mathbf{E}-\mathbf{E}^h)||_{0,K}], \\
(ii)\quad \eta_{T_K} &\le C\sum_{K\in UF}[h_K||(\mathbf{E}-\mathbf{E}^h)_{tt}||_{0,K} + h_K||\mathbf{E}-\mathbf{E}^h||_{0,K} \\
&\quad + h_K\int_0^t ||\mathbf{E}^h-\mathbf{E}||_{0,K}(s)ds + h_K||\mathbf{f}_h-\mathbf{f}||_{0,K} + ||\nabla\times(\mathbf{E}^h-\mathbf{E})||_{0,K}],
\end{aligned}
$$

$$(iii)\quad \eta_{D_K} \le C(||\mathbf{f}_h - \mathbf{f}||_{0,K} + ||(\mathbf{E}-\mathbf{E}^h)_{tt}||_{0,K} + ||\mathbf{E}-\mathbf{E}^h||_{0,K} + \int_0^t ||\mathbf{E}^h - \mathbf{E}||_{0,K}(s)ds),$$

$$(iv)\quad \eta_{N_K} \le C \sum_{K\in UF} (||\mathbf{f}_h - \mathbf{f}||_{0,K} + ||(\mathbf{E}-\mathbf{E}^h)_{tt}||_{0,K}$$
$$+||\mathbf{E}-\mathbf{E}^h||_{0,K} + \int_0^t ||\mathbf{E}^h - \mathbf{E}||_{0,K}(s)ds).$$

Proof. To give readers some ideas about how to prove these lower bounds, below we just show the proofs of (i) and (iv). Proofs of the rest can be found in the original paper [182].

(i) Let $\mathbf{v}_h = \mathbf{f}_h - \mathbf{E}^h_{tt} - \nabla\times\nabla\times\mathbf{E}^h - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h)$, and $\mathbf{v}_b = b_K\mathbf{v}_h$.
Using the governing equation (6.36), we have

$$\begin{aligned}
||b_K^{\frac{1}{2}}\mathbf{v}_h||^2_{0,K} &= \int_K (\mathbf{f}_h - \mathbf{E}^h_{tt} - \nabla\times\nabla\times\mathbf{E}^h - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h))\cdot\mathbf{v}_b dx\\
&= \int_K [(\mathbf{E}-\mathbf{E}^h)_{tt} + \nabla\times\nabla\times(\mathbf{E}-\mathbf{E}^h) + (\mathbf{E}-\mathbf{E}^h) - \mathbf{J}(\mathbf{E}-\mathbf{E}^h)]\cdot\mathbf{v}_b dx\\
&\quad + \int_K (\mathbf{f}_h - \mathbf{f})\cdot\mathbf{v}_b dx\\
&= \int_K [(\mathbf{E}-\mathbf{E}^h)_{tt} + (\mathbf{E}-\mathbf{E}^h) - \mathbf{J}(\mathbf{E}-\mathbf{E}^h) + (\mathbf{f}_h - \mathbf{f})]\cdot\mathbf{v}_b dx\\
&\quad + \int_K (\nabla\times(\mathbf{E}-\mathbf{E}^h))\cdot(\nabla\times\mathbf{v}_b)dx,
\end{aligned}$$

where in the last step we used integration by parts and the fact that $\mathbf{v}_b = 0$ on ∂K.

Then by Lemma 6.8, we have

$$||\mathbf{v}_h||_{0,K} \le C[||(\mathbf{E}-\mathbf{E}^h)_{tt}||_{0,K} + ||\mathbf{E}-\mathbf{E}^h||_{0,K} + \int_0^t ||\mathbf{E}-\mathbf{E}^h||_{0,K}(s)ds$$
$$+||\mathbf{f}_h - \mathbf{f}||_{0,K} + h_K^{-1}||\nabla\times(\mathbf{E}-\mathbf{E}^h)||_{0,K}],$$

which leads to

$$\eta_{R_K} \le C[h_K||(\mathbf{E}-\mathbf{E}^h)_{tt}||_{0,K} + h_K||\mathbf{E}-\mathbf{E}^h||_{0,K} + h_K\int_0^t ||\mathbf{E}-\mathbf{E}^h||_{0,K}(s)ds$$
$$+h_K||\mathbf{f}_h - \mathbf{f}||_{0,K} + ||\nabla\times(\mathbf{E}-\mathbf{E}^h)||_{0,K}],$$

which completes the proof of (i).

(iv) Let $\mathbf{v}_h = [[\mathbf{f}_h - \mathbf{E}^h_{tt} - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h)]]_N$, and $\mathbf{v}_b = b_F\mathbf{v}_h$. Using the facts that $[[\mathbf{f} - \mathbf{E}_{tt} - \mathbf{E} + \mathbf{J}(\mathbf{E})]]_N = 0$ on interior faces and $\nabla\cdot(\mathbf{f} - \mathbf{E}_{tt} - \mathbf{E} + \mathbf{J}(\mathbf{E})) = 0$ in K, we have

$$
\begin{aligned}
||b_F^{\frac{1}{2}}\mathbf{v}_h||_{0,F}^2 &= \int_F [[\mathbf{f}_h - \mathbf{E}_{tt}^h - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h)]]_N \cdot \mathbf{v}_b ds \\
&= \int_F [[\mathbf{f}_h - \mathbf{f} + (\mathbf{E}-\mathbf{E}^h)_{tt} + \mathbf{E} - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h - \mathbf{E})]]_N \cdot \mathbf{v}_b ds \\
&= \sum_{K\in UF} \int_K \nabla \cdot (\mathbf{f}_h - \mathbf{E}_{tt}^h - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h))\mathbf{v}_b dx \\
&\quad + \sum_{K\in UF} \int_K (\mathbf{f}_h - \mathbf{f} + (\mathbf{E}-\mathbf{E}^h)_{tt} + \mathbf{E} - \mathbf{E}^h + \mathbf{J}(\mathbf{E}^h - \mathbf{E})) \cdot \nabla \mathbf{v}_b dx \\
&\le C \sum_{K\in UF} h_K^{-1}\eta_{D_K}||\mathbf{v}_b||_{0,K} + \sum_{K\in UF} [||\mathbf{f}_h - \mathbf{f}||_{0,K} + ||(\mathbf{E}-\mathbf{E}^h)_{tt}||_{0,K} \\
&\quad + ||\mathbf{E}-\mathbf{E}^h||_{0,K} + \int_0^t ||\mathbf{E}^h - \mathbf{E}||_{0,K}(s)ds]||\nabla \mathbf{v}_b||_{0,K}.
\end{aligned}
$$

Using Lemma 6.4 and the estimate (iii), we have

$$
\begin{aligned}
&\eta_{N_K} = h_F^{\frac{1}{2}}||\mathbf{v}_h||_{0,F} \\
&\le C \sum_{K\in UF} (\eta_{D_K} + ||\mathbf{f}_h - \mathbf{f}||_{0,K} + ||(\mathbf{E}-\mathbf{E}^h)_{tt}||_{0,K} \\
&\quad + ||\mathbf{E}-\mathbf{E}^h||_{0,K} + \int_0^t ||\mathbf{E}^h - \mathbf{E}||_{0,K}(s)ds) \\
&\le C \sum_{K\in UF} (||\mathbf{f}_h - \mathbf{f}||_{0,K} + ||(\mathbf{E}-\mathbf{E}^h)_{tt}||_{0,K} + ||\mathbf{E}-\mathbf{E}^h||_{0,K} + \int_0^t ||\mathbf{E}^h - \mathbf{E}||_{0,K}(s)ds),
\end{aligned}
$$

which concludes the proof of (iv). □