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MIXED HP-DGFEM FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We consider mixed *hp*-discontinuous Galerkin approximations of the incompressible Navier-Stokes equations, and prove exponential rates of convergence in the number of degrees of freedom for piecewise analytic and small data.

Key words. Mixed *hp*-FEM, discontinuous Galerkin methods, exponential convergence

AMS subject classifications. 65N30, 65N35, 65N12, 65N15

1. Introduction. [2, 10]
[6, 3, 4, 5]
[8]
[14], [15]
[9]
[11]

2. The incompressible Navier-Stokes equations. Let Ω be a bounded Lipschitz polygon in $\mathbb{R}^2$. Given the source term $\boldsymbol{f} \in L^2(\Omega)^2$ and the constant kinematic viscosity $\nu > 0$, the incompressible Navier-Stokes equations consist in finding a velocity field $\boldsymbol{u}$ and a pressure p such that

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega,\\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

If we define the Sobolev spaces

$$\boldsymbol{V} := H_0^1(\Omega)^2, \qquad Q := L_0^2(\Omega) = \{\, q \in L^2(\Omega) : \int_\Omega q\, d\boldsymbol{x} = 0 \,\},$$

and introduce the forms

$$\begin{aligned} A(\boldsymbol{u},\boldsymbol{v}) &= \int_\Omega \nu\nabla\boldsymbol{u} :\, \nabla\boldsymbol{v}\, d\boldsymbol{x},\\ O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v} &= \int_\Omega ((\boldsymbol{w}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{v}\, d\boldsymbol{x},\\ B(\boldsymbol{u},p) &= -\int_\Omega p\,\nabla\cdot\boldsymbol{u}\, d\boldsymbol{x}, \end{aligned}$$

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then the corresponding variational problem is to find finding $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ such that

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v}) + O(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + B(\boldsymbol{v}, p) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x}, \\ B(\boldsymbol{u}, q) &= 0, \end{aligned} \tag{2.2}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ and $q \in Q$.

Clearly, the velocity $\boldsymbol{u}$ of a solution to (2.2) belongs to the continous kernel

$$\boldsymbol{Z} := \{ \boldsymbol{v} \in \boldsymbol{V} \, : \, B(\boldsymbol{v}, q) = 0 \; \forall q \in Q \} = \{ \boldsymbol{v} \in H_0^1(\Omega)^2 \, : \, \nabla \cdot \boldsymbol{v} \equiv 0 \text{ in } \Omega \}. \tag{2.3}$$

Moreover, it satisfies the stability bound

$$\|\nabla \boldsymbol{u}\|_0 \leq \frac{C_P \|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu}, \tag{2.4}$$

with C_P denoting the Poincaré constant in Ω. Moreover, it is well known that under the small data assumption

$$\frac{C_O C_P \|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu^2} < 1, \tag{2.5}$$

with C_O denoting the boundedness constant of the trilinear form O, problem (2.2) has a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$; see [12, 7, 13] and the references therein.

Let us also recall from [**?**], [8, Section 2] that, if $\boldsymbol{f} \in L^{4/3}(\Omega)^2 \supset L^2(\Omega)^2$, then we have the base regularity

$$(\boldsymbol{u}, p) \in W^{2,4/3}(\Omega)^2 \times W^{1,4/3}(\Omega). \tag{2.6}$$

In particular, standard embedding results then imply that $\boldsymbol{u} \in W^{1/4}(\Omega)^2$. [**DS: This is the base regularity which we need for the continuity of the convection form**]

3. A mixed interior penalty discretization.

We introduce a mixed hp-DG discretization of (2.2). It is based on an interior penalty discretization for the Stokes terms, see [14], combined with a discontinuous version of a skew-symmetrized form for the convection form; [11, 10].

3.1. Meshes and finite element spaces.

Let $\mathcal{T}_h$ be a family of shape-regular mesh of affine quadrilateral elements on Ω. Each element K is the affine image of the reference cube $\widehat{K} = (-1, 1)^2$ under an affine element mapping $F_K : \widehat{K} \to K$. We denote by h_K the diameter of the element $K \in \mathcal{T}_h$. Further, we assign to each element $K \in \mathcal{T}_h$ an approximation order $k_K \geq 1$. The local quantities h_K and k_K are stored in the vectors $\underline{h} = \{h_K\}_{K \in \mathcal{T}_h}$ and $\underline{k} = \{k_K\}_{K \in \mathcal{T}_h}$, respectively. We set $h = \max_{K \in \mathcal{T}_h} h_K$ and $|\underline{k}| = \max_{K \in \mathcal{T}_h} k_K$. We allow for 1-irregular meshes. Hence, the mesh sizes are of bounded local variation: there is a constant $\kappa_1 > 0$ such that

$$\kappa_1 h_K \leq h_{K'} \leq \kappa_1^{-1} h_K, \tag{3.1}$$

whenever K and K' share an interior edge, uniformly in the mesh family. We assume a similar property fo the approximation degrees: there is a constant $\kappa_2 > 0$ such that

$$\kappa_2 k_K \leq k_{K'} \leq \kappa_2^{-1} k_K, \tag{3.2}$$

whenever K and K' share an interior edge, uniformly in the mesh family.

An interior edge of $\mathcal{T}_h$ is the (non-empty) interior of $\partial K^+ \cap \partial K^-$, where K^+ and K^- are two adjacent elements of $\mathcal{T}_h$. Similarly, a boundary edge of $\mathcal{T}_h$ is the (non-empty) interior of $\partial K \cap \partial\Omega$ which consists of entire faces of ∂K. We denote by $\mathcal{E}_{\mathcal{I}}(\mathcal{T}_h)$ the set of all interior faces of $\mathcal{T}_h$, by $\mathcal{E}_{\mathcal{D}}(\mathcal{T}_h)$ the set of all boundary faces, and set $\mathcal{E}(\mathcal{T}_h) = \mathcal{E}_{\mathcal{I}}(\mathcal{T}_h) \cup \mathcal{E}_{\mathcal{D}}(\mathcal{T}_h)$.

For a given mesh $\mathcal{T}_h$ on Ω and a polynomial degree vector $\underline{k}$, we define the generic hp-version discontinuous Galerkin space

$$S^{\underline{k}}(\mathcal{T}_h) := \left\{\, v \in H^1(\mathcal{T}_h) \,:\, v|_K \in \mathbb{Q}_{k_K}(K),\ K \in \mathcal{T}_h \,\right\}, \tag{3.3}$$

with $\mathbb{Q}_k(K) := \{\, q = \widehat{q} \circ F_K^{-1} \,:\, \widehat{q} \in \widehat{\mathbb{Q}}_k \,\}$ and $\widehat{\mathbb{Q}}_k$ denoting the tensor product polynomials of degree less or equal than k in each coordinate direction. We wish to approximate the velocities and pressures in the discontinuous finite element spaces $\boldsymbol{V}_{\mathrm{DG}}$ and Q_{DG} given by

$$\boldsymbol{V}_{\mathrm{DG}} := \left[S^{\underline{k}}(\mathcal{T}_h)\right]^2, \qquad Q_{\mathrm{DG}} = Q \cap S^{\underline{k}-1}(\mathcal{T}_h). \tag{3.4}$$

where the degree vector $\underline{k}-1$ is given by $\{k_K - 1\}_{K\in\mathcal{T}_h}$. respectively, where $\mathbb{Q}_k(K)$ is the space of polynomials of maximum degree k in each variable on K.

For the derivation and analysis of the methods we will make use of the auxiliary space $\underline{\Sigma}_{\mathrm{DG}}$ defined by

$$\underline{\Sigma}_{\mathrm{DG}} := \left[S^{\underline{k}}(\mathcal{T}_h)\right]^{2\times 2}. \tag{3.5}$$

Note that $\nabla_h \boldsymbol{V}_{\mathrm{DG}} \subset \underline{\Sigma}_{\mathrm{DG}}$, where ∇_h is the broken gradient, taken elementwise and given by $[\nabla \boldsymbol{v}]_{ij} = \partial_j v_i = \frac{\partial v_i}{\partial x_j}$ on $K \in \mathcal{T}_h$.

3.2. Trace operators. In this section, we define the trace operators needed in our discontinuous Galerkin discretizations. To this end, for a partition $\mathcal{T}_h$ of Ω we introduce the broken Sobolev space

$$W^{1,p}(\mathcal{T}_h) := \{\, v \in L^2(\Omega) \,:\, v|_K \in W^{1,p}(K),\ K \in \mathcal{T}_h \,\}. \tag{3.6}$$

For simplicity, we also set $H^1(\mathcal{T}_h) := W^{1,2}(\mathcal{T}_h)$.

Let $\boldsymbol{v}$, q, and $\underline{\tau}$ be piecewise smooth functions in $H^1(\mathcal{T}_h)^2$, $H^1(\mathcal{T}_h)$, and $H^1(\mathcal{T}_h)^{2\times 2}$, respectively. Let $E \subset \mathcal{E}_{\mathcal{I}}(\mathcal{T}_h)$ be an interior face shared by K^+ and K^-. Let us denote by $\boldsymbol{n}^\pm$ the unit outward normals on $\partial K^\pm$, and by $(\boldsymbol{v}^\pm, q^\pm, \underline{\tau}^\pm)$ the traces of $(\boldsymbol{v}, q, \underline{\tau})$ on E from the interior of $K^\pm$. Then, we define the mean values $\{\!\!\{\cdot\}\!\!\}$ at $\boldsymbol{x} \in E$ as

$$\{\!\!\{\boldsymbol{v}\}\!\!\} := (\boldsymbol{v}^+ + \boldsymbol{v}^-)/2, \qquad \{\!\!\{q\}\!\!\} := (q^+ + q^-)/2, \qquad \{\!\!\{\underline{\tau}\}\!\!\} := (\underline{\tau}^+ + \underline{\tau}^-)/2.$$

Furthermore, we introduce the following jumps at $\boldsymbol{x} \in E$:

$$[\![q]\!] := q^+\,\boldsymbol{n}^+ + q^-\,\boldsymbol{n}^-, \quad [\![\boldsymbol{v}]\!] := \boldsymbol{v}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{v}^- \cdot \boldsymbol{n}^-, \quad \underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{v}^- \otimes \boldsymbol{n}^-,$$

where, for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we set $[\boldsymbol{a}\otimes\boldsymbol{b}]_{ij} = a_i b_j$. On a boundary face $E \subset \mathcal{E}_{\mathcal{D}}(\mathcal{T}_h)$ given by $E = \partial K \cap \partial\Omega$, we set accordingly $\{\!\!\{\boldsymbol{v}\}\!\!\} := \boldsymbol{v}$, $\{\!\!\{q\}\!\!\} := q$, $\{\!\!\{\underline{\tau}\}\!\!\} := \underline{\tau}$, as well as $[\![q]\!] := q\boldsymbol{n}$, $[\![\boldsymbol{v}]\!] := \boldsymbol{v} \cdot \boldsymbol{n}$, $\underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v} \otimes \boldsymbol{n}$, where $\boldsymbol{n}$ is the unit outward normal on $\partial\Omega$.

3.3. Discretization. Given forms A_{DG}, B_{DG}, and O_{DG}, chosen to discretize the vector Laplacian, the divergence operator, and the convection term, respectively, we consider mixed methods of the form: find $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$ such that

$$\begin{aligned} A_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}, \boldsymbol{v}) + O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}; \boldsymbol{u}_{\mathrm{DG}}, \boldsymbol{v}) + B_{\mathrm{DG}}(\boldsymbol{v}, p_{\mathrm{DG}}) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x}, \\ B_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}, q) &= 0, \end{aligned} \tag{3.7}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$.

Let us now specify the forms A_{DG}, B_{DG}, and O_{DG} involved in (3.7). In what follows, we shall use the notations $\int_{\mathcal{F}_h} g\, ds := \sum_{E\in\mathcal{F}_h} \int_E g\, ds$ and $\|g\|^p_{L^p(\mathcal{F}_h)} := \sum_{E\in\mathcal{F}_h} \|g\|^p_{L^p(E)}$ for any subset $\mathcal{F}_h \subseteq \mathcal{E}(\mathcal{T}_h)$.

3.3.1. The diffusion form. To discretize the diffusive terms, we take the symmetric interior penalty term written in terms of lifting operators [1, 14]. It is obtained by first defining the stabilization form $I^{\mathrm{j}}_{\mathrm{DG}}$ as

$$I^{\mathrm{j}}_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}) := \nu \int_{\mathcal{E}(\mathcal{T}_h)} \mathrm{j}\, \underline{[\![\boldsymbol{u}]\!]} : \underline{[\![\boldsymbol{v}]\!]}\, ds, \qquad \boldsymbol{u}, \boldsymbol{v} \in H^1(\mathcal{T}_h)^2, \tag{3.8}$$

where j is the interior penalty stabilization function. It is defined edgewise as

$$\mathrm{j}|_E = \mathrm{j}_E := \mathrm{j}_0 k_E^2 h_E^{-1}, \qquad E \in \mathcal{E}(\mathcal{T}_h), \tag{3.9}$$

with $\mathrm{j}_0 > 0$ sufficiently large, independently of $\underline{h}$, $\underline{k}$, and ν, and with h_E and p_E defined by

$$h_E := \begin{cases} \min\{h_K, h_{K'}\} & \text{if } E = \partial K \cap \partial K' \in \mathcal{E}_{\mathcal{I}}(\mathcal{T}_h), \\ h_K & \text{if } E = \partial K \cap \partial\Omega \in \mathcal{E}_{\mathcal{D}}(\mathcal{T}_h), \end{cases} \tag{3.10}$$

respectively,

$$k_E := \begin{cases} \max\{k_K, k_{K'}\} & \text{if } E = \partial K \cap \partial K' \in \mathcal{E}_{\mathcal{I}}(\mathcal{T}_h), \\ k_K & \text{if } E = \partial K \cap \partial\Omega \in \mathcal{E}_{\mathcal{D}}(\mathcal{T}_h). \end{cases} \tag{3.11}$$

Then, the form A_{DG} is chosen as

$$A_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \nu \big[\nabla_h \boldsymbol{u} : \nabla_h \boldsymbol{v} - \underline{\mathcal{L}}(\boldsymbol{u}) : \nabla_h \boldsymbol{v} - \underline{\mathcal{L}}(\boldsymbol{v}) : \nabla_h \boldsymbol{u}\big]\, d\boldsymbol{x} + I^{\mathrm{j}}_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}), \tag{3.12}$$

for $\boldsymbol{u}, \boldsymbol{v} \in H^1(\mathcal{T}_h)^2$. Here, $\underline{\mathcal{L}}$ is the lifting operator $\underline{\mathcal{L}} : H^1(\mathcal{T}_h)^2 \to \underline{\Sigma}_{\mathrm{DG}}$ defined by

$$\int_\Omega \underline{\mathcal{L}}(\boldsymbol{v}) : \underline{\tau}\, d\boldsymbol{x} = \int_{\mathcal{E}(\mathcal{T}_h)} \underline{[\![\boldsymbol{v}]\!]} : \{\!\{\underline{\tau}\}\!\}\, ds \qquad \forall \underline{\tau} \in \underline{\Sigma}_{\mathrm{DG}}, \tag{3.13}$$

see [14]. Notice that restricted to discrete functions $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}}$, we have

$$\begin{aligned} A_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nu \nabla_h \boldsymbol{u} : \nabla_h \boldsymbol{v}\, d\boldsymbol{x} - \int_{\mathcal{E}(\mathcal{T}_h)} \{\!\{\nu \nabla_h \boldsymbol{v}\}\!\} : \underline{[\![\boldsymbol{u}]\!]}\, ds \\ - \int_{\mathcal{E}(\mathcal{T}_h)} \{\!\{\nu \nabla_h \boldsymbol{u}\}\!\} : \underline{[\![\boldsymbol{v}]\!]}\big)\, ds + I^{\mathrm{j}}_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}). \end{aligned} \tag{3.14}$$

3.3.2. The divergence form. Following [14], the divergence form B_{DG} will be taken as

$$B_{\mathrm{DG}}(\boldsymbol{v},q) = -\int_\Omega q\,[\nabla_h\cdot\boldsymbol{v} - \mathcal{M}(\boldsymbol{v})]\,d\boldsymbol{x}, \qquad \boldsymbol{v}\in H^1(\mathcal{T}_h)^2,\ q\in Q, \tag{3.15}$$

where the lifting $\mathcal{M}: H^1(\mathcal{T}_h)^2 \to Q_{\mathrm{DG}}$ is given by

$$\int_\Omega \mathcal{M}(\boldsymbol{v})q\,d\boldsymbol{x} = \int_{\mathcal{E}(\mathcal{T}_h)} [\![\boldsymbol{v}]\!]\{\!\{q\}\!\}\,ds \qquad \forall q\in Q_{\mathrm{DG}}. \tag{3.16}$$

For discrete functions $(\boldsymbol{v},q)\in\boldsymbol{V}_{\mathrm{DG}}\times Q_{\mathrm{DG}}$, we have

$$B_{\mathrm{DG}}(\boldsymbol{v},q) = -\int_\Omega q\,\nabla_h\cdot\boldsymbol{v}\,d\boldsymbol{x} + \int_{\mathcal{E}(\mathcal{T}_h)} \{\!\{q\}\!\}[\![\boldsymbol{v}]\!]\,ds. \tag{3.17}$$

We notice that, if $(\boldsymbol{u},p)\in\boldsymbol{V}\times Q$ is a solution of (2.2), then there holds

$$B_{\mathrm{DG}}(\boldsymbol{u},q) = 0, \qquad q\in Q_{\mathrm{DG}}. \tag{3.18}$$

Hence, the form B_{DG} is consistent in enforcing the divergence constraint.

3.3.3. The convective form. We consider the following discontinuous convection form; cf. [11, 10]:

$$\begin{aligned} O_{\mathrm{DG}}(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = &\int_\Omega ((\boldsymbol{w}\cdot\nabla_h)\boldsymbol{u})\cdot\boldsymbol{v}\,d\boldsymbol{x} + \frac{1}{2}\int_\Omega (\nabla_h\cdot\boldsymbol{w})\boldsymbol{u}\cdot\boldsymbol{v}\,d\boldsymbol{x} \\ &- \int_{\mathcal{E}_{\mathcal{I}}(\mathcal{T}_h)} \{\!\{\boldsymbol{v}\}\!\}\cdot\underline{[\![\boldsymbol{u}]\!]}\cdot\{\!\{\boldsymbol{w}\}\!\}\,ds - \frac{1}{2}\int_{\mathcal{E}(\mathcal{T}_h)} [\![\boldsymbol{w}]\!]\{\!\{\boldsymbol{u}\cdot\boldsymbol{v}\}\!\}\,ds \end{aligned} \tag{3.19}$$

It is well defined on $W^{1/4}(\mathcal{T}_h)^2\times W^{1/4}(\mathcal{T}_h)^2\times W^{1/4}(\mathcal{T}_h)^2$, see Proposition 4.3 below. Clearly, the form is linear in each argument. Moreover, it is consistent in the sense that

$$O_{\mathrm{DG}}(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}), \qquad \boldsymbol{w}\in\boldsymbol{Z},\ \boldsymbol{u}\in\boldsymbol{V},\ \boldsymbol{v}\in\boldsymbol{V}_{\mathrm{DG}}. \tag{3.20}$$

4. Stability and existence and uniqueness of discrete solutions. We discuss the hp-version stability properties of the discrete forms involved in (3.7). Consequently, we shall establish the existence and uniqueness of solutions to (3.7) under a discrete version of the small data assumption (2.5).

4.1. Auxiliary results. We first show the embedding of the broken space $H^1(\mathcal{T}_h)$ into $L^p(\Omega)$ with constants independent of $\underline{h}$ and $\underline{k}$; see also [**?**, **?**] for related results in the context of h-version approximations. To that end, we introduce the broken norm

$$\|v\|^2_{1,\mathcal{T}_h} := \|\nabla_h v\|^2_{L^2(\Omega)} + \int_{\mathcal{E}(\mathcal{T}_h)} \mathtt{h}^{-1}\,|[\![v]\!]|^2\,ds, \tag{4.1}$$

where we set $\mathtt{h}|_E := h_E$ for $E\in\mathcal{E}(\mathcal{T}_h)$, with h_E defined in (3.10).

The following embedding result holds.

LEMMA 4.1. *For any $p\in[1,\infty)$, there is an embedding constant $C>0$ such that*

$$\|v\|_{L^p(\Omega)} \le C\|v\|_{1,\mathcal{T}_h}, \qquad v\in H^1(\mathcal{T}_h).$$

The constant $C > 0$ only depends on Ω, p, the shape-regularity of the meshes, and the bounded variation of the local mesh sizes in (3.1).

Proof. The proof follows along the lines of [**?**]. Consider first the case $p \geq 2$. For $v \in H^1(\mathcal{T}_h)$, let $v_0 := \pi_0 v$ be L^2-projection of v into the piecewise constants over the partition $\mathcal{T}_h$. By the triangle inequality, we have

$$\|v\|_{L^p(\Omega)} \leq \|v - v_0\|_{L^p(\Omega)} + \|v_0\|_{L^p(\Omega)}.$$

By [12, Theorem 5.3, item (ii)] and by adding and subtracting v, we obtain

$$\|v_0\|^2_{L^p(\Omega)} \leq C \int_{\mathcal{E}(\mathcal{T}_h)} \mathtt{h}^{-1} |[\![v_0]\!]|^2 \, ds \leq C \int_{\mathcal{E}(\mathcal{T}_h)} \mathtt{h}^{-1} |[\![v - v_0]\!]|^2 \, ds + C \|v\|^2_{1,\mathcal{T}_h}.$$

Then, by the shape-regularity of the meshes, the bounded variation of the local mesh sizes, and standard approximation results for π^0, we obtain

$$\begin{aligned} \int_{\mathcal{E}(\mathcal{T}_h)} \mathtt{h}^{-1} |[\![v - v_0]\!]|^2 \, ds &\leq C \sum_{K \in \mathcal{T}_h} h_K^{-1} \|v - v_0\|^2_{L^2(\partial K)} \\ &\leq C \sum_{K \in \mathcal{T}_h} \|\nabla v\|^2_{L^2(K)} \leq C \|v\|^2_{1,\mathcal{T}_h}. \end{aligned}$$

These bounds yield $\|v_0\|_{L^p(\Omega)} \leq C \|v\|_{1,\mathcal{T}_h}$.

To bound the second term $\|v - v_0\|_{L^p(\Omega)}$, we use the approximation properties of π^0 in L^p-spaces, see, e.g., [**?**], and conclude that

$$\begin{aligned} \|v - v_0\|_{L^p(\Omega)} &= \Big(\sum_{K \in \mathcal{T}_h} \|v - v_0\|^p_{L^p(K)} \Big)^{1/p} \\ &\leq C \Big(\sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla v\|^p_{L^2(K)} \Big)^{1/p} \leq C \Big(\sum_{K \in \mathcal{T}_h} \|\nabla v\|^p_{L^2(K)} \Big)^{1/p}. \end{aligned}$$

Since $\|\underline{x}\|_{l^p} \leq \|\underline{x}\|_{l^2}$ for any $p \geq 2$ and any sequence $\underline{x} \in \mathbb{R}^n_+$, it follows that

$$\|v - v_0\|_{L^p(\Omega)} \leq C \Big(\sum_{K \in \mathcal{T}_h} \|\nabla v\|^2_{L^2(K)} \Big)^{1/2} \leq C \|v\|_{1,\mathcal{T}_h}.$$

This yields the assertion for $p \in [2, \infty)$.

If now $1 \leq p < 2$, we have $\|v\|_{L^p(\Omega)} \leq C \|v\|_{L^2(\Omega)} \leq C \|v\|_{1,\mathcal{T}_h}$, due to the boundedness of Ω, and the above result for $p = 2$. This completes the proof. ◻

Lemma 4.2. *There is a constant $C > 0$ such that*

$$\Big(\sum_{K \in \mathcal{T}_h} h_K \|v\|^4_{L^4(\partial K)} \Big)^{1/4} \leq C \|v\|_{1,\mathcal{T}_h} \tag{4.2}$$

for any $v \in W^{1,4}(\mathcal{T}_h)$.

Proof. We recall the trace estimate of [10, Equation (7.7)]: there is a constant $C > 0$ independent of K such that

$$h_K^{1/4} \|v\|_{L^4(\partial K)} \leq C \left(\|v\|_{L^4(K)} + \|\nabla v\|_{L^2(K)} \right) \tag{4.3}$$

for any $v \in W^{1,4}(K)$. Thus, we obtain

$$\begin{aligned}\Big(\sum_{K\in\mathcal{T}_h} h_K \|v\|^4_{L^4(\partial K)}\Big)^{1/4} &\le C\Big(\sum_{K\in\mathcal{T}_h}(\|v\|^4_{L^4(K)} + \|\nabla v\|^4_{L^2(K)})\Big)^{1/4}\\ &\le C\|v\|_{L^4(\Omega)} + C\Big(\sum_{K\in\mathcal{T}_h}\|\nabla v\|^4_{L^2(K)}\Big)^{1/4}\\ &\le C\left(\|v\|_{L^4(\Omega)} + \|\nabla_h v\|_{L^2(\Omega)}\right),\end{aligned}$$

where in the last step we have used that $\|\underline{x}\|^2_{l^2} \le \|\underline{x}\|^2_{l^1}$ for any sequence $\underline{x} \in \mathbb{R}^n_+$. The embedding in Lemma 4.1 with $p = 4$ yields the assertion. ◻

4.2. Stability. We introduce the broken hp-version DG norm

$$\|\boldsymbol{v}\|^2_{\mathrm{DG}} = \|\nabla_h \boldsymbol{v}\|^2_{L^2(\Omega)} + \int_{E\in\mathcal{E}(\mathcal{T}_h)} \mathsf{j}\,|\underline{[\![\boldsymbol{v}]\!]}|^2\,ds, \tag{4.4}$$

where j is the edgewise constant interior penalty function defined in (3.9). From Lemma 4.1, we have

$$\|\boldsymbol{v}\|_{L^2(\Omega)} \le C_{\mathrm{poinc}}\|\boldsymbol{v}\|_{\mathrm{DG}}, \tag{4.5}$$

$$\|\boldsymbol{v}\|_{L^4(\Omega)} \le C_{\mathrm{emb}}\|\boldsymbol{v}\|_{\mathrm{DG}}, \tag{4.6}$$

for any $\boldsymbol{v} \in H^1(\mathcal{T}_h)^2$, with constants $C_{\mathrm{poinc}} > 0$ and $C_{\mathrm{emb}} > 0$ independent of $\underline{h}$, $\underline{k}$ and ν.

4.2.1. The elliptic forms. In [14], the elliptic forms A_{DG} and B_{DG} have been thoroughly studied in the context of the Stokes problem. First, we have the following continuity properties: there are constants $C_{\mathrm{a}} > 0$ and $C_{\mathrm{b}} > 0$ independent of $\underline{h}$, $\underline{k}$, and ν such that

$$|A_{\mathrm{DG}}(\boldsymbol{v},\boldsymbol{w})| \le C_{\mathrm{a}}\nu^{1/2}\|\boldsymbol{v}\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{w}\|_{\mathrm{DG}}, \qquad \boldsymbol{v},\ \boldsymbol{w} \in H^1(\mathcal{T}_h)^2, \tag{4.7}$$

$$|B_{\mathrm{DG}}(\boldsymbol{v},q)| \le C_{\mathrm{b}}\|\boldsymbol{v}\|_{\mathrm{DG}}\|q\|_{L^2(\Omega)}, \qquad \boldsymbol{v} \in H^1(\mathcal{T}_h)^2,\ q \in Q. \tag{4.8}$$

Then, the form A_{DG} is coercive over the discrete space $\boldsymbol{V}_{\mathrm{DG}}$: there exists a parameter $\gamma_{\min} > 0$ independent of $\underline{h}$, $\underline{k}$, and ν such that for any $\gamma \ge \gamma_{\min}$ there exists a coercivity constant $C_{\mathrm{coer}} > 0$ independent of $\underline{h}$, $\underline{k}$, and ν with

$$A_{\mathrm{DG}}(\boldsymbol{v},\boldsymbol{v}) \ge C_{\mathrm{coer}}\nu\|\boldsymbol{v}\|^2_{\mathrm{DG}}, \qquad \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}}. \tag{4.9}$$

Throughout, we shall assume that $\gamma \ge \gamma_{\min}$.

Finally, the divergence form B_{DG} satisfies the discrete inf-sup condition: for $k_K \ge 2$, the following discrete inf-sup condition for the finite element spaces $\boldsymbol{V}_{\mathrm{DG}}$ and Q_{DG} in (3.4) holds true:

$$\inf_{0\ne q\in Q_{\mathrm{DG}}}\ \sup_{\boldsymbol{0}\ne\boldsymbol{v}\in\boldsymbol{V}_{\mathrm{DG}}} \frac{B_{\mathrm{DG}}(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\mathrm{DG}}\|q\|_0} \ge C_{\mathrm{is}}|\underline{k}|^{-1} > 0, \tag{4.10}$$

with a constant $C_{\mathrm{is}} > 0$ independent of $\underline{h}$, $\underline{k}$, and ν.

4.3. The convection form. The next result shows two crucial properties of the convection form.

Proposition 4.3. *There holds:*
1. *For $\boldsymbol{w}, \boldsymbol{u} \in \boldsymbol{V}_{\mathrm{DG}}$, we have $O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{u}) = 0$.*
2. *There is a constant C_{o} independent of $\underline{h}$, $\underline{k}$, and ν such that*

$$|O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})| \leq C_{\mathrm{o}} \|\boldsymbol{w}\|_{\mathrm{DG}} \|\boldsymbol{u}\|_{\mathrm{DG}} \|\boldsymbol{v}\|_{\mathrm{DG}} \tag{4.11}$$

for all $\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v} \in W^{1/4}(\mathcal{T}_h)^2$. In particular,

$$|O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})| \leq C_{\mathrm{o}} \|\boldsymbol{w}\|_{\mathrm{DG}} \|\boldsymbol{u}\|_{\mathrm{DG}} \|\boldsymbol{v}\|_{\mathrm{DG}} \tag{4.12}$$

for all $\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}}$.

Remark 4.4. *Recall that in view of* (2.6)*, the velocity field of a solution $(\boldsymbol{u}, p)$ to* (2.2) *belongs to $W^{1/4}(\mathcal{T}_h)$.*

Proof. Item (i): To verify the first item, we note that, by integration by parts, there holds

$$\sum_{K \in \mathcal{T}_h} \int_K ((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{u} \, d\boldsymbol{x} = -\frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_K (\nabla \cdot \boldsymbol{w}) |\boldsymbol{u}|^2 \, d\boldsymbol{x} + \frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{w} \cdot \boldsymbol{n}_K |\boldsymbol{u}|^2 \, ds.$$

Then, by employing the formula in [1, WHERE] and since $[\![|\boldsymbol{u}|^2]\!]_j = 2 \sum_{i=1}^2 \{\!\{\boldsymbol{u}\}\!\}_i \underline{[\![\boldsymbol{u}]\!]}_{ij}$ for $j = 1, 2$, we find that

$$\begin{aligned}
\sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{w} \cdot \boldsymbol{n}_K |\boldsymbol{u}|^2 \, ds &= \frac{1}{2} \sum_{E \in \mathcal{E}(\mathcal{T}_h)} \int_E [\![\boldsymbol{w}]\!] \{\!\{|\boldsymbol{u}|^2\}\!\} \, ds + \sum_{E \in \mathcal{E}_I(\mathcal{T}_h)} \int_E \{\!\{\boldsymbol{w}\}\!\} \cdot [\![|\boldsymbol{u}|^2]\!] \, ds \\
&= \sum_{E \in \mathcal{E}(\mathcal{T}_h)} \int_E [\![\boldsymbol{w}]\!] \{\!\{|\boldsymbol{u}|^2\}\!\} \, ds + 2 \sum_{E \in \mathcal{E}_I(\mathcal{T}_h)} \int_E \{\!\{\boldsymbol{u}\}\!\} \cdot \underline{[\![\boldsymbol{u}]\!]} \cdot \{\!\{\boldsymbol{w}\}\!\} \, ds.
\end{aligned}$$

Using these auxiliary calculations in the expression for $O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{u})$, the assertion readily follows.

Item (ii): We write $O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = T_1 + T_2 + T_3$, where

$$\begin{aligned}
T_1 &= \int_\Omega ((\boldsymbol{w} \cdot \nabla_h)\boldsymbol{u}) \cdot \boldsymbol{v} \, d\boldsymbol{x} + \frac{1}{2} \int_\Omega (\nabla_h \cdot \boldsymbol{w}) \boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x}, \\
T_2 &= -\int_{\mathcal{E}_{\mathcal{I}}(\mathcal{T}_h)} \{\!\{\boldsymbol{v}\}\!\} \cdot \underline{[\![\boldsymbol{u}]\!]} \cdot \{\!\{\boldsymbol{w}\}\!\} \, ds, \\
T_3 &= -\frac{1}{2} \int_{\mathcal{E}(\mathcal{T}_h)} [\![\boldsymbol{w}]\!] \{\!\{\boldsymbol{u} \cdot \boldsymbol{v}\}\!\} \, ds.
\end{aligned}$$

The volume terms in T_1 can be readily bounded by employing Hölder's inequality and the embeddings in (4.5), (4.6). This results in

$$|T_1| \leq C \|\boldsymbol{w}\|_{\mathrm{DG}} \|\boldsymbol{u}\|_{\mathrm{DG}} \|\boldsymbol{v}\|_{\mathrm{DG}}.$$

To bound T_2, we apply Hölder's inequality over $\mathcal{E}_{\mathcal{I}}(\mathcal{T}_h)$. Since $k_K \geq 2$, we find that

$$\begin{aligned}
|T_2| &\leq \|\mathrm{j}^{1/2} |\underline{[\![\boldsymbol{u}]\!]}|\|_{L^2(\mathcal{E}_{\mathcal{I}}(\mathcal{T}_h))} \|\mathrm{j}^{-1/4} |\{\!\{\boldsymbol{v}\}\!\}|\|_{L^4(\mathcal{E}_{\mathcal{I}}(\mathcal{T}_h))} \|\mathrm{j}^{-1/4} |\{\!\{\boldsymbol{w}\}\!\}|\|_{L^4(\mathcal{E}_{\mathcal{I}}(\mathcal{T}_h))} \\
&\leq C \|\boldsymbol{u}\|_{\mathrm{DG}} \Big(\sum_{K \in \mathcal{T}_h} h_K \|\boldsymbol{v}\|_{L^4(\partial K)}^4 \Big)^{1/4} \Big(\sum_{K \in \mathcal{T}_h} h_K \|\boldsymbol{w}\|_{L^4(\partial K)}^4 \Big)^{1/4}.
\end{aligned}$$

Hence, by Lemma 4.2 we obtain

$$|T_2| \leq C\|\boldsymbol{u}\|_{\mathrm{DG}}\|\boldsymbol{v}\|_{\mathrm{DG}}\|\boldsymbol{w}\|_{\mathrm{DG}}.$$

Similarly, since $|[\![\boldsymbol{w}]\!]| \leq |\underline{[\![\boldsymbol{w}]\!]}|$, a repeated application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}
|T_3| &\leq \frac{1}{2}\|\mathrm{j}^{1/2}\underline{[\![\boldsymbol{w}]\!]}\|_{L^2(\mathcal{E}(\mathcal{T}_h))}\|\mathrm{j}^{-1/2}|\{\!\{\boldsymbol{u}\cdot\boldsymbol{v}\}\!\}\|_{L^2(\mathcal{E}(\mathcal{T}_h))} \\
&\leq C\|\boldsymbol{w}\|_{\mathrm{DG}}\Big(\sum_{K\in\mathcal{T}_h} h_K\||\boldsymbol{u}\cdot\boldsymbol{v}|\|^2_{L^2(\partial K)}\Big)^{1/2} \\
&\leq C\|\boldsymbol{w}\|_{\mathrm{DG}}\Big(\sum_{K\in\mathcal{T}_h} h_K\|\boldsymbol{u}\|^4_{L^4(\partial K)}\Big)^{1/4}\Big(\sum_{K\in\mathcal{T}_h} h_K\|\boldsymbol{v}\|^4_{L^4(\partial K)}\Big)^{1/4}.
\end{aligned}$$

Again with Lemma 4.2, we conclude that

$$|T_3| \leq \|\boldsymbol{w}\|_{\mathrm{DG}}\|\boldsymbol{u}\|_{\mathrm{DG}}\|\boldsymbol{v}\|_{\mathrm{DG}}.$$

This implies the desired continuity bound. □

4.4. Existence and uniqueness of discrete solutions. We introduce the discrete kernel

$$\boldsymbol{Z}_{\mathrm{DG}} := \{\, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}} \,:\, B_{\mathrm{DG}}(\boldsymbol{v}, q) = 0 \ \forall\, q \in Q_{\mathrm{DG}} \,\}. \tag{4.13}$$

With the stability results from Section 4.2, the following result is standard. It follows by proceeding as in the continuous case; see [12, 7, 13].

PROPOSITION 4.5. *Let* $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$ *be a solution of* (3.7). *Then* $\boldsymbol{u}_{\mathrm{DG}} \in \boldsymbol{Z}_{\mathrm{DG}}$, *and*

$$\|\boldsymbol{u}_{\mathrm{DG}}\|_{\mathrm{DG}} \leq \frac{C_{\mathrm{poinc}}\|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu C_{\mathrm{coer}}}. \tag{4.14}$$

Moreover, under the small data assumption

$$\frac{C_{\mathrm{o}}C_{\mathrm{poinc}}\|\boldsymbol{f}\|_{L^2(\Omega)}}{C^2_{\mathrm{coer}}\nu^2} < 1 \tag{4.15}$$

the discrete problem (3.7) *has a unique solution* $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$.

REMARK 4.6. *Assumption* (4.15) *(as well as the continuous analog in* (2.5)*) is a contraction property. By Banach's fixed point theorem, it implies that the Picard iteration: given* $(\boldsymbol{u}^{(m-1)}_{\mathrm{DG}}, p^{(m-1)}_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$, *find the next iterate* $(\boldsymbol{u}^{(m)}_{\mathrm{DG}}, p^{(m)}_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$ *by solving the linear Oseen problem*

$$\begin{aligned}
A_{\mathrm{DG}}(\boldsymbol{u}^{(m)}_{\mathrm{DG}}, \boldsymbol{v}) + O_{\mathrm{DG}}(\boldsymbol{u}^{(m-1)}_{\mathrm{DG}}; \boldsymbol{u}^{(m)}_{\mathrm{DG}}, \boldsymbol{v}) + B_{\mathrm{DG}}(\boldsymbol{v}, p^{(m)}_{\mathrm{DG}}) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,d\boldsymbol{x}, \\
B_{\mathrm{DG}}(\boldsymbol{u}^{(m)}_{\mathrm{DG}}, q) &= 0,
\end{aligned} \tag{4.16}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$, *converges linearly to the unique solution* $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}})$ *of the non-linear problem* (3.7) *for any initial guess* $(\boldsymbol{u}^{(0)}_{\mathrm{DG}}, p^{(0)}_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$.

5. Exponential convergence.

5.1. Abstract error estimates. Due to the use of the lifting operators, the DG forms A_{DG} and B_{DG} are not fully consistent; cf. [14]. As a measure for the inconsistency of a solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ of (2.2), we introduce the residual

$$R_{\mathrm{DG}}(\boldsymbol{u}, p; \boldsymbol{v}) := A_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}) + O_{\mathrm{DG}}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + B_{\mathrm{DG}}(\boldsymbol{v}, p) - \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x}, \tag{5.1}$$

for all $\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}}$. We point out that, due to the consistency of the convection form (3.20), the residual has the same form as in the Stokes case considered in [15]. [**more precise**]. Our abstract error estimates will then be expressed in terms of $\mathcal{R}_{\mathrm{DG}}(\boldsymbol{u}, p)$ given by

$$\mathcal{R}_{\mathrm{DG}}(\boldsymbol{u}, p) := \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}}} \frac{|R_{\mathrm{DG}}(\boldsymbol{u}, p; \boldsymbol{v})|}{\nu^{1/2} \|\boldsymbol{v}\|_{\mathrm{DG}}}. \tag{5.2}$$

We define

$$C_{\mathrm{sm}} := \frac{\max\{C_O, C_{\mathrm{o}}\} \max\{C_P, C_{\mathrm{poinc}}\}}{\min\{1, C_{\mathrm{coer}}^2\}} \tag{5.3}$$

Hence, under the condition $C_{\mathrm{sm}} \nu^{-2} \|\boldsymbol{f}\|_{L^2(\Omega)} < 1$, both the continuous and discrete solutions in (2.2) and (3.7) exist and are unique.

For simplicity, we further set

$$|||(\boldsymbol{u}, p)|||^2 := \nu \|\boldsymbol{u}\|_{\mathrm{DG}}^2 + \nu^{-1} \|p\|_{L^2(\Omega)}^2. \tag{5.4}$$

THEOREM 5.1. *Assume that*

$$C_{\mathrm{sm}} \nu^{-2} \|\boldsymbol{f}\|_{L^2(\Omega)} \leq \frac{1}{2}, \tag{5.5}$$

Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ *be the solution of* (2.2)*, and* $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$ *be the DG approximation in* (3.7) *obtained with* $\gamma \geq \gamma_{\min}$*. Then we have the error estimates*

$$|||(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{DG}}, p - p_{\mathrm{DG}})||| \leq C |\underline{k}|^{\alpha} \Big[\inf_{(\boldsymbol{v}, q) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}} |||(\boldsymbol{u} - \boldsymbol{v}, p - q)||| + \mathcal{R}_{\mathrm{DG}}(\boldsymbol{u}, p) \Big],$$

with a constant $C > 0$ independent of $\underline{h}$, $\underline{k}$, and ν.

Proof.

We proceed in standard steps.

Step 1: We first claim that

$$\nu^{1/2} \|\boldsymbol{u} - \boldsymbol{u}\|_{\mathrm{DG}} \leq C \left(\inf_{(\boldsymbol{z}, q) \in \boldsymbol{Z}_h \times Q_h} |||(\boldsymbol{u} - \boldsymbol{z}, p - q)|||_{\mathrm{DG}} + \mathcal{R}_{\mathrm{DG}}(\boldsymbol{u}, p) \right) \tag{5.6}$$

To show (5.6), fix $\boldsymbol{z} \in \boldsymbol{Z}_h$, and $q \in Q_h$. we write the errors as

$$\begin{aligned} \boldsymbol{u} - \boldsymbol{u}_{\mathrm{DG}} &= (\boldsymbol{u} - \boldsymbol{v}) + (\boldsymbol{v} - \boldsymbol{u}_{\mathrm{DG}}) =: \boldsymbol{\eta}_u + \boldsymbol{\xi}_u, \\ p - p_{\mathrm{DG}} &= (p - q) + (q - p_{\mathrm{DG}}) =: \eta_p + \xi_p, \end{aligned} \tag{5.7}$$

for fixed $\boldsymbol{v} \in \boldsymbol{V}_h$, $q \in Q_h$. By the triangle inequality

$$\begin{aligned} \nu^{1/2} \|\boldsymbol{u} - \boldsymbol{u}_{\mathrm{DG}}\|_{\mathrm{DG}} &\leq \nu^{1/2} \|\boldsymbol{\eta}_u\|_{\mathrm{DG}} + \nu^{1/2} \|\boldsymbol{\xi}_u\|_{\mathrm{DG}}, \\ \nu^{-1/2} \|p - p_{\mathrm{DG}}\|_{L^2(\Omega)} &\leq \nu^{-1/2} \|\eta_p\|_{L^2(\Omega)} + \nu^{-1/2} \|\xi_p\|_{L^2(\Omega)}. \end{aligned} \tag{5.8}$$

Hence, it is sufficient to bound the terms $\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}$ and $\|\xi_p\|_{L^2(\Omega)}$. To do so,

We first assume that $\boldsymbol{v}$ belong to the discrete kernel $\boldsymbol{Z}_h$ in (4.13). Then, we also have that $\boldsymbol{\xi}_u \in \boldsymbol{Z}_h$. From the coercivity of A_h in (4.9) and the definition of the residual R_{DG} in (5.1), we readily find that

$$\begin{aligned}\nu C_{\mathrm{coer}}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}^2 &\leq A_h(\boldsymbol{\xi}_u,\boldsymbol{\xi}_u)\\ &= -A_h(\boldsymbol{\eta}_u,\boldsymbol{\xi}_u)+A_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\xi}_u) =: T_1+T_2+T_3+T_4,\end{aligned} \tag{5.9}$$

with

$$\begin{aligned}&T_1 := -A_{\mathrm{DG}}(\boldsymbol{\eta}_u,\boldsymbol{\xi}_u), && T_2 := -O_{\mathrm{DG}}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{\xi}_u)+O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{u}_{\mathrm{DG}},\boldsymbol{\xi}_u),\\ &T_3 := -B_{\mathrm{DG}}(\boldsymbol{\xi}_u,p-p_h), && T_4 := R_{\mathrm{DG}}(\boldsymbol{u},p;\boldsymbol{\xi}_u).\end{aligned}$$

Obviously, by (4.7),

$$|T_1| \leq C_{\mathrm{a}}\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}. \tag{5.10}$$

For the term T_2, we first write

$$\begin{aligned}T_2 &= -O_{\mathrm{DG}}(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{u},\boldsymbol{\xi}_u)+O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{u}_{\mathrm{DG}}-\boldsymbol{u},\boldsymbol{\xi}_u),\\ &= -O_{\mathrm{DG}}(\boldsymbol{\eta}_u;\boldsymbol{u},\boldsymbol{\xi}_u)-O_{\mathrm{DG}}(\boldsymbol{\xi}_u;\boldsymbol{u},\boldsymbol{\xi}_u)-O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{\eta}_u,\boldsymbol{\xi}_u)-O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{\xi}_u,\boldsymbol{\xi}_u).\end{aligned}$$

Due to the first item in Proposition 4.3, we have $O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{\xi}_u,\boldsymbol{\xi}_u)=0$. Moreover, by the boundedness of O_{DG} in (4.11), the stability bound (4.14), and the small data assumption (5.5),

$$\begin{aligned}|O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{\eta}_u,\boldsymbol{\xi}_u)| &\leq C_{\mathrm{o}}\|\boldsymbol{u}_{\mathrm{DG}}\|_{\mathrm{DG}}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}\\ &\leq \frac{C_{\mathrm{o}}C_{\mathrm{poinc}}\|\boldsymbol{f}\|_{L^2(\Omega)}}{C_{\mathrm{coer}}\nu^2}\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}\\ &\leq \frac{1}{2}C_{\mathrm{coer}}\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}.\end{aligned}$$

Similarly, using the continuous stability bound (2.4) and (5.5), we obtain

$$|O_{\mathrm{DG}}(\boldsymbol{\xi}_u;\boldsymbol{u},\boldsymbol{\xi}_u)| \leq \frac{1}{2}\min\{1,C_{\mathrm{coer}}^2\}\nu\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}^2 \leq \frac{1}{2}C_{\mathrm{coer}}\nu\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}^2,$$

as well as

$$|O_{\mathrm{DG}}(\boldsymbol{\eta}_u;\boldsymbol{u},\boldsymbol{\xi}_u)| \leq \frac{1}{2}C_{\mathrm{coer}}\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}^2.$$

It follows that

$$|T_2| \leq C\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}+\frac{1}{2}C_{\mathrm{coer}}\nu\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}^2. \tag{5.11}$$

To bound T_3, we note that, since $\boldsymbol{\xi}_u \in \boldsymbol{Z}_h$,

$$T_3 = B_h(\boldsymbol{\xi}_u,p-p_h) = B_h(\boldsymbol{\xi},p) = B_h(\boldsymbol{\xi}_u,p-q).$$

Hence, from the boundedness of B_h in (4.8) we obtain

$$|T_3| \leq C_{\mathrm{b}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}\nu^{-1/2}\|\eta_p\|_{L^2(\Omega)}. \tag{5.12}$$

Finally, by the definition of $\mathcal{R}_{\mathrm{DG}}$, the term T_4 is bounded by

$$|T_4| \leq \mathcal{R}_{\mathrm{DG}}(\boldsymbol{u},p)\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}. \tag{5.13}$$

By combining (5.9) with the bounds for T_1 through T_4 in (5.10)–(5.13), respectively, and by bringing the term $\frac{1}{2}C_{\mathrm{coer}}\nu\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}^2$ in (5.11) to the left-hand side of the resulting inequality, we conclude that

$$\frac{1}{2}C_{\mathrm{coer}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}} \leq C\left(\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}} + \nu^{-1/2}\|\eta_p\|_{L^2(\Omega)} + \mathcal{R}_{\mathrm{DG}}(\boldsymbol{u},p)\right). \tag{5.14}$$

Step 2: Consider now an arbitrary function $\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}}$ in (5.7)

$\square$

REFERENCES

[1] D.N. Arnold, F. Brezzi, B. Cockburn, and L.D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39:1749–1779, 2001.

[2] G.A. Baker, W.N. Jureidini, and O.A. Karakashian. Piecewise solenoidal vector fields and the Stokes problem. *SIAM J. Numer. Anal.*, 27:1466–1485, 1990.

[3] B. Cockburn, G. Kanschat, and D. Schötzau. The local discontinuous Galerkin method for the Oseen equations. *Math. Comp.*, 73:569–593, 2004.

[4] B. Cockburn, G. Kanschat, and D. Schötzau. A locally conservative LDG method for the incompressible Navier-Stokes equations. *Math. Comp.*, 74:1067–1095, 2005.

[5] B. Cockburn, G. Kanschat, and D. Schötzau. A note on discontinuous Galerkin divergence-free solutions of the Navier-Stokes equations. *J. Sci. Comput.*, 31(112):61–73, 2007.

[6] B. Cockburn, G. Kanschat, D. Schötzau, and C. Schwab. Local discontinuous Galerkin methods for the Stokes system. *SIAM J. Numer. Anal.*, 40:319–343, 2002.

[7] V. Girault and P. A. Raviart. *Finite Element Methods for Navier-Stokes Equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer–Verlag, New York, 1986.

[8] V. Girault, B. Rivière, and M.F. Wheeler. A discontinuous Galerkin method with non-overlapping domain decomposition for the Stokes and Navier-Stokes problems. *Math. Comp.*, 74:53–84, 2005.

[9] P. Hansbo and M.G. Larson. Discontinuous finite element methods for incompressible and nearly incompressible elasticity by use of Nitsche's method. *Comput. Methods Appl. Mech. Engrg.*, 191:1895–1908, 2002.

[10] O.A. Karakashian and W.N. Jureidini. A nonconforming finite element method for the stationary Navier-Stokes equations. *SIAM J. Numer. Anal.*, 35:93–120, 1998.

[11] D. A. Di Pietro and A. Ern. Discrete functional analysis tools for discontinuous Galerkin methods with applications to the incompressible Navier-Stokes equations. *Math. Comp.*, 79:1301–1330, 2010.

[12] D. A. Di Pietro and A. Ern. *Mathematical Aspects of Discontinuous Galerkin Methods*, volume 69 of *Math/'ematiques et Applications*. Springer–Verlag, 2012.

[13] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*. Springer–Verlag, Berlin, 1994.

[14] D. Schötzau, C. Schwab, and A. Toselli. Mixed *hp*-DGFEM for incompressible flows. *SIAM J. Numer. Anal.*, 40:2171–2194, 2003.

[15] D. Schötzau and T.P. Wihler. Exponential convergence of mixed *hp*-DGFEM for Stokes flow in polygons. *Numer. Math.*, 96:339–361, 2003.