0.1 Schur complement preconditioner

We start our analysis with an ideal version of (**??**), $\mathcal{M}_{\text{schurMH}}$, which corresponds to combining $\mathcal{M}_{\text{NS}}$ in (**??**) with F replacing $\widehat{F}$, together with $\mathcal{M}_{\text{idealMX}}$ in (**??**), along with a Schur complement based on the coupling terms:

$$\mathcal{M}_{\text{schurMH}} = \begin{pmatrix} F + C^T(M + D^T L^{-1} D)^{-1} C & B^T & C^T & 0 \\ 0 & -\widehat{S} & 0 & 0 \\ 0 & 0 & M + D^T L^{-1} D & 0 \\ 0 & 0 & 0 & L \end{pmatrix}. \tag{1}$$

The matrix $\mathcal{M}_{\text{schurMH}}^{-1}\mathcal{K}_{\text{MH}}$ has an eigenvalue $\lambda = 1$ with algebraic multiplicity of at least $n_b + n_c$ where n_c is the dimension of the nullspace of C and an eigenvalue $\lambda = -1$ with algebraic multiplicity of at least m_b. The eigenvector corresponding to $\lambda = 1$ is: $(u_c, -\widehat{S}^{-1}Bu_c, b, L^{-1}Db)$ with u_c in the nullspace of C and b anything.

The corresponding eigenvalue problem is

$$\begin{pmatrix} F & B^T & C^T & 0 \\ B & 0 & 0 & 0 \\ -C & 0 & M & D^T \\ 0 & 0 & D & 0 \end{pmatrix} \begin{pmatrix} u \\ p \\ b \\ r \end{pmatrix} = \lambda \begin{pmatrix} F + C^T(M + D^T L^{-1} D)^{-1} C & B^T & C^T & 0 \\ 0 & -\widehat{S} & 0 & 0 \\ 0 & 0 & M + D^T L^{-1} D & 0 \\ 0 & 0 & 0 & L \end{pmatrix} \begin{pmatrix} u \\ p \\ b \\ r \end{pmatrix}.$$

The four block rows can be written as

$$(1-\lambda)(Fu + B^T p + C^T b) - \lambda C^T (M + D^T L^{-1} D)^{-1} Cu = 0, \tag{2}$$

$$Bu = -\lambda \widehat{S}\, p, \tag{3}$$

$$(\lambda - 1)Cu + (1-\lambda)Mb - \lambda D^T L^{-1} Db + D^T r = 0, \tag{4}$$

$$Db = \lambda L r. \tag{5}$$

If $\lambda = 1$, (**??**) is satisfied if

$$C^T (M + D^T L^{-1} D)^{-1} Cu = 0.$$

This only happens when $u = 0$ or $u \in \text{Null}(C)$. Using u_c to denote the nullspace vector of C then (**??**) simplifies to:

$$p = -\widehat{S}^{-1} B u_c.$$

Equation (**??**) leads to $r = L^{-1}Db$. If this holds, (**??**) is readily satisfied. Therefore, $(u_c, -\widehat{S}^{-1}Bu_c, b, L^{-1}Db)$ is an eigenvector corresponding to $\lambda = 1$. There exist n_c linearly independent such u's and n_b linearly independent such b's. There are at least $n_c + n_b$ linearly independent nonzero vectors satisfying the eigenvalue problem when $\lambda = 1$. It follows that $\lambda = 1$ is an eigenvalue with algebraic multiplicity of at least $n_c + n_b$.

If $\lambda = -1$, (**??**) leads to $r = -L^{-1}Db$. Substituting it into (**??**), we obtain $Cu = Mb$. If $b = Gs$ is a discrete gradient, with the gradient matrix G defined in (**??**), then $Mb = 0$ and $C^T b = 0$. If we take $u = 0$, then $Cu = 0$ and the requirement $Cu = Mb$ is satisfied. If $u = 0$ and $b = Gs$ is a discrete gradient, equation (**??**) becomes $B^T p = 0$. Since B has full row rank, this implies $p = 0$.

Therefore, if $b = Gs$ is a discrete gradient, then $(0, 0, Gs, -L^{-1}DGs)$ is an eigenvector corresponding to $\lambda = -1$. According to the discrete Helmholtz decomposition (**??**), there are m_b discrete gradients. Therefore $\lambda = -1$ is an eigenvalue with algebraic multiplicity at least m_b.

In our experiments, we have observed the eigenvalue $\lambda = 1$ has algebraic multiplicity of exactly $n_u + n_b$ and the eigenvalue $\lambda = -1$ has algebraic multiplicity of exactly m_b. Proving this may be difficult, though, due to complicated generalized eigenvalue problems that arise in the calculation.

Theorem **??** shows that $\mathcal{M}_{\text{idealMH}}^{-1}\mathcal{K}_{\text{MH}}$ has tightly clustered eigenvalues $\lambda = \pm 1$. Since we have provided explicit expressions for the eigenvectors associated with $\lambda = \pm 1$, we know that the geometric multiplicities of these two eigenvalues are equal to their algebraic multiplicity. We thus expect a good convergence behavior for $\mathcal{M}_{\text{schurMH}}^{-1}\mathcal{K}_{\text{MH}}$.