An $H(\mathrm{curl};\Omega)$-conforming FEM: Nédélec's elements of first type

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Abstract: The aim of this report is to give an introduction to Nédélec's $H(\mathrm{curl};\Omega)$-conforming finite element method of first type. As the name suggests, this method has been introduced in 1980 by J. C. Nédélec in [8]. In the first section, we present the model problem and introduce the framework for its variational formulation. In the second section, we present Nédélec's elements of first type for $H(\mathrm{curl};\Omega)$. We start by considering the case of affine grids in two and three space dimensions. We introduce the Piola transformation for vector fields and discuss the choice of function spaces and degrees of freedom. These results are then extendend to bi- and trilinear grids. We explain the practical construction of global shape functions and conclude this section with some remarks on approximation results.
Numerical results, which serve to illustrate the convergence of the method, are presented in the third section. In Appendix A, we demonstrate how solutions of the two-dimensional model problem can be constructed from solutions of the scalar Laplace equation.
In Appendix B we motivate the model problem studied in the report by considering the time-harmonic Maxwell's equations in the low-frequency case.

1 The model problem and the space $H(\mathrm{curl};\Omega)$

Consider the vector-valued model problem in a Lipschitz domain $\Omega \in \mathbb{R}^d$, $d = 2, 3$:

$$\mathrm{curl}\,\mathrm{curl}\,\underline{u} + c(x)\underline{u} = \underline{f} \quad \text{in} \quad \Omega\,, \tag{1}$$

with right hand side $\underline{f} \in L^2(\Omega)^d$.
We assume a homogeneous Dirichlet boundary condition on the tangential trace

$$\underline{u} \wedge \underline{n} = 0 \tag{2}$$

on the boundary $\partial\Omega$ of Ω.
The coefficient $c(x)$ is assumed to be bounded and uniform positive definite.
This type of problem typically arises in particular settings of Maxwell's equations. The boundary condition (2) then applies to a perfectly conducting boundary. For a derivation of the model problem (1), refer to Appendix B.

The subject of this section is to give an appropriate setting for a variational formulation of (1).
A more detailed treatment of the following notions and proofs can be found in [4].

1.1 Definitions

Convention 1 *In the following, the vector $\underline{t}$ will denote the unit tangent vector w. r. t. an edge of a triangle or quadrilateral, oriented counterclockwise with respect to the corresponding triangle or quadrilateral. (In 3d, the considered triangles or quadrilaterals will always be faces of a polyhedron, and the counterclockwise orientation has to be understood as induced by "outward unit normal of the face, plus right hand rule").*

Let us first consider the case of $d = 2$. For $\underline{v} = \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} \in [\mathcal{D}(\overline{\Omega})]^2$ and $\varphi \in \mathcal{D}(\overline{\Omega})$ we define the scalar- and the vector-valued curl-operators:

$$\mathrm{curl}\,\underline{v} := \partial_x v_2 - \partial_y v_1 \quad \text{and} \quad \underline{\mathrm{curl}}\,\varphi := \begin{pmatrix} \partial_y \varphi \\ -\partial_x \varphi \end{pmatrix}.$$

We note that the curl curl-operator in two dimensions has to be understood as $\underline{\text{curl}}\,\text{curl}$.

REMARK 1 *In the two dimensional case, the* curl *operator is simply the divergence of the rotated field* $\underline{v}$*. Similarly, the* $\underline{\text{curl}}$ *operator is the rotated gradient field of* φ*. Setting*

$$\boldsymbol{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we have

$$\text{curl}\,\underline{v} = \text{div}\,(\boldsymbol{R}\underline{v})$$

and

$$\underline{\text{curl}}\,\varphi = \boldsymbol{R}\nabla\varphi\,.$$

We further note that the tangential vector $\underline{t}$ *is just the rotated outward unit normal vector* $\underline{t} = \boldsymbol{R}^T\underline{n}$*. This will enable us to derive statements for the* curl*-operators in two dimensions from statements for the divergence and gradient operators in two dimensions.*

For the case of $d = 3$ and a vector field $\underline{v} \in [\mathcal{D}(\overline{\Omega})]^3$ we write

$$\text{curl}\,\underline{v} := \nabla \wedge \underline{v} := \begin{pmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{pmatrix}$$

DEFINITION 1 *For* $d = 2, 3$ *we write* $\tilde{d} = 1$ *if* $d = 2$ *and* $\tilde{d} = 3$ *if* $d = 3$*, and we define*

$$H(\text{curl};\Omega) := \{\underline{v} \in [L^2(\Omega)]^d : \text{curl}\,\underline{v} \in [L^2(\Omega)]^{\tilde{d}}\}$$

$H(\text{curl};\Omega)$ endowed with the inner product

$$(\underline{v},\underline{u})_{H(\text{curl};\Omega)} := (\underline{v},\underline{u})_{L^2(\Omega)} + (\text{curl}\,\underline{v}, \text{curl}\,\underline{u})_{L^2(\Omega)}$$

is a Hilbert space.

1.2 Trace theorem, integration by parts

The space $H(\text{curl};\Omega)$ will be the appropriate Sobolev space for a weak formulation of the model problem. In this section we provide a notion of trace of a $H(\text{curl};\Omega)$-function onto the boundary $\partial\Omega$ and we define intergation by parts on the space $H(\text{curl};\Omega)$.

THEOREM 1 (Approximation Property) *For* $d = 2, 3$, $[\mathcal{D}(\overline{\Omega})]^d$ *is dense in* $H(\text{curl};\Omega)$*.*

See [4] p.13, p.20 for the proof in the 2d-case and p.20 for a reference to the proof in 3d proposed in Duvaut & Lions, 1971.

Equipped with this approximation property of smooth functions to elements of $H(\text{curl};\Omega)$, we can state

THEOREM 2 (Green's Formula) *For the 2d case, let* $\underline{u}$ *be in* $[H(\text{curl};\Omega)]^2$ *and* φ *be a test function in* $H^1(\Omega)$*. We have*

$$\int_\Omega \text{curl}\,\underline{u}\,\varphi\,dx = \int_\Omega \underline{u}\cdot\underline{\text{curl}}\,\varphi\,dx + \int_{\partial\Omega} (\underline{u}\cdot\underline{t})\,\varphi\,ds\,,$$

For the 3d case, let $\underline{u}$ *be in* $[H(\text{curl};\Omega)]^3$ *and* $\underline{v}$ *be a test function in* $[H^1(\Omega)]^3$*. We then have*

$$\int_\Omega \underline{v}\cdot\text{curl}\,\underline{u}\,dx = \int_\Omega \underline{u}\cdot\text{curl}\,\underline{v}\,dx + \int_{\partial\Omega} (\underline{v}\wedge\underline{n})\cdot\underline{u}\,ds\,,$$

The boundary integrals are understood as duality pairings in $[H^{-\frac{1}{2}}(\partial\Omega)]^{\tilde{d}} \times H^{\frac{1}{2}}(\partial\Omega)$.

PROOF. For smooth functions, it is easy to see that the above Green's formula holds. In the 2d case this follows just from Gauss' divergence theorem and remark 1.
For the 3d case we use the identity

$$\operatorname{div}(\underline{u} \wedge \underline{v}) = \underline{v} \cdot \operatorname{curl} \underline{u} - \underline{u} \cdot \operatorname{curl} \underline{v}$$

together with Gauss' Divergence Theorem and the properties of the mixed product $(\underline{a} \wedge \underline{b}) \cdot \underline{c}$ to obtain

$$\int_\Omega \underline{v} \cdot \operatorname{curl} \underline{u} - \underline{u} \cdot \operatorname{curl} \underline{v} \, dx = \int_\Omega \operatorname{div}(\underline{u} \wedge \underline{v}) \, dx = \int_{\partial\Omega} (\underline{u} \wedge \underline{v}) \cdot \underline{n} \, ds = \int_{\partial\Omega} (\underline{v} \wedge \underline{n}) \cdot \underline{u} \, ds \; .$$

The extention to a pairing of $H(\text{curl})$ and $H^1(\Omega)$ functions follows with Theorem 1 by a density argument and is a result of the proof of the Trace Theorem. See [4] p.21 for details.

□

THEOREM 3 (Trace Theorem) *For $d = 3$, let $\underline{n}$ denote the outward unit normal to the boundary $\partial\Omega$. For $d = 2$, let $\underline{t}$ be as in convention 1*
For $d = 2$ the mapping

$$\gamma : \quad \underline{v} \mapsto \gamma(\underline{v}) \cdot \underline{t}$$

and for $d = 3$ the mapping

$$\gamma : \quad \underline{v} \mapsto \gamma(\underline{v}) \wedge \underline{n}$$

is contiuous and linear from $H(\text{curl}; \Omega)$ to $[H^{-\frac{1}{2}}(\partial\Omega)]^{\tilde{d}}$.

Note, that the trace of a $H(\text{curl}; \Omega)$-function is only defined in tangential direction. Its trace is in the dual space of traces of $[H^1(\Omega)]^{\tilde{d}}$ functions. Recall that traces of such functions are defined in every direction and are functions in $[H^{\frac{1}{2}}(\partial\Omega)]^{\tilde{d}}$.

PROOF. The proof of the trace theorem follows from Green's formula stated in theorem 2 applied to smooth functions and then by density arguments. See [4] p.21 for details.

□

Due to the Trace Theorem it makes sense to define a space of $H(\text{curl})$-functions with vanishing tangential components on the boundary.

DEFINITION 2

$$H_0(\text{curl}; \Omega) := \{\underline{v} \in H(\text{curl}; \Omega) : \quad \underline{v} \wedge \underline{n} = 0 \text{ on } \partial\Omega\}$$

REMARK 2 *For $d = 2, 3$, $[\mathcal{D}(\Omega)]^d$ is dense in $H_0(\text{curl}; \Omega)$.*

A consequence of Green's formula is the following important regularity property of $H(\text{curl}; \Omega)$-functions:

PROPOSITION 1 *Let K_- and K_+ be two polygonal (resp. polyhedral) Lipschitz domains in $\mathbb{R}^d$, with a common edge (resp. common edge or face) $e = \partial K_- \cap \partial K_+ \neq \emptyset$ and denote by $\Omega = \partial K_- \cup \partial K_+$ their union. A function v is in $H(\text{curl}; \Omega)$ if and only if the restricion v_- of v to K_- is in $H(\text{curl}; K_-)$, the restricion v_+ of v to K_+ is in $H(\text{curl}; K_+)$* and *the* tangential *jump over e vanishes: $v_- \wedge n_- + v_+ \wedge n_+ = 0$ on e.*

PROOF. The proposition follows from choosing an appropriate test function and integrating by parts (global and local). In order to *localise* the result of the Trace Theorem, we must choose a testfunction from the space $H_{00}^{\frac{1}{2}}(e)$. These functions vanish at the endpoints of e and can therefore be extended by zero to a $H^{\frac{1}{2}}(\partial\Omega)$-function. From the comparison of local (on K_- and K_+ separately) and global (on Ω) integration by parts it followas then that the tangential jump vanishes in the dual space of $H_{00}^{\frac{1}{2}}(e)$. By densitiy properties of $H_{00}^{\frac{1}{2}}(e)$ it follows that the tangential traces vanish in the "correct space" as well. The "correct space" would be $H^{-\frac{1}{2}}(e)$ if we have no further regularity of $\underline{v}_-$ and $\underline{v}_+$, and it would be $L^2(e)$ if $\underline{v}$ is elementwise in H^1 (e. g. for piecewise polynomial $\underline{v}$).

□

1.3 Variational formulation of the model problem

In the previous sections we introduced the space $H(\text{curl};\Omega)$, an integration-by-parts formula and the notion of trace for an $H(\text{curl};\Omega)$-function. In this framework, the variational formulation of the model problem (1) reads:

Find $\underline{u} \in H_0(\text{curl};\Omega)$ such that for all test functions $\underline{v} \in H_0(\text{curl};\Omega)$ holds

$$\int_\Omega \text{curl}\,\underline{u}\ \text{curl}\,\underline{v}\,dx + \int_\Omega c(x)\,\underline{u}\cdot\underline{v}\,dx = \int_\Omega \underline{f}\cdot\underline{v}\,dx \tag{3}$$

With our assumptions on the data, the forms

$$\begin{aligned} a(\underline{u},\underline{v}) &:= \int_\Omega \text{curl}\,\underline{u}\ \text{curl}\,\underline{v}\,dx + \int_\Omega c(x)\,\underline{u}\cdot\underline{v}\,dx \\ l(\underline{v}) &:= \int_\Omega \underline{f}\cdot\underline{v}\,dx \end{aligned}$$

are continuous and the bilinear form $a(\cdot,\cdot)$ is coercive on $H_0(\text{curl};\Omega)\times H_0(\text{curl};\Omega)$. By the Lax-Milgram lemma it follows, that there exists a unique solution $\underline{u} \in H_0(\text{curl};\Omega)$ of (3).

2 Nédélec's elements of first type for $H(\text{curl};\Omega)$

In this section we will present present $H(\text{curl};\Omega)$-conforming vector-valued finite elements, the Nédélec elements of first type (cf. [8]), which can be used to discretize the variational problem (3).

In order to define a finite element we must specify

the geometry We choose a reference element $\hat{K}$ and a change of variables $F_K(\hat{x})$, the element map. We set $K = F_K(\hat{K})$.

a function space We need a *finite dimensional* function space $\hat{R}$, typically a space of polynomials, on the reference cell, plus a transformation of $\hat{R}$ to a function space R_K on a general cell K.

dofs We have to define a set of dofs $\mathcal{A} = \{\alpha_i(\cdot)\}_{i=1}^N$. These are linear functionals on $\hat{R}$ and $N < \infty$ is the dimension of $\hat{R}$. $\mathcal{A}$ should be *unisolvent*, that is, the dofs $\alpha_i(\cdot)$ are linearly independent.

First, we observe that for a conforming discretization of (3) we cannot take vector-valued finite elements that are build by taking the standard nodal finite element spaces of globally continuous functions for each vector component. For $H(\text{curl};\Omega)$-functions, the only continuity condition is the continuity of the tangential component over cell boundaries. This fact will motivate the choice of appropriate degrees of freedom (abbreviated by dofs in the following).
We will give an outline of the construction of the finite element spaces described by Nédélec in [8]. In literature, they are also referred to as *Nédélec's elements of first type.*

2.1 Construction of Nédélec elements on tetrahedral grids

In this section, we denote by $\hat{K}$ the standard triangular or tetrahedral reference element.

2.1.1 Polynomial spaces on the reference element

In [8], Nédélec introduces the function spaces $\hat{R} = \mathcal{R}^k$, on which his finite element will be based. These spaces are subject to this section. For more details, consult [8].
We denote by $\mathbb{P}_k(\hat{\Sigma})$ the space of polynomials of degree k on $\hat{\Sigma}$, where $\hat{\Sigma}$ is an edge, a face of or the reference element itself.. The space $\tilde{\mathbb{P}}_k$ of homogeneous polynomials of degree k is the span of monomials of total degree k in d variables on $\hat{K}$.

Definition 3 *We define the auxiliary space*

$$\mathcal{S}^k := \{ \underline{p} \in (\tilde{\mathbb{P}}_k)^d : \underline{p} \cdot \underline{\hat{x}} = \sum_{i=1}^{d} p_i \, \hat{x}_i \equiv 0 \} \,, \tag{4}$$

with $\hat{x} \in \hat{K}$.
The dimension of this space is k in the case $d = 2$ and $k(k+2)$ for $d = 3$.

Nédélec's first family of $H(\text{curl}; \Omega)$-conforming finite elements is based on the polynomial spaces

Definition 4

$$\mathcal{R}^k = \left(\mathbb{P}_{k-1}(\hat{K}) \right)^d \oplus \mathcal{S}^k \,. \tag{5}$$

These spaces have dimension

$$\dim(\mathcal{R}^k) = k(k+2) \qquad \textit{for} \quad d = 2 \,,$$
$$\dim(\mathcal{R}^k) = \frac{(k+3)(k+2)k}{2} \qquad \textit{for} \quad d = 3 \,.$$

In the two-dimensional case, an equivalent characterization of the space $\mathcal{R}^k$ is

$$\mathcal{R}^k = \left(\mathbb{P}_{k-1}(\hat{K}) \right)^2 \oplus \tilde{\mathbb{P}}_{k-1} \begin{pmatrix} \hat{x}_2 \\ -\hat{x}_1 \end{pmatrix} \,. \tag{6}$$

This can be seen by noting that for $d = 2$

$$\tilde{\mathbb{P}}_{k-1} \begin{pmatrix} \hat{x}_2 \\ -\hat{x}_1 \end{pmatrix} \subseteq \mathcal{S}^k$$

obviously holds. Moreover, the dimension of the space $\tilde{\mathbb{P}}_{k-1}$ of homogeneous polynomials of degree $k-1$ in two variables is k and this is also the dimension $\mathcal{S}^k$. This proves the stated equivalent representation of the space $\mathcal{S}^k$.

We illustrate these definitions with some examples. We start with the case $d = 2$ and consider the spaces of polynomials of degree $k = 1$ and $k = 2$:

Example 1

$$\mathcal{R}^1 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \hat{x}_2 \\ -\hat{x}_1 \end{pmatrix} \right\rangle \tag{7}$$
$$\mathcal{R}^2 = \left(\mathbb{P}_1(\hat{K}) \right)^2 \oplus \left\langle \begin{pmatrix} \hat{x}_1 \, \hat{x}_2 \\ -\hat{x}_1^2 \end{pmatrix}, \begin{pmatrix} \hat{x}_2^2 \\ -\hat{x}_1 \, \hat{x}_2 \end{pmatrix} \right\rangle$$

To illustrate a case for $d = 3$, we consider the lowest polynomial degree $k = 1$:

Example 2 *We have to specify a basis for $\mathcal{S}^1$:*
Let $\underline{p}$ be a polynomial in $(\mathbb{P}_1(\hat{K}))^3$ with componentwise representation

$$p_i = \sum_{j=1}^{3} a_{ij} \hat{x}_j \,, \qquad i = 1, 2, 3 \,.$$

The condition for $\underline{p}$ being in $\mathcal{S}^1$ is

$$\underline{p} \cdot \underline{\hat{x}} = \sum_{i=1}^{3} a_{ii} \hat{x}_i^2 + \sum_{\substack{i,j=1 \\ j>i}}^{3} (a_{ij} + a_{ji}) \hat{x}_i \hat{x}_j \equiv 0 \,.$$

This leads to the condition on the coefficients of a polynomial in $\mathcal{S}^1$:

$$a_{11} = a_{22} = a_{33} = 0$$
$$a_{12} = -a_{21}\,, \quad a_{13} = -a_{31}\,, \quad a_{23} = -a_{32}\,.$$

With the basis of $\mathcal{S}^1$ which is obtained by choosing $a_{ij} = 1$, $i = 1,2,3$, $j > i$ and setting all the other coefficients to zero, we get

$$\mathcal{R}^1 = \left(\mathbb{P}_0(\hat{K})\right)^3 \oplus \left\langle \begin{pmatrix} 0 \\ \hat{x}_3 \\ \hat{x}_2 \end{pmatrix}, \begin{pmatrix} \hat{x}_3 \\ 0 \\ \hat{x}_1 \end{pmatrix}, \begin{pmatrix} \hat{x}_2 \\ \hat{x}_1 \\ 0 \end{pmatrix} \right\rangle$$

We remark at this point that the spaces $\mathcal{R}^k$ do not span the whole $(\mathbb{P}_k(\hat{K}))^d$. An $H(\mathrm{curl};\Omega)$-conforming FEM based on full polynomial spaces, the so called *Nédélec elements of second type*, was introduced in 1986 by Nédélec in [10].

REMARK 3 *The original, rather technical, representation of the spaces $\mathcal{R}^k$ is given in Definition 2 in [8]. Nédélec uses this representation in most of his proofs. We will not refer to it here.*

2.1.2 Degrees of freedom on the reference element

In this section we define the set $\mathcal{A}$ of dofs, which is a set of linear functionals on $\mathcal{R}^k$.

REMARK 4 *Recall that the dimension of the spaces of polynomials of degree k in n variables is $\binom{n+k+2}{n}$.*

DEFINITION 5 *Let $\hat{K}$ be the reference triangle and $\underline{\hat{t}}$ the tangent as defined in convention 1. The set of degrees of freedom $\mathcal{A}$ on $\mathcal{R}^k$ in the 2d case consists of the linear functionals*

edge dofs

$$\hat{\alpha}(\underline{\hat{u}}) := \int_{\hat{e}} (\underline{\hat{t}} \cdot \underline{\hat{u}})\, \hat{\varphi}\, d\hat{s} \quad \forall \hat{\varphi} \in \mathbb{P}_{k-1}(\hat{e})\,,$$

for every edge $\hat{e}$ of $\hat{K}$. We have a total of $3k$ of edge dofs.

inner dofs

$$\hat{\alpha}(\underline{\hat{u}}) := \int_{\hat{K}} \underline{\hat{u}} \cdot \underline{\hat{\varphi}}\, d\hat{x} \quad \forall \underline{\hat{\varphi}} \in (\mathbb{P}_{k-2}(\hat{K}))^2\,.$$

We have a total of $k(k-1)$ of inner dofs.

DEFINITION 6 *Let $\hat{K}$ be the reference tetrahedron, $\underline{\hat{t}}$ the tangent to an edge as defined in convention 1 and $\underline{\hat{n}}$ the outward unit normal vector to a face. The set of degrees of freedom $\mathcal{A}$ on $\mathcal{R}^k$ in the 3d case consists of the linear functionals*

edge dofs

$$\hat{\alpha}(\underline{\hat{u}}) := \int_{\hat{e}} (\underline{\hat{t}} \cdot \underline{\hat{u}})\, \hat{\varphi}\, d\hat{s} \quad \forall \hat{\varphi} \in \mathbb{P}_{k-1}(\hat{e})\,,$$

for every edge $\hat{e}$ of the tetrahedron $\hat{K}$. We have a total of $6k$ of edge dofs.

face dofs

$$\hat{\alpha}(\underline{\hat{u}}) := \int_{\hat{f}} (\underline{\hat{u}} \wedge \underline{\hat{n}}) \cdot \underline{\hat{\varphi}}\, d\hat{a} \quad \forall \hat{\varphi} \in (\mathbb{P}_{k-2}(\hat{f}))^2\,,$$

for every face $\hat{f}$ of the tetrahedron $\hat{K}$. We have a total of $4k(k-1)$ of face dofs.

inner dofs

$$\hat{\alpha}(\underline{\hat{u}}) := \int_{\hat{K}} \underline{\hat{u}} \cdot \underline{\hat{\varphi}}\, d\hat{x} \quad \forall \underline{\hat{\varphi}} \in (\mathbb{P}_{k-3}(\hat{K}))^3\,.$$

We have a total of $\frac{k(k-1)(k-2)}{2}$ of inner dofs.

We note that in the case of lowest order elements, i. e. $k = 1$, only edge dofs occur. This is not so for higher order elements. For $k = 2$ we additionally have inner dofs in the 2d case and face dofs in the 3d case. For $k \leq 3$ we have all types of dofs in both cases.
We also note that the total number of dofs equals the dimension of the spaces $\mathcal{R}^k$, as it should be.
The representation of the *interface* dofs, that is edge dofs in 2d, edge and face dofs in 3d, is motivated by the continuity condition on $H(\mathrm{curl};\Omega)$-functions stated in proposition 1.

PROPOSITION 2 *The set $\mathcal{A}$ of dofs befined above is unisolvent on $\mathcal{R}^k$. $\hat{\underline{u}} \in \mathcal{R}^k$ is uniquely defined by the moments $\hat{\alpha}(\hat{\underline{u}})$.*

PROOF. See [8], proof of theorem 1 and preceeding lemmas.

EXAMPLE 3 (Reference shape functions of lowest order for Nédélec elements on triangular meshes) *Let the reference element be the triangle $\hat{K} = \{(\hat{x},\hat{y}) \in \mathbb{R}^2 : \quad 0 \leq \hat{x} \leq 1\,,\ 0 \leq \hat{y} \leq 1 - \hat{x}\}$. Label the edges couterclockwise startung with $\hat{e}_0 = \overline{(0,0),(1,0)}$. The tangential vectors to the edges are (oriented counterclockwise)*

$$\hat{\underline{t}}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{\underline{t}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \hat{\underline{t}}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The underlying function space for lowest order Nédélec elements on a triangular mesh is $\mathcal{R}^1$ from (7).
In the case of $k = 1$ only egde-dofs occur. On $\hat{K}$ we have dofs of the type $\int_{\hat{e}_i} (\hat{\underline{t}} \cdot \hat{\underline{u}})\, \hat{\varphi}\, d\hat{s}\,, \forall \hat{\varphi} \in \mathbb{P}_0(\hat{e}_i)$. More precisely, since $\varphi \equiv 1$ is a basis for $\mathbb{P}_0(\hat{e}_i)$ we have the three dofs

$$\hat{\alpha}_i(\hat{\underline{u}}) = \int_{\hat{e}_i} (\hat{\underline{t}} \cdot \hat{\underline{u}})\, d\hat{s} \quad i = 0, 1, 2\,.$$

In order to construct a FE-basis $\hat{\underline{N}}_0, \hat{\underline{N}}_1, \hat{\underline{N}}_2$ for $\mathcal{R}^1$ with respect to these dofs, we require $\hat{\alpha}_i(\hat{\underline{N}}_j) = \delta_{ij}$. This leads to a linear system for the coefficients of the $\hat{\underline{N}}_i$ in a general basis of $\mathcal{R}^1$. In the case of lowest order elements, it is easy to verify that we have

$$\hat{\underline{N}}_0 = \begin{pmatrix} 1 - \hat{y} \\ \hat{x} \end{pmatrix}, \quad \hat{\underline{N}}_1 = \begin{pmatrix} -\hat{y} \\ \hat{x} \end{pmatrix}, \quad \hat{\underline{N}}_2 = \begin{pmatrix} -\hat{y} \\ \hat{x} - 1 \end{pmatrix}. \tag{8}$$

2.1.3 Piola transformation

An affine triangle or tetrahedron K is described by the affine element map

$$K \ni x = F_K(\hat{x}) = B_K \hat{x} + b_K$$

In standard $H^1(\Omega)$-conforming FEM, the shape functions N_i on a general cell K are obtained from the reference shape functions $\hat{N}_i$ on the reference element $\hat{K}$ by the pull-back

$$N_i(x) = \left(\hat{N}_i \circ F_K^{-1}\right)(x)$$

In the case of $H(\mathrm{curl};\Omega)$-conforming Nédélec FEM we cannot transforme our shape function in this way. The pull-back of a $H(\mathrm{curl};\hat{K})$-function needs not to be in $H(\mathrm{curl};K)$. In addition, the pull-back is not an $\mathcal{R}^k$-isomorphism and it does not lead to an $H(\mathrm{curl};\Omega)$-conforming method if prescribing the dofs by definitions 5 or 6.
In Nédélec's FEM (or, more general, in $H(\mathrm{curl};\Omega)$-conforming FEM), the shape functions are transformed by the following covariant transformation for vector-fields:

> The element shape functions $\underline{N}_i(x)$ on the element $K = F_K(\hat{K})$ are obtained from the reference shape functions by
> $$\underline{N}_i(x) = \mathcal{P}_K(\hat{\underline{N}}_i) = \left(\hat{D}F_K^{-T} \hat{\underline{N}}_i\right) \circ F_K^{-1}(x)\,, \tag{9}$$
> where $\hat{D}F_K$ is the jacobian $\frac{d}{d\hat{x}} F_K(\hat{x})$ of the element map.

In literature, an equivalent to this transformation for $H(\mathrm{div};\Omega)$-conforming FEM (which in that case is a contravariant map) is referred to as *Piola transformation*, cf. [3] pp. 97.
Here, we will refer to the transformation (9) of the vector field also as *Piola transformation.*
We note that the gradients of scalar nodal $H^1(\Omega)$-conforming finite elements transform according to the Piola transformation (9).
In the case of tetrahedral elements and affine element map $F_K(\hat{x}) = B_K\hat{x} + b_k$, the jacobian $\hat{D}F_K$ is just the constant matrix B_K and we have

$$\underline{v}(x) = \mathcal{P}_K(\hat{\underline{v}}) = B_K^{-T}\left(\hat{\underline{v}} \circ F_K^{-1}\right)(x)\,, \tag{10}$$

2.1.4 Transformation of the curl in 2d

For $\Omega \subset \mathbb{R}^2$, we noted in remark 1 that vector fields in $H(\mathrm{curl};\Omega)$ can be represented as rotated $H(\mathrm{div};\Omega)$ vector fields. Moreover, it is easy to verify that

$$B_K^{-T} = \det B_K^{-1}\, R^T B_K\, R\,, \tag{11}$$

where R is the rotation matrix from remark 1. Therefore, the properties of the Piola transformation (10) in the 2d case can be derived directly from the properties of the $H(\mathrm{div};\Omega)$-Piola transformation stated in [3] pp. 97.

THEOREM 4 (Some properties of 2d Piola transformation for affine element map) *Let* $\underline{v}(x) = \mathcal{P}_K(\hat{\underline{v}})$, $\varphi(x) = \left(\hat{\varphi} \circ F_K^{-1}\right)(x)$, $\hat{x} = F_K^{-1}(x)$, *with affine element map* F_K.

(i) The gradient $D\underline{v}$ *transforms according to*

$$D\underline{v} = B_K^{-T}\,\hat{D}\hat{\underline{v}}\,B_K^{-1}\,. \tag{12}$$

(ii) The curl transforms according to

$$\mathrm{curl}\,\underline{v} = \det B_K^{-1}\widehat{\mathrm{curl}}\hat{\underline{v}}\,. \tag{13}$$

As a consequence we see that $H(\mathrm{curl};K)$ *is isomorphic to* $H(\mathrm{curl};\hat{K})$ *under the Piola transformation (10).*

PROOF.

(i) Chain rule

(ii) We use that the 2d curl operator is just the trace of the rotated jacobian $R\,Dv$. By remark 11, we can replace B_K^{-T} and we get that $R\,Dv$ is affine-equivalent to $\det B_K^{-1}\,R\,\hat{D}\hat{v}$, which proves (ii).

□

COROLLARY 1 *From (ii) in theorem 4 we deduce*

$$\int_K \mathrm{curl}\,\underline{v}\,\varphi\,dx = \int_{\hat{K}} \widehat{\mathrm{curl}}\hat{\underline{v}}\,\hat{\varphi}\,d\hat{x}\,,$$

and we have, together with (ii) from theorem 4

$$\int_K \mathrm{curl}\,\underline{v}\;\mathrm{curl}\,\underline{u}\,dx = |B_K|^{-1}\int_{\hat{K}} \widehat{\mathrm{curl}}\hat{\underline{v}}\,\widehat{\mathrm{curl}}\hat{\underline{u}}\,d\hat{x}\,.$$

2.1.5 Transformation of the curl in 3d

In three dimensions, we cannot identify the curl-operator with the rotated gradient or with the divergence of a rotated vector field. We cannot, as in 2d, derive a transformation formula for the curl from the transformatin formula of the divergence.
By the chain rule, we obtain the transformation of the gradient of a vector field $\underline{v}$, defined by the Piola transformation (10) of a reference field $\hat{\underline{v}}$:

$$D\underline{v} = B_K^{-T}\, \hat{D}\hat{\underline{v}}\, B_K^{-1}\,. \tag{14}$$

We introduce the skew symmetric matrix $\operatorname{Curl} v$ as

$$(\operatorname{Curl} v)_{ij} = \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} \tag{15}$$

We see that $\operatorname{Curl} v = D\underline{v}^T - D\underline{v}$ and therefore by (14)

$$\operatorname{Curl} v = B_K^{-T}\widehat{\operatorname{Curl}}\,\hat{v}\, B_K^{-1} \tag{16}$$

Proposition 3 (Transformation of the curl in 3d) *Let $\hat{K}$ be the reference tetrahedron and $K = F_K(\hat{K})$ an affine image of it. The curl of a vector field $\underline{v}(x)$ on K, defined by the Piola transformation of a reference field $\hat{\underline{v}}(\hat{x})$ transforms according to*

$$(\operatorname{curl}\underline{v})_i\,(x) = \det \mathrm{M_i}(x)\,, \qquad i = 1,2,3 \tag{17}$$

We obtain the matrix $\mathrm{M_i}$ by replacing i-th column of the (constant) jacobian $D(F_K^{-1}) = B_K^{-1}$ by the vector $(\widehat{\operatorname{curl}}\,\hat{\underline{v}} \circ F_K^{-1})(x)$:

$$(\mathrm{M_i})_{kl}\,(x) := \begin{cases} (\widehat{\operatorname{curl}}\,\hat{v} \circ F_K^{-1})_k(x) & \text{if} \quad l = i \\ (B_K^{-1})_{kl} & \text{if} \quad l \neq i \end{cases}$$

(Note: an alternative, equivalent, transformation formula for the curl in 3d is given in proposition 4).

Proof. It holds

$$\operatorname{curl}\underline{v} = \begin{pmatrix} (\operatorname{Curl} v)_{23} \\ (\operatorname{Curl} v)_{31} \\ (\operatorname{Curl} v)_{12} \end{pmatrix}. \tag{18}$$

We demonstrate the statement of the proposition for the first component of the curl, which is $(\operatorname{curl}\underline{v})_1 = \operatorname{Curl} v_{23}$. Using the transformation (16), implicit summation over equal indices and the abbreviation $b_{ij} := (B_K^{-1})_{ij}$, we have

$$(\operatorname{Curl} v)_{23} = b_{k2}\,(\widehat{\operatorname{Curl}}\,\hat{v})_{kl}\, b_{l3}$$

Writing this out and recalling that $\operatorname{Curl} v$ is skew symmetric, yields

$$(\operatorname{Curl} v)_{23} = (b_{12}b_{23} - b_{22}b_{13})(\widehat{\operatorname{Curl}}\,\hat{v})_{12} - (b_{12}b_{33} - b_{32}b_{13})(\widehat{\operatorname{Curl}}\,\hat{v})_{31} + (b_{22}b_{33} - b_{32}b_{23})(\widehat{\operatorname{Curl}}\,\hat{v})_{23}\,,$$

and with (18) this is equal to the determinant of

$$\mathrm{M}_1 := \begin{pmatrix} (\widehat{\operatorname{curl}}\, v)_1 & b_{12} & b_{13} \\ (\widehat{\operatorname{curl}}\, v)_2 & b_{22} & b_{23} \\ (\widehat{\operatorname{curl}}\, v)_3 & b_{32} & b_{33} \end{pmatrix}.$$

The proof for the other components follows analogously.

□

In the next proposition, we state an alternative, equivalent, formula for the transformation of the curl (e. g. used by Demkovicz in [12])

PROPOSITION 4 *For a vector field $\underline{v}$ on the tetrahedron $K = F_K(\hat{K})$, defined by the Piola transformation (10) of a reference field $\hat{\underline{v}}$ on $\hat{K}$, we have*

$$\operatorname{curl} \underline{v} = \frac{1}{\det B_K} B_K \, (\widehat{\operatorname{curl}} \, \hat{\underline{v}} \circ F_K^{-1}) \,. \tag{19}$$

PROOF. The transformation formula (19) can be proven componentwise, and we will only carry out the proof for the first vector component $(\operatorname{curl} \underline{v})_1$. The proofs for the other components follow analogously.
The identity (19) reads for the first vector component

$$(\operatorname{curl} \underline{v})_1 = \frac{1}{\det B_K} (B_K)_{1j} ((\widehat{\operatorname{curl}} \, \hat{\underline{v}})_j \circ F^{-1}) \,. \tag{20}$$

Referring to (17), we show that the right hand side of (20) equals $\det \mathrm{M}_1$. For this, we expand $\det \mathrm{M}_1$ to

$$\det \mathrm{M}_1 = (\widehat{\operatorname{curl}} \, \hat{\underline{v}})_1 \det \mathcal{B}^{inv}_{11} - (\widehat{\operatorname{curl}} \, \hat{\underline{v}})_2 \det \mathcal{B}^{inv}_{21} + (\widehat{\operatorname{curl}} \, \hat{\underline{v}})_3 \det \mathcal{B}^{inv}_{31} \,,$$

where $\mathcal{B}^{inv}_{ij}$ is the 2×2-matrix arising from B_K^{-1} when cancelling its i-th row and its j-th column.
We recall the formula for the inverse of a matrix $A \in \mathbb{R}^{3\times 3}$

$$(A^{-1})_{ij} = \frac{1}{\det A} (-1)^{i+j} \det \mathcal{A}_{ji} \,, \tag{21}$$

where $\mathcal{A}_{ij}$ is the 2×2-matrix arising from A when cancelling its i-th row and its j-th column.
Replacing B_K in the right hand side of (19) by the expression (21) for $A = B_K^{-1}$, we get

$$\frac{1}{\det B_K} \frac{1}{\det B_K^{-1}} (-1)^{1+j} \det \mathcal{B}^{inv}_{j1} (\widehat{\operatorname{curl}} \, \hat{\underline{v}})_j = (\widehat{\operatorname{curl}} \, \hat{\underline{v}})_1 \det \mathcal{B}^{inv}_{11} - (\widehat{\operatorname{curl}} \, \hat{\underline{v}})_2 \det \mathcal{B}^{inv}_{21} + (\widehat{\operatorname{curl}} \, \hat{\underline{v}})_3 \det \mathcal{B}^{inv}_{31} = \det \mathrm{M}_1 \,.$$

□

2.2 Nédélec Elements on affine quadrilateral or hexahedral grids

We want to present the ingredients for Nédélec's finite elements of first type on grids consisiting of parallelograms (in 2d) or the respective objects in 3d, so called parallelotops (cf. section *FE built on cubes* in [8]). Such grids consist of elements C that are affine images of the square or cubic reference element $\hat{C} = [0,1]^d$:

$$C = F_C(\hat{C}) \quad C \ni x = B_C \hat{x} + \underline{b}_C \,, \hat{x} \in \hat{C} \,.$$

2.2.1 Polynomial spaces on the reference element

In order to introduce the function spaces needed for the construction of Nédélec's finite elements, let us define some spaces of vector-valued polynomials

DEFINITION 7 *$\mathcal{Q}_{l,m}$ are the spaces of polynomials on the reference square $\hat{C}$ with maximal degree l in $\hat{x}_1$ and m in $\hat{x}_2$.*
$\mathcal{Q}_{l,m,n}$ are the spaces of polynomials on the reference cube $\hat{C}$ with maximal degree l in $\hat{x}_1$, m in $\hat{x}_2$ and n in $\hat{x}_3$.

The spaces $\hat{R}$ for the reference shape functions now are in 2d

$$\mathcal{P}^k = \left\{ \underline{\hat{u}} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} : \quad \hat{u}_1 \in \mathcal{Q}_{k-1,k} \,, \hat{u}_2 \in \mathcal{Q}_{k,k-1} \right\} \,, \tag{22}$$

and in 3d

$$\mathcal{P}^k = \left\{ \underline{\hat{u}} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} : \quad \hat{u}_1 \in \mathcal{Q}_{k-1,k,k} \,, \hat{u}_2 \in \mathcal{Q}_{k,k-1,k} \,, \hat{u}_3 \in \mathcal{Q}_{k,k,k-1} \right\} \,. \tag{23}$$

We renounce an example, since it is quite evident, what these spaces look like for a specific k.

2.2.2 Degrees of freedom on the reference element

We start with the degrees of freedoms on the reference square $\hat{C} \subset \mathbb{R}^2$:

DEFINITION 8 *Let $\hat{C}$ denote the reference square and $\hat{\underline{t}}$ the tangent as defined in convention 1. The set of degrees of freedom $\mathcal{A}$ on $\mathcal{P}^k$ in the 2d case consists of the linear functionals*

edge dofs

$$\hat{\alpha}(\underline{\hat{u}}) := \int_{\hat{e}} (\hat{\underline{t}} \cdot \underline{\hat{u}})\, \hat{\varphi}\, d\hat{s}\,, \quad \forall\, \hat{\varphi} \in \mathbb{P}_{k-1}(\hat{e})\,,$$

for every edge $\hat{e}$ of $\hat{C}$. We have a total of $4k$ of edge dofs.

inner dofs

$$\hat{\alpha}(\underline{\hat{u}}) := \int_{\hat{C}} \underline{\hat{u}} \cdot \underline{\hat{\varphi}}\, d\hat{x}\,, \quad \forall\, \underline{\hat{\varphi}} = \begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{pmatrix}, \quad \hat{\varphi}_1 \in \mathcal{Q}_{k-2,k-1}\,, \quad \hat{\varphi}_2 \in \mathcal{Q}_{k-1,k-2}\,.$$

We have a total of $2k(k-1)$ of inner dofs.

DEFINITION 9 *Let $\hat{C}$ denote the reference cube, $\hat{\underline{t}}$ the tangent to an edge as defined in convention 1 and $\hat{\underline{n}}$ the outward unit normal vector to a face. The set of degrees of freedom $\mathcal{A}$ on $\mathcal{P}^k$ in the 3d case consists of the linear functionals*

edge dofs

$$\hat{\alpha}(\underline{\hat{u}}) := \int_{\hat{e}} (\hat{\underline{t}} \cdot \underline{\hat{u}})\, \hat{\varphi}\, d\hat{s}\,, \quad \forall\, \hat{\varphi} \in \mathbb{P}_{k-1}(\hat{e})\,,$$

for every edge $\hat{e}$ of $\hat{C}$. We have a total of $12k$ of edge dofs.

face dofs

$$\hat{\alpha}(\underline{\hat{u}}) := \int_{\hat{f}} (\underline{\hat{u}} \wedge \underline{\hat{n}}) \cdot \underline{\hat{\varphi}}\, d\hat{a}\,, \quad \forall\, \underline{\hat{\varphi}} = \begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{pmatrix}, \quad \hat{\varphi}_1 \in \mathcal{Q}_{k-2,k-1}(\hat{f})\,, \quad \hat{\varphi}_2 \in \mathcal{Q}_{k-1,k-2}(\hat{f})\,.$$

for every face $\hat{f}$ of $\hat{C}$. We have a total of $6 \cdot 2k(k-1)$ of face dofs.

inner dofs

$$\hat{\alpha}(\underline{\hat{u}}) := \int_{\hat{C}} \underline{\hat{u}} \cdot \underline{\hat{\varphi}}\, d\hat{x}\,, \quad \forall\, \underline{\hat{\varphi}} = \begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \\ \hat{\varphi}_3 \end{pmatrix}, \quad \hat{\varphi}_1 \in \mathcal{Q}_{k-1,k-2,k-2}\,, \quad \hat{\varphi}_2 \in \mathcal{Q}_{k-2,k-1,k-2}\,, \quad \hat{\varphi}_3 \in \mathcal{Q}_{k-2,k-2,k-1}\,.$$

We have a total of $3k(k-1)^2$ of inner dofs.

EXAMPLE 4 *Proceeding the same way as in example 3 for a triangular reference element, we obtain the reference shape functions of lowest order on the square $[0,1]^2$. For the unit tangents as in convention 1*

$$\hat{\underline{t}}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{\underline{t}}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \hat{\underline{t}}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \hat{\underline{t}}_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

they read

$$\underline{\hat{N}}_0 = \begin{pmatrix} 1-\hat{y} \\ 0 \end{pmatrix}, \quad \underline{\hat{N}}_1 = \begin{pmatrix} 0 \\ \hat{x} \end{pmatrix}, \quad \underline{\hat{N}}_2 = \begin{pmatrix} -\hat{y} \\ 0 \end{pmatrix}, \quad \underline{\hat{N}}_3 = \begin{pmatrix} 0 \\ \hat{x}-1 \end{pmatrix}. \tag{24}$$

2.2.3 Transformation of the vector field

Since the elements of the considered grids are still affine images of the reference element, we can use the Piola transformation (10) to transform vector fields and the results stated in sections 2.1.3 – 2.1.5 can be carried over one to one.

2.3 Construction of Nédélec elements on bi- or trilinear elements

We now want to consider grids that are composed of elements that are a bi- resp. trilinear images $F_C(\hat{C})$ of the reference element $\hat{C} = [0,1]^d$. The main difference here is, that the jacobian $\hat{D}F_C(\hat{x})$ of the element map F_C is not constant, and we have to use Piola transformation (9) to transform vector fields.

2.3.1 Bilinear elements in 2d

The polynomial spaces $\mathcal{P}^k$ and the dofs remain the same as in the case of affine quadrilateral elements. A transformed vector field on a general element is now defined by the Piola transformation (9)

$$\underline{v}(x) = (\hat{D}F_C^{-T}\hat{\underline{v}}_i) \circ F_C^{-1}(x)$$

of a vector field on the reference element. Note that the jacobian $\hat{D}F_C(\hat{x})$ is not constant in this case. In contrast to the case of affine elements, the gradient $D\underline{v}$ does not transform according to formula (12). Non-vanishing second derivatives of $\hat{D}F_C(\hat{x})$ appear in the transformation rule for gradients of vector fields. This requires a new approach to express $\operatorname{curl}\underline{v}$ in terms of $\widehat{\operatorname{curl}}\,\hat{\underline{v}}$. Nevertheless, it can be shown that the curl of a vector field transforms analogously to the case of affine elements.

PROPOSITION 5 *Let $\hat{C}$ be the reference element $[0,1]^2$ and C a bilinear image of $\hat{C}$. If the vector field $\underline{v}(x)$ transforms according to the Piola transformation (9), then the transformation of the curl obeys*

$$\operatorname{curl}\underline{v}(x) = (\det \hat{D}F)^{-1}\widehat{\operatorname{curl}}\,\hat{\underline{v}}(\hat{x}), \qquad x = F(\hat{x}),$$

as in the affine case.

PROOF. In this proof, the mapped element C will be fixed, so for simplicity we write F for F_C.
First note that $(\hat{D}F(F^{-1}(x)))^{-1} = D(F^{-1})(x)$. We use the notation $D(F^{-1})_{ij}(x) = \frac{\partial \hat{x}_i}{\partial x_j}(x)$ and imlicit summation to rewrite the Piola transformation of the vector field componentwise

$$v_i(x) = \frac{\partial \hat{x}_j}{\partial x_i}(x)\,\hat{\underline{v}}_j(F^{-1}(x)), \qquad i = 1,2.$$

In the case of affine elements, i. e. for constant jacobian, we have

$$\begin{aligned}
\frac{\partial v_2}{\partial x_1} &= \frac{\partial \hat{x}_i}{\partial x_2}(x)\,\frac{\partial}{\partial x_1}\hat{\underline{v}}_i(F^{-1}(x)) \\
\frac{\partial v_1}{\partial x_2} &= \frac{\partial \hat{x}_i}{\partial x_1}(x)\,\frac{\partial}{\partial x_2}\hat{\underline{v}}_i(F^{-1}(x)),
\end{aligned}$$

whereas for non-constant jacobian we have

$$\begin{aligned}
\frac{\partial v_2}{\partial x_1} &= \frac{\partial^2 \hat{x}_i}{\partial x_1 \partial x_2}(x)\,\hat{\underline{v}}_i(F^{-1}(x)) + \frac{\partial \hat{x}_i}{\partial x_2}(x)\,\frac{\partial}{\partial x_1}\hat{\underline{v}}_i(F^{-1}(x)) \\
\frac{\partial v_1}{\partial x_2} &= \frac{\partial^2 \hat{x}_i}{\partial x_1 \partial x_2}(x)\,\hat{\underline{v}}_i(F^{-1}(x)) + \frac{\partial \hat{x}_i}{\partial x_1}(x)\,\frac{\partial}{\partial x_2}\hat{\underline{v}}_i(F^{-1}(x)).
\end{aligned}$$

We see that in *both* cases we have

$$\operatorname{curl}\underline{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = \frac{\partial \hat{x}_i}{\partial x_2}(x)\,\frac{\partial}{\partial x_1}\hat{\underline{v}}_i(F^{-1}(x)) - \frac{\partial \hat{x}_i}{\partial x_1}(x)\,\frac{\partial}{\partial x_2}\hat{\underline{v}}_i(F^{-1}(x))$$

that is, the second derivatives cancel out in the expression for the curl and the curl in the non-affine case transforms equally to the curl in the affine case.

□

2.3.2 Trilinear elements in 3d

The polynomial spaces $\mathcal{P}^k$ and the dofs remain the same as in the case of affine hexahedral elements.
The vector field on a genereal element is defined by the Piola transformation (9).
The problem of the non-vanishing second derivatives of the jacobian $D(F_C^{-1})(x)$ arises again, and we cannot generalize the results from the affine case straight away.
But analogously to the 2d case, one can check that in in the transformation rule for expressions $\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}$, $i, j = 1, 2, 3$, which define the curl-operator, the terms containing second derivatives vanish. We have therefore again the transformation rule (16) for the skew matrix $\operatorname{Curl} v = Dv^T - Dv$:

$$\operatorname{Curl} v(x) = ((\hat{D}F_C^{-T}\widehat{\operatorname{Curl}}\,\hat{v}\,\hat{D}F_C^{-1}) \circ F_C^{-1})(x) = (DF_C^{-1})^T(x)\,(\widehat{\operatorname{Curl}}\,\hat{v} \circ F_C^{-1})(x)\,DF_C^{-1}(x)\,.$$

It follows that the following proposition can be proved analogously to the case of affine elements (replace there B_C by $\hat{D}F_C(\hat{(}x)))$.

PROPOSITION 6 *Let the vector field $\underline{v}(x)$ on a trilinear image $C = F_C(\hat{C})$ be defined by the Piola transformation of a reference field $\hat{\underline{v}}(\hat{x})$ on $\hat{C}$. The transformation formula for the curl then reads*

$$\operatorname{curl} \underline{v} = \left(\frac{1}{\det \hat{D}F_C}\,\hat{D}F_C\,\widehat{\operatorname{curl}}\,\hat{\underline{v}}\right) \circ F_C^{-1}\,.$$

2.4 Construction of global shape functions

In the previous sections we have introduced function spaces and degrees of freedom, which, together with the Piola transformation, will allow us to define an $H(\mathrm{curl};\Omega)$-conforming finite element method. Indeed, in [8], Nédélec shows the invariance of the spaces $\mathcal{R}^k$ and $\mathcal{Q}^k$ under Piola transformation of the vector field, as well as the unisolvence of the set of degrees of freedom $\mathcal{A}$ from sections 2.1.2 and 2.2.2 (for details, see [8], Section 1.2, Theorem 1 and Section 2, Theorem 5). This leads to the fact that $H(\mathrm{curl};\Omega)$-conforming global shape functions can be defined by mapping elementwise the reference shape functions with the Piola transformation $\mathcal{P}_K$. However, we must pay some care to the orientation of an interface on which the moments defining the degrees of freedom are based. For the 2d case, we will illustrate in this section how we must take into account the *orientation of an edge* in the definition of the respective element edge shape function, in order to get an $H(\mathrm{curl};\Omega)$-conforming finite element space of global shape functions.

Let $K = F(\hat{K})$ be an affine or bilinear image of a reference element, e one of its edges and $\hat{e}$ the corresponding edge on the reference element.
Let further $[0, |e|] \ni s \mapsto \underline{x}(s) \in e$ and $[0, |\hat{e}|] \ni \hat{s} \mapsto \hat{\underline{x}}(\hat{s}) \in \hat{e}$ be parametrizations with respect to the arc length of e and $\hat{e}$ respectively. We can assume that these parametrizations endow the edges with a counterclockwise orientation. Then, the unit tangent vectors $\underline{t}$ and $\hat{\underline{t}}$ are given by $\frac{d\underline{x}}{ds}$ and $\frac{d\hat{\underline{x}}}{d\hat{s}}$.

LEMMA 1 *Let $\hat{\underline{v}}(\hat{x})$ be a vector field on the reference element and $\underline{v}(x)$ be the corresponding vector field on K, defined by the Piola transformation (9). It then holds*

$$\underline{v} \cdot \underline{t} = \frac{|\hat{e}|}{|e|}(\hat{\underline{v}} \cdot \hat{\underline{t}})\,, \tag{25}$$

where $|\hat{e}|$ and $|e|$ denote the length of the edges $\hat{e}$ and e.

PROOF. With

$$(\underline{v}(x))_i = (D(F^{-1})^T\hat{\underline{v}})_i = \frac{\partial \hat{x}_j}{\partial x_i}(x)\hat{\underline{v}}_j(\hat{x})$$

and $\hat{x}_j = \hat{x}_j(\underline{x}(s))$ and $\hat{x}_j = \hat{x}_j(\hat{s}(s))$ on the edges, we have

$$\underline{v} \cdot \underline{t} = \underline{v} \cdot \frac{d\underline{x}}{ds} = \left(\hat{\underline{v}}_j \frac{\partial \hat{x}_j}{\partial x_i}\right)(x)\frac{dx_i}{ds} = \hat{\underline{v}}_j \frac{d\hat{x}_j}{ds} = \hat{\underline{v}}_j \frac{d\hat{x}_j}{d\hat{s}}\frac{d\hat{s}}{ds} = (\hat{\underline{v}} \cdot \hat{\underline{t}})\frac{d\hat{s}}{ds}$$

and with $\frac{d\hat{s}}{ds} = \frac{|\hat{e}|}{|e|}$ the lemma follows.

□

As a consequence, we have

PROPOSITION 7 (Invariance of the edge dofs) *Let the vector field $\underline{v}(x)$ on K be defined by the Piola transformation (9) of a reference vector field $\hat{\underline{v}}(\hat{x})$ on $\hat{K}$. Then, the functionals (*edge dofs*) $\alpha^{[K]}(\underline{u}) := \int_e (\underline{v} \cdot \underline{t})\varphi \, ds$ are invariant in the sense of*

$$\alpha^{[K]}(\underline{u}) = \int_e (\underline{v} \cdot \underline{t})\varphi \, ds = \int_{\hat{e}} (\hat{\underline{v}} \cdot \hat{\underline{t}})\hat{\varphi} \, d\hat{s} = \hat{\alpha}(\hat{\underline{u}}) \,, \qquad \forall \hat{\varphi} \in \mathbb{P}_{k-1}(\hat{e}) \,, \quad \varphi = \hat{\varphi} \circ F^{-1} \in \mathbb{P}_{k-1}(e) \,.$$

Let now $K_- = F_-(\hat{K})$ and $K_+ = F_+(\hat{K})$ be two neighbouring triangles with common edge e. Let $\underline{N}$ be the global edge shape function that 'lives' on e. By $\underline{N}_-$ and $\underline{N}_+$ we denote the restriction of $\underline{N}$ to K_+ and K_- respectively. Let $e_+ = F_+(\hat{e}_i)$ and $e_- = F_-(\hat{e}_j)$. We write $\underline{t}_+$ for the tangential unit vector to e, oriented counterclockwise with respect to K_+ and $\underline{t}_- = -\underline{t}_+$ for the respective from K_-. For line integrals over the edge e we write $\int_{e_+}$ if we chose the orientation induced by $\underline{t}_+$ and $\int_{e_-}$ for the orientation of e induced by $\underline{t}_-$. In order to obtain an $H(\text{curl};\Omega)$-conforming method, proposition 1 tells us that we must ensure the continuity of the tangential components of the global shape functions, that is

$$\underline{N}_+ \cdot \underline{t}_+ + \underline{N}_- \cdot \underline{t}_- = 0 \,. \tag{26}$$

The following lemma will justify the choice of the moments describing the edge dofs. A consequence of the lemma will be, that the matching of the local edge dofs $\alpha^{[K+]}$ and $\alpha^{[K-]}$ guarantees the pointwise condition (26).

LEMMA 2 *Let $\hat{K}$ denote the reference triangle and $\hat{e}$ one of its edges, parametrized by $\hat{e} \ni \hat{x}(s) := \underline{a} + s\,\hat{\underline{t}}$. Let $\hat{\underline{p}} \in \mathcal{S}^k$, $\mathcal{S}^k$ as defined in (4). It then holds*

$$(\hat{\underline{p}} \cdot \hat{\underline{t}})|_{\hat{e}} \in \mathbb{P}_{k-1}(\hat{e}) \,.$$

PROOF.

$$\hat{\underline{p}} \in \mathcal{S}^k \quad \Longrightarrow \quad \text{for } i = 1,2,3: \quad \hat{p}_i(\hat{x}) = \prod_{j=1}^{3} \hat{x}_j^{k_{ij}} \,, \quad \text{where } \sum_{j=1}^{3} k_{ij} = k \,.$$

Hence, with the parametrization of $\hat{e}$ by $\hat{x}(s)$

$$\hat{p}_i(\hat{x}(s)) = \prod_{j=1}^{3} (a_j + s\,\hat{t}_j)^{k_{ij}} = s^k \prod_{j=1}^{3} \hat{t}_j^{k_{ij}} + \hat{\varphi}_{k-1}(s) \,,$$

with $\hat{\varphi}_{k-1}(s) \in \mathbb{P}_{k-1}(\hat{e})$, and

$$(\hat{\underline{p}} \cdot \hat{\underline{t}})|_{\hat{e}} = s^k \sum_{i=1}^{3} \hat{t}_i \left(\prod_{j=1}^{3} \hat{t}_j^{k_{ij}} \right) + \hat{\varphi}_{k-1}(s) \,.$$

We observe that the coefficient of s^k is exactly $\hat{\underline{p}}(\hat{\underline{t}}) \cdot \hat{\underline{t}}$. By the definition of the space $\mathcal{S}^k$, this expression must vanish.

□

REMARK 5 *In the case of $\hat{K}$ being a quadrilateral, we have $\hat{R} = \mathcal{P}^k$. By the definition of $\mathcal{P}^k$ we see immediately that here also $(\hat{\underline{v}} \cdot \hat{\underline{t}})|_{\hat{e}} \in \mathbb{P}_{k-1}(\hat{e})$.*

The next proposition tells us how exactly to define element shape functions on a mapped element K in order to get $H(\text{curl};\Omega)$-conforming global shape functions.

PROPOSITION 8 *Condition (26) is satiesfied, if we define the element shape functions $\underline{N}_+$ and $\underline{N}_-$ by the Piola transformation (10) and take into account the orientation of the edge e:*

$$\underline{N}_+ := \mathcal{P}_+(\underline{\hat{N}}_i) = \hat{D}F_+^{-T}\underline{\hat{N}}_i\,, \qquad \underline{N}_- := -\,\mathcal{P}_-(\underline{\hat{N}}_j) = -\hat{D}F_-^{-T}\underline{\hat{N}}_j\,. \tag{27}$$

PROOF. Let $\hat{K}$ be the reference element and K its affine or bilinear image. Let $\underline{v} := \mathcal{P}_K(\underline{\hat{v}})$ be a vector field on K, defined by the Piola transformation of a reference vector field $\underline{\hat{v}} \in \hat{R}$. Let e be one of the edges of K and $\underline{t}$ the tangent according to convention 1.
In the case of $\hat{K}$ being a triangle, we have $\hat{R} = \mathcal{R}^k$. By the definition of the space $\mathcal{R}^k$, lemma 2 and 1 we can conclude that $(\underline{v}\cdot\underline{t})|_e \in \mathbb{P}_{k-1}(e)$.
If $\hat{K}$ is a quadrilateral, the previous remark and 1 also tell us that $(\underline{v}\cdot\underline{t})|_e \in \mathbb{P}_{k-1}(e)$.
Hence the condition

$$\int_{e_+} \left((\underline{N}_+\cdot\underline{t}_+) + (\underline{N}_-\cdot\underline{t}_-)\right)\,\varphi\, ds\,, \qquad \forall\,\varphi \in \mathbb{P}_{k-1}(e)$$

on the edge moments is sufficient for the global edge shape functions to satiesfy (26). Note that $\int_{e_+}(\underline{N}_-\cdot\underline{t}_-)\,\varphi\,ds = -\int_{e_-}(\underline{N}_-\cdot\underline{t}_-)\,\varphi\,ds$. So, by the definition (27) of the element shape functions on K_+ resp. on K_-, by the invariance of the dofs (proposition 7) and by the definition of the reference shape functions (example 3) we have

$$\int_{e_+}(\underline{N}_+\cdot\underline{t}_+)\varphi\,ds = \int_{\hat{e}_i}(\underline{\hat{N}}_i\cdot\underline{\hat{t}}_i)\hat{\varphi}\,d\hat{s} = 1 \qquad \text{and} \qquad \int_{e_-}(\underline{N}_-\cdot\underline{t}_-)\varphi\,ds = -\int_{\hat{e}_j}(\underline{\hat{N}}_j\cdot\underline{\hat{t}}_j)\hat{\varphi}\,d\hat{s} = -1\,.$$

□

To close this section, let us make a note on the interpretation of the dofs on an element K in the case of lowest order polynomial degree. In this case, all dofs are edge dofs, the degrees of freedom are $\hat{\alpha}_j(\underline{\hat{v}}) = \int_{\hat{e}_j}\underline{\hat{v}}\cdot\underline{\hat{t}}_j\,d\hat{s}$ and the tangential traces of shape functions are constant on each edge. Since we require $\hat{\alpha}_j(\underline{\hat{N}}_i) = \delta_{ij}$ for the reference shape functions, we have

$$v_j = \hat{\alpha}_j(\underline{\hat{v}}) = (\underline{\hat{N}}_j\cdot\underline{\hat{t}})\,|\hat{e}_j| = (\underline{N}_j\cdot\underline{t}_j)\,|e_j|\,,$$

where for the last equality we have used lemma 1. We see that the dof $\alpha_j(\underline{v})$ 'sitting' on the edge e_j is the value of the *scaled* tangential component $|e_j|\,(\underline{v}\cdot\underline{t}_j)\,|_e$.

REMARK 6 *For the invariance of the edge dofs it is essential that the moments $\alpha^{[K]}$ on K are defined by using the* unit *tangent vector $\underline{t} = \frac{|\hat{e}|}{|e|}\,(\hat{D}F)\,\underline{\hat{t}}$ on K. If not, e. g. if we just used the tangent $\underline{\tilde{t}} = (\hat{D}F)\,\underline{\hat{t}}$, we would lose the invariance of the dofs. In that case the dofs would scale by a factor depending on the size of the edge or face ([8], remark on p. 326).*

2.5 Approximation and convergence results

Without going into details, we will cite here some results on approximation properties and convergence of Nédélec FEM of first type.
We are in the setting of a *conforming* FEM and have quasi-optimal approximation properties of the FE-spaces $V_h \subset H(\mathrm{curl};\Omega)$

$$\|\underline{u} - \Pi_h^k\underline{u}\|_{H(\mathrm{curl};\Omega)} = C\,\inf_{w\in V_h}\|\underline{u} - \underline{w}\|_{H(\mathrm{curl};\Omega)}\,,$$

where $\Pi_h^k\underline{u} \in \mathcal{R}^k$ or $\Pi_h^k\underline{u} \in \mathcal{P}^k$ respectively, denotes the interpolate of $\underline{u}$ with regard to the Nédélec dofs: $\alpha(\underline{u}) = \alpha(\Pi_h^k\underline{u})$ for all dofs α. The interpolation operator Π_h^k is defined for sufficiently smooth vector fields, namely for all $\underline{v} \in H^r(\mathrm{curl})$ for any $r > \frac{1}{2}$ (see [1], Lemma 5.1., [7] and references therein).
For Nédélec's FEM of first type we state (without proof) the following optimal estimate in the curl-norm:

THEOREM 5 *If $\mathcal{T}_h$, $h > 0$, is a regular family of triangulations on Ω and $r > \frac{1}{2}$, then there exists a constant $C > 0$, depending on r but not on h or $\underline{v}$, such that*

$$\|\underline{v} - \Pi_h^k \underline{v}\|_{H(\text{curl};\Omega)} \leq C\, h^{\min\{r,k\}} \|\underline{v}\|_{H^r(\text{curl};\Omega)}\,, \tag{28}$$

for all $\underline{v} \in H^r(\text{curl};\Omega)$.

The result in (28) was obtained by Alonso and Valli in [1], extending earlier interpolation results by Nédélec in [8] and Monk in [6].
Optimal convergence in the $H(\text{curl};\Omega)$-norm for the error of the FE-approximation of the model problem (3) by Nédélec's elements of first type follows from (28) by Céa's lemma. This result has been verified in numerical experiments with a `MATLAB` code, which uses lowest order Nédélec elements on affine triangular meshes for 2d problems, as well as with a `deal.II` code, which uses lowest order Nédélec elements on bilinear resp. trilinear meshes for 2d resp. 3d problems.
As for the $L^2(\Omega)$-approximation properties of FE spaces based on $\mathcal{R}^k$ or $\mathcal{P}^k$, we could hope for a better order than $\mathcal{O}(h^k)$ at first sight: still, we have $[\mathbb{P}^{k-1}(K)]^d \subseteq \mathcal{R}^k(K)$. However, Nédélec shows in [8] that only suboptimality can be expected:

$$\|\underline{v} - \Pi_h^k \underline{v}\|_{L^2(\Omega)} \leq Ch^k |\underline{v}|_{H^k(\Omega)}\,. \tag{29}$$

Nédélec uses a standard scaling and Bramble-Hilbert argument to derive (29). Since $[\mathbb{P}^{k-1}(K)]^d \subseteq \mathcal{R}^k(K) \subsetneq [\mathbb{P}^k(K)]^d$, the Bramble-Hilbert argument only guarantees an elementwise approximation of order k of $H^k(K)$-functions from the space $\mathcal{R}^k(K)$.
However, in a recent paper Hiptmair uses a duality technique to state optimal convergence of the $L^2(\Omega)$-error $\|\underline{u}-\underline{u}_h\|_{L^2(\Omega)}$ for the 3d case and Nédélec's elements of first type of order k on tetrahedral meshes (see Section 5.3, Theorem 5.8 in [5]):

THEOREM 6 *There is an $s > \frac{1}{2}$ such that*

$$\|\underline{u} - \underline{u}_h\|_{L^2(\Omega)} \leq Ch^s \|\underline{u} - \underline{u}_h\|_{H(\text{curl};\Omega)}\,. \tag{30}$$

Under the assumption that the boundary $\partial\Omega$ is smooth or convex, $s = 1$ can be chosen.

Several key arguments of the proof in [5] make explicitely use of features that are limited to 3d problems and the family of finite elements based on tetrahedrons. They cannot be modified trivially to apply to 2d problems or 3d problems on hexahedral meshes. Even worse, it is suggested by the results of numerical experiments that one cannot hope to obtain a result similar to (30).
A possibility to overcome this deficiency of convergence is to use Nédélec elements of second type, where the full $[\mathbb{P}_k]^d$ are used as polynomial spaces (see [10]).

3 Numerical results

The numerical results in this section provide some samples of the quality of the $H(\text{curl};\Omega)$-conforming FEM with Nédélec elements of first type and lowest order (polynomial degree $k = 1$).
We considered the model problem (1) in $\Omega = [-1,1]^d$, $d = 2,3$, with homogeneous Dirichlet boundary condition (2).
The first few results for the two-dimensional problem have been obtained by a `MATLAB` code. For the first example we used the data

$$c \equiv 1\,, \qquad \underline{f}(x,y) = \begin{pmatrix} 3 - y^2 \\ 3 - x^2 \end{pmatrix}. \tag{31}$$

For the second example we have followed the outlines from Appendix A and taken the data from example 5

$$c \equiv 1\,, \qquad \underline{f}(x,y) = (2\pi^2 + 1) \begin{pmatrix} \cos\pi x \sin\pi y \\ -\sin\pi x \cos\pi y \end{pmatrix}. \tag{32}$$

The finite element solution has been computed using Nédélec elements of first type and of polynomial degree $k = 1$ on a family of affine triangular grids. The initial coarse grid consisted of 2^5 triangles. The finest grid

with 2^{13} triangles results after five global refinements.
In Table 1 we see that for both examples we have optimal convergence in the $H(\text{curl};\Omega)$-semiorm, as we would expect from the theoretical results of the previous section. As for the $L^2(\Omega)$-norm, it appears that in both examples the convergence of the numerical solution is not optimal for our choice of finite elements. In the case of Nédélec elements of first type and of polynomial degree $k=1$, we got only $\mathcal{O}(h)$-convergence of the L^2-error. However, this order of convergence is consistent with the result (29) obtained by Nédélec in [8].

	grid	# cells	H(curl)-error		L^2-error	
example 1	1	32	6.66e-01	–	4.66e-01	–
	2	128	3.33e-01	1.00	2.35e-01	0.98
	3	512	1.66e-01	1.00	1.17e-01	0.99
	4	2048	8.33e-02	1.00	5.89e-02	0.99
	5	8192	4.17e-02	1.00	2.95e-02	0.99
example 2	1	32	3.05e+00	–	6.48e-01	–
	2	128	1.61e+00	0.91	3.22e-01	1.00
	3	512	0.81e-01	0.97	1.60e-01	1.00
	4	2048	0.41e-01	0.99	8.02e-02	1.00
	5	8192	2.05e-01	0.99	4.01e-02	1.00

Table 1: Errors and convergence rates in the $L^2(\Omega)$-norm and $H(\text{curl};\Omega)$-seminorm for the two `MATLAB` examples.

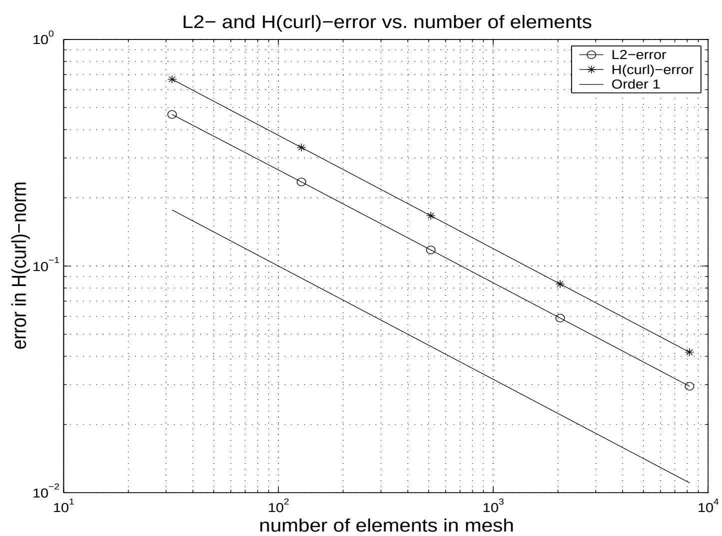

Figure 1: Convergence of the FE-approximation to the smooth solution of the `MATLAB` example (31) in the $L^2(\Omega)$-norm and the $H(\text{curl};(\Omega))$-seminorm

Remark 7 *The mesh generation and refinement was done by* `PDE-toolbox` *commands. Since the* `PDE-toolbox` *does not support three dimensional grids, we restricted ourselves to 2d problems, and we have so far no numerical results for the case of tetrahedral grids in 3d.*

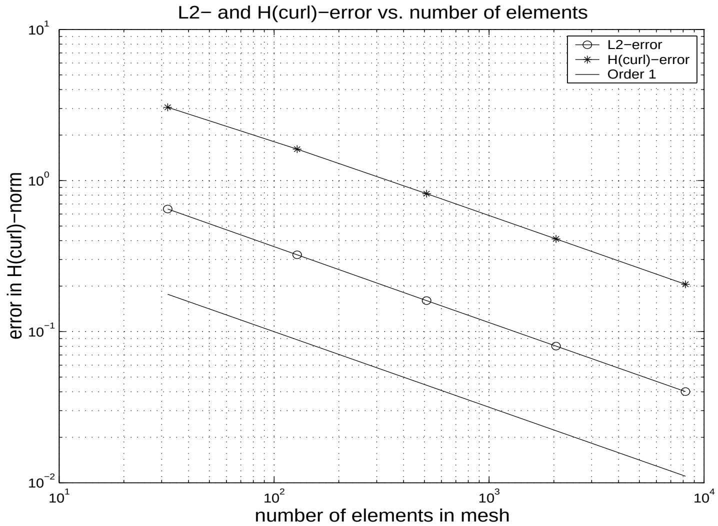

Figure 2: Convergence of the FE-approximation to the smooth solution of the `MATLAB` example (32) in the $L^2(\Omega)$-norm and the $H(\text{curl}; (\Omega))$-seminorm

As for meshes with quadrilateral cells, numerical results were obtained with a `deal.II` code, using the finite element class `fe/fe_nedelec.cc`. This class provides Nédélec's $H(\text{curl}; \Omega)$-conforming element of first type and lowest order in two and three space dimensions, on bilinear quadrilateral, resp. trilinear hexahedral grids. For details about `deal.II`, see [2]. In the following results were obtained for the model problem (1) in two dimensions using the data (32). We computed the solution on five successive non-affine bilinear grids (figure 3), each of which was obtained by global refinement of the previous one.

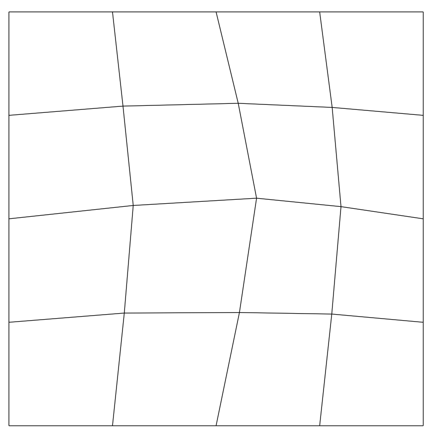

Figure 3: Non-affine bilinear grid used in the `deal.II` code, after one refinement step.

Again, in Table 2 we can observe optimal convergence of order $\mathcal{O}(h)$ in the $H(\text{curl}; \Omega)$-norm. The same order of convergence is obtained for the error in the $L^2(\Omega)$-norm.

With `deal.II`, we are also able to treat 3d problems on hexahedral grids. For our type of problem, Nédélec's $H(\text{curl}; \Omega)$-conforming elements of first type and lowest order, based on a cubic reference element, are available.

grid	# cells	$H(\mathrm{curl})$-error		L^2-error	
1	4	6.112e+00	-	1.442e+00	-
2	16	3.688e+00	0.73	6.765e-01	1.09
3	64	1.991e+00	0.89	3.280e-01	1.04
4	256	1.015e+00	0.97	1.617e-01	1.02
5	1024	5.098e-01	0.99	8.049e-02	1.01

Table 2: Errors and convergence rates in the $H(\mathrm{curl};\Omega)$- and $L^2(\Omega)$-norm for the 2d-example solved with `deal.II`.

We computed an approximation to the model problem (1) in 3d using the data

$$c \equiv 1\,, \qquad \underline{f}(x,y,z) = \begin{pmatrix} xy(1-y^2)(1-z^2) + 2xy(1-z^2) \\ y^2(1-x^2)(1-z^2) + (1-y^2)(2-x^2-z^2) \\ yz(1-x^2)(1-y^2) + 2yz(1-x^2) \end{pmatrix}\,. \tag{33}$$

In a first experiment, the finite element solution was computed on five successive globally refined affine grids. In a second computation, we approximated the solution of the same problem on five successive globally refined non-affine trilinear grids.
We see in Table 3 that in both cases we observe again convergence of order $\mathcal{O}(h)$ in the $H(\mathrm{curl};\Omega)$- and the $L^2(\Omega)$-norm.

	grid	# cells	$H(\mathrm{curl})$-error		L^2-error	
affine grids	1	8	7.696e-01	-	6.609e-01	-
	2	64	4.088e-01	0.91	2.943e-01	1.17
	3	512	2.075e-01	0.98	1.408e-01	1.06
	4	4096	1.041e-01	0.99	6.955e-02	1.02
	5	32768	5.210e-02	1.00	3.467e-02	1.00
non-affine grids	1	8	7.716e-01	-	6.611e-01	-
	2	64	4.108e-01	0.91	2.955e-01	1.16
	3	512	2.085e-01	0.98	1.413e-01	1.06
	4	4096	1.046e-01	0.99	6.982e-02	1.02
	5	32768	5.237e-02	1.00	3.480e-02	1.00

Table 3: Errors and convergence rates in the $H(\mathrm{curl};\Omega)$- and $L^2(\Omega)$-norm for the 3d-example solved with `deal.II`. The first data set is for the computation on a family of affine grids, the second set of data is for non-affine trilinear grids.

The conclusion that can be drawn from these numerical experiments is, that the restriction to three-dimensional tetrahedral grids of Hiptmair's result on the L^2-convergence of the error (6) cannot be relaxed.

Finally, here are some pretty pictures: the vector field plots from the `MATLAB` computations.

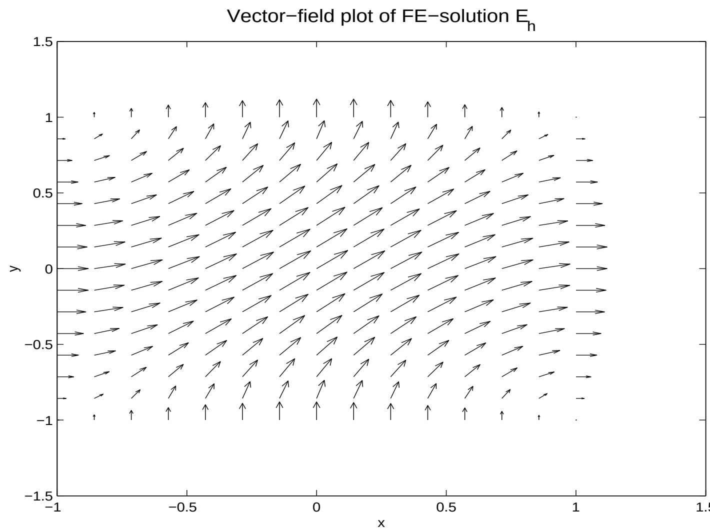

Figure 4: Vector-field plot of the FE-solution of example (31).

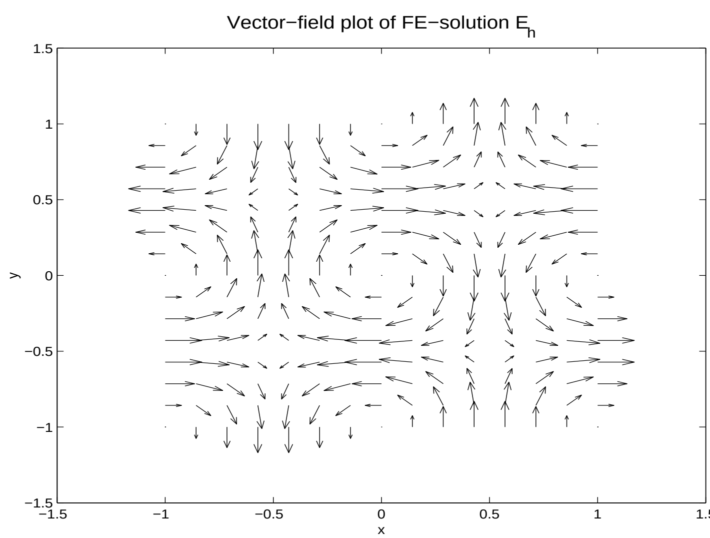

Figure 5: Vector-field plot of the FE-solution of example (32).

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A Construction of solutions in 2d

We present how divergence-free solutions of the model problem (1) on a domain $\Omega \subset \mathbb{R}^2$ with perfectly conducting boundary can be constructed from solutions of the scalar Laplace equation.

PROPOSITION 9 *Let Ω be a sufficiently smooth domain in $\mathbb{R}^2$, $\varphi(x,y)$ a sufficently smooth scalar function on Ω and the coefficient $c > 0$ globally constant.*
Let w be a solution of the scalar equation

$$\begin{aligned} -\Delta w + c\,w &= \varphi \quad \text{in} \quad \Omega \\ \underline{n} \cdot \nabla w &= 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned} \tag{34}$$

Then, $\underline{E} := \nabla^{\perp} w$ is a solution of the model equation

$$\begin{aligned} \underline{\text{curl}}\, \text{curl}\, \underline{E} + c\,\underline{E} &= \underline{f} \quad \text{in} \quad \Omega\,, \\ \underline{E} \wedge \underline{n} &= 0 \quad \text{on} \quad \partial\Omega\,, \end{aligned}$$

with right hand side $\underline{f} := \nabla^{\perp}\varphi$.
We use the notation $\nabla^{\perp}\varphi := \boldsymbol{R}\nabla\varphi = \begin{pmatrix} \partial_y \varphi \\ -\partial_x \varphi \end{pmatrix}$.

PROOF. We first show the correspondence of the boundary conditions. With the definition $\underline{E} := \nabla^{\perp} w$ it holds

$$\underline{E} \wedge \underline{n} = \underline{E} \cdot \underline{t} = \nabla w^T \boldsymbol{R}^T \boldsymbol{R} \underline{n} = \nabla w \cdot \underline{n}\,.$$

It remains to show that $\underline{E}$ solves the model problem for an appropriate right hand side. First, note that $\underline{E}$ is divergence-free: $\nabla \cdot \nabla^{\perp} w = 0$ for all w. Hence, the identity $\underline{\text{curl}}\,\text{curl}\,\underline{E} = \nabla(\nabla \cdot \underline{E}) - \Delta \underline{E}$ reduces to $\underline{\text{curl}}\,\text{curl}\,\underline{E} = -\Delta \underline{E}$. The observation that for smooth data $\nabla^{\perp} w$ solves the Laplace equation (34) with right hand side $\nabla^{\perp}\varphi$ concludes the proof.

□

EXAMPLE 5 (Solutions from eigenfunctions of the Laplacian) *Choose w to be a solution of the eigenvalue problem*

$$\begin{aligned} -\Delta w &= \lambda\, w \quad \text{in} \quad \Omega \\ \underline{n} \cdot \nabla w &= 0 \quad \text{on} \quad \partial\Omega\,, \end{aligned}$$

and set $\varphi = (\lambda + c)\, w$.
As an example, take $\Omega = [-1,1]^2$ and $\lambda = 2\pi^2$. Then, $w = \cos\pi x \cos\pi y$ is an eigenfunction and we compute

$$\underline{f} = (2\pi^2 + c)\pi \begin{pmatrix} \cos\pi x \sin\pi y \\ -\sin\pi x \cos\pi y \end{pmatrix}, \qquad \underline{E} = \pi \begin{pmatrix} \cos\pi x \sin\pi y \\ -\sin\pi x \cos\pi y \end{pmatrix}.$$

EXAMPLE 6 (Solutions from any scalar function satiesfying the boundary condition) *Take again $\Omega = [-1,1]^2$. We have to find a scalar function w which satiesfies the homogeneous Neumann boundary condition. Take for example $w(x,y) = (1-x^2)^2(1-y^2)^2$, for which we have $\underline{n} \cdot \nabla w = 0$ on $\partial[-1,1]^2$. The right hand side is then $\varphi = -\Delta w + cw$.*

B Time-harmonic Maxwell's equations with low-frequency approximation

We show, how the model problem can be derived from the time-harmonic Maxwell's equations in the low-frequency case. We follow the outline of [1]:
We consider the following primal formulation of Maxwell's equations:

$$\begin{aligned} \varepsilon \frac{\partial \mathcal{E}}{\partial t} &= \operatorname{curl} \mathcal{H} - \sigma \mathcal{E}\,, \\ \mu \frac{\partial \mathcal{H}}{\partial t} &= -\operatorname{curl} \mathcal{E}\,, \end{aligned} \tag{35}$$

where $\mathcal{E}$ and $\mathcal{H}$ are the electric and magnetic field. $\varepsilon(x), \mu(x)$ are the dielectric and magnetic permeability coefficients, and $\sigma(x)$ denotes the electric conductivity. $\varepsilon(x), \mu(x)$ and $\sigma(x)$ are assumed to be symmetric matrices in $L^\infty(\Omega)^{d\times d}$, and $\varepsilon(x)$ and $\mu(x)$ are positive definite. $\sigma(x)$ is positive definite in a conductor and vanishes in an insulator.

Time-harmonic, low-frequency case

We assume that $\mathcal{E}(x,t)$ and $\mathcal{H}(x,t)$ are *time-harmonic*, i. e. they can be represented as

$$\begin{aligned} \mathcal{E}(x,t) &= \operatorname{Re}\left(E(x)\exp(i\omega t)\right)\,, \\ \mathcal{H}(x,t) &= \operatorname{Re}\left(H(x)\exp(i\omega t)\right)\,. \end{aligned}$$

Here, $E(x), H(x)$ are complex-valued vector fields and $\omega \neq 0$ is a given angular frequency.

REMARK 8 *For example, a monofrequent laser can be described by the time-harmonic Maxwell's equations.*

In the time-harmonic case the space and time variables decouple and we can eliminate the time dependency. For this, we ask $E(x)\exp(i\omega t)$ and $H(x)\exp(i\omega t)$ to satiesfy (35). By then inserting the second equation of (35) into the first one, we can eliminate the magnetic field $H(x)$. This yields

$$\operatorname{curl}(\mu^{-1}\operatorname{curl} E) - \omega^2 \varepsilon E + i\omega\sigma E = 0$$

In the *low-frequency case* where $|\omega|$ is small, it is known that for general materials the material parameters are such that

$$\omega^2\varepsilon \ll \mu^{-1}\,, \quad \omega^2\varepsilon \ll \omega\sigma\,.$$

Hence, neglecting the expression $\omega^2\varepsilon E(x)$ is reasonable and it brings us to the low-frequency approximation of the time-harmonic Maxwell's equations:

$$\operatorname{curl}(\mu^{-1}\operatorname{curl} E) + i\omega\sigma E = 0$$

We consider this equation in a conductor Ω ($\sigma(x)$ pos. def.) and a impose Dirichlet boundary condition on the tangential trace of the field:

$$E \wedge n = \Phi \quad \text{on} \quad \partial\Omega\,. \tag{36}$$

Proceeding as in [1], we assume that a vector funciton $\tilde{E}$ is known, satiesfying (36), and we end up with the following boundary value problem for $\underline{u} = E - \tilde{E}$

$$\begin{aligned} \operatorname{curl}(\mu^{-1}\operatorname{curl} u) + i\omega\sigma u &= F \quad \text{in} \quad \Omega\,, \\ u \wedge n &= 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned} \tag{37}$$

Although problem (37) is complex-valued, finding a finite element method to approximate (37) basically boils down to finding a finite element method for the real valued model problem (1).