Chapter 5
Superconvergence Analysis for Metamaterials

In this chapter, we first give a quick review of superconvergence analysis in Sect. 5.1. Then we carry out the superclose analysis for 3-D metamaterial Maxwell's equations represented by the Drude model. The analysis for a semi-discrete scheme is presented in Sect. 5.2, which is followed by the analysis for two fully-discrete schemes in Sect. 5.3. In Sect. 5.4, a superconvergence result in the discrete l_2 norm is proved. Finally, the superconvergence analysis is extended to the 2-D case in Sect. 5.5.

5.1 A Brief Overview of Superconvergence Analysis

In finite element methods, when the underlying differential equations have smooth solutions and the differential equations are solved on very structured meshes such as rectangular grids or strongly regular triangular grids, we often see that the obtained convergence rates have higher order than the theoretical approximation results suggested. Such a phenomenon is called superconvergence. Study of the superconvergence phenomenon started in the early 1970s, and many interesting results have been obtained for problems described by elliptic equations [22,23,128], parabolic equations [292], the second-order wave equations, and porous media flows [113]. Detailed superconvergence analysis can be found in classic books [67, 201, 289]. A detailed bibliography on superconvergence by 1996 can be found in a review paper by Krizek and Neittaanmaki [170].

Compared to those widely studied equations, there are not many superconvergence results existing for Maxwell's equations. In 1994, Monk [215] obtained the first superconvergence result for Maxwell's equations in vacuum. Later, Brandts [50] presented another superconvergence analysis for 2-D Maxwell's equations in vacuum. Also Lin and his collaborators [199, 200, 202] systematically obtained many global superconvergence results using the so-called Lin's Integral Identity technique [203, 204, 308] developed in the early 1990s. More details on Lin's Integral Identity technique can be found in books [201, 297]. In 2008, Lin

J. Li and Y. Huang, *Time-Domain Finite Element Methods for Maxwell's Equations in Metamaterials*, Springer Series in Computational Mathematics 43,
DOI 10.1007/978-3-642-33789-5_5,

and Li [198] extended the superconvergence result for vacuum to three popular dispersive media models. Some superconvergence work has been recently carried out for metamaterial models [153, 156]. In this chapter, we will present detailed superconvergence analysis for both semi-discrete and fully-discrete schemes on cubic and rectangular meshes.

5.2 Superclose Analysis for a Semi-discrete Scheme

To simplify the presentation, we consider the non-dimensionalized Drude model equations

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J}, \tag{5.1}$$

$$\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E} - \mathbf{K}, \tag{5.2}$$

$$\frac{\partial \mathbf{J}}{\partial t} + \Gamma_e \mathbf{J} = \omega_e^2 \mathbf{E}, \tag{5.3}$$

$$\frac{\partial \mathbf{K}}{\partial t} + \Gamma_m \mathbf{K} = \omega_m^2 \mathbf{H}, \tag{5.4}$$

subject to the perfect conducting boundary condition (3.59) and the initial conditions (3.60) and (3.61). Derivation of (5.1)–(5.4) can be found in Sect. 4.4. Here for clarity, all tildes are dropped.

For superconvergence analysis, we assume that the domain Ω is a rectangular cuboid, which is partitioned by a family of regular cubic meshes T_h with maximum mesh size h. Recall that the Raviart-Thomas-Nédélec cubic elements (the pair of divergence and curl conforming elements) are defined as (cf. Chap. 3):

$$\mathbf{U}_h = \{\psi_h \in H(\mathrm{div};\Omega) : \ \psi_h|_K \in Q_{k,k-1,k-1} \times Q_{k-1,k,k-1} \times Q_{k-1,k-1,k}, \ \forall\, K \in T_h\},$$

$$\mathbf{V}_h = \{\phi_h \in H(\mathrm{curl};\Omega) : \ \phi_h|_K \in Q_{k-1,k,k} \times Q_{k,k-1,k} \times Q_{k,k,k-1}, \ \forall\, K \in T_h\}.$$

Furthermore, we need the so-called Nédélec interpolation operator Π_h, which has been defined in Chap. 3.

The superclose analysis depends on the following two fundamental results.

Lemma 5.1 ([202, Lemma 3.1]). *On any cubic element K, for any $\mathbf{E} \in (H^{k+2}(K))^3$, we have*

$$\int_K \nabla\times(\mathbf{E}-\Pi_h\mathbf{E})\cdot\psi_h dxdydz = O(h^{k+1})||\mathbf{E}||_{k+2,K}||\psi_h||_{0,K}, \quad \forall\, \psi_h|_K \in \mathbf{U}_h(K).$$

Lemma 5.2 ([202, Lemma 3.2]). *On any cubic element K, for any $\mathbf{E} \in (H^{k+1}(K))^3$, we have*

$$\int_K (\mathbf{E} - \Pi_h\mathbf{E}) \cdot \phi_h dxdydz = O(h^{k+1})||\mathbf{E}||_{k+1,K}||\phi_h||_{0,K}, \quad \forall\, \phi_h|_K \in \mathbf{V}_h(K).$$

Though Lemmas 5.1 and 5.2 were stated for the whole domain Ω in [202], the proofs of [202] actually show that the results hold true element-wisely.

A corresponding weak formulation for the system (5.1)–(5.4) is: For any $t \in (0, T]$, find the solutions $\mathbf{E} \in H_0(\text{curl}; \Omega)$, $\mathbf{J} \in H(\text{curl}; \Omega)$, $\mathbf{H}$ and $\mathbf{K} \in (L^2(\Omega))^3$ such that

$$(\mathbf{E}_t, \phi) - (\mathbf{H}, \nabla \times \phi) + (\mathbf{J}, \phi) = 0, \quad \forall\, \phi \in H_0(\text{curl}; \Omega), \tag{5.5}$$

$$(\mathbf{H}_t, \psi) + (\nabla \times \mathbf{E}, \psi) + (\mathbf{K}, \psi) = 0, \quad \forall\, \psi \in (L^2(\Omega))^3, \tag{5.6}$$

$$(\mathbf{J}_t, \tilde{\phi}) + \Gamma_e(\mathbf{J}, \tilde{\phi}) - \omega_e^2(\mathbf{E}, \tilde{\phi}) = 0, \quad \forall\, \tilde{\phi} \in H(\text{curl}; \Omega), \tag{5.7}$$

$$(\mathbf{K}_t, \tilde{\psi}) + \Gamma_m(\mathbf{K}, \tilde{\psi}) - \omega_m^2(\mathbf{H}, \tilde{\psi}) = 0, \quad \forall\, \tilde{\psi} \in (L^2(\Omega))^3, \tag{5.8}$$

subject to the initial conditions (3.60) and (3.61), i.e.,

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}), \quad \mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x}),$$
$$\mathbf{J}(\mathbf{x}, 0) = \mathbf{J}_0(\mathbf{x}), \quad \mathbf{K}(\mathbf{x}, 0) = \mathbf{K}_0(\mathbf{x}).$$

Now a semi-discrete mixed method can be constructed for solving (5.5)–(5.8): For any $t \in (0, T]$, find the solutions $\mathbf{E}^h \in \mathbf{V}_h^0, \mathbf{J}^h \in \mathbf{V}_h, \mathbf{H}^h, \mathbf{K}^h \in \mathbf{U}_h$ such that

$$(\mathbf{E}_t^h, \phi_h) - (\mathbf{H}^h, \nabla \times \phi_h) + (\mathbf{J}^h, \phi_h) = 0, \quad \forall\, \phi_h \in \mathbf{V}_h^0, \tag{5.9}$$

$$(\mathbf{H}_t^h, \psi_h) + (\nabla \times \mathbf{E}^h, \psi_h) + (\mathbf{K}^h, \psi_h) = 0, \quad \forall\, \psi_h \in \mathbf{U}_h, \tag{5.10}$$

$$(\mathbf{J}_t^h, \tilde{\phi}_h) + \Gamma_e(\mathbf{J}^h, \tilde{\phi}_h) - \omega_e^2(\mathbf{E}^h, \tilde{\phi}_h) = 0, \quad \forall\, \tilde{\phi}_h \in \mathbf{V}_h, \tag{5.11}$$

$$(\mathbf{K}_t^h, \tilde{\psi}_h) + \Gamma_m(\mathbf{K}^h, \tilde{\psi}_h) - \omega_m^2(\mathbf{H}^h, \tilde{\psi}_h) = 0, \quad \forall\, \tilde{\psi}_h \in \mathbf{U}_h, \tag{5.12}$$

with the initial approximations

$$\mathbf{E}_h^0(\mathbf{x}) = \Pi_h \mathbf{E}_0(\mathbf{x}), \quad \mathbf{H}_h^0(\mathbf{x}) = P_h \mathbf{H}_0(\mathbf{x}), \tag{5.13}$$

$$\mathbf{J}_h^0(\mathbf{x}) = \Pi_h \mathbf{J}_0(\mathbf{x}), \quad \mathbf{K}_h^0(\mathbf{x}) = P_h \mathbf{K}_0(\mathbf{x}). \tag{5.14}$$

Recall that P_h denotes the standard L^2 projection operator onto space $\mathbf{U}_h$, and $\mathbf{V}_h^0 = \{\mathbf{v}_h \in \mathbf{V}_h : \mathbf{v}_h \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}$.

For this scheme, we have the following superclose result.

Theorem 5.1. *Let* $(\mathbf{E}, \mathbf{H}, \mathbf{J}, \mathbf{K})$ *and* $(\mathbf{E}^h, \mathbf{H}^h, \mathbf{J}^h, \mathbf{K}^h)$ *be the analytic and finite element solutions of (5.5)–(5.8) and (5.9)–(5.12) at time* $t \in (0, T]$, *respectively. Under the regularity assumptions*

$$\mathbf{E}_t, \mathbf{J}_t, \mathbf{J} \in L^\infty(0, T; (H^{k+1}(\Omega))^3), \quad \mathbf{E} \in L^\infty(0, T; (H^{k+2}(\Omega))^3),$$

there exists a constant $C > 0$ *independent of* h *but linearly dependent on* T *such that*

$$
\begin{aligned}
&||\varPi_h\mathbf{E}-\mathbf{E}^h||_{L^\infty(0,T;(L^2(\Omega))^3)}+||P_h\mathbf{H}-\mathbf{H}^h||_{L^\infty(0,T;(L^2(\Omega))^3)}\\
&+\frac{1}{\omega_e}||\varPi_h\mathbf{J}-\mathbf{J}^h||_{L^\infty(0,T;(L^2(\Omega))^3)}+\frac{1}{\omega_m}||P_h\mathbf{K}-\mathbf{K}^h||_{L^\infty(0,T;(L^2(\Omega))^3)}\\
\le\ & Ch^{k+1}(||\mathbf{E}_t||_{L^\infty(0,T;(H^{k+1}(\Omega))^3)}+||\mathbf{J}||_{L^\infty(0,T;(H^{k+1}(\Omega))^3)}\\
&+||\mathbf{E}||_{L^\infty(0,T;(H^{k+2}(\Omega))^3)}+||\mathbf{J}_t||_{L^\infty(0,T;(H^{k+1}(\Omega))^3)}),
\end{aligned}
$$

where $k\ge 1$ is the order of the basis functions in spaces $\mathbf{U}_h$ and $\mathbf{V}_h$.

Proof. Denote $\xi=\varPi_h\mathbf{E}-\mathbf{E}^h,\eta=P_h\mathbf{H}-\mathbf{H}^h,\tilde{\xi}=\varPi_h\mathbf{J}-\mathbf{J}^h,\tilde{\eta}=P_h\mathbf{K}-\mathbf{K}^h$. Choosing $\phi=\phi_h=\xi$ in (5.5) and (5.9), $\psi=\psi_h=\eta$ in (5.6) and (5.10), $\tilde{\phi}=\tilde{\phi}_h=\tilde{\xi}$ in (5.7) and (5.11), $\tilde{\psi}=\tilde{\psi}_h=\tilde{\eta}$ in (5.8) and (5.12), respectively, and rearranging the resultants, we obtain the error equations

$$
\begin{aligned}
(i)\quad & (\xi_t,\xi)-(\eta,\nabla\times\xi)+(\tilde{\xi},\xi)\\
&=((\varPi_h\mathbf{E}-\mathbf{E})_t,\xi)-(P_h\mathbf{H}-\mathbf{H},\nabla\times\xi)+(\varPi_h\mathbf{J}-\mathbf{J},\xi),\\
(ii)\quad & (\eta_t,\eta)+(\nabla\times\xi,\eta)+(\tilde{\eta},\eta)\\
&=((P_h\mathbf{H}-\mathbf{H})_t,\eta)+(\nabla\times(\varPi_h\mathbf{E}-\mathbf{E}),\eta)+(P_h\mathbf{K}-\mathbf{K},\eta),\\
(iii)\quad & (\tilde{\xi}_t,\tilde{\xi})+\varGamma_e(\tilde{\xi},\tilde{\xi})-\omega_e^2(\xi,\tilde{\xi})\\
&=((\varPi_h\mathbf{J}-\mathbf{J})_t,\tilde{\xi})+\varGamma_e(\varPi_h\mathbf{J}-\mathbf{J},\tilde{\xi})-\omega_e^2(\varPi_h\mathbf{E}-\mathbf{E},\tilde{\xi}),\\
(iv)\quad & (\tilde{\eta}_t,\tilde{\eta})+\varGamma_m(\tilde{\eta},\tilde{\eta})-\omega_m^2(\eta,\tilde{\eta})\\
&=((P_h\mathbf{K}-\mathbf{K})_t,\tilde{\eta})+\varGamma_m(P_h\mathbf{K}-\mathbf{K},\tilde{\eta})-\omega_m^2(P_h\mathbf{H}-\mathbf{H},\tilde{\eta}).
\end{aligned}
$$

Dividing the last two equations by ω_e^2 and ω_m^2, respectively, then adding the above four equations together, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\xi||_0^2+||\eta||_0^2+\frac{1}{\omega_e^2}||\tilde{\xi}||_0^2+\frac{1}{\omega_m^2}||\tilde{\eta}||_0^2)+\frac{\varGamma_e}{\omega_e^2}||\tilde{\xi}||_0^2+\frac{\varGamma_m}{\omega_m^2}||\tilde{\eta}||_0^2\\
=\ &((\varPi_h\mathbf{E}-\mathbf{E})_t,\xi)+(\varPi_h\mathbf{J}-\mathbf{J},\xi)+(\nabla\times(\varPi_h\mathbf{E}-\mathbf{E}),\eta)\\
&+\frac{1}{\omega_e^2}((\varPi_h\mathbf{J}-\mathbf{J})_t,\tilde{\xi})+\frac{\varGamma_e}{\omega_e^2}(\varPi_h\mathbf{J}-\mathbf{J},\tilde{\xi})-(\varPi_h\mathbf{E}-\mathbf{E},\tilde{\xi}),
\end{aligned}
\tag{5.15}
$$

where we used the L^2-projection property in the above derivation.

Using Lemmas 5.1 and 5.2 and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&((\varPi_h\mathbf{E}-\mathbf{E})_t,\xi)\le Ch^{k+1}||\mathbf{E}_t||_{L^\infty(0,T;\mathbf{H}^{k+1}(\Omega))}||\xi||_{L^\infty(0,T;\mathbf{L}^2(\Omega))},\\
&(\varPi_h\mathbf{J}-\mathbf{J},\xi)\le Ch^{k+1}||\mathbf{J}||_{L^\infty(0,T;\mathbf{H}^{k+1}(\Omega))}||\xi||_{L^\infty(0,T;\mathbf{L}^2(\Omega))},\\
&(\nabla\times(\varPi_h\mathbf{E}-\mathbf{E}),\eta)\le Ch^{k+1}||\mathbf{E}||_{L^\infty(0,T;\mathbf{H}^{k+2}(\Omega))}||\eta||_{L^\infty(0,T;\mathbf{L}^2(\Omega))},
\end{aligned}
$$

$$\frac{1}{\omega_e^2}((\Pi_h \mathbf{J} - \mathbf{J})_t, \tilde{\xi}) \leq \frac{1}{\omega_e^2} \cdot Ch^{k+1} ||\mathbf{J}_t||_{L^\infty(0,T;\mathbf{H}^{k+1}(\Omega))} ||\tilde{\xi}||_{L^\infty(0,T;\mathbf{L}^2(\Omega))},$$
$$\frac{\Gamma_e}{\omega_e^2}(\Pi_h \mathbf{J} - \mathbf{J}, \tilde{\xi}) \leq \frac{\Gamma_e}{\omega_e^2} \cdot Ch^{k+1} ||\mathbf{J}||_{L^\infty(0,T;\mathbf{H}^{k+1}(\Omega))} ||\tilde{\xi}||_{L^\infty(0,T;\mathbf{L}^2(\Omega))},$$
$$(\Pi_h \mathbf{E} - \mathbf{E}, \tilde{\xi}) \leq Ch^{k+1} ||\mathbf{E}||_{L^\infty(0,T;\mathbf{H}^{k+1}(\Omega))} ||\tilde{\xi}||_{L^\infty(0,T;\mathbf{L}^2(\Omega))}.$$

Substituting the above estimates into (5.15), and using Gronwall inequality, we obtain

$$\begin{aligned}
&||\xi||^2_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + ||\eta||^2_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{\omega_e^2}||\tilde{\xi}||^2_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{\omega_m^2}||\tilde{\eta}||^2_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \\
\leq\ & Ch^{2(k+1)}(||\mathbf{E}_t||^2_{L^\infty(0,T;\mathbf{H}^{k+1}(\Omega))} + ||\mathbf{J}||^2_{L^\infty(0,T;\mathbf{H}^{k+1}(\Omega))} \\
&+ ||\mathbf{E}||^2_{L^\infty(0,T;\mathbf{H}^{k+2}(\Omega))} + ||\mathbf{J}_t||^2_{L^\infty(0,T;\mathbf{H}^{k+1}(\Omega))}),
\end{aligned}$$

where the constant C is linearly dependent on T^2. The proof is completed by using the triangle inequality, and the interpolation error (3.78) and projection error (3.79). □

5.3 Superclose Analysis for Fully-Discrete Schemes

Now we can formulate the Crank-Nicolson mixed finite element scheme for solving (5.5)–(5.8): For $m = 1, 2, \cdots, M$, find $\mathbf{E}_h^m \in \mathbf{V}_h^0, \mathbf{J}_h^m \in \mathbf{V}_h, \mathbf{H}_h^m, \mathbf{K}_h^m \in \mathbf{U}_h$ such that

$$(\delta_\tau \mathbf{E}_h^m, \phi_h) - (\overline{\mathbf{H}}_h^m, \nabla \times \phi_h) + (\overline{\mathbf{J}}_h^m, \phi_h) = 0, \quad \forall\, \phi_h \in \mathbf{V}_h^0, \tag{5.16}$$
$$(\delta_\tau \mathbf{H}_h^m, \psi_h) + (\nabla \times \overline{\mathbf{E}}_h^m, \psi_h) + (\overline{\mathbf{K}}_h^m, \psi_h) = 0, \quad \forall\, \psi_h \in \mathbf{U}_h, \tag{5.17}$$
$$(\delta_\tau \mathbf{J}_h^m, \tilde{\phi}_h) + \Gamma_e(\overline{\mathbf{J}}_h^m, \tilde{\phi}_h) - \omega_e^2(\overline{\mathbf{E}}_h^m, \tilde{\phi}_h) = 0, \quad \forall\, \tilde{\phi}_h \in \mathbf{V}_h, \tag{5.18}$$
$$(\delta_\tau \mathbf{K}_h^m, \tilde{\psi}_h) + \Gamma_m(\overline{\mathbf{K}}_h^m, \tilde{\psi}_h) - \omega_m^2(\overline{\mathbf{H}}_h^m, \tilde{\psi}_h) = 0, \quad \forall\, \tilde{\psi}_h \in \mathbf{U}_h, \tag{5.19}$$

subject to the initial approximations (5.13) and (5.14). As before, we denote

$$\delta_\tau \mathbf{E}_h^m = (\mathbf{E}_h^m - \mathbf{E}_h^{m-1})/\tau, \quad \overline{\mathbf{H}}_h^m = \frac{1}{2}(\mathbf{H}_h^m + \mathbf{H}_h^{m-1}).$$

For this fully-discrete scheme, we have the following superclose result.

Theorem 5.2. *Let* $(\mathbf{E}^m, \mathbf{H}^m, \mathbf{J}^m, \mathbf{K}^m)$ *and* $(\mathbf{E}_h^m, \mathbf{H}_h^m, \mathbf{J}_h^m, \mathbf{K}_h^m)$ *be the analytic and finite element solutions of (5.5)–(5.8) and (5.16)–(5.19) at time* t_m*, respectively. Under the regularity assumptions*

$$\mathbf{E}_t, \mathbf{J}_t, \mathbf{J} \in L^\infty(0,T;(H^{k+1}(\Omega))^3), \quad \mathbf{E} \in L^\infty(0,T;(H^{k+2}(\Omega))^3),$$
$$\mathbf{E}_{tt}, \mathbf{H}_{tt}, \mathbf{J}_{tt}, \mathbf{K}_{tt}, \nabla\times\mathbf{E}_{tt}, \nabla\times\mathbf{H}_{tt} \in L^\infty(0,T;(L^2(\Omega))^3),$$

there exists a constant $C > 0$, independent of h but linearly dependent on T such that

$$\begin{aligned}
&\max_{1\le m\le M}(||\Pi_h\mathbf{E}^m - \mathbf{E}_h^m||_0 + ||P_h\mathbf{H}^m - \mathbf{H}_h^m||_0 + ||\Pi_h\mathbf{J}^m - \mathbf{J}_h^m||_0 + ||P_h\mathbf{K}^m - \mathbf{K}_h^m||_0)\\
&\le Ch^{k+1}(||\mathbf{E}_t||_{L^\infty(0,T;(H^{k+1}(\Omega))^3)} + ||\mathbf{J}||_{L^\infty(0,T;(H^{k+1}(\Omega))^3)}\\
&\quad +||\mathbf{E}||_{L^\infty(0,T;(H^{k+2}(\Omega))^3)} + ||\mathbf{J}_t||_{L^\infty(0,T;(H^{k+1}(\Omega))^3)})\\
&\quad +C\tau^2(||\nabla\times\mathbf{H}_{tt}||_{L^\infty(0,T;(L^2(\Omega))^3)} + ||\mathbf{J}_{tt}||_{L^\infty(0,T;(L^2(\Omega))^3)} + ||\nabla\times\mathbf{E}_{tt}||_{L^\infty(0,T;(L^2(\Omega))^3)}\\
&\quad +||\mathbf{K}_{tt}||_{L^\infty(0,T;(L^2(\Omega))^3)} + ||\mathbf{E}_{tt}||_{L^\infty(0,T;(L^2(\Omega))^3)} + ||\mathbf{H}_{tt}||_{L^\infty(0,T;(L^2(\Omega))^3)}),
\end{aligned}$$

where $k \ge 1$ is the order of the basis functions in spaces $\mathbf{U}_h$ and $\mathbf{V}_h$.

Proof. Integrating (5.5)–(5.8) in time over $I_m = [t_{m-1}, t_m]$ and dividing all by τ, we have

$$(\delta_\tau\mathbf{E}^m, \phi) - (\frac{1}{\tau}\int_{I_m}\mathbf{H}(s)ds, \nabla\times\phi) + (\frac{1}{\tau}\int_{I_m}\mathbf{J}(s)ds, \phi) = 0, \tag{5.20}$$

$$(\delta_\tau\mathbf{H}^m, \psi) + (\nabla\times\frac{1}{\tau}\int_{I_m}\mathbf{E}(s)ds, \psi) + (\frac{1}{\tau}\int_{I_m}\mathbf{K}(s)ds, \psi) = 0, \tag{5.21}$$

$$(\delta_\tau\mathbf{J}^m, \tilde{\phi}) + \Gamma_e(\frac{1}{\tau}\int_{I_m}\mathbf{J}(s)ds, \tilde{\phi}) - \omega_e^2(\frac{1}{\tau}\int_{I_m}\mathbf{E}(s)ds, \tilde{\phi}) = 0, \tag{5.22}$$

$$(\delta_\tau\mathbf{K}^m, \tilde{\psi}) + \Gamma_m(\frac{1}{\tau}\int_{I_m}\mathbf{K}(s)ds, \tilde{\psi}) - \omega_m^2(\frac{1}{\tau}\int_{I_m}\mathbf{H}(s)ds, \tilde{\psi}) = 0. \tag{5.23}$$

Denote $\xi_h^m = \Pi_h\mathbf{E}^m - \mathbf{E}_h^m$, $\eta_h^m = P_h\mathbf{H}^m - \mathbf{H}_h^m$, $\tilde{\xi}_h^m = \Pi_h\mathbf{J}^m - \mathbf{J}_h^m$, $\tilde{\eta}_h^m = P_h\mathbf{K}^m - \mathbf{K}_h^m$. Subtracting (5.16)–(5.19) from (5.20)–(5.23) with $\phi = \phi_h$, $\psi = \psi_h$, $\tilde{\phi} = \tilde{\phi}_h$, and $\tilde{\psi} = \tilde{\psi}_h$, we can obtain the error equations

$$\begin{aligned}
(i)\quad &(\delta_\tau\xi_h^m, \phi_h) - (\overline{\eta}_h^m, \nabla\times\phi_h) + (\overline{\tilde{\xi}}{}_h^m, \phi_h) = (\delta_\tau(\Pi_h\mathbf{E}^m - \mathbf{E}^m), \phi_h)\\
&-(P_h\overline{\mathbf{H}}^m - \frac{1}{\tau}\int_{I_m}\mathbf{H}(s)ds, \nabla\times\phi_h) + (\Pi_h\overline{\mathbf{J}}^m - \frac{1}{\tau}\int_{I_m}\mathbf{J}(s)ds, \phi_h),\\
(ii)\quad &(\delta_\tau\eta_h^m, \psi_h) + (\nabla\times\overline{\xi}_h^m, \psi_h) + (\overline{\tilde{\eta}}{}_h^m, \psi_h) = (\delta_\tau(P_h\mathbf{H}^m - \mathbf{H}^m), \psi_h)\\
&+(\nabla\times(\Pi_h\overline{\mathbf{E}}^m - \frac{1}{\tau}\int_{I_m}\mathbf{E}(s)ds), \psi_h) + (P_h\overline{\mathbf{K}}_h^m - \frac{1}{\tau}\int_{I_m}\mathbf{K}(s)ds, \psi_h),
\end{aligned}$$

$$(iii)\quad (\delta_\tau \tilde{\xi}_h^m, \tilde{\phi}_h) + \Gamma_e(\overline{\tilde{\xi}}_h^m, \tilde{\phi}_h) - \omega_e^2(\overline{\xi}_h^m, \tilde{\phi}_h) = (\delta_\tau(\Pi_h \mathbf{J}^m - \mathbf{J}^m), \tilde{\phi}_h)$$
$$+\Gamma_e(\Pi_h \overline{\mathbf{J}}^m - \frac{1}{\tau}\int_{I_m} \mathbf{J}(s)ds, \tilde{\phi}_h) - \omega_e^2(\Pi_h \overline{\mathbf{E}}_h^m - \frac{1}{\tau}\int_{I_m} \mathbf{E}(s)ds, \tilde{\phi}_h),$$
$$(iv)\quad (\delta_\tau \tilde{\eta}_h^m, \tilde{\psi}_h) + \Gamma_m(\overline{\tilde{\eta}}_h^m, \tilde{\psi}_h) - \omega_m^2(\overline{\eta}_h^m, \tilde{\psi}_h) = (\delta_\tau(P_h \mathbf{K}^m - \mathbf{K}^m), \tilde{\psi}_h)$$
$$+\Gamma_m(P_h \overline{\mathbf{K}}^m - \frac{1}{\tau}\int_{I_m} \mathbf{K}(s)ds, \tilde{\psi}_h) - \omega_m^2(P_h \overline{\mathbf{H}}_h^m - \frac{1}{\tau}\int_{I_m} \mathbf{H}(s)ds, \tilde{\psi}_h).$$

Choosing $\phi_h = \tau\overline{\xi}_h^m, \psi_h = \tau\overline{\eta}_h^m, \tilde{\phi}_h = \tau\overline{\tilde{\xi}}_h^m, \tilde{\psi}_h = \tau\overline{\tilde{\eta}}_h^m$ in the above error equations, dividing the last two equations by ω_e^2 and ω_m^2, adding the resultants together, and using the property of operator P_h, we obtain

$$\begin{aligned}
&\frac{1}{2}[||\xi_h^m||_0^2 - ||\xi_h^{m-1}||_0^2 + ||\eta_h^m||_0^2 - ||\eta_h^{m-1}||_0^2 \\
&+\frac{1}{\omega_e^2}(||\tilde{\xi}_h^m||_0^2 - ||\tilde{\xi}_h^{m-1}||_0^2) + \frac{1}{\omega_m^2}(||\tilde{\eta}_h^m||_0^2 - ||\tilde{\eta}_h^{m-1}||_0^2)] \\
\le\ & \tau(\delta_\tau(\Pi_h \mathbf{E}^m - \mathbf{E}^m), \overline{\xi}_h^m) - \tau(\overline{\mathbf{H}}^m - \frac{1}{\tau}\int_{I_m} \mathbf{H}(s)ds, \nabla \times \overline{\xi}_h^m) \\
&+\tau(\Pi_h \overline{\mathbf{J}}^m - \frac{1}{\tau}\int_{I_m} \mathbf{J}(s)ds, \overline{\xi}_h^m) + \tau(\nabla \times (\Pi_h \overline{\mathbf{E}}^m - \frac{1}{\tau}\int_{I_m} \mathbf{E}(s)ds), \overline{\eta}_h^m) \\
&+\tau(\overline{\mathbf{K}}_h^m - \frac{1}{\tau}\int_{I_m} \mathbf{K}(s)ds, \overline{\eta}_h^m) + \frac{\tau}{\omega_e^2}(\delta_\tau(\Pi_h \mathbf{J}^m - \mathbf{J}^m), \overline{\tilde{\xi}}_h^m) \\
&+\frac{\tau\Gamma_e}{\omega_e^2}(\Pi_h \overline{\mathbf{J}}^m - \frac{1}{\tau}\int_{I_m} \mathbf{J}(s)ds, \overline{\tilde{\xi}}_h^m) - \tau(\Pi_h \overline{\mathbf{E}}_h^m - \frac{1}{\tau}\int_{I_m} \mathbf{E}(s)ds, \overline{\tilde{\xi}}_h^m) \\
&+\frac{\tau\Gamma_m}{\omega_m^2}(\overline{\mathbf{K}}^m - \frac{1}{\tau}\int_{I_m} \mathbf{K}(s)ds, \overline{\tilde{\eta}}_h^m) - \tau(\overline{\mathbf{H}}_h^m - \frac{1}{\tau}\int_{I_m} \mathbf{H}(s)ds, \overline{\tilde{\eta}}_h^m) \\
=\ & \sum_{i=1}^{10} Err_i.
\end{aligned} \tag{5.24}$$

After careful estimating of $Err_i, i = 1, \cdots, 10$ (details can be found in the original paper [153]), and substituting them into (5.24), summing up the result from $m = 1$ to any $n \le M$ with the fact that

$$\xi_h^0 = \eta_h^0 = \tilde{\xi}_h^0 = \tilde{\eta}_h^0 = 0,$$

then using the arithmetic-geometric mean inequality and taking the maximum with respect to n, we obtain

$$
\begin{aligned}
& ||\xi_h||^2_{l^\infty(L^2)} + ||\eta_h||^2_{l^\infty(L^2)} + ||\tilde{\xi}_h||^2_{l^\infty(L^2)} + ||\tilde{\eta}_h||^2_{l^\infty(L^2)} \\
\leq\ & Ch^{2(k+1)}(||\mathbf{E}_t||^2_{L^\infty(0,T;(H^{k+1}(\Omega))^3)} + ||\mathbf{J}||^2_{L^\infty(0,T;(H^{k+1}(\Omega))^3)} \\
& + ||\mathbf{E}||^2_{L^\infty(0,T;(H^{k+2}(\Omega))^3)} + ||\mathbf{J}_t||^2_{L^\infty(0,T;(H^{k+1}(\Omega))^3)}) \\
& + C\tau^4(||\nabla\times\mathbf{H}_{tt}||^2_{L^\infty(0,T;(L^2(\Omega))^3)} + ||\mathbf{J}_{tt}||^2_{L^\infty(0,T;(L^2(\Omega))^3)} + ||\nabla\times\mathbf{E}_{tt}||^2_{L^\infty(0,T;(L^2(\Omega))^3)} \\
& + ||\mathbf{K}_{tt}||^2_{L^\infty(0,T;(L^2(\Omega))^3)} + ||\mathbf{E}_{tt}||^2_{L^\infty(0,T;(L^2(\Omega))^3)} + ||\mathbf{H}_{tt}||^2_{L^\infty(0,T;(L^2(\Omega))^3)}),
\end{aligned}
$$

which concludes the proof. Note that C linearly depends on T^2. □

Remark 5.1. Similarly, we can formulate a leap-frog mixed finite element scheme for solving (5.5)–(5.8): Given initial approximations $\mathbf{E}_h^0, \mathbf{K}_h^0, \mathbf{H}_h^{\frac{1}{2}}, \mathbf{J}_h^{\frac{1}{2}}$, for $m \geq 1$, find $\mathbf{E}_h^m \in \mathbf{V}_h^0, \mathbf{J}_h^{m+\frac{1}{2}} \in \mathbf{V}_h, \mathbf{H}_h^{m+\frac{1}{2}}, \mathbf{K}_h^m \in \mathbf{U}_h$ such that

$$
\begin{aligned}
& (\frac{\mathbf{E}_h^m - \mathbf{E}_h^{m-1}}{\tau}, \phi_h) - (\mathbf{H}_h^{m-\frac{1}{2}}, \nabla\times\phi_h) + (\mathbf{J}_h^{m-\frac{1}{2}}, \phi_h) = 0, \\
& (\frac{\mathbf{H}_h^{m+\frac{1}{2}} - \mathbf{H}_h^{m-\frac{1}{2}}}{\tau}, \psi_h) + (\nabla\times\mathbf{E}_h^m, \psi_h) + (\mathbf{K}_h^m, \psi_h) = 0, \\
& (\frac{\mathbf{J}_h^{m+\frac{1}{2}} - \mathbf{J}_h^{m-\frac{1}{2}}}{\tau}, \tilde{\phi}_h) + \Gamma_e(\frac{1}{2}(\mathbf{J}_h^{m+\frac{1}{2}} + \mathbf{J}_h^{m-\frac{1}{2}}), \tilde{\phi}_h) - \omega_e^2(\mathbf{E}_h^m, \tilde{\phi}_h) = 0, \\
& (\frac{\mathbf{K}_h^m - \mathbf{K}_h^{m-1}}{\tau}, \tilde{\psi}_h) + \Gamma_m(\frac{1}{2}(\mathbf{K}_h^m + \mathbf{K}_h^{m-1}), \tilde{\psi}_h) - \omega_m^2(\mathbf{H}_h^{m-\frac{1}{2}}, \tilde{\psi}_h) = 0,
\end{aligned}
$$

hold true for test functions $\phi_h \in \mathbf{V}_h^0, \psi_h \in \mathbf{U}_h, \tilde{\phi}_h \in \mathbf{V}_h, \tilde{\psi}_h \in \mathbf{U}_h$. Combining the above proof techniques with those developed for the leap-frog scheme [183], we can obtain the following superclose result:

$$
\begin{aligned}
& \max_{1\leq m}(||\Pi_h\mathbf{E}^m - \mathbf{E}_h^m||_0 + ||P_h\mathbf{H}^{m+\frac{1}{2}} - \mathbf{H}_h^{m+\frac{1}{2}}||_0 \\
& \quad + ||\Pi_h\mathbf{J}^{m+\frac{1}{2}} - \mathbf{J}_h^{m+\frac{1}{2}}||_0 + ||P_h\mathbf{K}^m - \mathbf{K}_h^m||_0) \leq C(\tau^2 + h^{k+1}).
\end{aligned}
$$

5.4 Superconvergence in the Discrete l_2 Norm

In this section, we first prove a superconvergence interpolation result obtained at element centers for the lowest order cubic edge element (i.e., $k = 1$ in spaces $\mathbf{U}_h$ and $\mathbf{V}_h$). Then we use that to obtain a global superconvergence result in the discrete l_2 norm.

Lemma 5.3. *Let* $K = [x_c - h_x, x_c + h_x] \times [y_c - h_y, y_c + h_y] \times [z_c - h_z, z_c + h_z]$ *be an arbitrary cubic element with maximum length* h*. Then for any* $\mathbf{u} \in \mathbf{W}^{2,\infty}(K)$ *and its corresponding Nédélec interpolation* $\Pi_K^c \mathbf{u} \in Q_{0,1,1} \times Q_{1,0,1} \times Q_{1,1,0}$*, we have*

$$\left(\mathbf{u} - \Pi_K^c \mathbf{u}\right)(x_c, y_c, z_c) \leq C h^2. \tag{5.25}$$

Proof. Note that the lowest order $H(curl)$ interpolation $\Pi_K^c \mathbf{u}$ can be written explicitly as (cf. Example 3.5)

$$\Pi_K^c \mathbf{u}(x, y, z) = (\Pi_K^c u_1, \Pi_K^c u_2, \Pi_K^c u_3) = \sum_{i=1}^{12} \left(\int_{l_i} \mathbf{u} \cdot \tau_i \, dl \right) \mathbf{N}_i(x, y, z), \tag{5.26}$$

where l_i are the 12 edges of the element, and τ_i represent the unit tangent vector along l_i (cf. Fig. 5.1), and $\mathbf{N}_i \in Q_{0,1,1} \times Q_{1,0,1} \times Q_{1,1,0}$ are the basis functions.

The first component of $\Pi_K^c \mathbf{u}$ is

$$\begin{aligned}
(\Pi_K^c u)_1 = & \left(\int_{l_1} u_1(x, y_c - h_y, z_c - h_z) dl \right) \cdot \frac{1}{8 h_x h_y h_z} (y_c + h_y - y)(z_c + h_z - z) \\
& + \left(\int_{l_2} u_1(x, y_c + h_y, z_c - h_z) dl \right) \cdot \frac{1}{8 h_x h_y h_z} (y + h_y - y_c)(z_c + h_z - z) \\
& + \left(\int_{l_3} u_1(x, y_c - h_y, z_c + h_z) dl \right) \cdot \frac{1}{8 h_x h_y h_z} (y_c + h_y - y)(z + h_z - z_c) \\
& + \left(\int_{l_4} u_1(x, y_c + h_y, z_c + h_z) dl \right) \cdot \frac{1}{8 h_x h_y h_z} (y + h_y - y_c)(z + h_z - z_c),
\end{aligned}$$

from which we see that the value at the element center is

$$\begin{aligned}
(\Pi_K^c u)_1(x_c, y_c, z_c) = & \frac{1}{8 h_x} \left(\int_{l_1} u_1(x, y_c - h_y, z_c - h_z) dl + \int_{l_2} u_1(x, y_c + h_y, z_c - h_z) dl \right) \\
& + \frac{1}{8 h_x} \left(\int_{l_3} u_1(x, y_c - h_y, z_c + h_z) dl + \int_{l_4} u_1(x, y_c + h_y, z_c + h_z) dl \right).
\end{aligned}$$

By Taylor expansion at x_c and the fact that $\int_{x_c - h_x}^{x_c + h_x} (x - x_c) dx = 0$, we easily have

$$\begin{aligned}
& \int_{l_1} u_1(x, y_c - h_y, z_c - h_z) \\
= & \int_{l_1} [u_1(x_c, y_c - h_y, z_c - h_z) + O(h_x^2) \partial_{xx} u_1(x_*, y_c - h_y, z_c - h_z)] dl \\
= & \, 2 h_x u_1(x_c, y_c - h_y, z_c - h_z) + 2 h_x O(h_x^2) \partial_{xx} u_1(x_*, y_c - h_y, z_c - h_z),
\end{aligned}$$

where x_* is some number between x and x_c.

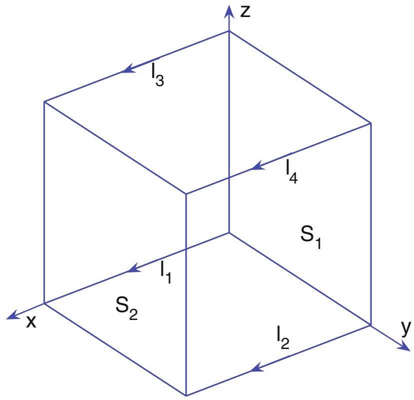

Fig. 5.1 The exemplary cubic edge element

Similar estimates can be obtained for other line integrals. Hence we have

$$(\Pi_K^c u)_1(x_c, y_c, z_c) = \frac{1}{4}[u_1(x_c, y_c - h_y, z_c - h_z) + u_1(x_c, y_c + h_y, z_c - h_z) \\ + u_1(x_c, y_c - h_y, z_c + h_z) + u_1(x_c, y_c + h_y, z_c + h_z)] + O(h_x^2).$$

Using Taylor expansion at (x_c, y_c, z_c) again, we can easily see that

$$(\Pi_K^c u)_1(x_c, y_c, z_c) = u_1(x_c, y_c, z_c) + O(h_x^2 + h_y^2 + h_z^2).$$

By symmetry, the same estimates can be proved for the second and third components of $\Pi_K^c \mathbf{u}$. □

With the above estimates, we can now obtain a superconvergence result in the discrete l_2 norm, which is one-order higher compared to the optimal error estimate obtained in the continuous L_2 norm.

Theorem 5.3. *Let $\mathbf{x}_c^K = (x_c, y_c, z_c)$ be the center of a cubic element $K = [x_c - h_x, x_c + h_x] \times [y_c - h_y, y_c + h_y] \times [z_c - h_z, z_c + h_z]$, $(\mathbf{E}^m, \mathbf{H}^m)$ and $(\mathbf{E}_h^m, \mathbf{H}_h^m)$ be the analytical and numerical solutions of (5.1)–(5.4) and (5.16)–(5.19), respectively. Under the assumptions of Theorem 5.2 (with $m = 1$) and Lemma 5.3, we have*

$$\max_{1 \le m \le M} \left(||\mathbf{E}^m - \mathbf{E}_h^m||_{l^2} + ||\mathbf{H}^m - \mathbf{H}_h^m||_{l^2} \right) \le C(\tau^2 + h^2),$$

where we denote $||u||_{l_2} = \left(\sum_e |u(\mathbf{x}_c^K)|^2 \cdot |K| \right)^{\frac{1}{2}}$, and $|K|$ for the volume of element K.

Proof. Note that any $u_h \in Q_{0,1,1}$ can be written as $(c_1 + c_2 y)(c_3 + c_4 z)$, which satisfies the identity

$$u_h(\mathbf{x}_c^K) = (c_1 + c_2 y_c)(c_3 + c_4 z_c) = \frac{1}{|K|} \int_K u_h dxdydz. \qquad (5.27)$$

For the lowest order edge element space $\mathbf{V}_h$, both $\Pi_h \mathbf{E}^m$ (simplified notation of $\Pi_h^c \mathbf{E}^m$) and $\mathbf{E}_h^m \in Q_{0,1,1} \times Q_{1,0,1} \times Q_{1,1,0}$, hence applying (5.27) to the first component of $\Pi_h \mathbf{E}^m - \mathbf{E}_h^m$, we have

$$(\Pi_h \mathbf{E}^m - \mathbf{E}_h^m)_1(\mathbf{x}_c^K) = \frac{1}{|K|} \int_K (\Pi_K^c \mathbf{E}^m - \mathbf{E}_h^m)_1 dxdydz.$$

Using the Cauchy-Schwarz inequality and Theorem 5.2 with $k = 1$, we obtain

$$\begin{aligned}\sum_{K \in T^h} |(\Pi_h \mathbf{E}^k - \mathbf{E}_h^k)_1(\mathbf{x}_c^K)|^2 \cdot |K| &= \sum_{K \in T^h} \frac{1}{|K|} \left(\int_K (\Pi_K^c \mathbf{E}^m - \mathbf{E}_h^m)_1 dxdydz \right)^2 \\ &\leq \int_\Omega (\Pi_h \mathbf{E}^m - \mathbf{E}_h^m)_1^2 dxdydz \leq C(\tau^4 + h^4).\end{aligned}$$

Same estimates can be proved for the other two components. Then by the triangle inequality and Lemma 5.3, we have

$$||\mathbf{E}^m - \mathbf{E}_h^m||_{l_2} \leq ||\mathbf{E}^m - \Pi_h \mathbf{E}^m||_{l_2} + ||\Pi_h \mathbf{E}^m - \mathbf{E}_h^m||_{l_2} \leq C(\tau^2 + h^2). \quad (5.28)$$

The estimate $||\mathbf{H}^m - \mathbf{H}_h^m||_{l_2} \leq C(\tau^2 + h^2)$ can be proved similarly (cf. [156]) □

5.5 Extensions to 2-D Superconvergence Analysis

In this section, we want to prove similar superclose results for the 2-D Maxwell's equations. Note that in some sense the 2-D case is more complicated than the 3-D case, since in the 2-D Maxwell's equations, one field is a 2-D vector, while the other field becomes a scalar. Without loss of generality, here we assume that the electrical field $\mathbf{E}$ is a vector, while the magnetic field H is a scalar. To make the extension clearly, we define the 2-D vector and scalar curl operators:

$$\nabla \times H = (\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x})', \quad \nabla \times \mathbf{E} = \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}, \quad \forall\, \mathbf{E} \equiv (E_1, E_2)'. \quad (5.29)$$

5.5.1 Superconvergence on Rectangular Edge Elements

For a 2-D domain Ω, we partition it by a family of regular rectangular meshes T_h with maximum mesh size h. The corresponding Raviart-Thomas-Nédélec rectangular elements are:

$$\begin{aligned}U_h &= \{\psi_h \in L^2(\Omega) : \ \psi_h|_K \in Q_{k-1,k-1}, \ \forall\, K \in T_h\}, \\ \mathbf{V}_h &= \{\phi_h \in H(\mathrm{curl}; \Omega) : \ \phi_h|_K \in Q_{k-1,k} \times Q_{k,k-1}, \ \forall\, K \in T_h\},\end{aligned}$$

for any $k \geq 1$. Recall that $Q_{i,j}$ denotes the space of polynomials whose degrees are less than or equal to i, j in variables x, y, respectively. It is easy to see that $\nabla \times \mathbf{V}_h \subset U_h$ still holds.

In the 2-D case, the Nédélec operator $\Pi_h \mathbf{E} \in \mathbf{V}_h$ is defined as:

$$\int_{l_i} (\mathbf{E} - \Pi_h \mathbf{E}) \cdot \tau_i q dl = 0, \quad \forall\, q \in P_{k-1}(l_i),\ i = 1, \cdots, 4, \tag{5.30}$$

$$\int_K (\mathbf{E} - \Pi_h \mathbf{E}) \cdot \mathbf{q} dxdy = 0, \quad \forall\, \mathbf{q} \in Q_{k-1,k-2} \times Q_{k-2,k-1}, \tag{5.31}$$

where l_i denotes the i-th edge of an element K, and τ_i is the unit tangent vector along the edge l_i. When $k = 1$ (the lowest-order rectangular edge element), $\Pi_h \mathbf{E}$ is defined by (5.30) only.

The 2-D superclose analysis depends on the following fundamental results.

Lemma 5.4. *For any* $\mathbf{u} \in H(curl; K)$ *and* $q \in Q_{k-1,k-1}(K), k \geq 1$, *we have*

$$\int_K \nabla \times (\mathbf{u} - \Pi_h \mathbf{u}) \cdot q dxdy = 0.$$

Proof. The proof follows from the Stokes' formula

$$\int_K \nabla \times (\mathbf{u} - \Pi_h \mathbf{u}) \cdot q dxdy = \int_{\partial K} (\mathbf{u} - \Pi_h \mathbf{u}) \cdot \tau q dl + \int_K (\mathbf{u} - \Pi_h \mathbf{u}) \cdot (\nabla \times q) dxdy$$

and the property (5.30) and (5.31) for the operator Π_h. □

Let P_h be the L^2-projection operator onto the space U_h. By the property $\nabla \times \mathbf{V}_h \subset U_h$, we immediately have

Lemma 5.5. *For any* $w \in L^2(K)$ *and* $\phi_h|_K \in Q_{k-1,k} \times Q_{k,k-1}, k \geq 1$, *we have*

$$\int_K (w - P_h w) \cdot \nabla \times \phi_h dxdy = 0.$$

Lemma 5.6. *Let* $K = [x_c - h_x, x_c + h_x] \times [y_c - h_y, y_c + h_y]$ *be an arbitrary rectangular element. Then for any* $\mathbf{u} \in H(curl; K)$ *and* $\phi_h|_K \in Q_{k-1,k} \times Q_{k,k-1}, k \geq 1$, *we have*

$$\int_K (u_1 - (\Pi_h \mathbf{u})_1)\phi_1 dxdy = O(h_y^{k+1})||\partial_y^{k+1} u_1||_{0,K}||\phi_1||_{0,K}, \tag{5.32}$$

$$\int_K (u_2 - (\Pi_h \mathbf{u})_2)\phi_2 dxdy = O(h_x^{k+1})||\partial_x^{k+1} u_2||_{0,K}||\phi_2||_{0,K}, \tag{5.33}$$

where u_1, u_2 and ϕ_1, ϕ_2 are the two components of $\mathbf{u}$ and ϕ_h, respectively. Hence, we have

$$\int_K (\mathbf{u} - \Pi_h \mathbf{u}) \cdot \phi_h dxdy = O(h^{k+1})||\mathbf{u}||_{k+1,K}||\phi_h||_{0,K}.$$

Proof. Since

$$\int_K (\mathbf{u} - \Pi_h \mathbf{u}) \cdot \phi_h dxdy = \int_K (u_1 - (\Pi_h \mathbf{u})_1)\phi_1 dxdy + \int_K (u_2 - (\Pi_h \mathbf{u})_2)\phi_2 dxdy,$$

we just need to consider the first inner product. For simplicity, below we just present the proof for the $k = 1$ case. For $k \geq 2$ case, interested readers can find the detailed proof in the original paper [153].

By definition, when $k = 1$, $\phi_1 \in Q_{0,1}$. Then by the Taylor expansion, we obtain

$$\begin{aligned} &\int_K (u_1 - (\Pi_h \mathbf{u})_1)\phi_1 dxdy \\ &= \int_K (u_1 - (\Pi_h \mathbf{u})_1)[\phi_1(x_c, y_c) + (y - y_c)\partial_y \phi_1(x_c, y_c)]dxdy. \end{aligned} \tag{5.34}$$

Denote the functions

$$A(x) = \frac{1}{2}[(x - x_c)^2 - h_x^2], \quad B(y) = \frac{1}{2}[(y - y_c)^2 - h_y^2]. \tag{5.35}$$

Note that in the proof below we will constantly use the facts that:

$$A(x) = 0 \quad \text{on } x = x_c \pm h_x, \quad B(y) = 0 \quad \text{on } y = y_c \pm h_y. \tag{5.36}$$

Using integration by parts and the identity $\partial_{yy} B(y) = 1$, (5.30) and (5.36), we have

$$\begin{aligned} &\int_K (u_1 - (\Pi_h \mathbf{u})_1)dxdy = \int_K (u_1 - (\Pi_h \mathbf{u})_1)\partial_{yy} B(y)dxdy \\ &= \int_{x=x_c-h_x}^{x_c+h_x} (u_1 - (\Pi_h \mathbf{u})_1)\partial_y B(y)|_{y=y_c-h_y}^{y_c+h_y} dx - \int_K (u_1 - (\Pi_h \mathbf{u})_1)_y \partial_y B(y)dxdy \\ &= \int_K (u_1 - (\Pi_h \mathbf{u})_1)_{yy} B(y)dxdy = \int_K \partial_{yy} u_1 \cdot B(y)dxdy, \end{aligned}$$

where in the last step we used the fact that $(\Pi_h \mathbf{u})_1 \in Q_{0,1}$.

Similarly, by the identity $y - y_c = \frac{1}{6}\partial_y^3(B^2(y))$ and integration by parts, we obtain

$$\int_K (u_1 - (\Pi_h \mathbf{u})_1)(y - y_c)dxdy = \int_K (u_1 - (\Pi_h \mathbf{u})_1) \cdot \frac{1}{6}\partial_y^3(B^2(y))dxdy$$
$$= \int_{x=x_c-h_x}^{x_c+h_x} (u_1 - (\Pi_h \mathbf{u})_1) \cdot \frac{1}{6}\partial_y^2(B^2(y))|_{y=y_c-h_y}^{y_c+h_y} dx$$
$$- \int_K (u_1 - (\Pi_h \mathbf{u})_1)_y \cdot \frac{1}{6}\partial_y^2(B^2(y))dxdy$$
$$= \int_K (u_1 - (\Pi_h \mathbf{u})_1)_{yy} \cdot \frac{1}{6}\partial_y(B^2(y))dxdy = \int_K \partial_{yy}u_1 \cdot \frac{1}{6}(B^2(y))_y dxdy.$$

Substituting the above integral identities into (5.34) and using the inverse estimate, we have

$$\int_K (u_1 - (\Pi_h \mathbf{u})_1)\phi_1 dxdy$$
$$= \int_K \partial_{yy}u_1 \cdot B(y) \cdot \phi_1(x_c, y_c)dxdy + \int_K \partial_{yy}u_1 \cdot \frac{1}{6}(B^2(y))_y \cdot \partial_y\phi_1(x_c, y_c)dxdy$$
$$= \int_K \partial_{yy}u_1 \cdot B(y) \cdot [\phi_1(x, y) - (y - y_c)\partial_y\phi_1(x, y)]dxdy$$
$$+ \int_K \partial_{yy}u_1 \cdot \frac{1}{3}B(y) \cdot (y - y_c)\partial_y\phi_1(x, y)dxdy$$
$$= O(h_y^2)||\partial_{yy}u_1||_{0,K}||\phi_1||_{0,K}.$$

Using the same arguments, we can prove

$$\int_K (u_2 - (\Pi_h \mathbf{u})_2)\phi_2 dxdy = O(h_x^2)||\partial_{xx}u_2||_{0,K}||\phi_2||_{0,K},$$

which completes our proof for the $k = 1$ case. □

With Lemmas 5.4–5.6, we can see that Theorems 5.1 and 5.2 hold true for 2-D rectangular elements. Below we want to show that for the lowest-order edge element (i.e., $k = 1$ in U_h and $\mathbf{V}_h$), we have one-order higher superconvergence in the L^∞-norm at rectangular element centers.

Lemma 5.7. *Let $K = [x_c - h_x, x_c + h_x] \times [y_c - h_y, y_c + h_y]$ be an arbitrary rectangular element. Then for any $\mathbf{u} \in H(curl; K)$ and $\Pi_h \mathbf{u}|_K \in Q_{0,1} \times Q_{1,0}$, we have*

$$(\mathbf{u} - \Pi_K^c \mathbf{u})(x_c, y_c) = O(h^2). \tag{5.37}$$

Proof. For the lowest-order edge element $Q_{0,1} \times Q_{1,0}$, the interpolation $\Pi_K^c \mathbf{u}$ of any $\mathbf{u} \in H(\text{curl}; K)$ can be written as (cf. Example 3.6):

$$\Pi_K^c \mathbf{u}(x, y) = \sum_{j=1}^{4} (\int_{l_j} \mathbf{u} \cdot \tau_j \, dl) \mathbf{N}_j(x, y), \tag{5.38}$$

where we denote l_j the four edges of the element, which start from the bottom edge and are oriented counterclockwise. Furthermore, we denote τ_j for the unit tangent vector along l_j. Recall that the edge element basis functions $\mathbf{N}_j$ are as follows (cf. Example 3.6):

$$\mathbf{N}_1 = \begin{pmatrix} \frac{(y_c+h_y)-y}{4h_x h_y} \\ 0 \end{pmatrix}, \quad \mathbf{N}_2 = \begin{pmatrix} 0 \\ \frac{x-(x_c-h_x)}{4h_x h_y} \end{pmatrix},$$

$$\mathbf{N}_3 = \begin{pmatrix} \frac{(y_c-h_y)-y}{4h_x h_y} \\ 0 \end{pmatrix}, \quad \mathbf{N}_4 = \begin{pmatrix} 0 \\ \frac{x-(x_c+h_x)}{4h_x h_y} \end{pmatrix}.$$

By (5.38) and the notation $\mathbf{u} = (u_1, u_2)'$, we have

$$\begin{aligned}
&\Pi_K^c \mathbf{u}(x_c, y_c) \\
&= \int_{l_1} u_1(x, y_c - h_y) dx \cdot \begin{pmatrix} \frac{(y_c+h_y)-y_c}{4h_x h_y} \\ 0 \end{pmatrix} + \int_{l_2} u_2(x_c + h_x, y) dy \cdot \begin{pmatrix} 0 \\ \frac{x_c-(x_c-h_x)}{4h_x h_y} \end{pmatrix} \\
&\quad - \int_{l_3} u_1(x, y_c + h_y) dx \cdot \begin{pmatrix} \frac{(y_c-h_y)-y_c}{4h_x h_y} \\ 0 \end{pmatrix} - \int_{l_4} u_2(x_c - h_x, y) dy \cdot \begin{pmatrix} 0 \\ \frac{x_c-(x_c+h_x)}{4h_x h_y} \end{pmatrix},
\end{aligned}$$

from which we obtain the first component

$$\begin{aligned}
&\frac{1}{4h_x} (\int_{x_c-h_x}^{x_c+h_x} u_1(x, y_c - h_y) dx + \int_{x_c-h_x}^{x_c+h_x} u_1(x, y_c + h_y) dx) \\
&= \frac{1}{4h_x} (\int_{x_c-h_x}^{x_c+h_x} [u_1(x_c, y_c - h_y) + (x - x_c)\partial_x u_1(x_c, y_c - h_y) + O(h_x^2)] dx \\
&\quad + \int_{x_c-h_x}^{x_c+h_x} [u_1(x_c, y_c + h_y) + (x - x_c)\partial_x u_1(x_c, y_c + h_y) + O(h_x^2)] dx) \\
&= \frac{1}{2} [u_1(x_c, y_c - h_y) + u_1(x_c, y_c + h_y)] + O(h_x^2),
\end{aligned}$$

where we used the Taylor expansion and the fact that $\int_{x_c-h_x}^{x_c+h_x} (x - x_c) dx = 0$. Using the Taylor expansion one more time, we can easily see that

$$
\begin{aligned}
&((\Pi_K^c \mathbf{u})_1 - u_1)(x_c, y_c) \\
&= \frac{1}{2}[u_1(x_c, y_c - h_y) + u_1(x_c, y_c + h_y)] - u_1(x_c, y_c) + O(h_x^2) \\
&= O(h_x^2) + O(h_y^2).
\end{aligned}
$$

By the same arguments, we can obtain the same estimate for the second component:

$$((\Pi_K^c \mathbf{u})_2 - u_2)(x_c, y_c) = O(h_x^2) + O(h_y^2),$$

which completes the proof. □

With the above preparations, finally we can prove the following L^∞ superconvergence result.

Theorem 5.4. *Let (x_c, y_c) be the center of a rectangular element $K = [x_c - h_x, x_c + h_x] \times [y_c - h_y, y_c + h_y]$, and $\mathbf{E}^h$ and H^h be the lowest-order finite element solution of (5.9)–(5.12), i.e., $\mathbf{E}^h|_K \in Q_{0,1} \times Q_{1,0}$ and $H^h|_K \in Q_{0,0}$. Under the assumption that the L^2 norms of $\Pi_h \mathbf{E} - \mathbf{E}^h$ and $P_h H - H^h$ are almost uniformly distributed, i.e.,*

$$\int_K |\Pi_h \mathbf{E} - \mathbf{E}^h|^2 dK \le \frac{C}{N} \int_\Omega |\Pi_h \mathbf{E} - \mathbf{E}^h|^2 dK, \tag{5.39}$$

$$\int_K |P_h H - H^h|^2 dK \le \frac{C}{N} \int_\Omega |P_h H - H^h|^2 dK, \tag{5.40}$$

where N denotes the total number of elements over Ω. Then on a quasi-uniform mesh we have the L^∞ superconvergence

$$|(\mathbf{E} - \mathbf{E}^h)(x_c, y_c)| + |(H - H^h)(x_c, y_c)| \le C h^2. \tag{5.41}$$

Proof. Using the fact that the m-point Gaussian quadrature holds exactly for all polynomials up to degree $2m - 1$, and the Cauchy-Schwarz inequality, for the first component of error $\Pi_h \mathbf{E} - \mathbf{E}^h$ we easily have

$$
\begin{aligned}
&|(\Pi_h \mathbf{E} - \mathbf{E}^h)_1(x_c, y_c)| = |\frac{1}{|K|} \int_K (\Pi_h \mathbf{E} - \mathbf{E}^h)_1 dxdy| \\
&\le \frac{1}{|K|} (\int_K |(\Pi_h \mathbf{E} - \mathbf{E}^h)_1|^2 dxdy)^{1/2} (\int_K 1^2 dxdy)^{1/2} \\
&\le \frac{1}{|K|^{1/2}} (\frac{1}{N} \int_\Omega |(\Pi_h \mathbf{E} - \mathbf{E}^h)_1|^2 dxdy)^{1/2} \\
&\le \frac{1}{(N|K|)^{1/2}} \cdot Ch^2 \le Ch^2,
\end{aligned} \tag{5.42}
$$

where we used Theorem 5.1 and the fact that $N|K| \approx \text{meas}(\Omega) = O(1)$. Similar estimate can be obtained for the second component, i.e.,

$$|(\Pi_h \mathbf{E} - \mathbf{E}^h)_2(x_c, y_c)| = O(h^2),$$

from which and Lemma 5.4, we obtain

$$(\mathbf{E} - \mathbf{E}^h)(x_c, y_c) = (\mathbf{E} - \Pi_h \mathbf{E})(x_c, y_c) + (\Pi_h \mathbf{E} - \mathbf{E}^h)(x_c, y_c) = O(h^2).$$

Note that for any function $f(x, y)$, by Taylor expansion, we have

$$\begin{aligned} &\frac{1}{|K|}\int_K f(x,y)dxdy - f(x_c, y_c) = \frac{1}{|K|}\int_K (f(x,y) - f(x_c, y_c))dxdy \\ &= \frac{1}{|K|}\int_K [(x - x_c)\partial_x f(x_c, y_c) + (y - y_c)\partial_y f(x_c, y_c) + O(h^2)]dxdy \\ &= O(h^2), \end{aligned} \tag{5.43}$$

using which, the fact that $\int_K (P_h H - H)dxdy = 0$ and similar arguments used in (5.42), we have

$$\begin{aligned} &(H - H^h)(x_c, y_c) \approx \frac{1}{|K|}\int_K (H - H^h)(x,y)dxdy + O(h^2) \\ &= \frac{1}{|K|}\int_K (P_h H - H^h)(x,y)dxdy + O(h^2) \\ &\le \frac{1}{|K|}(\int_K |P_h H - H^h|^2 dxdy)^{1/2}(\int_K 1^2 dxdy)^{1/2} + O(h^2) \le Ch^2, \end{aligned}$$

which concludes the proof. □

By similar arguments, for the fully-discrete scheme (5.16)–(5.19), under the constraints (5.39) and (5.40), we can prove

$$\max_{1\le m\le M}(|(\mathbf{E}^m - \mathbf{E}_h^m)(x_c, y_c)| + |(H^m - H_h^m)(x_c, y_c)|) \le C(h^2 + \tau^2).$$

Without imposing the constraints (5.39) and (5.40), we can similarly prove the discrete l_2 superconvergence as Theorem 5.3. More specifically, we have

Theorem 5.5. *Let $\mathbf{x}_c^K = (x_c, y_c)$ be the center of a rectangular element $K = [x_c - h_x, x_c + h_x] \times [y_c - h_y, y_c + h_y]$, $(\mathbf{E}^m, H^m)$ and $(\mathbf{E}_h^m, H_h^m)$ be the 2-D analytical and numerical solutions of (5.1)–(5.4) and (5.16)–(5.19), respectively. Then we have*

$$\max_{1\le m\le M}\left(||\mathbf{E}^m - \mathbf{E}_h^m||_{l^2} + ||H^m - H_h^m||_{l^2}\right) \le C(\tau^2 + h^2),$$

where we denote $||u||_{l_2} = \left(\sum_e |u(\mathbf{x}_c^K)|^2 \cdot |K|\right)^{\frac{1}{2}}$*, and* $|K|$ *for the area of element* K.

Numerical results demonstrating L^∞ convergence rate $O(h^2)$ at rectangular element centers are indeed observed for the lowest-order rectangular edge element. Detailed results are presented in Chap. 7.

5.5.2 Superconvergence on Triangular Edge Elements

In this section, we would like to show that some superconvergence results as Sect. 5.5.1 hold true for the lowest-order triangular edge element. We assume that the domain Ω is partitioned by a family of regular triangular meshes T_h with maximum mesh size h, in which case the mixed finite element spaces used to solve (5.16)–(5.19) are:

$$U_h = \{\psi_h \in L^2(\Omega) : \quad \psi_h = \text{ piecewise constant}, \quad \forall\ K \in T_h\},$$
$$\mathbf{V}_h = \{\phi_h \in H(\text{curl}; \Omega) : \quad \phi_h|_K = \text{span}(\lambda_i \nabla\lambda_j - \lambda_j \nabla\lambda_i),\ i, j = 1, 2, 3,\ \forall\ K \in T_h\},$$

where λ_i is the barycentric coordinate at the i-th vertex of the triangle K.

By the Stokes' formula, it is easy to see that:

Lemma 5.8. *For any* $\mathbf{u} \in H(curl; K)$*, we have*

$$\int_K \nabla \times (\mathbf{u} - \Pi_h \mathbf{u}) dxdy = 0.$$

Note that for any $\phi_h|_K \in \mathbf{V}_h$, $\nabla \times \phi_h$ is a constant, hence we easily have the following result.

Lemma 5.9. *For any* $w \in L^2(K)$ *and* $\phi_h|_K \in \mathbf{V}_h$*, we have*

$$\int_K (w - P_h w) \cdot \nabla \times \phi_h dxdy = 0.$$

Since there exists no natural superconvergence point for the numerical solution of (5.16)–(5.19) obtained with the lowest-order triangular edge element, we consider a special triangular mesh formed by parallelograms such as Fig. 5.2.

Below is a superclose result between a function and its Nédélec interpolation on a parallelogram.

Theorem 5.6 ([154, Theorem 3.3]). *On a parallelogram* $\diamondsuit$ *formed by two triangles, if* $\mathbf{u} \in H(curl; \diamondsuit) \cap H^3(\diamondsuit)$ *and* $\phi_h \in \mathbf{V}_h$*, then we have*

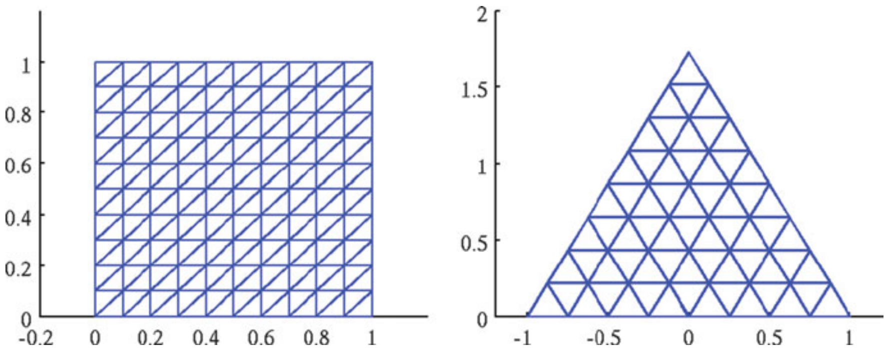

Fig. 5.2 Exemplary triangular meshes formed by parallelograms

$$\int_{\diamond} (\mathbf{u} - (\Pi_h \mathbf{u})) \cdot \phi_h \, \mathrm{d}x\mathrm{d}y = O(h^2)||\partial^3 \mathbf{u}||_{0,\diamond}||\phi_h||_{0,\diamond}. \tag{5.44}$$

Note that by the standard interpolation estimate [217], we only have

$$\int_{\diamond} (\mathbf{u} - (\Pi_h \mathbf{u})) \cdot \phi_h = O(h)||\mathbf{u}||_{H(\mathrm{curl};\diamond)}||\phi_h||_{0,\diamond},$$

which is one order less than (5.44).

Through some technical calculation, a pointwise superconvergence result at the center of each parallelogram can be proved by taking an average of the interpolations from those two neighboring triangles.

Theorem 5.7 ([154, Theorem 3.4]). *Assume that (x_c, y_c) is the center of one parallelogram $\diamond$ formed by two triangles L and R, then for any $\mathbf{u} = (u_1, u_2) \in C^2(\overline{\diamond})$, we have*

$$[\mathbf{u} - \frac{1}{2}((\Pi_h \mathbf{u})|_L + (\Pi_h \mathbf{u})|_R)](x_c, y_c) = O(h^2).$$

Another interesting result for the lowest-order triangular edge element is that the average of a function over a parallelogram is equal to the function value at the parallelogram center.

Lemma 5.10. *Consider a parallelogram $\diamond$ formed by vertices $A(x_c - l_3\cos\alpha, y_c - l_3\sin\alpha)$, $B(x_c - l_3\cos\alpha + 2l_1, y_c - l_3\sin\alpha)$, $C(x_c + l_3\cos\alpha, y_c + l_3\sin\alpha)$, and $D(x_c + l_3\cos\alpha - 2l_1, y_c + l_3\sin\alpha)$, where $O(x_c, y_c)$ denotes the midpoint of AC, $\alpha = \angle CAB$, $2l_1, 2l_2$ and $2l_3$ are the lengths of AB, BC and CA, respectively. The following holds true (for any parallelogram in Fig. 5.3):*

$$\frac{1}{|\diamond|}\int_{\diamond} \mathbf{u}_h dxdy = \mathbf{u}_h(x_c, y_c) \quad \forall\, \mathbf{u}_h \in \mathbf{V}_h. \tag{5.45}$$

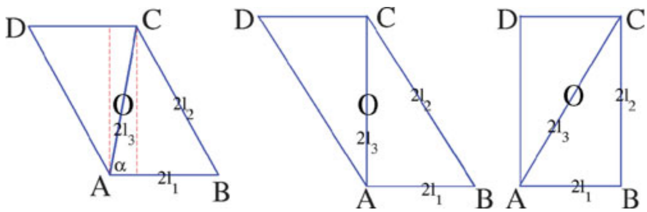

Fig. 5.3 The exemplary parallelograms

Proof. By definition of $\mathbf{V}_h$, the first component of $\mathbf{u}_h \in \mathbf{V}_h$ can be written as (cf. [154, Lemma 2.1]):

$$u_h^1 = c_1 + c_2 y,$$

where c_1 and c_2 are some constants. Below we just prove (5.45) for the first component on the general parallelogram (the left one in Fig. 5.3), since proofs of the other cases are easier.

From Fig. 5.3, we can write the line equations of AD and BC respectively:

$$l_{AD}: \ y - (y_c - l_3 \sin\alpha) = \frac{l_3 \sin\alpha}{l_3 \cos\alpha - 2l_1}[x - (x_c - l_3 \cos\alpha)],$$

$$l_{BC}: \ y - (y_c - l_3 \sin\alpha) = \frac{l_3 \sin\alpha}{l_3 \cos\alpha - 2l_1}[x - (x_c - l_3 \cos\alpha + 2l_1)],$$

solving which for x, we obtain

$$x_{l_{AD}} = \frac{l_3 \cos\alpha - 2l_1}{l_3 \sin\alpha}[y - (y_c - l_3 \sin\alpha)] + (x_c - l_3 \cos\alpha),$$

$$x_{l_{BC}} = \frac{l_3 \cos\alpha - 2l_1}{l_3 \sin\alpha}[y - (y_c - l_3 \sin\alpha)] + (x_c - l_3 \cos\alpha + 2l_1).$$

Therefore, we have

$$\int_\diamond u_h^1 dxdy = \int_\diamond (c_1 + c_2 y) dxdy$$

$$= \int_{y_c - l_3 \sin\alpha}^{y_c + l_3 \sin\alpha} \int_{x_{l_{AD}}}^{x_{l_{BC}}} (c_1 + c_2 y) dxdy = \int_{y_c - l_3 \sin\alpha}^{y_c + l_3 \sin\alpha} 2l_1 (c_1 + c_2 y) dy$$

$$= 2l_1 [c_1 \cdot 2l_3 \sin\alpha + c_2 \cdot 2 y_c l_3 \sin\alpha] = 2l_1 \cdot 2l_3 \sin\alpha (c_1 + c_2 y_c) = |\diamond| u_h^1 (x_c, y_c).$$

By the same technique, we can prove that $\int_\diamond u_h^2 dxdy = |\diamond| u_h^2 (x_c, y_c)$, which concludes our proof. $\square$

Using the above results, we can prove that the averaged solutions have pointwise superconvergence at parallelogram centers (cf. [154, Theorem 4.3]).

Theorem 5.8. *Let* (x_c, y_c) *be the center of a parallelogram* $\diamond$ *shown in Fig. 5.3, and* $\mathbf{E}_h^m$ *and* H_h^m *be the finite element solution of (5.16)–(5.19) at time level* t_m*. If the* L^2 *estimates of* $\Pi_h \mathbf{E}^m - \mathbf{E}_h^m$ *and* $P_h H^m - H_h^m$ *are almost uniformly distributed over* Ω*, i.e.,*

$$\int_{\diamond} |\Pi_h \mathbf{E}^m - \mathbf{E}_h^m|^2 dxdy \leq \frac{C}{N} \int_{\Omega} |\Pi_h \mathbf{E}^m - \mathbf{E}_h^m|^2 dxdy, \tag{5.46}$$

$$\int_{\diamond} |P_h H^m - H_h^m|^2 dxdy \leq \frac{C}{N} \int_{\Omega} |P_h H^m - H_h^m|^2 dxdy, \tag{5.47}$$

where N *denotes the total number of elements over* Ω*, then we have*

$$\max_{m \geq 1} (|(\mathbf{E}^m - \mathbf{E}_{*h}^m)(x_c, y_c)| + |(H^m - H_{*h}^m)(x_c, y_c)|) \leq C(h^2 + \tau^2),$$

where $\mathbf{E}_{*h}^m$ *and* H_{*h}^m *are the averaged values at the parallelogram centers:*

$$\mathbf{E}_{*h}^m = \frac{1}{2}(\mathbf{E}_h^m|_L + \mathbf{E}_h^m|_R)(x_c, y_c), \quad H_{*h}^m = \frac{1}{2}(H_h^m|_L + H_h^m|_R)(x_c, y_c).$$