Block Preconditioners for an Incompressible Magnetohydrodynamics Problem

Chen Greif, Dan Li, Dominik Schötzau, Michael Wathen

The University of British Columbia

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Problem background

- MHD models electrically conductive fluids (such as liquid metals, plasma, salt water, etc) in an electic field
- Applications: electromagnetic pumping, aluminium electrolysis, the Earth's molten core and solar flares
- MHD models couple electromagnetism (governed by Maxwell's equations) and fluid dynamics (governed by the Navier-Stokes equations)
- Movement of the conductive material that induces and modifies any existing electromagnetic field
- Magnetic and electric fields generate a mechanical force on the fluid

Steady-state Navier-Stokes equations

Incompressible Navier-Stokes Equations:

$$
\begin{aligned}
-\nu\,\Delta u + (u\cdot\nabla)u + \nabla p &= f && \text{in } \Omega,\\
\nabla\cdot u &= 0 && \text{in } \Omega,\\
u &= u_D && \text{on } \partial\Omega
\end{aligned}
$$

where u is the fluids velocity; p is the fluids pressure; f is the body force acting on the fluid and ν the kinematic viscosity.

Steady-state Navier-Stokes equations

Corresponding linear system:

$$\mathcal{K}_{\mathrm{NS}} = \begin{pmatrix} A+0 & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

where $A \in \mathbb{R}^{n_u \times n_u}$ is the discrete Laplacian; $O \in \mathbb{R}^{n_u \times n_u}$ is the convection operator and $B \in \mathbb{R}^{m_u \times n_u}$ is a discrete divergence operator with full row rank.

Note: due to convection term the linear system is non-symmetric

For an extensive discussion of preconditioners we refer to Elman, Silvester and Wathen 2005/2014.

Time-Harmonic Maxwell in mixed form

Maxwell operator in mixed form:

$$
\begin{aligned}
\nabla \times (\nabla \times b) - k^2 b + \nabla r &= g && \text{in } \Omega, \\
\nabla \cdot b &= 0 && \text{in } \Omega, \\
b \times n &= b_D && \text{on } \partial\Omega, \\
r &= 0 && \text{on } \partial\Omega,
\end{aligned}
$$

where b is the magnetic vector field, r is the scalar multiplier and k is the wave number.

Note: for the MHD system $k = 0$

Discretised and linearised Incompressible Navier-Stokes system

$$\begin{pmatrix} M - k^2 X & D^T \\ D & 0 \end{pmatrix} \begin{pmatrix} b \\ r \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix},$$

where $M \in \mathbb{R}^{n_b \times n_b}$ is the discrete curl-curl operator; $X \in \mathbb{R}^{n_b \times n_b}$ the discrete mass matrix; $D \in \mathbb{R}^{m_b \times n_b}$ the discrete divergence operator

Note: M is semidefinate with nulity m_b.

A significant amount of literature on time-harmonic Maxwell: notably for us, work of Hiptmair 1999 and Hiptmair & Xu 2007 for solving shifted curl-curl equations in a fully scalable fashion.

MHD model: coupled Navier-Stokes and Maxwell's equations

$$
\begin{aligned}
-\nu\,\Delta u + (u\cdot\nabla)u + \nabla p - \kappa\,(\nabla\times b)\times b &= f && \text{in } \Omega,\\
\nabla\cdot u &= 0 && \text{in } \Omega,\\
\kappa\nu_m\,\nabla\times(\nabla\times b) + \nabla r - \kappa\,\nabla\times(u\times b) &= g && \text{in } \Omega,\\
\nabla\cdot b &= 0 && \text{in } \Omega,
\end{aligned}
$$

with appropriate boundary conditions.

- $(\nabla\times b)\times b$: Lorentz force accelerates the fluid particles in the direction normal to the electric and magnetic fields.
- $\nabla\times(u\times b)$: electromotive force modifying the magnetic field

Discretisation

- Finite element discretisation based on the formulation in Schötzau 2004
- Fluid variables: lowest order Taylor-Hood ($\mathcal{P}_2/\mathcal{P}_1$)
- Magnetic variables: mixed Nédélec element approximation
- Nédélec elements capture solutions correctly on non-convex domains

Discretised and linearised MHD model:

$$\begin{pmatrix} A+O(u) & B^T & C(b)^T & 0 \\ B & 0 & 0 & 0 \\ -C(b) & 0 & M & D^T \\ 0 & 0 & D & 0 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta p \\ \delta b \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_p \\ r_b \\ r_r \end{pmatrix},$$

with

$$\begin{aligned} r_u &= f - Au - O(u)u - C(b)^T b - B^T p, \\ r_p &= -Bu, \\ r_b &= g - Mu + C(b)b - D^T r, \\ r_r &= -Db. \end{aligned}$$

A: discrete Laplacian operator, O: discrete convection operator,
B: discrete divergence operator, M: discrete curl-curl operator,
C: coupling terms, D: discrete divergence operator.

Decoupling schemes

Magnetic Decoupling (MD):

$$\mathcal{K}_{\mathrm{MD}} = \left(\begin{array}{cc|cc} A + O(u) & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right)$$

Complete Decoupling (CD):

$$\mathcal{K}_{\mathrm{CD}} = \left(\begin{array}{cc|cc} A & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right)$$

Linear solver and Preconditioning

Consider

$$Ax = b,$$

to iteratively solve:

$$\text{find } x_k \in x_0 + \mathrm{span}\{r_0, Ar_0, \ldots, A^{k-1}r_0\}$$

where $r_0 = b - Ax_0$ and x_0 is the initial guess.

Key for success: preconditioning

1. the preconditioner P approximates A
2. P easy to solve for than A

Want eigenvalues of $P^{-1}A$ to be clusters

Ideal preconditioning

Non-singular $(1,1)$ block

$$\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F & B^T \\ 0 & BF^{-1}B^T \end{pmatrix}$$

Murphy, Golub & Wathen 2000 showed exactly two eigenvalues: ± 1

Singular $(1,1)$ block

$$\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F + B^T W^{-1} B & 0 \\ 0 & W \end{pmatrix},$$ where W is SPD

Greif & Schötzau 2006 showed exactly two eigenvalues: ± 1

Navier-Stokes subproblem

$$\mathcal{K}_{\mathrm{NS}} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}$$

where $F = A + O$ is the discrete convection diffusion operator. Shown in Elman, Silvester & Wathen 2005/2014 that

$$\mathcal{P}_{\mathrm{NS}} = \begin{pmatrix} F & B^T \\ 0 & S \end{pmatrix}, \quad S = A_p F_p^{-1} Q_p$$

is a good approximation to the Schur complement preconditioner.
A_p: pressure space Laplacian, F_p: pressure space convection-diffusion operator, Q_p: pressure space mass matrix

Maxwell subproblem

$$\mathcal{K}_{\mathrm{NS}} = \begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix}$$

*N*ote: M is highly rank defficient.

Greif & Schötzau 2007 shows that L (scalar Laplacian) is the appropriate choice for W

$$\mathcal{P}_{\mathrm{iM}} = \begin{pmatrix} M + B^T L^{-1} B & 0 \\ 0 & L \end{pmatrix}$$

Practical preconditioner:

$$\mathcal{P}_{\mathrm{M}} = \begin{pmatrix} M + X & 0 \\ 0 & L \end{pmatrix}$$

where X vector mass matrix is spectrally equivalent to $B^T L^{-1} B$

MHD problem

Combining the Navier-Stokes and Maxwell preconditioners

$$\mathcal{P}_{\mathrm{MH}} = \begin{pmatrix} F & B^T & C^T & 0 \\ 0 & -S & 0 & 0 \\ -C & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

Note: $\mathcal{P}_{\mathrm{MH}}$ remains impractical due to coupling terms. Dual Schur complement approximation for velocity-magnetic unknowns

$$\mathcal{P}_{\mathrm{schurMH}} = \begin{pmatrix} F+N_C & B^T & 0 & 0 \\ 0 & -S & 0 & 0 \\ 0 & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

where $N_C = C^T(M+X)^{-1}C$

Approximation for N_C

Summary of decoupling scheme preconditioners

Iteration scheme	Coefficient matrix	Preconditioner
(MD)	$\left(\begin{array}{cc\|cc} F & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right)$	$\left(\begin{array}{cc\|cc} F & B^T & 0 & 0 \\ 0 & -S & 0 & 0 \\ \hline 0 & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{array}\right)$
(CD)	$\left(\begin{array}{cc\|cc} A & B^T & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & M & D^T \\ 0 & 0 & D & 0 \end{array}\right)$	$\left(\begin{array}{cc\|cc} A & 0 & 0 & 0 \\ 0 & \frac{1}{\nu}Q_p & 0 & 0 \\ \hline 0 & 0 & M+X & 0 \\ 0 & 0 & 0 & L \end{array}\right)$

TABLE: Summary of coefficient matrices and corresponding preconditioners for each the decoupling scheme

Numerical software used

- Finite element software FEniCS: core libraries are the problem-solving interface DOLFIN, the compiler for finite element variational forms FFC, the finite element tabulator FIAT for creating finite element function spaces, the just-in-time compiler Instant, the code generator UFC and the form language UFL.
- Linear algebra software: HYPRE as a multigrid solver and the sparse direct solvers UMFPACK, PASTIX, SuperLU and MUMPS

Future work:

- Scalable inner solvers
- Release code on a public repository
- Parallelisation of the code
- Robustness with respect to kinematic viscosity
- Other non-linear solvers
- Different mixed finite element discretisations