PhD Research Proficiency Exam:

Block Preconditioners for an Incompressible Magnetohydrodynamics Problem

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25^{th} November 2015

Outline

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Problem background

- MHD models electrically conductive fluids (such as liquid metals, plasma, salt water, etc) in an electromagnetic field
- Applications: electromagnetic pumping, aluminium electrolysis, the Earth's molten core and solar flares
- MHD couples electromagnetism (governed by Maxwell's equations) and fluid dynamics (governed by the Navier-Stokes equations)

MHD model: coupled Navier-Stokes and Maxwell's equations

$$
\begin{aligned}
-\nu\,\Delta u + (u\cdot\nabla)u + \nabla p - \kappa\,(\nabla\times b)\times b &= f,\\
\nabla\cdot u &= 0,\\
\kappa\nu_m\,\nabla\times(\nabla\times b) + \nabla r - \kappa\,\nabla\times(u\times b) &= g,\\
\nabla\cdot b &= 0,
\end{aligned}
$$

with appropriate boundary conditions

- $(\nabla\times b)\times b$: Lorentz force accelerates the fluid particles in the direction normal to the electric and magnetic fields
- $\nabla\times(u\times b)$: electromotive force modifying the magnetic field

Outline

Discretisation

- Finite element discretisation based on the formulation in Schötzau 2004
- Fluid variables: Taylor-Hood P2/P1
- Magnetic variables: mixed Nédélec element approximation N1/P1
- Nédélec elements capture solutions correctly on non-convex domains

Discretised and linearised MHD model:

$$\begin{pmatrix} F(u) & B^T & C(b)^T & 0 \\ B & 0 & 0 & 0 \\ -C(b) & 0 & M & D^T \\ 0 & 0 & D & 0 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta p \\ \delta b \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_p \\ r_b \\ r_r \end{pmatrix}$$

with

$$\begin{aligned} r_u &= f - F(u)u - C(b)^T b - B^T p, \\ r_p &= -Bu, \\ r_b &= g - Mu + C(b)b - D^T r, \\ r_r &= -Db \end{aligned}$$

where F is the discrete convection-diffusion operator; B is a discrete divergence operator; M is the discrete curl-curl operator; D the discrete divergence operator and C discrete coupling terms

Discretised and linearised MHD model:

$$\begin{pmatrix} F(u) & B^T & C(b)^T & 0 \\ B & 0 & 0 & 0 \\ -C(b) & 0 & M & D^T \\ 0 & 0 & D & 0 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta p \\ \delta b \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_p \\ r_b \\ r_r \end{pmatrix}$$

Note: We re-order solution vector to (u, b, p, r) to obtain a saddle-point form

$$\left(\begin{array}{cc|cc} F(u) & C(b)^T & B^T & 0 \\ -C(b) & M & 0 & D^T \\ \hline B & 0 & 0 & 0 \\ 0 & D & 0 & 0 \end{array}\right) \begin{pmatrix} \delta u \\ \delta b \\ \delta p \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_b \\ r_p \\ r_r \end{pmatrix}$$

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Preconditioning related work

- Little has been done with respect to a preconditioned iterative solution method
- Phillips, Elman, Cyr, Shadid, and Pawlowski 2014: block preconditioners for an exact penalty formulation, using nodal elements; resulting system is block 3-by-3
- Results show good scalability with respect to the mesh
- Our formulation: Nédélec (edge) elements, giving rise to a richer finite element space, 4-by-4 system, but with a different set of challenges

Ideal preconditioning

Non-singular $(1,1)$ block (as in Navier-Stokes)

$$\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F & B^T \\ 0 & -BF^{-1}B^T \end{pmatrix}$$

Murphy, Golub & Wathen 2000 showed exactly one eigenvalue: 1

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M singular with nullity m (as in time-harmonic Maxwell)

$$\mathcal{K} = \begin{pmatrix} M & D^T \\ D & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} M + D^T W^{-1} D & 0 \\ 0 & W \end{pmatrix}, \text{ where } W \text{ is SPD}$$

Greif & Schötzau 2006 showed exactly two eigenvalues: ± 1

Subproblem preconditioning

Navier-Stokes:
Using PCD from Elman, Silvester & Wathen 2014

$$\mathcal{K}_{\mathrm{NS}} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}, \quad \mathcal{P}_{\mathrm{NS}} = \begin{pmatrix} F & B^T \\ 0 & -S \end{pmatrix}, \quad S = A_p F_p^{-1} Q_p$$

Mixed-Maxwell:
Using augmentation technique from Greif & Schötzau 2007

$$\mathcal{K}_{\mathrm{M}} = \begin{pmatrix} M & D^T \\ D & 0 \end{pmatrix}, \quad \mathcal{P}_{\mathrm{M}} = \begin{pmatrix} M + X & 0 \\ 0 & L \end{pmatrix}$$

MHD problem

Combining the Navier-Stokes and Maxwell preconditioners

$$\mathcal{P}_{\mathrm{MH}} = \begin{array}{cl} \begin{pmatrix} F & C^T & B^T & 0 \\ -C & M+X & 0 & 0 \\ 0 & 0 & -S & 0 \\ 0 & 0 & 0 & L \end{pmatrix} & \begin{array}{l} \}n_u \text{ rows} \\ \}n_b \text{ rows} \\ \}m_u \text{ rows} \\ \}m_b \text{ rows} \end{array} \end{array}$$

Note: $\mathcal{P}_{\mathrm{MH}}$ remains challenging to solve due to coupling terms.
Schur complement approximation for velocity-magnetic unknowns

$$\mathcal{P}_{\mathrm{schurMH}} = \begin{pmatrix} F+M_C & C^T & B^T & 0 \\ 0 & M+X & 0 & 0 \\ 0 & 0 & -S & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

where $M_C = C^T(M+X)^{-1}C$

Spectral analysis (ideal preconditioner)

Note: using $X = D^T L^{-1} D$ for eigenvalue analysis

Theorem

The matrix $\mathcal{P}_{\mathrm{MH}}^{-1}\mathcal{K}_{\mathrm{MH}}$ has an eigenvalue $\lambda = 1$ with algebraic multiplicity of (at least) $n_u + n_b$ and an eigenvalue $\lambda = -1$ with algebraic multiplicity of (at least) m_b.

Theorem

The matrix $\mathcal{P}_{\mathrm{schurMH}}^{-1}\mathcal{K}_{\mathrm{MH}}$ has an eigenvalue $\lambda = 1$ with algebraic multiplicity of (at least) $n_b + n_c$ where n_c is the dimension of the nullspace of C and an eigenvalue $\lambda = -1$ with algebraic multiplicity of (at least) m_b.

$n_u = \dim(u),\ n_b = \dim(b),\ m_b = \dim(r),\ n_c = \dim(\mathrm{null}(C))$

Eigenvalue distribution

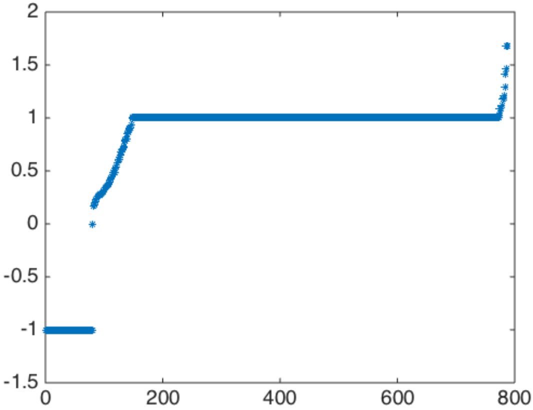


Figure: Real part of eigenvalues of preconditioned matrix $\mathcal{P}_{\mathrm{schurMH}}^{-1}\mathcal{K}_{\mathrm{MH}}$ (imaginary parts small)

Approximation for $M_C = C^T(M+X)^{-1}C$

- Want to use identity

$$\nabla \times \Delta^{-1} \nabla \times g = -g$$

proved in Phillips, Elman, Cyr, Shadid & Pawlowski 2014

- Challenge with our formulation is the shifted curl-curl operator
- We approximate the Laplacian by a shifted curl-curl to yield

$$M_C \approx Q_s = \kappa {\nu_m}^{-1} b \times (u \times b)$$

Approximation for $M_C = C^T(M+X)^{-1}C$

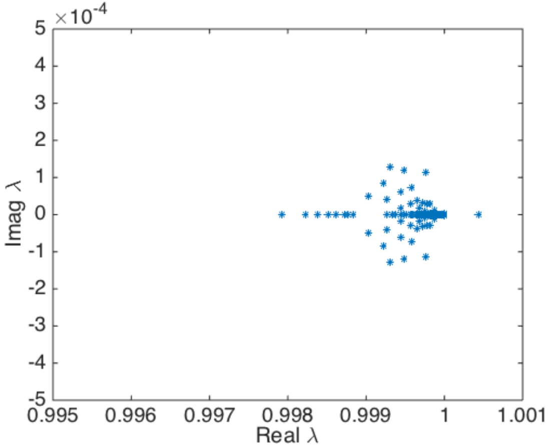


Figure: Eigenvalues of preconditioned matrix $(F+Q_S)^{-1}(F+M_C)$

Practical preconditioner

After all approximations the preconditioner is:

$$\mathcal{P} = \begin{pmatrix} F + Q_S & C^T & B^T & 0 \\ 0 & M + X & 0 & 0 \\ 0 & 0 & -A_p F_p^{-1} Q_p & 0 \\ 0 & 0 & 0 & L \end{pmatrix}$$

Outline

Numerical setup

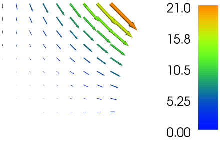
Software:

- Finite element software FEniCS:
- Linear algebra software:
  - PETSc linear algebra wrapper features
  - HYPRE as a multigrid solver
  - MUMPS sparse direct solver

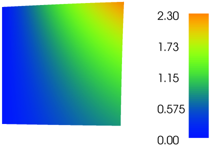
Scalable inner solvers:

- Fluid matrices: AMG from HYPRE
- Magnetic matrices: Auxiliary Space Preconditioner from Hiptmair & Xu 2007

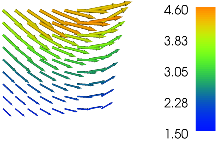
2D: smooth solution



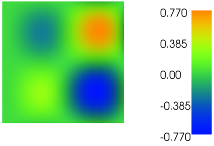
(a) Velocity Solution



(b) Pressure Solution



(c) Magnetic Solution



(d) Multiplier Solution

2D: smooth solution

ℓ	DoF	$\text{time}_{\text{solve}}$	time_{NL}	it_{NL}	$\text{it}_{\text{av}}^{\text{D}}$
4	3,556	0.33	2.7	7	20.1
5	13,764	1.11	9.2	7	20.4
6	54,148	4.48	37.2	7	20.9
7	214,788	20.32	166.4	7	21.4
8	855,556	94.29	762.0	7	21.8
9	3,415,044	486.53	3835.0	7	-
10	13,645,828	2231.71	17944.6	7	-

Table: 2D smooth: Number of nonlinear iterations and number of iterations to solve the MHD system with Tol = 1e-4, $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$

2D: smooth solution

ℓ	DoF	$\text{time}_{\text{solve}}$	time_{NL}	it_{NL}	$\text{it}_{\text{av}}^{\text{D}}$	$\text{it}_{\text{av}}^{\text{I}}$
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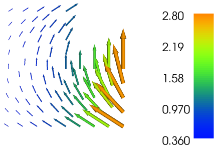
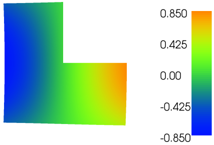
5	13,764	1.11	9.2	7	20.4	25.9
6	54,148	4.48	37.2	7	20.9	27.1
7	214,788	20.32	166.4	7	21.4	28.4
8	855,556	94.29	762.0	7	21.8	31.3
9	3,415,044	486.53	3835.0	7	-	34.3
10	13,645,828	2231.71	17944.6	7	-	34.0

Table: 2D smooth: Number of nonlinear iterations and number of iterations to solve the MHD system with Tol = 1e-4, $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$

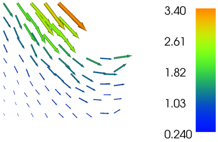
2D: smooth solution on L-shaped domain



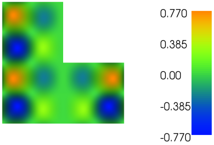
(e) Velocity Solution



(f) Pressure Solution



(g) Magnetic Solution



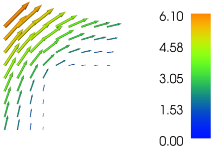
(h) Multiplier Solution

2D: smooth solution on an L-shaped domain

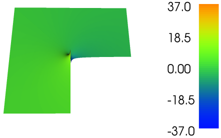
ℓ	DoF	$\mathrm{it}_{\mathrm{NL}}$	$\mathrm{it}_{\mathrm{av}}^{\mathrm{D}}$
5	12,880	5	24.4
6	51,678	5	26.0
7	203,712	5	27.4
8	809,705	5	29.6
9	3,219,082	-	-

Table: 2D unstructured L-shaped: Number of nonlinear iterations and number of iterations to solve the MHD system with Tol = 1e-4, $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$. The iteration was terminated before completion for $\ell = 9$ due to the computation reaching the prescribed time limit

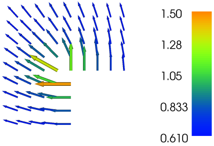
2D: singular solution on an L-shaped domain



(i) Velocity Solution



(j) Pressure Solution



(k) Magnetic Solution

2D: singular solution on an L-shaped domain

ℓ	DoF	$\mathrm{it}_{\mathrm{NL}}$	$\mathrm{it}_{\mathrm{av}}^{\mathrm{D}}$
4	2,724	4	14.5
5	10,436	4	15.8
6	40,836	4	17.5
7	161,540	4	18.5
8	642,564	4	20.0
9	2,563,076	4	21.8

Table: 2D singular solution on L-shaped: Number of nonlinear iterations and number of iterations to solve the MHD system with Tol = 1e-4, $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$

3D: smooth solution

ℓ	DoF	$\mathrm{time}_{\mathrm{solve}}$	$\mathrm{time}_{\mathrm{NL}}$	$\mathrm{it}_{\mathrm{NL}}$	$\mathrm{it}_{\mathrm{av}}^{\mathrm{D}}$
1	527	0.03	0.9	4	18.0
2	3,041	0.22	3.5	3	22.3
3	20,381	1.77	26.6	3	24.7
4	148,661	22.11	237.0	3	26.0
5	1,134,437	206.43	2032.7	3	-
6	8,861,381	2274.28	19662.0	3	-

Table: 3D smooth: Number of nonlinear iterations and number of iterations to solve the MHD system with Tol = 1e-4, $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$

3D: smooth solution

ℓ	DoF	$\mathrm{time}_{\mathrm{solve}}$	$\mathrm{time}_{\mathrm{NL}}$	$\mathrm{it}_{\mathrm{NL}}$	$\mathrm{it}_{\mathrm{av}}^{\mathrm{D}}$	$\mathrm{it}_{\mathrm{av}}^{\mathrm{I}}$
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4	148,661	22.11	237.0	3	26.0	40.7
5	1,134,437	206.43	2032.7	3	-	44.3
6	8,861,381	2274.28	19662.0	3	-	50.0

Table: 3D smooth: Number of nonlinear iterations and number of iterations to solve the MHD system with Tol = 1e-4, $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$

Note: things not quite as good in terms of scalability

Outline

Future work

- Further develop code and release on a public repository
  - Black-box solvers for fluid/magnetic flows
  - Several PDE examples built in
- Scalability for three-dimensional problems
- Other non-linear solvers
- Different mixed finite element discretisations

Paper

- Currently being written up
- Preconditioners for Mixed Finite Element Discretizations of Incompressible MHD Equations, Wathen, Greif, Schötzau submitting early 2016

References

H. C. Elman, D. J. Silvester and A. J. Wathen
Finite Elements and Fast Iterative Solvers: with Applications in Incompressible Fluid Dynamics.
Oxford University Press 2014

C. Greif and D. Schötzau
Preconditioners for saddle point linear systems with highly singular (1, 1) blocks.
Electronic Transactions on Numerical Analysis, Special Volume on Saddle Point Problems 2006

C. Greif and D. Schötzau
Preconditioners for the discretized time-harmonic Maxwell equations in mixed form.
Numerical Linear Algebra with Applications 2007

R. Hiptmair and J. Xu
Nodal auxiliary space preconditioning in $H(curl)$ and $H(div)$ spaces.
SIAM Journal on Numerical Analysis 2007

M. F. Murphy, G. H. Golub and A. J. Wathen
A note on preconditioning for indefinite linear systems.
SIAM Journal on Scientific Computing 2000

E. G. Phillips, H. C. Elman, E. C. Cyr, J. N. Shadid and R. P. Pawlowski
A Block Preconditioner for an Exact Penalty Formulation for Stationary MHD.
SIAM Journal on Scientific Computing 2014

D. Schötzau
Mixed finite element methods for stationary incompressible magneto–hydrodynamics.
Numerische Mathematik 2004