CPSC 517, Fall 2013

Assignment 2, due Thursday November 21st (soft deadline)

0. Hand in a proposal, approximately half a page to a page long, describing briefly what you are planning to do for your course project. If unsure, please come and talk with me. Check out the Project Guidelines document posted on the course webpage.

1. Arnoldi gives us $AQ_k = Q_{k+1}H_{k+1,k}$. Suppose we carry out the algorithm and at step j we get $h_{j+1,j} = 0$. As a result the algorithm is terminated to avoid division by zero. (Check out the "Arnoldi/Lanczos" pdf file on the course webpage.)
   (a) Explain when this could happen in general, that is, for the Arnoldi procedure with a given initial guess, not necessarily connected to the solution of a linear system or an eigenvalue problem.
   (b) If Arnoldi is incorporated into GMRES for solving $Ax = b$ when this happens, what can you say about how close is the iterate x_j to the actual solution of the linear system? Ignore roundoff errors.
   (c) Determine the computational cost of a k-step Arnoldi procedure for an $n \times n$ matrix A, with an average of m nonzero entries per row, where m is small and does not depend on the size of the matrix. State this in terms of overall number of floating point operations, depending on m, n and k. Try to be as precise as possible in terms of the operations count, but if it is difficult, then at the very least provide a 'Big O' estimate.
   (d) Suppose GMRES is applied to $Ax = b$ with a zero initial guess and by some kind of an amazing coincidence, the right hand side b happens to be an eigenvector of A. What can we say about the number of iterations GMRES will take to converge to the solution? You may ignore roundoff errors.

2. Write a Lanczos procedure and apply it to the discrete Laplacian. Compute the eigenvalues of $T_{k,k}$ (namely, the Ritz values) for a few values of n and a few values of $k \ll n$. Now, compute the eigenvalues of A. Compare the three largest Ritz values and the three smallest Ritz values to the three largest and three smallest eigenvalues of A, respectively. Make a few observations on the quality of your results. In particular, comment on the difference between convergence for the largest and the smallest values, and on how increasing k for a fixed n affects accuracy. Feel free to choose values of k and n that make sense to you.

3. Work on this question is intended to be done primarily by hand, but you may write code if you prefer (by this what is meant a self-contained piece of code, not the use of a built-in MATLAB command). Consider a matrix whose nonzero pattern is given as follows:

$$A = \begin{pmatrix}
1 & & & & \times & & \times & \times & & \\
 & 2 & \times & & & & & & \times & \\
 & \times & 3 & & & & & & & \times \\
 & & & 4 & & \times & & & \times & \\
\times & & & & 5 & & & \times & & \times \\
 & & & \times & & 6 & & & \times & \\
\times & & & & & & 7 & & & \\
\times & & & & \times & & & 8 & \times & \times \\
 & \times & & \times & & \times & & \times & 9 & \times \\
 & & \times & & \times & & & \times & \times & 10
\end{pmatrix}.$$

(a) Write down the adjacency graph of A.

(b) Find the permutation vector and the nonzero pattern of the matrix corresponding to the RCM ordering, using node 7 as a starting node. Find the nonzero pattern of the LU factorization of this matrix and determine how many nonzero entries $L + U$ has.

(c) Write down the elimination graphs, quotient graphs, and sparsity patterns of the matrices that arise in the process of minimum degree. You may use any tie breaking strategy you wish. Find the nonzero pattern of the LU factorization of this matrix and determine how many nonzero entries $L + U$ has.

(d) **(Bonus)** Determine the ordering in the approximate minimum degree algorithm. Note that this entails computing approximate external degrees, which could be labour intensive (here you may want to resort to writing code). Find the nonzero pattern of the LU factorization of this matrix and determine how many nonzero entries $L + U$ has.

4. Let A be a real matrix and assume we have calculated an eigenvector $\vec{x}_1$ associated with a real eigenvalue, λ_1. The eigenvector, $\vec{x}_1$ is normalized so that its first entry is equal to 1. Now, consider the matrix $A^{(1)} = A - \vec{x}_1 \vec{a}_1^T$ where the first row of A is $\vec{a}_1^T$. Suppose $A^{(1)}\vec{y}_j = \mu_j \vec{y}_j$.

   (a) Show that $\mu_1 = 0$ and $\mu_j = \lambda_j \qquad \text{for } j = 2, ..., n.$

   (b) Describe the relationship between the eigenvectors of A and $A^{(1)}$.

   (c) **(Hard)** Now suppose A is real, but $\lambda_1 = \alpha + \imath\beta$, $\lambda_2 = \alpha - \imath\beta$, and $\vec{x}_1 = \vec{p} + \imath\vec{q}$, $\vec{x}_2 = \vec{p} - \imath\vec{q}$. Here $\imath = \sqrt{-1}$. Show how to construct a real matrix, $A^{(2)}$, so that the eigenvalues and eigenvectors are related to A in a fashion similar to parts (a) and (b).

   (d) Explain how the above procedure is useful for eigenvalue computations.