October 10, 2013

Preconditioning:

$r_k = p_k(A) r_0$, $p_k(0) = 1$

want $p_k(\lambda_i(A))$ small in magnitude

GMRES, basic form:

$x_k = x_0 + Q_k z_k$

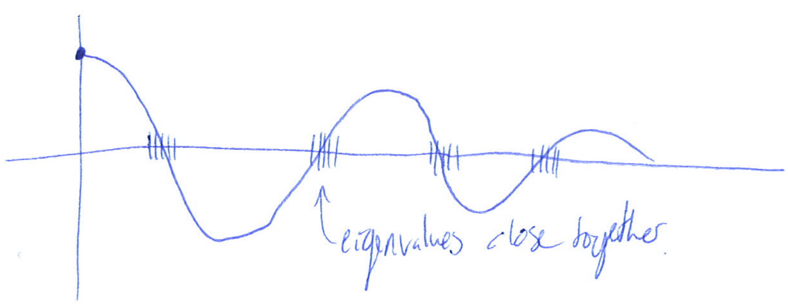
where $z_k = \underset{z}{\arg\min} \|\beta e_1 - H_{k+1,k} z\|_2$ $\qquad \beta = \|r_0\|_2$

So we use a preconditioner M

$M^{-1}Ax = M^{-1}b$ left preconditioning (could also $AM^{-1}u = b$, $x = M^{-1}u$ right preconditioning)

hope: $p_k(M^{-1}A)$ or $p_k(AM^{-1})$ is "good"

$p_k(\lambda_i(AM^{-1}))$ or $p_k(\lambda_i(M^{-1}A))$ small for all λ_i



left + right preconditioning

l: $M^{-1}Ax = M^{-1}b$

r: $AM^{-1}u = b$, $x = M^{-1}u$

- Notice that the corresponding preconditioned Krylov subspaces are a bit different

right : span$\{b, AM^{-1}b, \ldots\}$

left : span$\{M^{-1}b, (M^{-1}A)M^{-1}b$

doesn't make a huge difference

M is $n \times n$

M should be <u>much</u> easier to invert than A, but still $M^{-1} \simeq A^{-1}$ in a certain way ... (clustering or less important, good condition number of $M^{-1}A$)

Note: instead of a matrix-vector product w/ A originally, now we have:

mat-vec prod

plus need to solve $Mz_k = s_k$

mat-vec prod w/ $M^{-1}A$

↳ given y, need $M^{-1}\underbrace{Ay}_{s} = z$

* Compute $s = Ay$
* solve $Mz = s$.

Also have split preconditioning: $\overbrace{M_1^{-1} A M_2^{-1}}^{\text{pre-conditioned matrix}} \overbrace{(M_2 x)}^{u} = \overbrace{M_1^{-1} b}^{\tilde{b}}$

$M = M_1 M_2$ ← in factored form

eg: M is SPD

$M = FF^T$ (Cholesky)

Split preconditioning + CG:

$Ax = b$

A is SPD ; need M SPD

recall CG: Lanczos

$T_{kk} z = \beta e_1$

$x_k = x_0 + Q_k z$.

PCG: $M^{-1}A$ is not even symmetric

but if M is SPD: $M = Q \Lambda Q^T$

$M^{1/2} = Q \Lambda^{1/2} Q^T$ $\qquad$ $M^{1/2} M^{1/2} = Q\Lambda^{1/2}\Lambda^{1/2}Q^T = Q\Lambda Q^T = M$

$(M^{-1/2} A M^{-1/2})(M^{1/2} x) = F^{-1/2} b$

if $M = FF^T$

$$x^T F^{-T} A F^{-1} x = (F^{-1}x)^T A (F^{-1}x)$$
$$= y^T A y$$
$$> 0.$$

if A is not symmetric

$Ax = b$

$A^T A x = A^T b$

then $A^T A$ is SPD if A is nonsingular.

CGLS

but if the condition number of A is large $\Rightarrow$ the condition number of $A^T A$ squares it.

$A^T A \rightarrow$ apply CG only if A is well-conditioned

↳ A: nonsymmetric

<u>Preconditioners</u>:

we want $M^{-1} \approx A^{-1}$

but much easier to invert

Simple preconditioners:

$M = I$ or αI : Richardson

$M = \mathrm{diag}(A)$: Jacobi

$M = \mathrm{tril}(A)$: Gauss Seidel

Incomplete LU:

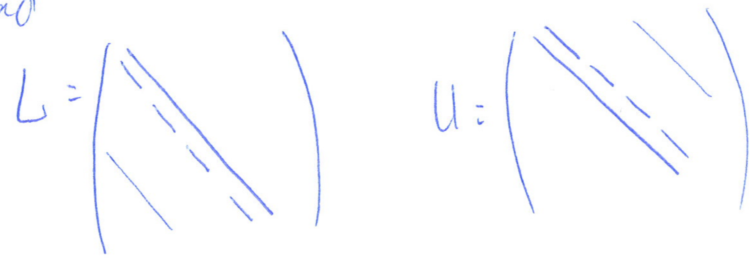
$A \approx LU$

Static pattern ILU } find sparse L, U
dynamic pattern ILU } s.t. $A = LU$ in some places, $A \approx LU$ in some places, $A \neq LU$ in places we don't care about...

$A = LU$:

$A =$ 

instead



Impose: $A = LU$ only for $\{(i,j)\}$ for which $A_{ij} \neq 0$