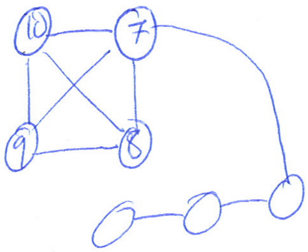


$$\binom{6}{2} = \frac{6\cdot 5}{2} = 15 \quad \text{degree}$$

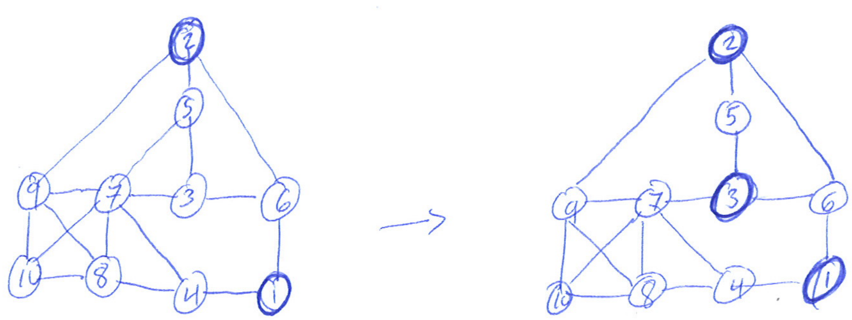
$$\binom{3}{2} = \frac{3\cdot 2}{2} = 3 \quad \text{external degree}$$

$\{7, 8, 9, 10\}$ → super variable (they are a clique, can treat them as one thing).

Transition from G^2 to G^3

node 3 is selected as pivot

~~elimination graph~~
quotient graph



- $L_3 = A_3 = \{5, 6, 7\}$

represents pairwise adjacency of variables 5,6,7

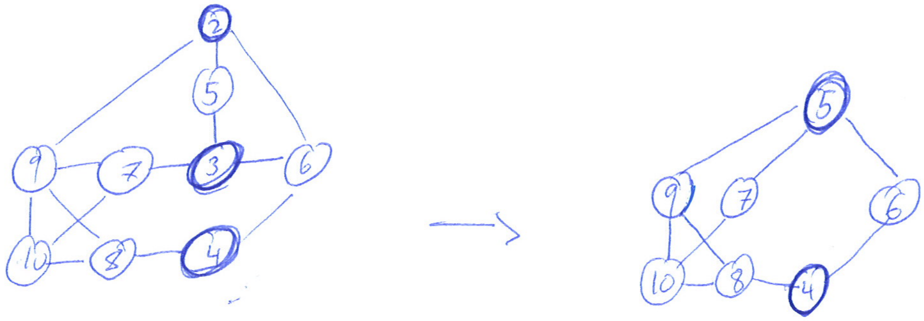
(5,7) is now redundant, remove from A_5, A_7

* A: connections b/wn variables

* L: connections from element.

Transforming from G^4 to G^5

- variable 5 is chosen as a pivot



A_5 is empty

$\mathcal{E}_5 = \{2,3\}$

$$L_5 = (A_5 \cup L_2 \cup L_3) \setminus \{5\}$$
$$= \phi \cup \{5,6,9\} \cup \{5,6,7\} \setminus \{5\}$$
$$= \{6,7,9\}$$

- The pair $(7,9)$ is now redundant, so eliminate from A_7, A_9
- can also eliminate elements 2+3 b/c they don't add information

In G^4:

$A_6 = \phi$	$\mathcal{E}_6 = \{2,3,4\}$
$A_7 = \{9,10\}$	$\mathcal{E}_7 = \{3,4\}$
$A_9 = \{7,8,10\}$	$\mathcal{E}_9 = \{2\}$

In G^5:

$A_6 = \phi$	$\mathcal{E}_6 = \{4,5\}$	
$A_7 = \{10\}$	$\mathcal{E}_7 = \{4,5\}$	$L_5 = \{6,7,9\}$
$A_9 = \{8,10\}$	$\mathcal{E}_9 = \{5\}$	

Approximate Minimum Degree

Approximate degree of node 6:
(in G^4)

$L_4 = \{6, 7, 8\}$

$L_2 = \{5, 6, 9\}$

$L_3 = \{5, 6, 7\}$

$\bar{d}_6^4 = |A_6 \setminus \{6\}| + |L_4 \setminus \{6\}| + |L_2 \setminus L_4| + |L_3 \setminus L_4|$

allow common guys btwn L_2 + L_3

$= \phi \setminus \{6\} + |\{6,7,8\} \setminus \{6\}| + |\{5,6,9\} \setminus \{6,7,8\}| + |\{5,6,7\} \setminus \{6,7,8\}|$

$= \quad 2 + 2 + 1$

$= 5$

The actual degree of 6 is

$d_6^4 = 4$

Eigenvalues:

November 12, 2013

$Ax = b$ love

$Ax = \lambda x$ peace

for linear systems

$Ax = b$

↳ direct — compute in finite number of operations

↳ iterative - could be infinite

now → infinite

$Ax = \lambda x$ ($Ax = \lambda x$: right eigenvector, $y^* A = \lambda y^*$: left eigenvector)

↳ direct — all of them are iterative, but direct are decompositional (QR iteration)

↳ iterative — iterative are mat-vec product based (Lanczos/Arnoldi, Jacobi/Davidson)

$AX = X\Lambda$

where

X : matrix w/ eigenvectors in its columns

Λ : diagonal

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$A = X \Lambda X^{-1}$

- diagonalizable

$A = Q \Lambda Q^*$ ($AA^* = A^*A$, normal - symm. included)

- unitarily diagonalizable

non-diagonalizable

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$ $n = 10$, e.g.

eigenvectors? → 1 eigenvector of geometric multiplicity 1

$x = e_1$

$\lambda = 0$ is an eigenvalue w/ algebraic multiplicity n

$$\det(\lambda I - A) = 0 = \lambda^n$$

but now, if

$$A_\varepsilon = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ \varepsilon & & & 0 \end{pmatrix}$$

$$\det(\lambda I - A_\varepsilon) = 0$$

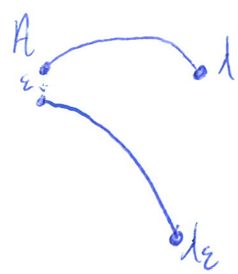
$$\lambda^n - \varepsilon = 0$$

$$|\lambda| = \sqrt[n]{\varepsilon}$$

if $\varepsilon = 10^{-10}$

$$|\lambda_i| = 0.1$$

Small perturbation in the matrix $\to$ huge perturbation in the solution



$$Ax = \lambda x \quad , \quad y^* A = \lambda y^*$$

$$(A + \delta A)(x + \delta x) = (\lambda + \delta\lambda)(x + \delta x) \qquad \text{where } \|\delta A\| \approx \varepsilon \|A\|$$

$$Ax + \delta A x + A\delta x + \delta A \delta x = \lambda x + \delta\lambda\, x + \lambda \delta x + \delta\lambda\, \delta x$$

assume we can ignore second order terms

estimate $\delta\lambda$ ~~and~~

$$\delta A x + A \delta x = \delta\lambda\, x + \lambda \delta x$$

mult. by y^*

$$y^* \delta A x + y^* A \delta x + y^* \delta\lambda\, x + y^* \lambda \delta x$$

$$\boxed{\delta\lambda = \frac{y^* \delta A x}{y^* x}}$$

If y^*, x orthogonal $\to$ blows up.

$\frac{1}{y^* x}$: condition number of the eigenvalue problem

If A is symmetric

$\delta\lambda \sim \varepsilon \|A\|$

since

$$\|\delta\lambda\| \leq \frac{\|y\| \, \|\delta A\| \, \|x\|}{\|y^* x\|} \lesssim \varepsilon \|A\|$$

Forward + Backward error

ex:

$\sqrt{2} \rightarrow 1.4 \quad \rightarrow 1.4 = \sqrt{1.96}$

($x^2 - 2 = 0$: newton's)

Forward error: $|1.4 - 1.414\ldots|$

Backward error: $|2 - 1.96|$

Power method:

- orthogonal / simultaneous iteration (QR it).
- Lanczos / Arnoldi
- gives us the dominant eigenpair

$$Ax_i = \lambda_i x_i \qquad |\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

V: initial guess

$$V = \sum_{i=1}^{n} \alpha_i x_i$$

$$A^k v = \sum_{i=1}^{n} \alpha_i \underbrace{A^k x_i}_{\lambda_i^k x_i} = \alpha_1 \lambda_1^k x_1 + \sum_{i=2}^{n} \alpha_i (\lambda_i^k x_i)$$

$$= \lambda_1^k \left(\alpha_1 x_1 + \sum_{i=2}^{n} \alpha_i \left(\frac{\lambda_i}{\lambda_1}\right)^k x_i\right)$$

$\left(\frac{\lambda_i}{\lambda_1}\right)^k \to 0$ as $k \to \infty$

initial v_0 $i = 1, \ldots$

repeat

$v_i = Av_{i-1}$

$v_i \leftarrow v_i / \|v_i\|$

eventually

$v_i \rightarrow x_1$

eigenvalue

$$r(v_i) = \frac{v_i^T A v_i}{v_i^T v_i}$$ Rayleigh Quotient

best estimate for an eigenvalue in the L-S sense: given x

$$\min_{\alpha} \| Ax - \lambda x \|_2$$

$$= \min \| c - x\alpha \|_2$$

where $c = Ax$ given

(recall $\min \|b - Ax\|$ NE $A^TAx = A^Tb$)

so the normal equations are

$$x^Tc = x^Tx\alpha$$

$$\Rightarrow \alpha = \frac{x^Tc}{x^Tx} = \frac{x^TAx}{x^Tx}$$

Terribly slow method!

↳ converges linearly - proportional to λ_2/λ_1

↳ if initial guess has no component in the direction of x_1

but → round-off errors save the day!

Shift & invert (inverse power method)

$(A - \alpha I)^{-1}$ $A: \lambda_1, \dots, \lambda_n$

- need to solve a system
- need a good way to guess α

eigenvalues of

$$\frac{1}{\lambda_1 - \alpha}, \frac{1}{\lambda_2 - \alpha}, \dots\dots, \frac{1}{\lambda_n - \alpha}$$

apply power method on $(A - \alpha I)^{-1}$

say $\alpha \approx \lambda_2$

$$\frac{1}{\lambda_2 - \alpha} \gg \frac{1}{\lambda_i - \alpha} \quad i \neq 2$$