Thm Hints: October 3, 2013

Block Jacobi: try not to invert anything

$M^{-1}N$

try to think of individual blocks

Neumann boundary conditions:

approx to 2nd deriv

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2)$$

recommended to use a second order approx also for first deriv

$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$ A SPD, pos + neg: Think Schur complements $BA^{-1}B^T$

Reform Block G.E. $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} = LDL^T$

2x2 block matrices.

## 1D Projection:

MR, SD

$x \in x_0 + Vy$ $\quad$ V is a basis for K - search space

L constraint space

if $L = K$, A SPD $\rightarrow$ CG $\|e_k\|_A$

$L = AK$, A nonsingular $\rightarrow$ MINRES, GMRES $\|r_k\|_2$

W is a basis for L

$W^TAV$ needs to be inverted

$$\boxed{x = x_0 + V(W^TAV)^{-1}W^Tr_0}$$

## Krylov Subspace Methods:

$K^k(A, r_0) = \text{Span}\{r_0, Ar_0, \ldots, A^{k-1}r_0\}$

① optimality condition

② forming a basis for $K^k(A, r_0)$

③ preconditioning

focus on ②

↳ suppose $V = [r_0, Ar_0, \dots, A^{k-1} r_0]$

bad basis b/c $A^k r_0$ large $k$ is similar to, say, $A^{k-1} r_0$

(Precisely why power method is good - later)

• Want a good basis

orthogonal

QR decomposition

w.l.o.g.: assume $x_0 = 0$ (if not, then solve $Ay = b - Ax_0$ w/ $y_0 = 0$)

so: $r_0 = b$, $x_0 + K^k(A, r_0) \longrightarrow K^k(A, b)$

$$u_j = A^{j-1} r_0 = A^{j-1} b$$

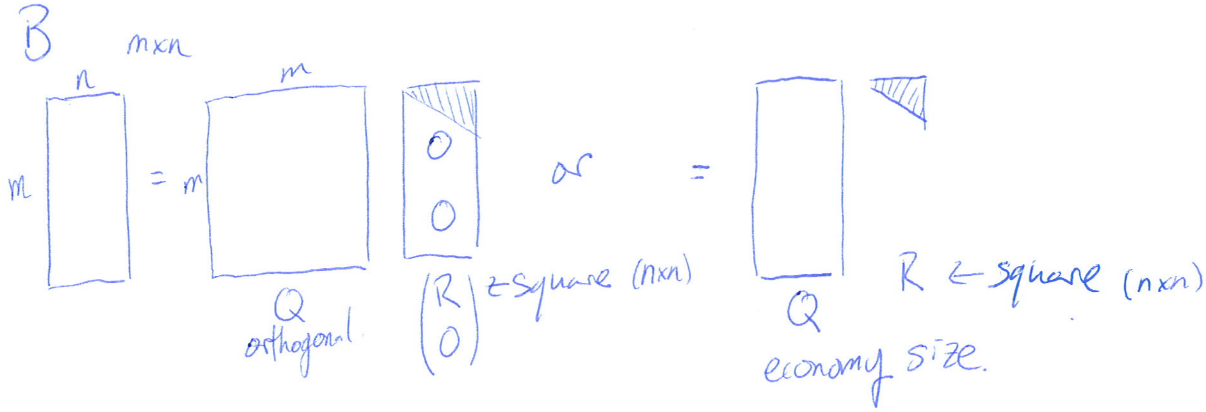
$$U_k = [u_1, u_2, \dots u_k]$$

$$AU_k = [u_2, \dots, u_{k+1}] = U_k \cdot B_k + u_{k+1} e_k^T$$

$$B_k = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \ddots & \vdots \\ 0 & 1 & 0 \end{pmatrix}$$

$$\underbrace{Au_j}_{A \cdot A^{j-1} r_0} = \underbrace{u_{j+1}}_{A^j r_0}$$

QR factorization:



Three ways to find QR

Grahm-Schmidt

Householder

Gibbons Rotations

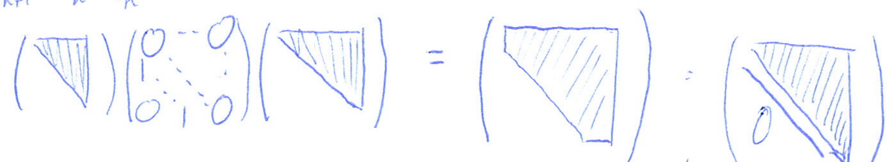
$U_k = Q_k R_k$

$A Q_k R_k = Q_k R_k B_k + u_{k+1} e_k^T$

mult. by $R_k^{-1}$ on the right

$A Q_k = Q_k \underset{\sim}{C_k} + u_{k+1} e_k^T R_k^{-1}$

$R_k B_k R_k^{-1}$ :



$= H_k$

upper triangular + 1

upper Hessenberg

$A Q_k = Q_k H_k +$ correction $\leftarrow$ 1 column @ k-th position

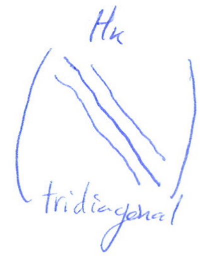
$= Q_{k+1} H_{k+1,k}$

if $A$ is symmetric $Q_k^T A Q_k = H_k$

$H_k^T = (Q_k^T A Q_k)^T = Q_k^T A^T Q_k$

$= Q_k^T A Q_k$

$= H_k$




JDI:

$A Q_k = \ldots$

column-by column : Arnoldi-Process

$A q_1 = (q_1 \; q_2) \begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} = h_{11} q_1 + h_{21} q_2 \leftarrow$ 4 unknowns $q_1, q_2, h_{11}, h_{21}$

$\leftarrow$ orthogonality:

$q_1^T q_1 = 1$

$q_2^T q_1 = 0$

$q_2^T q_2 = 0$

$A (q_1 \; q_2) = (q_1 \; q_2 \; q_3) \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ 0 & h_{32} \end{pmatrix}$

Start taking inner products

$q_1^T A q_1 = h_{11} q_1^T q_1 + h_{21} q_1^T q_2$

$= h_{11}$

what is $q_1$? $q_1$ spans $r_0$

$\{q_1, q_2\}$ span $\{r_0, A r_0\}$ $\Rightarrow q_1 = \frac{r_0}{\|r_0\|}$

$$h_{21} q_2 = Aq_1 - h_{11} q_1$$

$$h_{21} = \| Aq_1 - h_{11} q_1 \|$$