CHAPTER 1

Common and Unusual Finite Elements

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This chapter provides a glimpse of the considerable range of finite elements in the literature and the challenges that may be involved with automating "all" the elements. Many of the elements presented here are included in the FEniCS project already; some are future work.

1.1 Ciarlet's Finite Element Definition

As discussed in Chapter **??**, a finite element is defined by a triple $(K, \mathcal{P}_K, \mathcal{L}_K)$, where

- $K \subset \mathbb{R}^d$ is a bounded closed subset of $\mathbb{R}^d$ with nonempty interior and piecewise smooth boundary;
- $\mathcal{P}_K$ is a function space on K of dimension $n_K < \infty$;
- $\mathcal{L}_K = \{\ell_1^K, \ell_2^K, \ldots, \ell_{n_K}^K\}$ is a basis for $\mathcal{P}_K'$ (the bounded linear functionals on $\mathcal{P}_K$).

This definition was first introduced by Ciarlet in a set of lecture notes [Cia75] and became popular after his 1978 book [Cia78, Cia02]. It remains the standard definition today, see for example [BS08]. Similar ideas were introduced earlier in [CR72] which discusses unisolvence of a set of interpolation points $\Sigma = \{a_i\}_i$. This is closely related to the unisolvence of $\mathcal{L}_K$. In fact, the set of functionals $\mathcal{L}_K$ is given by $\ell_i^K(v) = v(a_i)$. It is also interesting to note that the Ciarlet triple was originally written as (K, P, Σ) with Σ denoting $\mathcal{L}_K$. Conditions for uniquely determining a polynomial based on interpolation of function values and derivatives at a set of points was also discussed in [BZ70], although the term unisolvence was not used.

1.2 Notation

It is common to refer to the space of linear functionals $\mathcal{L}_K$ as the *degrees of freedom* of the element $(K, \mathcal{P}_K, \mathcal{L}_K)$. The degrees of freedom are typically given by point evaluation or moments of function values or derivatives. Other commonly used degrees of freedom are point evaluation or moments of certain components of function values, such as normal or tangential components, but also directional derivatives. We summarize the notation used to indicate degrees of freedom graphically in Figure 1.1. A filled circle at a point $\bar{x}$ denotes point evaluation at that point,

$$\ell(v) = v(\bar{x}).$$

We note that for a vector valued function v with d components, a filled circle denotes evaluation of all components and thus corresponds to d degrees of freedom,

$$\begin{aligned} \ell_1(v) &= v_1(\bar{x}), \\ \ell_2(v) &= v_2(\bar{x}), \\ \ell_3(v) &= v_3(\bar{x}). \end{aligned}$$

An arrow denotes evaluation of a component of a function value in a given direction, such as a normal component $\ell(v) = v(\bar{x}) \cdot n$ or tangential component $\ell(v) = v(\bar{x}) \cdot t$. A plain circle denotes evaluation of all first derivatives, a line denotes evaluation of a directional first derivative such as a normal derivative $\ell(v) = \nabla v(\bar{x}) \cdot n$. A dotted circle denotes evaluation of all second derivatives. Finally, a circle with a number indicates a number of interior moments (integration against functions over the domain K).

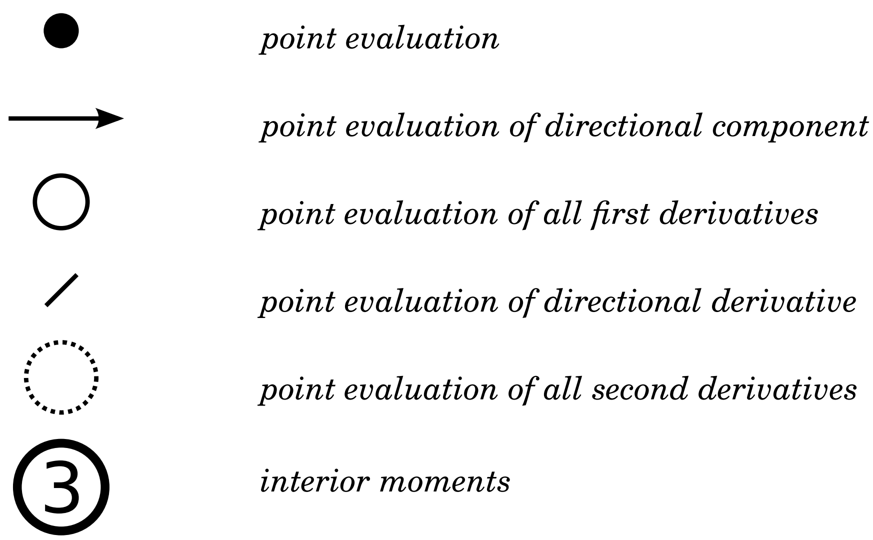

Figure 1.1: Notation

1.3 The Argyris Element

1.3.1 *Definition*

The Argyris triangle [AFS68, Cia02] is based on the space $\mathcal{P}_K = P_5(K)$ of quintic polynomials over some triangle K. It can be pieced together with full C^1 continuity between elements with C^2 continuity at the vertices of a triangulation. Quintic polynomials in $\mathbb{R}^2$ are a 21-dimensional space, and the dual basis $\mathcal{L}_K$ consists of six degrees of freedom per vertex and one per each edge. The vertex degrees of freedom are the function value, two first derivatives to specify the gradient, and three second derivatives to specify the unique components of the (symmetric) Hessian matrix.

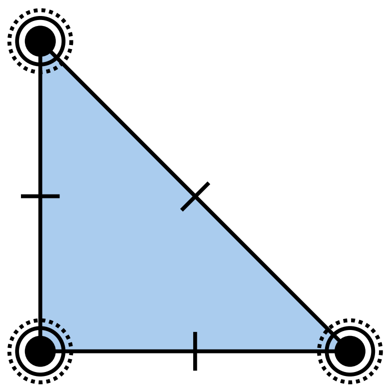

Figure 1.2: The quintic Argyris triangle.

1.3.2 *Historical notes*

The Argyris element [AFS68] was first called the TUBA element and was applied to fourth-order plate-bending problems. In fact, as Ciarlet points out [Cia02], the element also appeared in an earlier work by Felippa [Fel66].

The normal derivatives in the dual basis for the Argyris element prevent it from being affine-interpolation equivalent. This prevents the nodal basis from being constructed on a reference cell and affinely mapped. Recent work by Dominguez and Sayas [DS08] has developed a transformation that corrects this issue and requires less computational effort than directly forming the basis on each cell in a mesh.

The Argyris element can be generalized to polynomial degrees higher than quintic, still giving C^1 continuity with C^2 continuity at the vertices [ŠSD04]. The Argyris element also makes an appearance in exact sequences of finite elements, where differential complexes are used to explain the stability of many kinds of finite elements and derive new ones [AFW06a].

1.4 The Brezzi–Douglas–Marini element

1.4.1 Definition

The Brezzi–Douglas–Marini element [BDM85b] discretizes $H(\mathrm{div})$. That is, it provides a vector field that may be assembled with continuous normal components so that global divergences are well-defined. The BDM space on a simplex in d dimensions ($d = 2, 3$) consists of vectors of length d whose components are polynomials of degree q for $q \geq 1$.

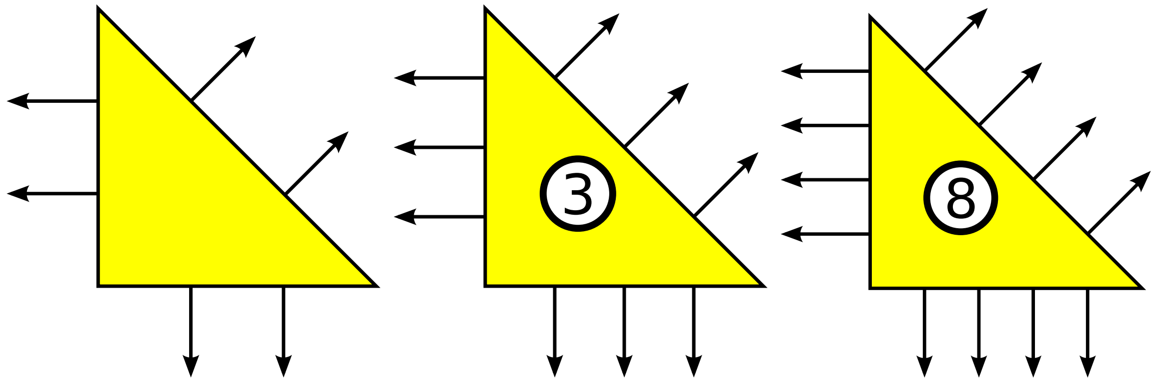

Figure 1.3: The linear, quadratic and cubic Brezzi–Douglas–Marini triangles.

The degrees of freedom for the BDM triangle include the normal component on each edge, specified either by integral moments against P_q or the value of the normal component at $q + 1$ points per edge. For $q > 1$, the degrees of freedom also include integration against gradients of $P_q(K)$ over K. For $q > 2$, the degrees of freedom also include integration against curls of $b_K P_{q-2}(K)$ over K, where b_K is the cubic bubble function associated with K.

► <u>Author note:</u> *What about tets? Will also make up for the empty space on the next page.*

The BDM element is also defined on rectangles and boxes, although it has quite a different flavor. Unusually for rectangular domains, it is not defined using tensor products of one-dimensional polynomials, but instead by supplementing polynomials of complete degree $[P_q(K)]^d$ with extra functions to make the divergence onto $P_q(K)$. The boundary degrees of freedom are similar to the simplicial case, but the internal degrees of freedom are integral moments against $[P_q(K)]^d$.

1.4.2 Historical notes

The BDM element was originally derived in two dimensions [BDM85b] as an alternative to the Raviart–Thomas element using a complete polynomial space. Extensions to tetrahedra came via the "second-kind" elements of Nédélec [Néd86] as well as in Brezzi and Fortin [BF91]. While Nédélec uses quite different internal degrees of freedom (integral moments against the Raviart–Thomas spaces), the degrees of freedom in Brezzi and Fortin are quite similar to [BDM85b].

A slight modification of the BDM element constrains the normal components on the boundary to be of degree $q - 1$ rather than q. This is called the Brezzi–Douglas–Fortin–Marini or BDFM element [BF91]. In similar spirit, elements with differing orders on the boundary suitable for varying the polynomial degree between triangles were derived in [BDM85a]. Besides mixed formulations of second-order scalar elliptic equations, the BDM element also appears in elasticity [AFW07], where it is seen that each row of the stress tensor may be approximated in a BDM space with the symmetry of the stress tensor imposed weakly.

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1.5 The Crouzeix–Raviart element

1.5.1 Definition

The Crouzeix–Raviart element [CR73] most commonly refers to a linear non-conforming element. It uses piecewise linear polynomials, but unlike the Lagrange element, the degrees of freedom are located at edge midpoints rather than at vertices. This gives rise to a weaker form of continuity, but it is still a suitable C^0-nonconforming element. The extension to tetrahedra in $\mathbb{R}^3$ replaces the degrees of freedom on edge midpoints by degrees of freedom on face midpoints.

► Author note: *What other element does it refer to? Sounds like there may be several, but I just know about this one.*

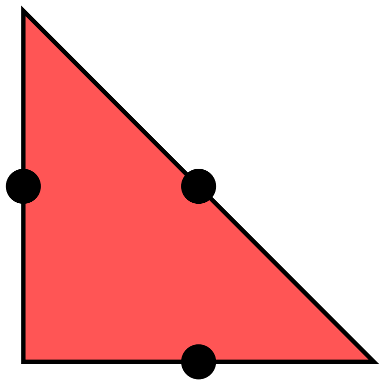

Figure 1.4: The linear Crouzeix–Raviart triangle.

1.5.2 Historical notes

Crouzeix and Raviart developed two simple Stokes elements, both using pointwise evaluation for degrees of freedom. The second element used extra bubble functions to enrich the typical Lagrange element, but the work of Crouzeix and Falk [CF89] later showed that the bubble functions were in fact not necessary for quadratic and higher orders.

► Author note: *The discussion in the previous paragraph should be expanded so it states more explicitly what this has to do with the CR element.*

The element is usually associated with solving the Stokes problem but has been used for linear elasticity [HL03] and Reissner-Mindlin plates [AF89] as a remedy for locking. There is an odd order extension of the element from Arnold and Falk.

► Author note: *Missing reference here to odd order extension.*

1.6 The Hermite Element

1.6.1 Definition

The Hermite element [Cia02] generalizes the classic cubic Hermite interpolating polynomials on the line segment. On the triangle, the space of cubic polynomials is ten-dimensional, and the ten degrees of freedom are point evaluation at the triangle vertices and barycenter, together with the components of the gradient evaluated at the vertices. The generalization to tetrahedra is analagous.

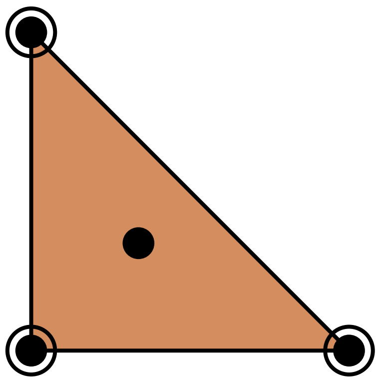

Figure 1.5: The cubic Hermite triangle.

Unlike the cubic Hermite functions on a line segment, the cubic Hermite triangle and tetrahedron cannot be patched together in a fully C^1 fashion.

1.6.2 Historical notes

Hermite-type elements appear in the finite element literature almost from the beginning, appearing at least as early as the classic paper of Ciarlet and Raviart [CR72]. They have long been known as useful C^1-nonconforming elements [Bra07, Cia02]. Under affine mapping, the Hermite elements form *affine-interpolation equivalent* families. [BS08].

1.7 The Lagrange Element

1.7.1 Definition

The best-known and most widely used finite element is the Lagrange P_1 element. In general, the Lagrange element uses $\mathcal{P}_K = P_q(K)$, polynomials of degree q on K, and the degrees of freedom are simply pointwise evaluation at an array of points. While numerical conditioning and interpolation properties can be dramatically improved by choosing these points in a clever way [**?**], for the purposes of this chapter the points may be assumed to lie on an equispaced lattice.

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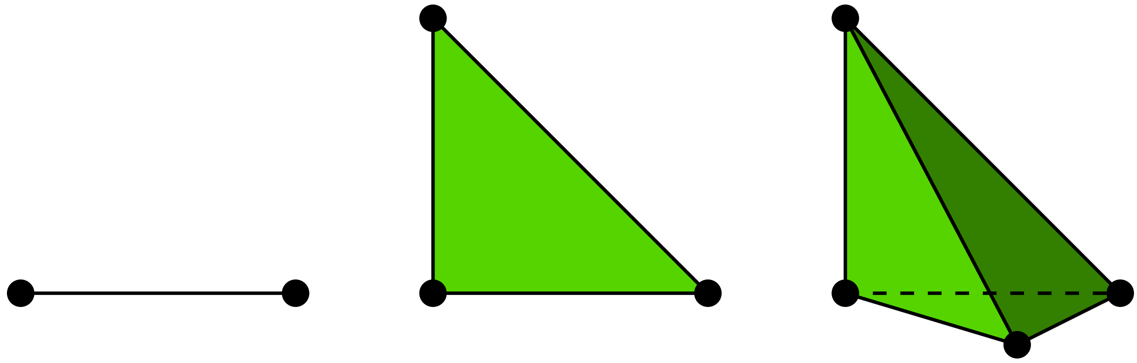


Figure 1.6: The linear Lagrange interval, triangle and tetrahedron.

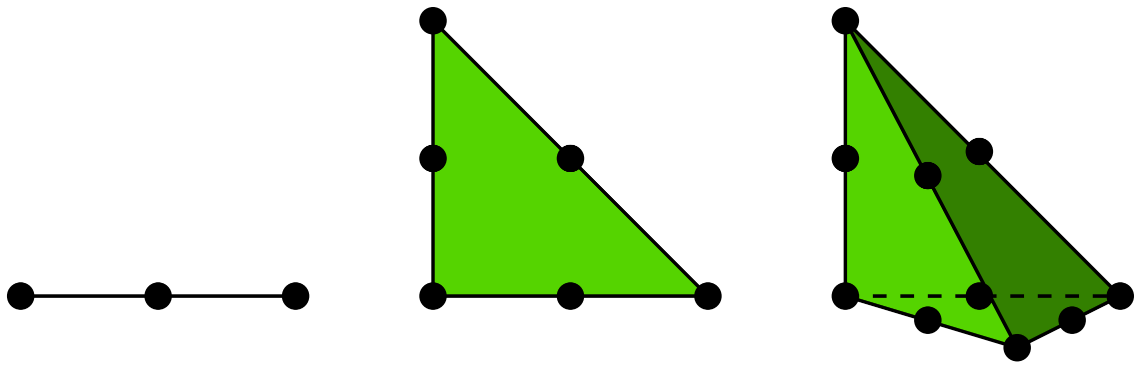


Figure 1.7: The quadratic Lagrange interval, triangle and tetrahedron.

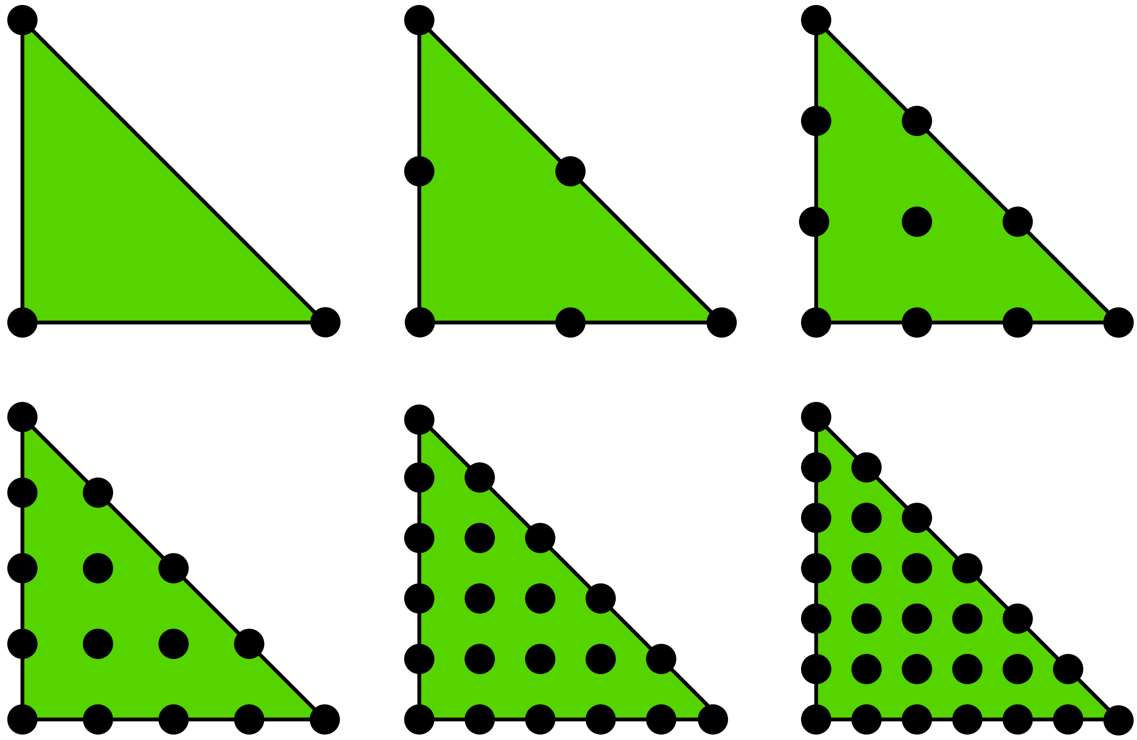


Figure 1.8: The Lagrange P_q triangle for $q = 1, 2, 3, 4, 5, 6$.

1.7.2 Historical notes

Reams could be filled with all the uses of the Lagrange elements. The Lagrange element predates the modern study of finite elements. The lowest-order triangle is sometimes called the *Courant* triangle, after the seminal paper [Cou43] in which variational techniques are considered and the P_1 triangle is used to derive a finite difference method. The rest is history.

► Author note: *Expand the historical notes for the Lagrange element. As far as I can see, Bramble and Zlamal don't seem to be aware of the higher order Lagrange elements (only the Courant triangle). Their paper from 1970 focuses only on Hermite interpolation.*

1.8 The Morley Element

1.8.1 *Definition*

The Morley triangle [Mor68] is a simple H^2-nonconforming quadratic element that is used in fourth-order problems. The function space is simply $\mathcal{P}_K = P_2(K)$, the six-dimensional space of quadratics. The degrees of freedom consist of pointwise evaluation at each vertex and the normal derivative at each edge midpoint. It is interesting that the Morley triangle is neither C^1 nor even C^0, yet it is suitable for fourth-order problems, and is the simplest known element for this purpose.

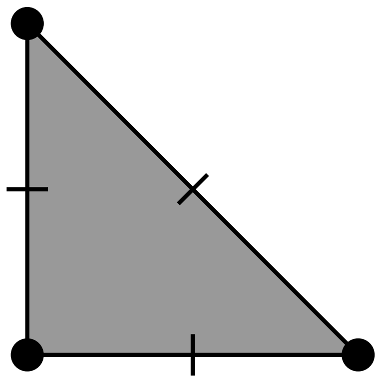


Figure 1.9: The quadratic Morley triangle.

1.8.2 *Historical notes*

The Morley element was first introduced to the engineering literature by Morley in 1968 [Mor68]. In the mathematical literature, Lascaux and Lesaint [LL75] considered it in the context of the patch test in a study of plate-bending elements.

► Author note: *Fill up page.*

1.9 The Nédélec Element

1.9.1 Definition

The widely celebrated $H(\mathrm{curl})$-conforming elements of Nédélec [Néd80, Néd86] are much used in electromagnetic calculations and stand as a premier example of the power of "nonstandard" (meaning not lowest-order Lagrange) finite elements.

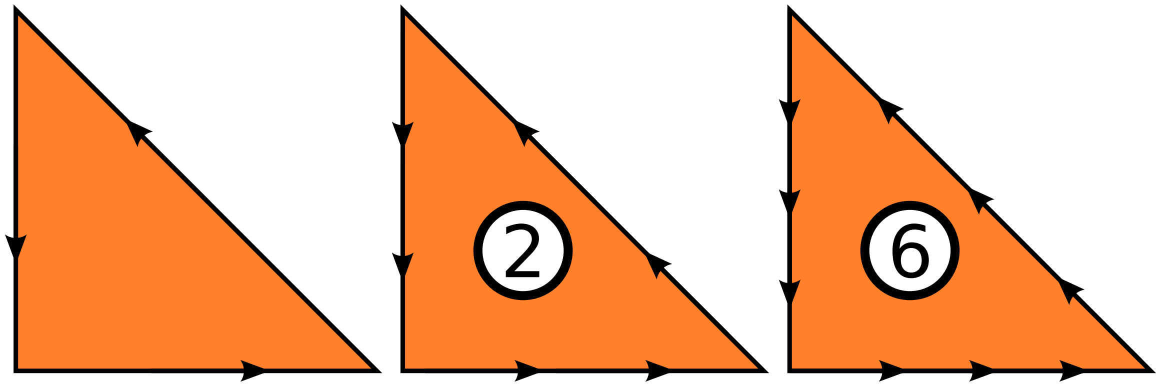


Figure 1.10: The linear, quadratic and cubic Nédélec triangles.

On triangles, the function space $\mathcal{P}_K$ may be obtained by a simple rotation of the Raviart–Thomas basis functions, but the construction of the tetrahedral element is substantially different. In the lowest order case $q = 1$, the space $\mathcal{P}_K$ may be written as functions of the form

$$v(x) = \alpha + \beta \times x,$$

where α and β are vectors in $\mathbb{R}^3$. Hence, $\mathcal{P}_K$ contains all vector-valued constant functions and some but not all linears. In the higher order case, the function space may be written as the direct sum

$$\mathcal{P}_K = [P_{q-1}(K)]^3 \oplus S_q,$$

where

$$S_q = \{v \in [\tilde{P}_q(K)]^3 : v \cdot x = 0\}.$$

Here, $\tilde{P}_q(K)$ is the space of homogeneous polynomials of degree q on K. An alternate characterization of $\mathcal{P}_K$ is that it is the space of polynomials of degree $q + 1$ on which the qth power of the elastic stress tensor vanishes. The dimension of $\mathcal{P}_q$ is exactly

$$n_K = \frac{q(q+2)(q+3)}{2}.$$

▶ Author note: *What is the qth power of the elastic stress tensor?*

▶ Author note: *What is the dimension on triangles?*

The degrees of freedom are chosen to ensure tangential continuity between elements and thus a well-defined global curl. In the lowest order case, the six degrees of freedom are the average value of the tangential component along each edge of the tetrahedron, hence the term "edge elements". In the more general case, the degrees of freedom are the $q-1$ tangential moments along each edge, moments of the tangential components against $(P_{q-2})^2$ on each face, and moments against $(P_{q-3})^3$ in the interior of the tetrahedron.

For tetrahedra, there also exists another family of elements known as Nedelec elements of the second kind, appearing in [Néd86]. These have a simpler function space at the expense of more complicated degrees of freedom. The second kind space of order q is simply vectors of polynomials of degree q. The degrees of freedom are integral moments of degree q along each edge together with integral moments against lower-order first-kind bases on the faces and interior.

▶ Author note: *Note different numbering compared to RT, starting at 1, not zero.*

1.9.2 Historical notes

Nédélec's original paper [Néd80] provided rectangular and simplicial elements for $H(\mathrm{div})$ and $H(\mathrm{curl})$ based on incomplete function spaces. This built on earlier two-dimensional work for Maxwell's equations [AGSNR80] and extended the work of Raviart and Thomas for $H(\mathrm{div})$ to three dimensions. The second kind elements, appearing in [Néd86], extend the Brezzi–Douglas–Marini triangle [BDM85b] to three dimensions and curl-conforming spaces. We summarize the relation between the Nedelec elements of first and second kind with the Raviart–Thomas and Brezzi–Douglas–Marini elements in Table 1.1.

In many ways, Nédélec's work anticipates the recently introduced finite element exterior calculus [AFW06a], where the first kind spaces appear as $\mathcal{P}_q^-\Lambda^k$ spaces and the second kind as $\mathcal{P}_q\Lambda^k$. Moreover, the use of a differential operator (the elastic strain) in [Néd80] to characterize the function space foreshadows the use of differential complexes [AFW06b].

Simplex	$H(\mathrm{div})$		$H(\mathrm{curl})$	
$K \subset \mathbb{R}^2$	RT_{q-1}	$\mathcal{P}_q^- \Lambda^1(K)$	$\mathrm{NED}_{q-1}(\mathrm{curl})$	—
	BDM_q	$\mathcal{P}_q \Lambda^1(K)$		
$K \subset \mathbb{R}^3$	$\mathrm{RT}_{q-1} = \mathrm{NED}^1_{q-1}(\mathrm{div})$	$\mathcal{P}_q^- \Lambda^2(K)$	$\mathrm{NED}^1_{q-1}(\mathrm{curl})$	$\mathcal{P}_q^- \Lambda^1(K)$
	$\mathrm{BDM}_q = \mathrm{NED}^2_q(\mathrm{div})$	$\mathcal{P}_q \Lambda^2(K)$	$\mathrm{NED}^2_q(\mathrm{curl})$	$\mathcal{P}_q \Lambda^1(K)$

Table 1.1: Nedelec elements of the first and second kind and their relation to the Raviart–Thomas and Brezzi–Douglas–Marini elements as well as to the notation of finite element exterior calculus.

► Author note: *Should we change the numbering of the Nedelec elements and Raviart–Thomas elements to start at* $q = 1$*?*

1.10 The PEERS Element

1.10.1 *Definition*

The PEERS element [ABD84] provides a stable tensor space for discretizing stress in two-dimensional mixed elasticity problems. The stress tensor σ is represented as a 2×2 matrix, each row of which is discretized with a vector-valued finite element. Normally, one expects the stress tensor to be symmetric, although the PEERS element works with a variational formulation that enforces this condition weakly.

The PEERS element is based on the Raviart–Thomas element described in Section 1.11. If $\mathrm{RT}_0(K)$ is the lowest-order Raviart-Thomas function space on a triangle K and b_K is the cubic bubble function that vanishes on ∂K, then the function space for the PEERS element is given by

$$\mathcal{P}_K = \left[\mathrm{RT}_0(K) \oplus \mathrm{span}\{\mathrm{curl}(b_K)\}\right]^2 .$$

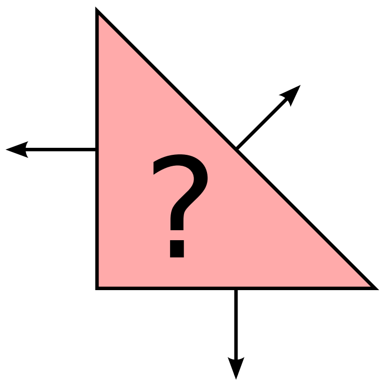


Figure 1.11: The PEERS triangle. One vector-valued component is shown.

▶ <u>Author note:</u> *Which degrees of freedom in the interior? The curl?*

▶ <u>Author note:</u> *Is this really an element? We could also introduce other mixed elements like Taylor–Hood. But perhaps it's suitable to include it since it is not a trivial combination of existing elements (the extra curl part).*

1.10.2 Historical notes

Discretizing the mixed form of planar elasticity is quite a difficult task. Polynomial spaces of symmetric tensors providing inf-sup stability are quite rare, only appearing in the last decade [AW02]. A common technique is to relax the symmetry requirement of the tensor, imposing it weakly in a variational formulation. This extended variational form requires the introduction of a new field discretizing the assymetric portion of the stress tensor. When the PEERS element is used for the stress, the displacement is discretized in the space of piecewise constants, and the asymmetric part is discretized in the standard space of continuous piecewise linear elements.

The PEERS element was introduced in [ABD84], and some practical details, including postprocessing and hybridization strategies, are discussed in [AB85].

1.11 The Raviart–Thomas Element

1.11.1 Definition

The Raviart–Thomas element, like the Brezzi–Douglas–Marini and Brezzi–Douglas–Fortin–Marini elements, is an $H(\mathrm{div})$-conforming element. The space of order q is constructed to be the smallest polynomial space such that the divergence maps $\mathrm{RT}_q(K)$ onto $P_q(K)$. The function space $\mathcal{P}_K$ is given by

$$\mathcal{P}_K = P_{q-1}(K) + xP_{q-1}(K).$$

The lowest order Raviart–Thomas space thus consists of vector-valued functions of the form

$$v(x) = \alpha + \beta x,$$

where α is a vector-valued constant and β is a scalar constant.

On triangles, the degrees of freedom are the moments of the normal component up to degree q, or, alternatively, the normal component at $q + 1$ points per edge. For higher order spaces, these degrees of freedom are supplemented with integrals against a basis for $[P_{q-1}(K)]^2$.

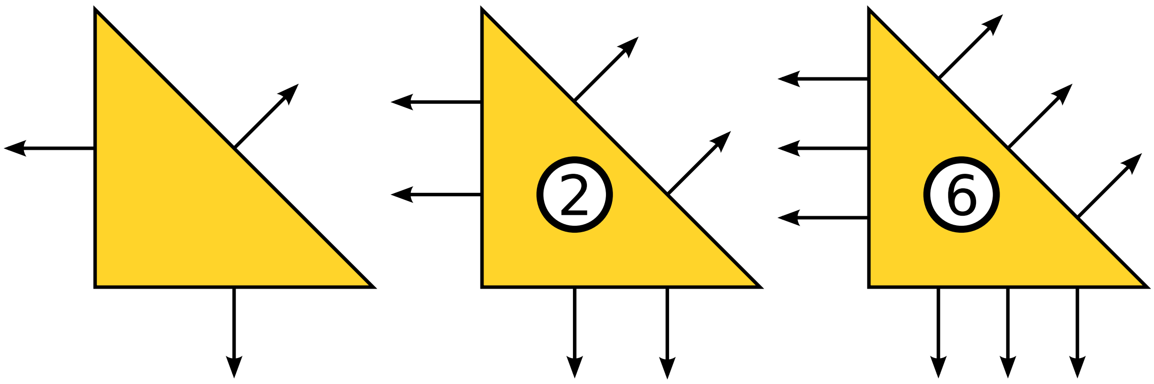


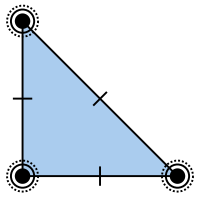
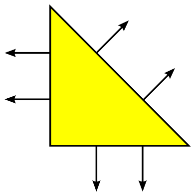
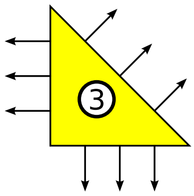
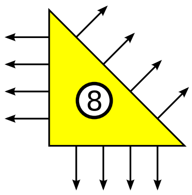
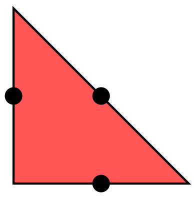
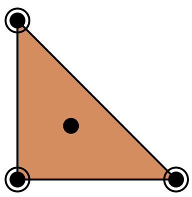
Figure 1.12: The zeroth order, linear and quadratic Raviart–Thomas triangles.

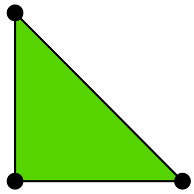
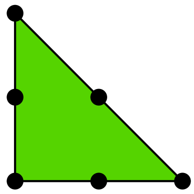
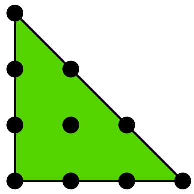
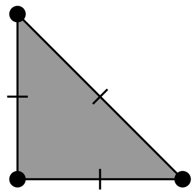
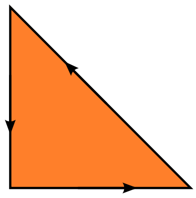
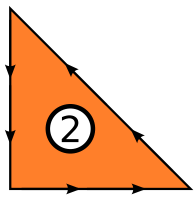
1.11.2 Historical notes

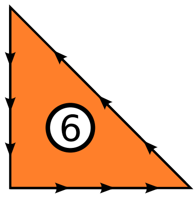
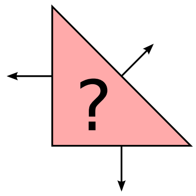
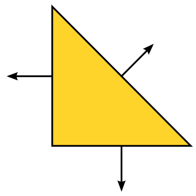
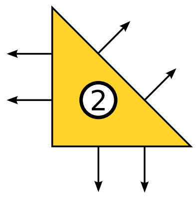
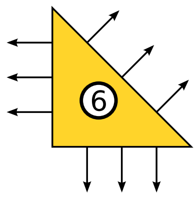
The Raviart–Thomas element was introduced in [RT77] in the late 1970's, the first element to discretize the mixed form of second order elliptic equations. Shortly thereafter, it was extended to tetrahedra and boxes by Nédélec [Néd80] and so is sometimes referred to as the Raviart–Thomas–Nédélec element or a first kind $H(\mathrm{div})$ element.

On rectangles and boxes, there is a natural relation between the lowest order Raviart–Thomas element and cell-centered finite differences. This was explored in [RW83], where a special quadrature rule was used to diagonalize the mass matrix and eliminate the flux unknowns. Similar techniques are known for triangles [ADK+98], although the stencils are more complicated.

1.12 Summary

Notation	Element family	$\mathcal{L}_K$	$\dim \mathcal{P}_K$	References
ARG_5	Quintic Argyris		21	
BDM_1	Brezzi–Douglas–Marini		6	
BDM_2	Brezzi–Douglas–Marini		12	
BDM_3	Brezzi–Douglas–Marini		20	
CR_1	Crouzeix–Raviart		3	
HERM_q	Hermite		10	

P_1	**Lagrange**		3	
P_2	**Lagrange**		6	
P_3	**Lagrange**		10	
MOR_1	**Morley**		6	
NED_1	**Nédélec**		3	
NED_2	**Nédélec**		8	

NED_3	**Nédélec**		15	
PEERS	**PEERS**		?	
RT_0	**Raviart–Thomas**		3	
RT_0	**Raviart–Thomas**		8	
RT_0	**Raviart–Thomas**		15	

► Author note: *Add references to table.*

► Author note: *Indicate which elements are supported by FIAT and SyFi.*

► Author note: *Include formula for space dimension as function of q for all elements.*

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