NUMERICAL LINEAR ALGEBRA WITH APPLICATIONS
Numer. Linear Algebra Appl. 2002; **9**:223–238 (DOI: 10.1002/nla.271)

An algebraic multigrid method for finite element discretizations with edge elements

S. Reitzinger*,†, J. Schöberl‡

Special Research Program (SFB) F013, Numerical and Symbolic Scientific Computing, University of Linz, Austria

SUMMARY

This paper presents an algebraic multigrid method for the efficient solution of the linear system arising from a finite element discretization of variational problems in $H_0(\mathrm{curl}, \Omega)$. The finite element spaces are generated by Nédélec's edge elements.

A coarsening technique is presented, which allows the construction of suitable coarse finite element spaces, corresponding transfer operators and appropriate smoothers. The prolongation operator is designed such that coarse grid kernel functions of the curl-operator are mapped to fine grid kernel functions. Furthermore, coarse grid kernel functions are 'discrete' gradients. The smoothers proposed by Hiptmair and Arnold, Falk and Winther are directly used in the algebraic framework.

Numerical studies are presented for 3D problems to show the high efficiency of the proposed technique. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: Maxwell's equations; finite element method; Nédélec's edge elements; algebraic multigrid

1. INTRODUCTION

Numerical simulations based on 3D Maxwell's equations are important tools for many applications in science and engineering. Efficient algorithms enable large-scale computations on nowadays midrange computers. This paper is concerned with the development of an efficient iterative solver for the linear system of equations

$$K_h^e \mathbf{u}_h = \mathbf{f}_h \tag{1}$$

arising from the discretization of Maxwell's equation with Nédélec's edge elements [1] (further referred to as edge elements) for variational problems in the function space $H_0(\mathrm{curl}, \Omega)$. In

*Correspondence to: S. Reitzinger, Johannes Kepler University of Linz, Special Research Program SFB F013, Freistädter Strasse 313, A-4040 Linz, Austria.
†E-mail: reitz@sfb013.uni-linz.ac.at
‡E-mail: joachim@sfb013.uni-linz.ac.at

Contract/grant sponsor: Austrian Science Fund; contract/grant number: SFB F013

Received 28 September 2000
Revised 1 August 2001

particular we consider the linear, magnetostatic case of Maxwell's equations. In Equation (1), $K_h^e \in \mathbb{R}^{N_h^e \times N_h^e}$ is a symmetric and positive definite (SPD) sparse matrix, $\mathbf{f}_h \in \mathbb{R}^{N_h^e}$ the right-hand side, and $\mathbf{u}_h \in \mathbb{R}^{N_h^e}$ the solution vector. The superscript 'e' is used for edge quantities, and the subscript 'h' indicates the maximal mesh-width parameter. To achieve the desired accuracy, the number N_h^e of unknowns may become very large for 3D problems as h tends to zero. Additionally, the system matrix K_h^e is usually very ill conditioned, which is the reason for exhibiting slow convergence of standard iterative solution methods. Consequently, the construction of efficient and robust preconditioning techniques for (1) is an important aspect of the finite element (FE) method.

For the design of multilevel methods for the linear system (1), the Helmholtz decomposition of the vector field into a solenoidal and a gradient field is the key point. Especially, the rotation free functions need special treatment in the smoother. This idea lead a geometric multilevel method which was discussed by Hiptmair in Reference [2] the first time and a different multigrid approach is due to Arnold *et al.* [3]. Many references are found in Reference [4]. For numerical results using geometric multigrid methods we refer to References [4–6].

Geometric multigrid methods come along with the requirement of hierarchical FE-meshes, which are not always available. On the other hand, the coarse grid might be too large to be solved efficiently by a classical direct or iterative method. In such cases, an algebraic multigrid (AMG) method provides a way out. The AMG approach requires (in contrast to geometric multigrid) only a single grid information, i.e. at least the system matrix K_h^e and the right-hand side $\mathbf{f}_h$. In order to perform a multigrid cycle (see Reference [7]) a matrix hierarchy (coarse grid operators), corresponding transfer operators and appropriate smoothers have to be defined properly. The crucial point of the setup process consists in the definition of the prolongation operators, since they have to be designed with pure algebraic knowledge. Once the prolongation operator is assembled the coarse grid operator is computed by Galerkin's method.

A proper construction of an AMG method for (1) requires a careful treatment of all multigrid components. In spite of the fact that the FE-matrix K_h^e is SPD, the classical approaches of References [8–15] and variants of them fail for problem (1) at hand. All these methods are designed for SPD problems which either stem from FE-discretizations of $H^1(\Omega)$-elliptic problems, or need beside the SPD property special characteristics of the system matrix (e.g. M-matrix property). A first AMG approach to solve (1) can be found in Reference [16] by Beck. The key idea there is to split an $H_0(\mathrm{curl}, \Omega)$ function into an $(H^1(\Omega))^3$ function and a gradient function, and apply classical AMG for all components. In this way an appropriate preconditioner was constructed for (1).

Though our AMG method is exclusively used as a preconditioner in the preconditioned conjugate gradient method, it could be used as a solver too. Since we assume to have access to a single grid information, we extend the classical concept of AMG in a very natural way. An auxiliary matrix is introduced that is given on the finest grid by an FE-discretization of the potential equation, or by a simple nodal distance matrix. This auxiliary matrix allows us to construct prolongation operators which preserve the discrete kernel. This is essential for multigrid methods since the smoother can hardly reduce the smooth error components efficiently. Thus, the coarse grid correction must reduce these error components. Moreover, we are able to assemble appropriate smoothers for the considered problem class with the aid of the auxiliary matrix.

The remainder of the paper is organized as follows. In Section 2 we define the problem class to be considered. In addition, the FE-discretization is specified. Section 3 introduces the main components of an AMG method. Further we propose a generalization of standard AMG methods. Numerical studies are presented in Section 4 showing the efficiency of the proposed technique. Finally, further remarks are given and conclusions are drawn.

2. PROBLEM FORMULATION

Let us consider the linear, magnetostatic special case of the Maxwell equations (e.g. see Reference [17]), namely

$$\mathbf{B} = \operatorname{curl} \mathbf{u}, \quad \mathbf{H} = \nu \mathbf{B}, \quad \operatorname{curl} \mathbf{H} = \mathbf{J}$$

where $\mathbf{u}$ is the vector potential, $\mathbf{B}$ is the magnetic induction, $\mathbf{H}$ is the magnetic field strength and $\mathbf{J}$ is the given current density. The coefficient ν is bounded from below and from above. We pose the equations on the bounded, polyhedral domain $\Omega \subset \mathbb{R}^3$, and close them, e.g. with the boundary conditions $\mathbf{B} \cdot \mathbf{n} = 0$ on the boundary $\partial\Omega$. We use Coulomb gauging $\operatorname{div} \mathbf{u} = 0$ to select a unique vector potential.

The variational form is a saddle point problem [18]. It is posed on the spaces

$$\mathbb{V} := H_0(\operatorname{curl}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 \mid \operatorname{curl} \mathbf{v} \in (L^2(\Omega))^3 \text{ and } \mathbf{v} \times \mathbf{n} = 0 \text{ on } \partial\Omega\} \tag{2}$$

equipped with its canonical norm

$$\|\mathbf{v}\|_{\mathbb{V}} = (\|\mathbf{v}\|_0^2 + \|\operatorname{curl} \mathbf{v}\|_0^2)^{1/2}$$

with $\|\cdot\|_0$ the usual $L^2(\Omega)$ norm and the Sobolev space

$$\mathbb{Q} = H_0^1(\Omega)$$

We search for $\mathbf{u} \in \mathbb{V}$ and $\varphi \in \mathbb{Q}$ which fulfill

$$\int_\Omega \nu \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} \, \mathrm{d}x + \int_\Omega \mathbf{v} \cdot \operatorname{grad} \varphi \, \mathrm{d}x = \int_\Omega \mathbf{J} \cdot \mathbf{v} \, \mathrm{d}x \quad \forall \mathbf{v} \in \mathbb{V} \tag{3}$$

$$\int_\Omega \mathbf{u} \cdot \operatorname{grad} \psi \, \mathrm{d}x = 0 \quad \forall \psi \in \mathbb{Q} \tag{4}$$

To apply the abstract theory of Brezzi and Fortin [19, Chapter 2], several conditions are necessary. First, continuity of the bilinear forms and the linear form are obvious, also the LBB condition is easily seen. The kernel ellipticity, i.e.

$$\alpha \|\mathbf{u}\|_{\mathbb{V}}^2 \leqslant \int_\Omega \nu |\operatorname{curl} \mathbf{u}|^2 \, \mathrm{d}x \quad \forall \mathbf{u} \in \mathbb{V} \text{ s.t. } \operatorname{div} \mathbf{u} = 0 \tag{5}$$

is non-trivial, and follows from [20, Section 1.a]. This proves a unique solution, and, furthermore the stability estimate

$$\|\mathbf{u}\|_{\mathbb{V}} \leqslant \alpha^{-1} \|\mathbf{J}\|_0 \tag{6}$$

[19, Proposition 1.3]. By physical reasons, the right-hand side has vanishing divergence in the sense of

$$\int_\Omega \mathbf{J} \cdot \operatorname{grad} \psi \, \mathrm{d}x = 0 \quad \forall \psi \in \mathbb{Q}$$

Thus, using the trial function $\mathbf{v} = \operatorname{grad} \varphi$ (which is in $\mathbb{V}$) for Equation (3) gives $\varphi = 0$.

Instead of dealing with the saddle point problem, we prefer to solve a regularized elliptic problem, namely,

$$\text{find } \mathbf{u}^\sigma \in \mathbb{V}: \quad a_\sigma(\mathbf{u}^\sigma, \mathbf{v}) = \int_\Omega \mathbf{J} \cdot \mathbf{v} \, \mathrm{d}x \quad \forall \mathbf{v} \in \mathbb{V} \tag{7}$$

with the bilinear form

$$a_\sigma(\mathbf{u}, \mathbf{v}) := \int_\Omega \nu \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} + \sigma \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}x$$

Here, $\sigma > 0$ is a small regularization parameter. The following lemma establishes convergence as $\sigma \to 0$:

Lemma 2.1. *Let* $\mathbf{u}$ *and* $\mathbf{u}^\sigma$ *be the solutions of (3) and (7), respectively. Then the error estimate*

$$\|\mathbf{u} - \mathbf{u}^\sigma\|_{\mathbb{V}} \leqslant \sigma \alpha^{-1} \|\mathbf{u}\|_0$$

is valid, where α *is the constant from Equation (5).*

Proof

By choosing trial functions $v = \operatorname{grad} \psi$ in Equation (7), we observe that also $\mathbf{u}^\sigma$ fulfills

$$\int_\Omega \mathbf{u}^\sigma \cdot \operatorname{grad} \psi \, \mathrm{d}x = 0 \quad \forall \psi \in \mathbb{Q}$$

i.e. $\operatorname{div} \mathbf{u}^\sigma = 0$. Subtracting Equation (7) from (3) leads to the error equation

$$a_\sigma(\mathbf{u} - \mathbf{u}^\sigma, \mathbf{v}) = \sigma(\mathbf{u}, \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathbb{V}$$

Since $\operatorname{div}(\mathbf{u}^\sigma - \mathbf{u}) = 0$, ellipticity (5) applies. Together with the choice $\mathbf{v} = \mathbf{u} - \mathbf{u}^\sigma$ we obtain

$$\alpha \|\mathbf{u} - \mathbf{u}^\sigma\|_{\mathbb{V}}^2 \leqslant a_\sigma(\mathbf{u} - \mathbf{u}^\sigma, \mathbf{u} - \mathbf{u}^\sigma) = \sigma(\mathbf{u}, \mathbf{u} - \mathbf{u}^\sigma)_0 \leqslant \sigma \|\mathbf{u}\|_0 \|\mathbf{u} - \mathbf{u}^\sigma\|_0$$

Dividing by $\alpha \|\mathbf{u} - \mathbf{u}^\sigma\|_{\mathbb{V}}$ gives the result.

We want to emphasize, that only if robust solvers with respect to $\sigma \to 0$ are applied, the regularized method is efficient. The geometric multigrid methods by Hiptmair [2] and Arnold *et al.* [3] are robust, so is the proposed AMG method.

The FE-discretization is based on the regular partitioning τ_h of the domain Ω into tetrahedral elements $\{T\}$ in the sense of Reference [21]. Canonical finite elements for the approximation of $H(\operatorname{curl}, \Omega)$ are the one by Nédélec [21]. The lowest order of the family are known as edge elements. For each element T, the space

$$\mathbb{V}_{h,T} = \{(\mathbf{a} + \mathbf{b} \times \mathbf{x})|_T \mid \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\}$$

is defined and we set

$$\mathbb{V}_h = \{\mathbf{v} \in H(\mathrm{curl}, \Omega) \mid \mathbf{v}|_T \in \mathbb{V}_{h,T}\} \subset \mathbb{V}$$

The integrals of the tangential components along edges give the proper degrees of freedom. The edge elements have the property that the tangential component is continuous while the normal component is free to jump across elements, i.e. an edge element discretization is $H_0(\mathrm{curl}, \Omega)$-conform. The finite element space $\mathbb{Q}_h \subset \mathbb{Q}$ is based on the standard, linear Lagrangian (nodal) elements.

For the multigrid methods, two properties are essential [2, 3]. One is (a part of) the complete sequence property, namely,

$$\mathrm{grad}\, \mathbb{Q} = \mathbb{V}_0 := \{\mathbf{v} \in \mathbb{V}: \ \mathrm{curl}\, \mathbf{v} = 0\}$$

and its discrete counterpart for the above choice of elements

$$\mathrm{grad}\, \mathbb{Q}_h = \mathbb{V}_{0h} := \{v_h \in \mathbb{V}_h: \ \mathrm{curl}\, v_h = 0\}$$

The other is the commuting diagram property $I_h^e\, \mathrm{grad} = \mathrm{grad}\, I_h^n$, for the canonical edge and nodal interpolation operators I_h^e and I_h^n:

$$\begin{array}{ccc} \mathbb{Q} \cap C^1 & \xrightarrow{\mathrm{grad}} & \mathbb{V} \cap C \\ \downarrow I_h^n & & \downarrow I_h^e \\ \mathbb{Q}_h & \xrightarrow{\mathrm{grad}} & \mathbb{V}_h \end{array}$$

Our AMG approach is motivated by these two properties, too.

Later on, we will use the notation of the FE-isomorphisms

$$G_h^e: V_h \to \mathbb{V}_h \quad \text{and} \quad G_h^n: Q_h \to \mathbb{Q}_h$$

with $V_h = \mathbb{R}^{N_h^e}$ and $Q_h = \mathbb{R}^{N_h^n}$.

The discrete kernel of the curl-operator is defined by

$$V_{0h} = \{\mathbf{v}_h \in V_h \mid \mathrm{curl}\, G_h^e \mathbf{v}_h = 0\} = \mathrm{grad}_h\, Q_h \tag{8}$$

with the discrete gradient operator $\mathrm{grad}_h: Q_h \to V_{0h}$

$$\mathrm{grad}_h\, \mathbf{q}_h = (G_h^e)^{-1}\, \mathrm{grad}\, G_h^n\, \mathbf{q}_h \quad \forall \mathbf{q}_h \in Q_h \tag{9}$$

3. CONSTRUCTION OF AN AMG METHOD

In order to solve Equation (1) by means of a multigrid cycle several constituents have to be defined properly. For further discussion the basic ingredients for an AMG method are discussed by a two grid method, for which the indices h and H are related to the fine and coarse grid quantities, respectively.

According to geometric multigrid methods the efficient interplay of smoother and coarse grid correction is again the key idea for an AMG method. The main difference to geometric multigrid methods is the missing grid hierarchy. In order to come along with that deficiency

a coarsening strategy is introduced which decreases the number of degrees of freedom on the fine level. Most coarsening techniques are based on the matrix graph, see e.g. References [8, 9, 12, 13, 15]. Once the coarsening is done the prolongation operator

$$P_h^e : V_H \mapsto V_h$$

is constructed. We assume on P_h^e full-rank and $\dim(V_H) < \dim(V_h)$, with V_H the coarse space. For the required restriction operator, the transposed of P_h^e is taken. The next step is to construct the coarse grid operator by

$$K_H^e = (P_h^e)^{\mathrm{T}} K_h^e P_h^e$$

A recursive application of this process leads immediately a matrix hierarchy with corresponding transfer operators. If an appropriate smoothing operator is defined, then a usual multigrid cycle can be assembled, see Algorithm 1.

Algorithm 1. V(v_F, v_B)-cycle: MG($\mathbf{u}_\ell, \mathbf{f}_\ell, \ell$)
  if $\ell = COARSELEVEL$ **then**
    $u_\ell = (K_\ell^e)^{-1}\mathbf{f}_\ell$ with a direct solver
  else
    Smooth v_F times on $K_\ell^e \mathbf{u}_\ell = \mathbf{f}_\ell$
    Calculate the defect $\mathbf{d}_\ell = \mathbf{f}_\ell - K_\ell^e \mathbf{u}_\ell$
    Restrict the defect to the next coarser level $\ell + 1$: $\mathbf{d}_{\ell+1} = (P_\ell^e)^{\mathrm{T}} \mathbf{d}_\ell$
    Set $\mathbf{s}_{\ell+1} \equiv 0$
    Apply MG($\mathbf{s}_{\ell+1}, \mathbf{d}_{\ell+1}, \ell + 1$)
    Prolongate the correction $\mathbf{s}_\ell = P_\ell^e \mathbf{s}_{\ell+1}$
    Update the solution $\mathbf{u}_\ell = \mathbf{u}_\ell + \mathbf{s}_\ell$
    Smooth v_B times on $K_\ell^e \mathbf{u}_\ell = \mathbf{f}_\ell$
  end if

3.1. The construction of 'Virtual' FE-meshes

Standard AMG methods are not reasonable for our problem class, therefore we suggest the following strategy.

- Identify connected pairs of nodes with the connecting edge, i.e. set up a 'node-to-edge' map.
- Perform a coarsening technique such that the 'node to edge' map hands over to the coarse level.
- Define a prolongation operator compatible with the Helmholtz decomposition.
- Calculate the coarse grid matrix by Galerkin's method.
- Take an appropriate smoother for the considered problem class.

A pivotal point is the construction of the 'node-to-edge' map to be able to construct an appropriate prolongation operator and smoother for K_h^e. Since we are concerned with an FE-discretization, a feasible 'node-to-edge' map is given by an auxiliary matrix $K_h^n \in \mathbb{R}^{N_h^n \times N_h^n}$, which provides the opportunity to be interpreted as a 'virtual' FE-mesh in the following way: The diagonal entry $(K_h^n)_{ii}$ is related to the grid point i and a non-zero entry $(K_h^n)_{ij}$, $i \neq j$, is related to an edge (i, j). The set of grid points on the fine level h is given by $\omega_h^n = \{1, 2, \ldots, N_h^n\}$,

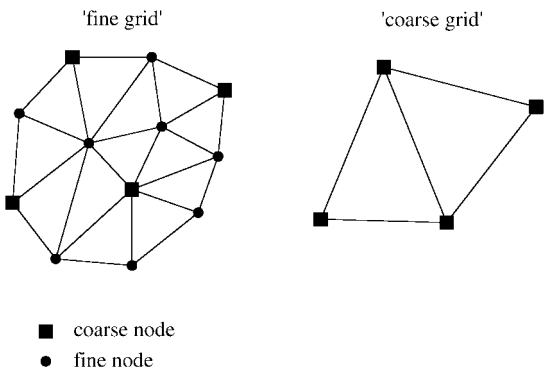

Figure 1. Detail view of a fine and coarse 'virtual' FE-mesh.

with cardinality $\text{card}(\omega_h^n) = N_h^n$. Consequently, the 'virtual' FE-mesh provides a 'node-to-edge' map in a natural way. For instance, such auxiliary matrix K_h^n is assembled from an FE-discretization of a potential equation with linear, nodal FE-functions.

Now, we are able to define the following relations on a pure algebraic level:

$$\begin{aligned} \omega_h^e &= \{(i,j) \,|\, |(K_h^n)_{ij}| \neq 0, i \neq j\} \\ \text{card}(\omega_h^e) &= N_h^e \\ N_h^i &= \{\, j \in \omega_h^n \,|\, |(K_h^n)_{ij}| \neq 0, i \neq j\} \\ T_h^i &= \{(i,j) \,|\, j \in N_h^i\} \end{aligned}$$

which are related to the set of edges, the number of edges, the set of neighbours around a node $i \in \omega_h^n$, and the set of edges belonging to a node $i \in \omega_h^n$, respectively. In an analogous way, the above relations are defined on the coarse grid H for a given matrix K_H^n.

The next step consists in a standard coarsening on K_h^n. Motivated from an FE-grid (see Figure 1), a 'virtual' FE-grid can be split into coarse grid nodes ω_C^n and fine grid nodes ω_F^n, i.e.

$$\omega_h^n = \omega_C^n \cup \omega_F^n, \quad \omega_C^n \cap \omega_F^n = \emptyset$$

such that there are (almost) no direct connections between any two coarse grid nodes and the resulting number of coarse grid nodes is as large as possible. The coarse grid is defined by identifying each coarse grid node $j \in \omega_C^n$ with an index $k \in \omega_H^n$. This is expressed by the index map $\text{ind}(\cdot)$ as

$$\omega_H^n = \text{ind}(\omega_C^n)$$

Remarks

(1) The coarse grid selection can be done by several different coarsening strategies (see References [8, 12, 14, 15]). On the one hand, a pure matrix graph-based method can be used

or on the other, a coarsening method depending on the matrix entries. The latter case has chances to detect parameter jumps and anisotropies.

(2) With Galerkin's method, we compute a coarse grid matrix K_H^n, which can be related to a 'virtual' FE-mesh and consequently, K_H^n gives rise to a 'node-to-edge' map on the coarse grid. The resulting coarse edges on the 'virtual' coarse grid are degrees of freedom on that.

(3) As we will see later, the smoothing operators [3, 2] can be easily assembled via the auxiliary matrix.

A 'useful' set of coarse grid edges ω_H^e can be constructed if we invest in a special prolongation operator $P_h^n : Q_H \to Q_h$ for the auxiliary matrix K_h^n. The prolongation operator P_h^n is constructed such that each fine grid variable prolongates exactly from one coarse grid variable. We extend the index map $\mathrm{ind} : \omega_C^n \to \omega_H^n$ defined above onto the whole fine space ω_h^n by assigning the coarse grid index of the representative of the cluster

$$\mathrm{ind} : \omega_h^n \to \omega_H^n$$

A consequence is that $\mathrm{ind}(i) = \mathrm{ind}(j)$ if and only if $i, j \in \omega_h^n$ prolongate from the same coarse grid variable. We define an agglomerate I_h^i of a grid point $i \in \omega_h^n$ by

$$I_h^i = \{ j \in \omega_h^n \mid \mathrm{ind}(j) = \mathrm{ind}(i) \} \subset N_h^i$$

and the set of coarse grid nodes can be written as

$$\omega_H^n = \{ \mathrm{ind}(i) \mid i \in \omega_h^n \}$$

The prolongation operator P_h^n has only 0 and 1 entries by construction, i.e.

$$(P_h^n)_{ij} = p_{ij}^n = \begin{cases} 1 & i \in \omega_h^n,\ j = \mathrm{ind}(i) \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

The coarse grid matrix K_H^n calculated by Galerkin's method ($K_H^n = (P_h^n)^{\mathrm{T}} K_h^n P_h^n$) which is equivalent to the formula

$$(K_H^n)_{kl} = \sum_{i \in I_h^{\tilde{k}}} \sum_{j \in I_h^{\tilde{l}}} p_{ik}^n \cdot (K_h^n)_{ij} \cdot p_{il}^n \tag{11}$$

with $k = \mathrm{ind}(\tilde{k})$, $l = \mathrm{ind}(\tilde{l})$, and $\tilde{k}, \tilde{l} \in \omega_H^n$. K_H^n has useful properties, because of the prolongation operator defined in Equation (10). This is the content of the next lemma.

Lemma 3.1. *Let $\tilde{k}, \tilde{l} \in \omega_C^n$, $\tilde{k} \neq \tilde{l}$ and $k = \mathrm{ind}(\tilde{k}) \in \omega_H^n$, $l = \mathrm{ind}(\tilde{l}) \in \omega_H^n$. Further let K_h^n stems from an FE-discretization of a potential equation with linear nodal FE-functions and let P_h^n be defined by Equation (10). $K_H^n = (P_h^n)^{\mathrm{T}} K_h^n P_h^n$. If for all $i \in I_h^{\tilde{k}}$ and for all $j \in I_h^{\tilde{l}}$*

$$(K_h^n)_{ij} = 0$$

then

$$(K_H^n)_{kl} = 0$$

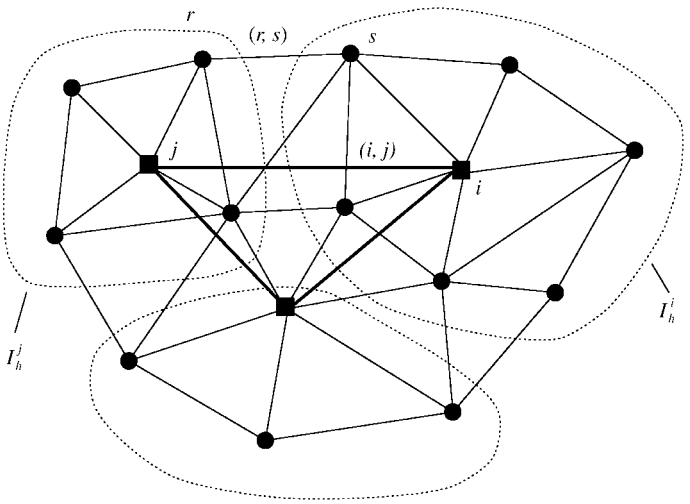

Figure 2. 'Virtual' FE-mesh with a feasible agglomeration.

Proof
The proof follows immediately by using Equation (11). ■

Remarks

(1) The essence of Lemma 3.1 is that a coarse grid edge exists only if there is at least one fine edge connecting the agglomerates I_h^i and I_h^j $(i \neq j)$, i.e.

$$\exists r \in I_h^i, \ \exists s \in I_h^j \quad \text{such that } (r,s) \in \omega_h^e$$

(see Figure 2).

(2) The constructed coarse matrix K_H^n gives rise to a virtual FE-mesh with nodes ω_H^n and edges ω_H^e.

(3) A decrease of edges in the coarsening process is not proved in general, but heuristically a decrease is given, if the average number of non-zero entries of K_h^n grows not too fast.

3.2. The construction of coarse FE-spaces

The construction of the prolongation operator $P_h^e : V_H \to V_h$, $P_h^e \in \mathbb{R}^{N_h^e \times N_H^e}$, is delicate because of the kernel of the curl-operator. Thus, the challenge is to cope with the kernel of the curl-operator, because the smooth error components can hardly be reduced by the smoother. Consequently, if we do not care about the kernel the optimal convergence rate of the multigrid method is destroyed in general.

Subsequently, we define the prolongation operator for the system matrix and after that we show that the prolongation operator is appropriate for the considered problem class. P_h^e is

defined for a fine grid edge $i=(i_1,i_2)\in\omega_h^e$ and a coarse grid edge $j=(j_1,j_2)\in\omega_H^e$ as

$$(P_h^e)_{ij}=\begin{cases}1 & \text{if } j=(\mathrm{ind}(i_1),\mathrm{ind}(i_2))\\ -1 & \text{if } j=(\mathrm{ind}(i_2),\mathrm{ind}(i_1))\\ 0 & \text{otherwise}\end{cases} \tag{12}$$

by assuming a positive orientation of an edge $j=(j_1,j_2)$ from j_1 to j_2 if $j_1<j_2$. The constructed prolongation operator P_h^e has full-rank, because the coarse grid edges prolongate to N_H^e distinct fine grid edges by construction. Moreover, the prolongation operator is only able to prolongate constants exactly, i.e. optimal convergence rates cannot be expected.

Next we note that the operator $\mathrm{grad}_h: Q_h\to V_h$ defined in Equation (9) has the representation (with $i=(i_1,i_2)\in\omega_h^e$ and $\mathbf{q}_h\in Q_h$)

$$(\mathrm{grad}_h\,\mathbf{q}_h)_i=q_{h,i_2}-q_{h,i_1} \tag{13}$$

This can be seen by evaluating Equation (9) for a $\mathbf{q}_h\in Q_h$ and using the facts that $\mathbb{Q}_h$ is piecewise linear FE-space and the degree of freedom of the edge element discretization is the path integral on the edge (i_1,i_2). In analogy, we define $\mathrm{grad}_H: Q_H\to V_H$ on the coarse level. Since we use a Galerkin approach, the coarse grid kernel is a subspace of the fine grid kernel, namely,

$$V_{0H}=\{\mathbf{v}_H\in V_H\mid P_h^e\mathbf{v}_H\in V_{0h}\} \tag{14}$$

with V_{0h} defined in Equation (8) and $V_H=\mathbb{R}^{N_H^e}$. The crux is that P_h^e prolongates discrete gradients of the coarse space to discrete gradients of the fine space, which is shown in the next two lemmas.

Lemma 3.2. *For $\mathbf{q}_H\in Q_H$ there holds*

$$P_h^e\,\mathrm{grad}_H\,\mathbf{q}_H=\mathrm{grad}_h\,P_h^n\mathbf{q}_H \tag{15}$$

This means the commuting diagram

$$\begin{array}{ccc} Q_H & \xrightarrow{\mathrm{grad}_H} & V_H \\ \downarrow{\scriptstyle P_h^n} & & \downarrow{\scriptstyle P_h^e} \\ Q_h & \xrightarrow{\mathrm{grad}_h} & V_h \end{array}$$

is valid.

Proof

We consider the edge $i=(i_1,i_2)\in\omega_h^e$. We have to distinguish two cases. First, let us assume the edge is inside one agglomerate, i.e. $\mathrm{ind}(i_1)=\mathrm{ind}(i_2)$. Then both sides of Equation (15) vanish. The left-hand side vanishes by definition of the prolongation operator P_h^e, the right-hand side vanishes since $(P_h^n\mathbf{q}_H)_{i_1}=(P_h^n\mathbf{q}_H)_{i_2}$.

Now, we assume that $\mathrm{ind}(i_1)\neq\mathrm{ind}(i_2)$. Thus, there exists a coarse grid edge $j=(j_1,j_2)$ such that either $j_1=\mathrm{ind}(i_1)$, $j_2=\mathrm{ind}(i_2)$ or $j_1=\mathrm{ind}(i_2)$, $j_2=\mathrm{ind}(i_1)$. In both cases there holds $(\mathrm{grad}_H\,\mathbf{q}_H)_j=\pm(q_{H,j_1}-q_{H,j_2})$. The sign in the prolongation compensates, such that $(P_h^e\mathrm{grad}_H\,\mathbf{q}_H)_i=q_{H,\mathrm{ind}(i_1)}-q_{H,\mathrm{ind}(i_2)}$. Evaluating $(\mathrm{grad}_h\,P_h^n\mathbf{q}_H)_i$ gives the same result. ■

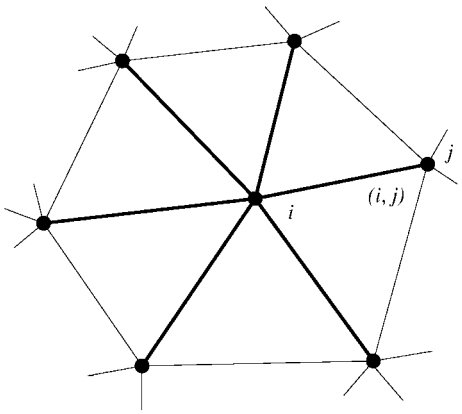

Figure 3. Detail view of a virtual FE-mesh.

Lemma 3.3. *The coarse grid kernel functions are exactly gradient functions, i.e. there holds*

$$V_{0H} = \mathrm{grad}_H\, Q_H \tag{16}$$

Proof
First, we show the inclusion $\mathrm{grad}_H\, Q_H \subset V_{0H}$. We fix a $\mathbf{q}_H \in Q_H$ and define $\mathbf{v}_H = \mathrm{grad}_H\, \mathbf{q}_H$. Using Lemma 3.2 we obtain

$$P_h^e \mathbf{v}_H = P_h^e\, \mathrm{grad}_H\, \mathbf{q}_H = \mathrm{grad}_h\, P_h^n \mathbf{q}_H$$

From $\mathrm{grad}_h\, Q_h = V_{0h}$ there follows $P_h^e \mathbf{v}_H \in V_{0h}$, and from definition (14) of V_{0H} there follows $\mathbf{v}_H \in V_{0H}$.

Now, we verify $V_{0H} \subset \mathrm{grad}_H\, Q_H$. Therefore, we fix a $\mathbf{v}_H \in V_{0H}$. Since the kernels are nested, $\mathbf{v}_h = P_h^e \mathbf{v}_H$ is in V_{0h}, and thus there exists a $\mathbf{q}_h \in Q_h$ such that

$$\mathbf{v}_h = \mathrm{grad}_h\, \mathbf{q}_h$$

By the definition of the prolongation P_h^e, the values of $\mathbf{v}_h$ inside an agglomerate vanish, i.e. $(\mathbf{v}_h)_i = 0$ for $i = (i_1, i_2)$ and $\mathrm{ind}(i_1) = \mathrm{ind}(i_2)$. Since $(\mathbf{v}_h)_i = q_{h,i_1} - q_{h,i_2}$, the potential $\mathbf{q}_h$ is constant inside an agglomerate. Thus there exists a $\mathbf{q}_H \in Q_H$ such that $\mathbf{q}_h = P_h^n \mathbf{q}_H$. Combining the steps and using Lemma 3.2 we obtain

$$\mathbf{v}_h = P_h^e \mathbf{v}_H = \mathrm{grad}_h\, \mathbf{q}_h = \mathrm{grad}_h\, P_h^n \mathbf{q}_H = P_h^e\, \mathrm{grad}_H\, \mathbf{q}_H$$

Since P_h^e has full rank, we can conclude that $\mathbf{v}_H = \mathrm{grad}_H\, \mathbf{q}_H$. ■

3.3. The smoothing operator

To complete the ingredients for an AMG method for edge element FE-discretizations, we need an appropriate smoother. We consider two different types of smoothers for K_h^e. The first one was suggested in Reference [3] by Arnold *et al.* This is a block Gauß-Seidel where all edges are smoothed simultaneously which belong to T_h^i for all $i \in \omega_h^n$ (see Figure 3).

Another kind of smoother was suggested in Reference [2] by Hiptmair. A mathematical equivalent formulation is outlined in Algorithm 2. Therein the vector $\mathbf{g}_h^{e,i} \in V_h$ is defined by

$$\mathbf{g}_h^{e,i} = \mathrm{grad}_h\, \mathbf{g}_h^{n,i} = \begin{cases} 1, & j < i \quad (i,j) \in T_h^i \\ -1, & j > i \quad (i,j) \in T_h^i \\ 0, & \text{otherwise} \end{cases}$$

Algorithm 2. *Hybrid smoother of Hiptmair*: Smooth(K_h^e,$\mathbf{u}_h, \mathbf{f}_h$)
Perform a Gauß-Seidel sweep on K_h^e
i.e. GS($K_h^e, \mathbf{f}_h, \mathbf{u}_h$)
Update the solution $\mathbf{u}_h$
for all $i \in \omega_h^n$ **do**

$$\mathbf{u}_h = \mathbf{u}_h + \frac{(\mathbf{g}_h^{e,i})^{\mathrm{T}}(\mathbf{f}_h - K_h^e \mathbf{u}_h)}{(\mathbf{g}_h^{e,i})^{\mathrm{T}} K_h^e \mathbf{g}_h^{e,i}} \cdot \mathbf{g}_h^{e,i}$$

end for

with the vector $\mathbf{g}_h^{n,i} \in Q_h$, $(\mathbf{g}_h^{n,i})_j = \delta_{ij}$.

Now, the 'setup process' (see Algorithm 3) can be defined, where in Algorithm 3 the parameter COARSEGRID is an appropriately value for which a factorization is applicable in a reasonable CPU-time, and COARSELEVEL stores the number of levels which has been used.

Algorithm 3. *Setup process for edge element AMG*: Setup(K_ℓ^e, K_ℓ^n, ℓ)
if card(ω_ℓ^e) > *COARSEGRID* **then**
Split ω_ℓ^n into disjoint sets ω_C^n and ω_F^n
Set $\omega_{\ell+1}^n = \omega_C^n$
Define the interpolation operator P_ℓ^n
Calculate the coarse grid matrix $K_{\ell+1}^n$ by the Galerkin method
$K_{\ell+1}^n = (P_\ell^n)^{\mathrm{T}} K_\ell^n P_\ell^n$

Define the interpolation operator P_ℓ^e
Calculate the coarse grid matrix $K_{\ell+1}^e$ by the Galerkin method
$K_{\ell+1}^e = (P_\ell^e)^{\mathrm{T}} K_\ell^e P_\ell^e$
Define the index set for the block-smoother

Setup($K_{\ell+1}^e, K_{\ell+1}^n, \ell+1$)
else
Perform a factorization of K_ℓ^e
COARSELEVEL = ℓ
end if

4. NUMERICAL STUDIES

The proposed AMG technique is used as a preconditioner for the preconditioned conjugate gradient (PCG) method (see References [22, 23]). All calculations were done on an SGI Octane, 300 MHz, workstation.

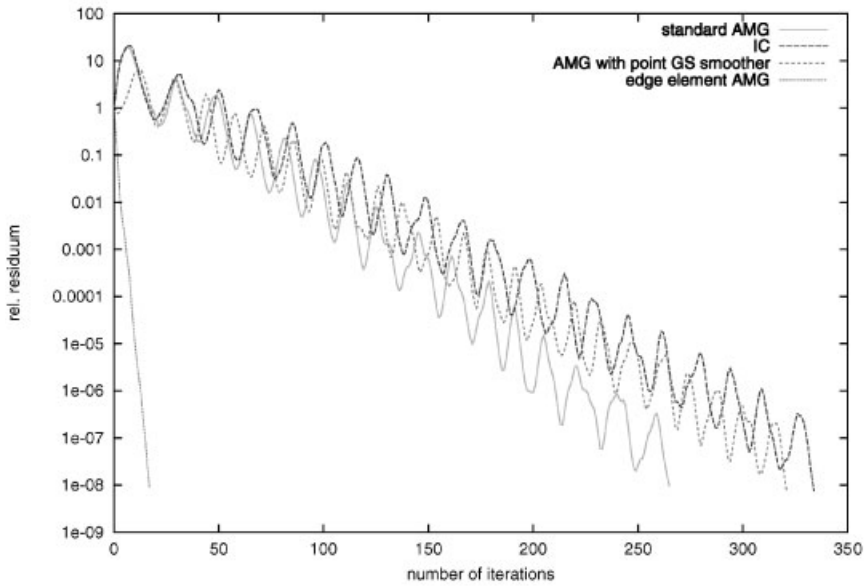

Figure 4. Comparison of different solvers.

First, we show that all ingredients of the proposed AMG method for edge elements are necessary. Let the domain Ω be the unit cube and let us assume homogeneous Dirichlet boundary conditions on $\partial\Omega$. We solve a linear system with $N_h^e = 4184$ unknowns arising for a parameter setting of $\nu = 1$ and $\sigma = 10^{-4}$ in Ω. The results are depict in Figure 4. It can be seen that the standard AMG, which was constructed for $H^1(\Omega)$-elliptic problems fails as well as the incomplete Cholesky preconditioner in the sense of robustness. An interesting observation is that an AMG method with the 'correct' prolongation operator (defined in Section 3), but with a point Gauß–Seidel smoother fails as well.

The domain of the second example is given by $\Omega = \bar{\Omega}_1 \cup \bar{\Omega}_2 \cup \Omega_3$ with $\Omega_1 = (-0.5 + 0.5)^3$, $\Omega_2 = (-1+1)^3 \setminus \bar{\Omega}_1$ and $\Omega_3 = (-5+5)^3 \setminus (\bar{\Omega}_1 \cup \bar{\Omega}_2)$. We use the parameter setting $\nu_1 = 1$, $\nu_2 = 10^{-3}$, $\nu_3 = 1$ for the corresponding domains Ω_1, Ω_2 and Ω_3, respectively, and set the artificial conductivity $\sigma = \nu \cdot 10^{-6}$. The right-hand side is defined by $f = (0,0,1)^{\mathrm{T}}$ in $\bar{\Omega}_1$ and zero else. Again we assume homogeneous Dirichlet boundary conditions on $\partial\Omega$. The following short cuts are used:

- 'setup': CPU-time of the setup process in seconds,
- 'solver': CPU-time for the iterations of the PCG method in seconds,
- 'solution': overall CPU-time, i.e. setup and solver, in seconds,
- 'iteration': number of iterations in the PCG method,
- 'level': number of generated levels in the AMG method, i.e. COARSEGRID $\leqslant 500$.

The iteration was stopped if an error reduction in the $K_h^e C_h^{-1} K_h^e$-energy norm by a factor of 10^{-6} had been achieved, where C_h^{-1} denotes the preconditioner. The subsequent example is calculated with the proposed AMG method for edge elements constructed in the previous sections. First of all, the number of edges and nodes of the fine grid are given in Table I.

Table I. Number of edges and nodes on the finest level and number of generated levels.

N_h^e	N_h^n	level
2473	333	2
19174	2871	3
151260	22045	5

Table II. Smoother of Arnold *et al.*

N_h^e	setup (s)	solver (s)	solution (s)	iteration
2473	0.28	0.31	0.59	8
		0.32	0.60	8
19174	1.34	5.01	6.35	12
		4.91	6.25	11
151260	10.52	80.51	91.03	22
		75.50	86.02	18

Table III. Smoother of Hiptmair.

N_h^e	setup (s)	solver (s)	solution (s)	iteration
2473	0.23	0.50	0.73	12
		0.51	0.74	12
19174	0.95	8.85	9.80	18
		7.94	8.89	16
151260	7.23	142.7	149.9	35
		126.7	133.9	27

In addition, the number of constructed levels is included therein. The intrinsic results of the AMG method are given in Table II for the smoothing iteration of Arnold *et al.* and in Table III for the smoother of Hiptmair. Every row of Tables II and III consists of two sub-rows. The first one is directed to a $V(2,2)$-cycle and the second one to a generalized V-cycle with 2^ℓ smoothing steps on level ℓ. In both cases, a slight increase in the number of iterations can be detected with respect to the number of unknowns. This is an effect of the designed prolongation operator, since it suffer from the poor approximation property. As it can be expected the generalized V-cycle performs better for both smoothers compared to the V(2,2)-cycle. Actually there are no big differences between the different cycles with respect to CPU-time. The overall computation time is better with the smoother of Arnold *et al.* than with the smoother of Hiptmair for the considered example.

Table IV. Number of iterations for different values of σ and mesh sizes.

N_h^e	10^{-4}	10^{-6}	10^{-8}
2473	8	8	7
19174	13	12	11
151260	24	22	20

Next, we consider different values of the artificial conductivity σ and the corresponding number of iterations, see Table IV. These computations are done with the smoother of Arnold *et al*. As expected, the number of iterations stays constant with respect to σ, whereas the number of iterations grows slightly with the mesh size.

5. CONCLUSIONS AND FURTHER REMARKS

A new AMG approach was proposed for the solution of $H_0(\mathrm{curl}, \Omega)$-conforming FE-discretization with edge elements. Therefore, a coarsening technique based on an auxiliary matrix was introduced in order to setup a proper prolongation operator and an appropriate smoother for the system matrix. The numerical studies show the independence of the method of parameter jumps and the artificial conductivity, but the number of required PCG iterations slightly depend on the mesh size. Nevertheless it performs much better than standard preconditioners (i.e. incomplete Cholesky preconditioner). An improvement of the proposed prolongation operator is under current research, in order to get better convergence rates.

ACKNOWLEDGEMENTS

The authors would like to thank Dr. M. Kuhn for many helpful discussions. This work has been supported by the Austrian Science Fund under the grand 'Fonds zur Förderung der wissenschaftlichen Forschung (FWF)'—under the project SFB F013 'Numerical and Symbolic Scientific Computing'.

REFERENCES

1. Nédélec J. A new family of mixed finite elements in R^3, *Numerische Mathematik* 1986; **50**:57–81.
2. Hiptmair R. Multigrid method for Maxwell's equations. *SIAM Journal on Numerical Analysis* 1999; **36**(1): 204–225.
3. Arnold D, Falk R, Winther R. Multigrid in $H(\mathrm{div})$ and $H(\mathrm{curl})$. *Numerische Mathematik* 2000; **85**:197–218.
4. Beck R, Deuflhard P, Hiptmair R, Hoppe, R Wohlmuth B. Adaptive multilevel methods for edge element discretizations of Maxwell's equations. *Surveys on Mathematics for Industry* 1999; **8**:271–312.
5. Kuhn M, Langer U, Schöberl J. Scientific computing tools for 3D magnetic field problems, In Whiteman JR (ed.). *The Mathematics of Finite Elements and Applications X*. Elsevier: Amsterdam, 2000; 239–258.
6. Schinnerl M, Schöberl J, Kaltenbacher M. Nested multigrid methods for the fast numerical computation of 3D magnetic fields, *IEEE Transactions on Magnetics*, 2000; **36**(4):1557–1560.
7. Hackbusch W. *Multigrid Methods and Application*. Springer: Berlin, Heidelberg, New York, 1985.
8. Braess D. Towards algebraic multigrid for elliptic problems of second order. *Computing* 1995; **55**:379–393.
9. Brandt A. Algebraic multigrid theory: the symmetric case. *Applied Mathematics and Computation* 1986; **19**: 23–56.
10. Brandt A, McCormick S, Ruge JW. Algebraic multigrid (AMG) for automatic multigrid solution with application to geodetic computations. *Report*, *Inst. Comp. Studies Colorado State Univ.*, *1982*.

11. Brandt A, McCormick S, Ruge JW. *Algebraic multigrid* (*AMG*) *for sparse matrix equations*. In Evans DJ (ed.). *Sparsity and its Application*, Cambridge: Cambridge University Press, 1985; 257–284.
12. Kickinger F. Algebraic multigrid for discrete elliptic second-order problems, Multigrid Methods V. *Proceedings of the 5th European Multigrid conference*, Hackbusch W (ed.). Springer Lecture Notes in Computational Science and Engineering, vol. 3. Springer: Berlin, 1998; 157–172.
13. Ruge JW, Stüben K. Efficient solution of finite difference and finite element equations. In *Multigrid Methods for integral and differential equations*, Paddon D, Holstein H (eds), vol. 3. Clarendon Press: Oxford, 1985; 169 –212.
14. Ruge JW, Stüben K. Algebraic multigrid (AMG), multigrid methods, In *Frontiers in Applied Mathematics*, McCormick S (ed.) vol. 5. SIAM: Philadelphia, 1986:73–130.
15. Vanek P, Mandel J, Brezina M. Algebraic multigrid by smoothed aggregation for second and fourth order elliptic problems. *Computing* 1996; **56**:179–196.
16. Beck R. Algebraic multigrid by component splitting for edge elements on simplicial triangulations. *Preprint* SC 99-40, Konrad-Zuse-Zentrum für Informationstechnik Berlin, December 1999.
17. Ida N, Bastos PA. *Electromagnetics and Calculation of Fields*, Springer: Berlin, 1997.
18. Demkowicz L, Vardapetyan L. Modeling of Electromagnetic Absoption/Scattering Problems Using hp-Adaptive Finite Elements. *Computer Methods in Applied Mechanics and Engineering* 1998; **1**:103–124.
19. Brezzi F, Fortin F. *Mixed and Hybrid Finite Element Methods.* Springer: Berlin, 1991.
20. Costabel M, Dauge M. Singularities of electromagnetic fields in polyhedral domains. *Archive for Rational Mechanics and Analysis* 2000; **151**:221–276.
21. Ciarlet PG. *The Finite Element Method for Elliptic Problems*. North-Holland Publishing Company, Amsterdam: New York, Oxford, 1987.
22. Jung M, Langer U. Applications of multilevel methods to practical problems. *Surveys on Mathematics for Industry* 1991; **1**:217–257.
23. Jung M, Langer U, Meyer A, Queck W, Schneider M. Multigrid preconditioners and their application. *Proceedings of the 3rd GDR Multigrid Seminar held at Biesenthal*, Karl-Weierstraß-Institut für Mathematik, May 1989:11–52.

