Algebraic Multigrid

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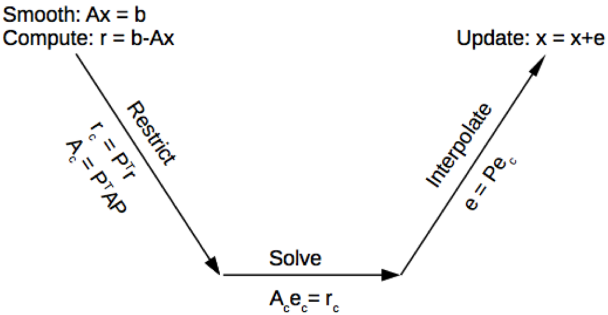
- Solving $n \times n$ linear system

$$Ax = b$$

- P prolongation (maps $\mathbb{R}^m \to \mathbb{R}^n$ where $m < n$)
- $P^\top$ restriction (maps $\mathbb{R}^n \to \mathbb{R}^m$)
- coarse grid operator $A_c = P^\top A P$ (Galerkin operator)


Smooth: Ax = b
Compute: r = b-Ax
Restrict
r_c = P^T r
A_c = P^T AP
Solve
A_c e_c = r_c
Interpolate
e = Pe_c
Update: x = x+e

Smoothness:

$$e^{\mathsf{T}} A e = \lambda \ll 1$$

$$e^{\mathsf{T}} A e = \sum_{i<j} (-a_{ij})(e_i - e_j)^2 \ll 1$$

Strength of Connection

$$-a_{ij} \geq \theta \max_{k \neq i} \{-a_{ik}\} \quad \text{where } \theta \in (0, 1]$$

Choose grid

1. Define strength matrix A_s
2. Choose set of fine points based on A_s
3. Choose extra points to satisfy interpolation requirements

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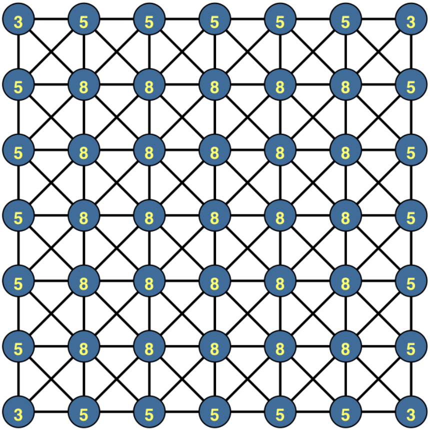
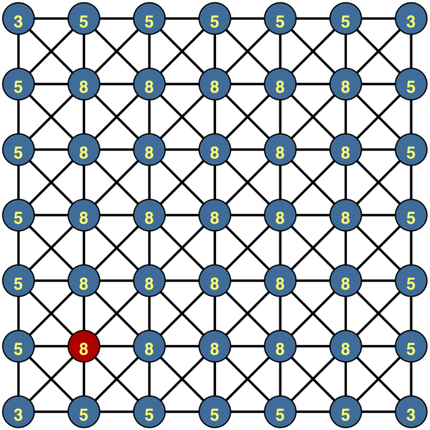
FE Poisson stencil:

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$



- select C-pt with maximal measure
- select neighbours as F-pts
- update measures of F-pt neighbours



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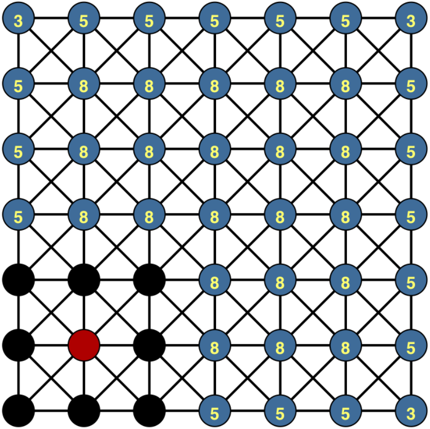

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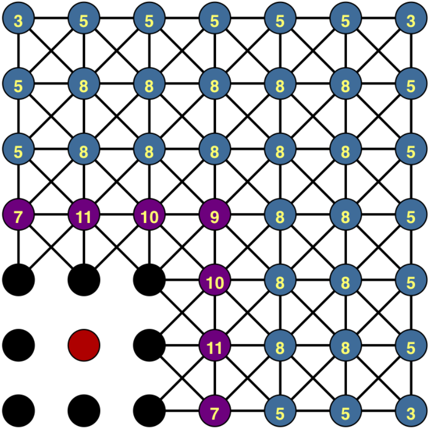

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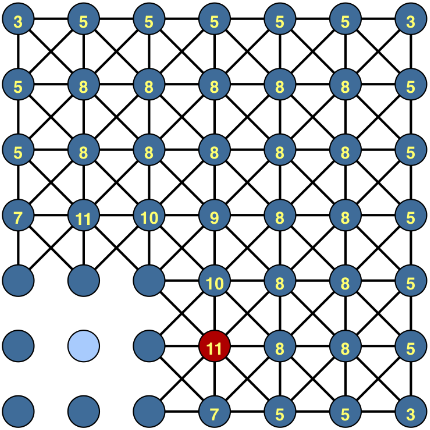

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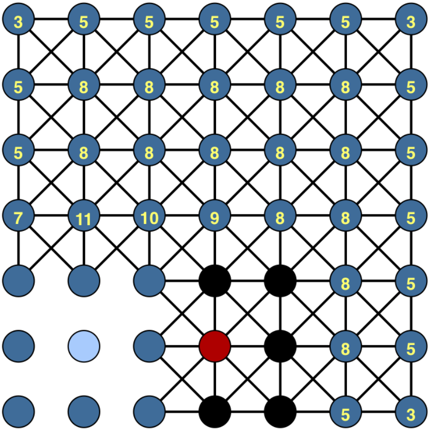

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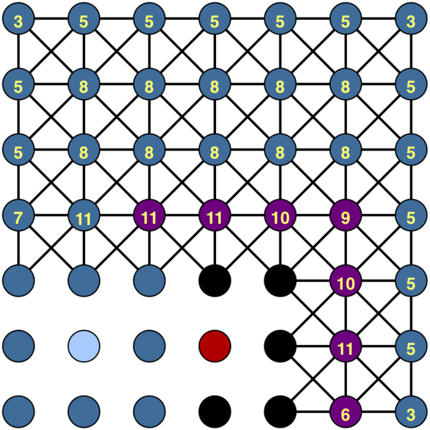

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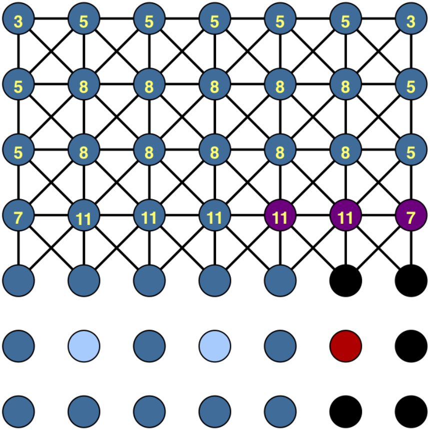

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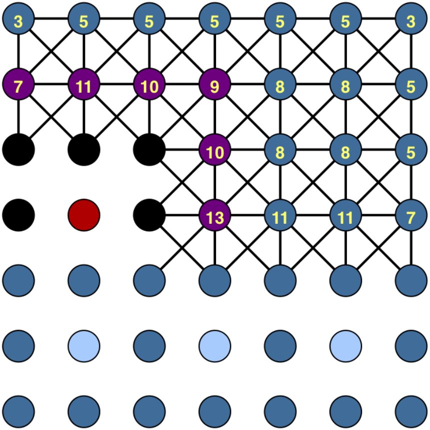

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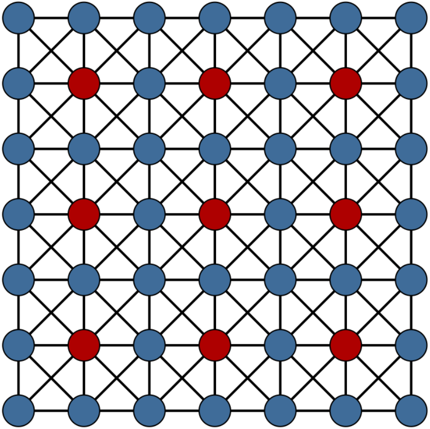


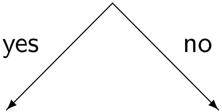
Falgout (2006)

- select C-pt with maximal measure
- select neighbours as F-pts
- update measures of F-pt neighbours

Second pass

Classical AMG:

- Loop though F-points
- find pairs of F-points that are strongly connected
- check F-point pair strongly connected to C-point



check next F-point pair

make one F-point a C-point

Interpolate

Smooth error:

$$\lambda^2 = e^\mathsf{T} A^\mathsf{T} A e = r^\mathsf{T} r = ||r|| \ll 1$$

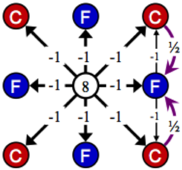
Derive interpolation:

$$r_i = (Ae)_i = 0$$

$$a_{ii} e_i = -\sum_{j \in C_i} a_{ij} e_j - \sum_{j \in F_i} a_{ij} e_j - \sum_{j \in N_i} a_{ij} e_j$$

C_i: C-points strongly connected to i
F_i: F-points strongly connected to i
N_i: all points weakly connected to i

Collapse stencil



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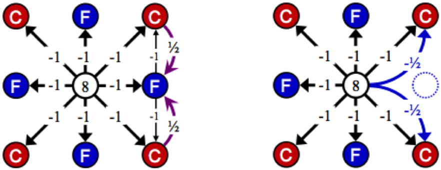

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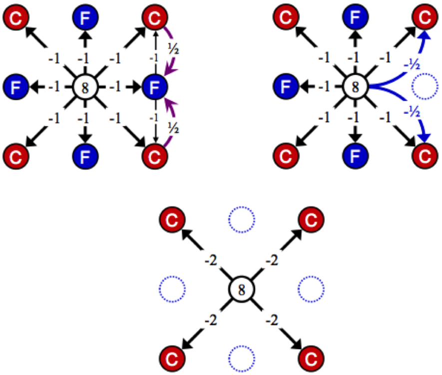

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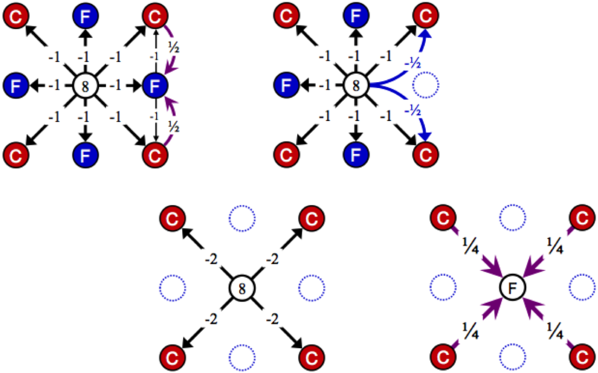

Falgout (2006)

Poisson

$$-\Delta \vec{u} = \vec{f} \quad \text{in } \Omega$$
$$\vec{u} = \vec{0} \quad \text{on } \partial\Omega$$

Grid size	DoF	AMG # iters	AMG Soln Time	ILU # iters	ILU Soln Time	Direct (MUMPS) Soln Time
2^2	18	1	1.31e-05	1	6.91e-06	5.29e-04
4^2	50	2	3.60e-05	5	1.22e-05	4.99e-04
8^2	162	3	1.25e-04	8	5.41e-05	8.80e-04
16^2	578	4	5.80e-04	14	1.96e-04	2.89e-03
32^2	2178	4	1.94e-03	25	1.38e-03	9.74e-03
64^2	8450	4	7.73e-03	48	1.06e-02	6.84e-02
128^2	33282	4	3.01e-02	93	7.92e-02	3.38e-01
256^2	132098	4	1.36e-01	181	6.99e-01	1.77e+00
512^2	526338	4	6.05e-01	349	5.81e+00	9.76e+00
1024^2	2101250	4	2.49e+00	668	4.62e+01	6.33e+01
2048^2	8396802	4	9.98e+00	1272	3.47e+02	5.66e+02

3 Dimensional example

Grid size	DoF	AMG # iters	AMG Soln Time	ILU # iters	ILU Soln Time	Direct (MUMPS) Soln Time
2^3	81	1	1.81e-05	1	7.87e-06	7.22e-04
4^3	375	2	1.19e-04	4	3.60e-05	1.50e-03
8^3	2187	3	1.37e-03	8	3.85e-04	9.22e-03
16^3	14739	3	1.24e-02	14	5.35e-03	2.44e-01
32^3	107811	3	1.26e-01	26	8.98e-02	1.29e+01
64^3	823875	4	1.63e+00	45	1.31e+00	1.04e+03
128^3	6440067	4	1.60e+01	84	1.94e+01	-

Summary

- Tries to mimic GMG
- Relies on matrix coefficients
- No geometric information needed
- Black box for elliptic problems

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