Direct Methods

October 31, 2013

$Ax=b$

Direct methods - black box

LAPACK - dense matrices

Sparse matrizies - Sparskit, PetSc

All Gaussian Elimination (G.E.)

$A=LU$, $A=FF^T$ Cholesky (A SPD)

A symm. indefinite

Bunch-Kaufman : $A=PLDL^TP^T$ D-block diagonal with 1×1 + 2×2 blocks

MA 57

$$C=\begin{pmatrix} \varepsilon & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\varepsilon} & 1 \end{pmatrix}\begin{pmatrix} \varepsilon & 0 \\ 0 & -\frac{1}{\varepsilon} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{\varepsilon} \\ 0 & 1 \end{pmatrix}$$

$$\begin{array}{c} n \\ m \end{array}\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ BA^{-1} & I \end{pmatrix}\underbrace{\begin{pmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{pmatrix}}_{\substack{n \text{ pos e.g.,} \\ m \text{ eig eig if } A \text{ is SPD}}}\begin{pmatrix} I & A^{-1}B^T \\ 0 & I \end{pmatrix}$$

$\uparrow$

LDL^T decomp.

$$x=L^{-T}(D^{-1}(L^{-1}b))$$

typically follow

$Ax=b$

$A=LU$

$L\underbrace{Ux}_{y}=b$

$$\begin{cases} Ly=b \\ Ux=y \end{cases}$$

what if ε small?

$C=\begin{pmatrix} \varepsilon & 1 \\ 1 & 0 \end{pmatrix}$ - just a permuted identity, nearly perfectly conditioned.

but when we decompose $\rightarrow$ poor condition numbers

D is not something we want to invert

This is why we allow D to have 2×2 blocks (Bunch-Kaufman).

1) a few flavors

LU

LDL^T, LTL^T (A asym)

FF^T

2) Pivoting: swap rows to keep largest elt. in diagonal (pivot) → partial pivoting (GEPP - not stab

: complete pivoting, swapping rows + columns

Take $A \to U$

swapping rows → relabel RHS

cols → relabel unknowns

: rook pivoting, hybrid btwn complete + partial.

Symmetric Permutations:

$$Ax = b$$

$$(P^TAP)(P^Tx) = P^Tb$$ → important for pre-ordering for sparse matrices

$$\tilde{A}\tilde{x} = \tilde{b}$$

3) order of operat^ns in G.E.

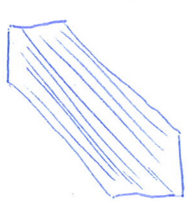
- eg KIJ order

(see, eg. Saad Ch 10).

- significant in ILU factorizations.

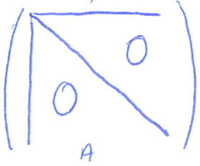
Sparse Matrices

• narrow-banded

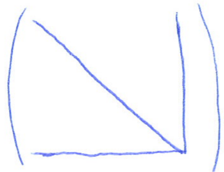


no fill, but if mtx is sparse w/in the band, then there is fill w/in the band

• arrow matrix

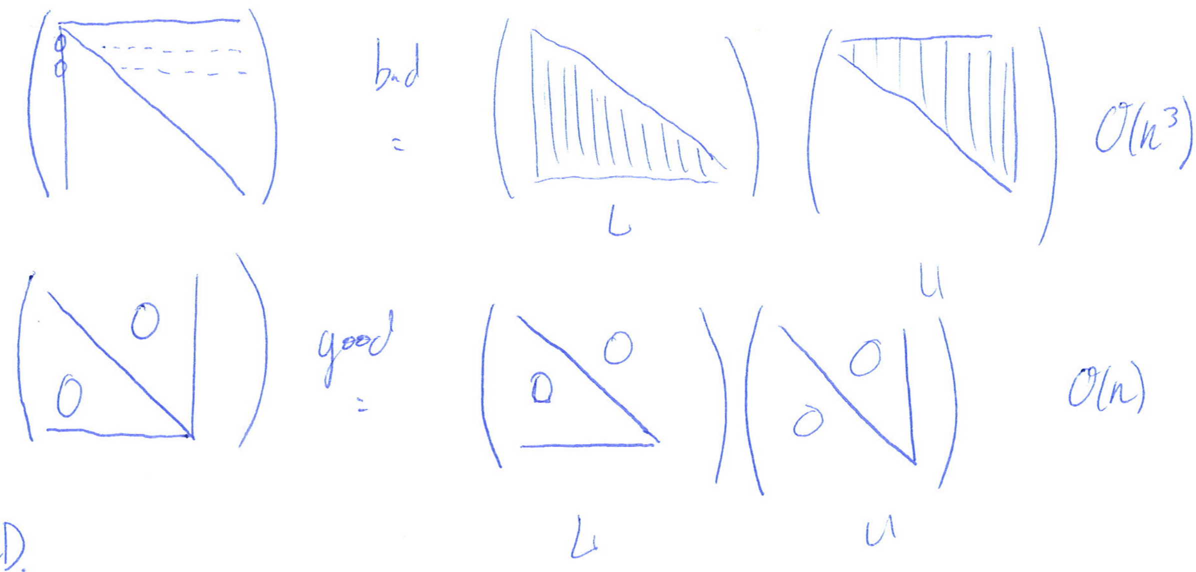
A = 

$$P = \begin{pmatrix} 0 & & & & 1 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & & & & 0 \end{pmatrix}$$

$PAP^T =$ 

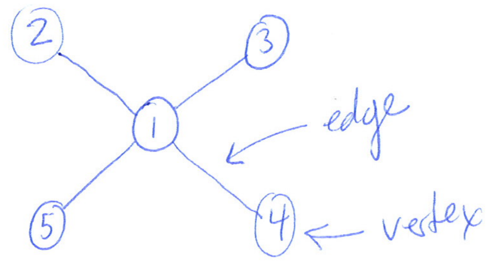
Which one is better for GE?
↳ PAP^T is better

for A get fill in L, U



RCM, AMD.

Graph of a matrix

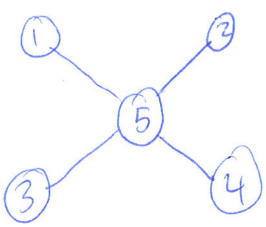
A: 

a_{14}, a_{41}
a_{12}, a_{21}
a_{13}, a_{31}
a_{15}, a_{51}

all nonzer.

(Structurally symmetric)

PAP^T



← Symmetric permutation is equivalent to relabelling nodes

degree of a node: numbers of nodes connected to the node

We want (maybe) to order nodes w/ lowest degree first.

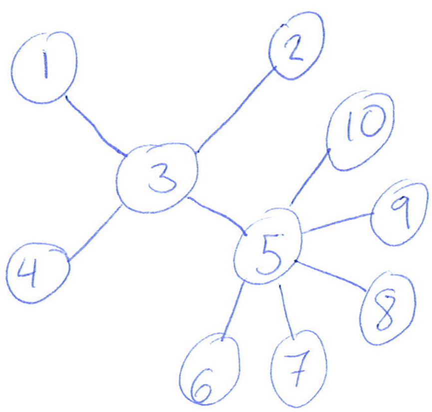
Orderings -

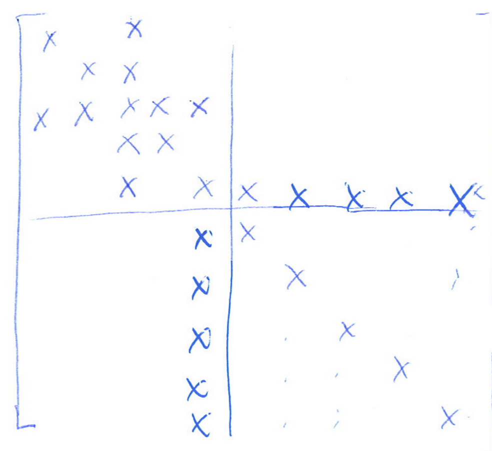
① Try to reduce bandwidth

② Try to see how to order nodes so that the fill is reduced

③ Try to "cut" graphs into 2, recursively

BFS : breadth first search





1) Locally, do the best we can

level set   level set   level set   level set

1 , 3 , 2 , 4 , 5 , 6 , 7 , 8 , 9 , 10

reverse the order

RCM : 10 , 9 , 8 , 7 , 6 , 5 , 4 , 2 , 3 , 1