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$x_{k+1} = x_k + \alpha_k r_k$

$\min \|r_{k+1}\|_2$

$b - A \cdot [x_{k+1} = x_k + \alpha_k r_k]$

$r_{k+1} = r_k - \alpha_k A r_k$

$\|r_{k+1}\|_2^2 = r_{k+1}^T r_{k+1} = (r_k - \alpha_k A r_k)^T (r_k - \alpha_k A r_k)$

: quadratic in $\alpha_k$ (scalar quantity)

$\rightarrow$ minimize (differentiate + equate to zero)

$$\alpha_k = \frac{(r_k, Ar_k)}{(Ar_k, Ar_k)}$$ where $(r_k, r_k) = r_k^T r_k$

Algorithm: MR: Minimum Residual Scheme

$r = b - Ax$, $p = Ar$ initialization

repeat

$\alpha \leftarrow \frac{(r, Ar)}{(p, p)}$

$x \leftarrow x + \alpha r$

$r \leftarrow r - \alpha p$

$p \leftarrow Ar$ only 1 matrix-vector product

## Projection Methods

$Ax = b$

Search space : $K$, m-dimensional

subspace of constraints : $L$, m-dimensional

Can ask : $b - Ax \perp L$ - Petrov-Galerkin

if $K = L$ : orthogonal projection

$K \neq L$ : oblique projection.

General Framework: $\boxed{\tilde{x} \in x_0 + K \quad \text{s.t.} \quad b - A\tilde{x} \perp L}$

$K$ would be, for example: Krylov Subspace

$$\mathcal{K}^k(A;r_0) = \text{Span}\{r_0, Ar_0, \ldots, A^{k-1}r_0\}$$

Suppose

$V = [v_1, \ldots, v_m]$ is a basis for $K$

$W = [w_1, \ldots, w_m]$ is a basis for $L$

$x = x_0 + y_1 v_1 + y_2 v_2 + \ldots + y_m v_m = x_0 + Vy$

$\{y_i\}$: scalars

orthogonality

$W^T(b - Ax) = 0$

$W^T b = W^T A(x_0 + Vy)$

$W^T r_0 = W^T A V y$

$y = (W^T A V)^{-1} W^T r_0$

$$\boxed{x = x_0 + V(W^T A V)^{-1} W^T r_0}$$

Choice of Subspaces $L, K$

- $K = L$, $A$ is SPD $\longrightarrow$ Conjugate Gradient $\rightarrow \min \|e\|_A$
- $L = AK$, $A$ is nonsingular $\rightarrow$ GMRES, MINRES $\rightarrow \min \|r\|_2$

recall: $\|x\|_A = \sqrt{x^T A x}$

A-norm

energy-norm

Note: need $W^T A V$ nonsingular

Easy to show, for example, $L = K$, $A$ SPD

$W = VG$, $G$ nonsingular

$W^T A V = \underset{\text{nonsingular}}{G^T} \underbrace{V^T A V}_{\text{SPD}}$ $\therefore$ $W^T A V$ nonsingular

$\hookrightarrow$ since $x^T V^T A V x = y^T A y$ : $A$ SPD, $V$ basis

done!

$\min \|e_k\|_A$

$\|x\|_A = \sqrt{x^T A x}$

why is this a norm?

$$\|e_k\|_A^2 = e_k^T A e_k = \begin{cases} >0 & e_k \neq 0 \\ =0 & e_k = 0 \end{cases}$$

We don't know the error! But...

$$\|e_k\|_A = \|r_k\|_B$$

$$A e_k = r_k$$

$$e_k = A^{-1} r_k$$

$$e_k^T A e_k = r_k^T A^{-1} A A^{-1} r_k$$

$$= r_k^T A^{-1} r_k \quad \Rightarrow B = A^{-1}$$

1-D Projection

$K = \text{span}\{v\}$ $\qquad$ $L = \text{span}\{w\}$

$$x \leftarrow x + \alpha v$$

$$x = x_0 + v(w^T A v)^{-1} w^T r_0$$

$$\alpha = \frac{w^T r_0}{w^T A v} \Big\} \text{ scalars}$$

if

$$\left.\begin{array}{l} v = r \text{ (residual)} \\ w = Ar \end{array}\right\} \Rightarrow \alpha = \frac{r^T A r}{r^T A^T A r} = \frac{(r,p)}{(p,p)} \quad \text{where } p = Ar$$

$\min \|r_k\|_2$

MR

Steepest descent:

$v = r$

$A$ is SPD

$w = v = r$

$$\alpha = \frac{(r,r)}{(p,r)} \quad p = Ar$$