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FAST ITERATIVE SOLUTION OF STABILISED STOKES SYSTEMS PART II: USING GENERAL BLOCK PRECONDITIONERS*

DAVID SILVESTER† AND ANDREW WATHEN‡

Abstract. Mixed finite element approximation of the classical Stokes problem describing slow viscous incompressible flow gives rise to symmetric indefinite systems for the discrete velocity and pressure variables. Iterative solution of such indefinite systems is feasible and is an attractive approach for large problems. Part I of this work described a conjugate gradient-like method (the method of preconditioned conjugate residuals) which is applicable to symmetric indefinite problems [A. J. Wathen and D. J. Silvester, *SIAM J. Numer. Anal.*, 30 (1993), pp. 630–649]. Using simple arguments, estimates for the eigenvalue distribution of the discrete Stokes operator on which the convergence rate of the iteration depends are easily derived. Part I discussed the important case of diagonal preconditioning (scaling). This paper considers the more general class of block preconditioners, where the partitioning into blocks corresponds to the natural partitioning into the velocity and pressure variables. It is shown that, provided the appropriate scaling is used for the blocks corresponding to the pressure variables, the preconditioning of the Laplacian (viscous) terms determines the complete eigenvalue spectrum of the preconditioned Stokes operator.

All conventional preconditioners for the Laplacian including (modified) incomplete factorisation, hierarchical basis, multigrid and domain decomposition methods are covered by the analysis. It is shown that if a spectrally equivalent preconditioner is used for the Laplacian terms then the convergence rate of iterative solution algorithms is independent of the mesh-size. The results apply to both locally and globally stabilised mixed approximations as well as to mixed methods that are inherently stable.

Key words. Stokes problem, mixed finite elements, preconditioning

AMS subject classifications. 65N30, 65F10

1. Introduction.

The aim of this work is to investigate the behaviour of a particular Krylov subspace iteration, the method of preconditioned conjugate residuals (PCR), when used to solve discretised Stokes systems arising in the modeling of viscous incompressible flow. In so doing, we derive results that would be required for convergence estimates for other iterative solution strategies.

In recent years, new discretisation methods have been developed for the Stokes and Navier–Stokes equations. These so-called stabilised formulations allow the approximation of the component velocity and pressure variables to be chosen independently. For a review of the possibilities, see [FHS] or [T]. Working in a conventional unstabilised framework, the component approximations of velocity and pressure have to be compatible if convergence to the true solution under mesh refinement is to be guaranteed. The condition for compatibility is popularly referred to as the LBB stability condition. Typical computationally convenient methods are not LBB stable, for example, approximating the velocity and pressure fields using a single grid using equal-order interpolation is notoriously unstable. Finite difference and finite volume schemes for solving Stokes and Navier–Stokes equations are subject to similar considerations, forcing practitioners to construct approximations using staggered grids; see, for example, [STW].

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†Department of Mathematics, University of Manchester Institute of Science and Technology, P.O. Box 88, Manchester M60 1QD, United Kingdom (na.silvester@na-net.ornl.gov).

‡Department of Mathematics, University of Bristol, Bristol BS8 1TW, United Kingdom.

A feature of the new methodology is that the resulting methods typically involve one or more "stabilisation parameters." Although there has been some work on the optimal choice of such parameters from the point of view of solution accuracy (see [P], for example), the effect of such parameters on the rate of convergence of iterative solution methods has received little attention.

In some previous work [WS], henceforth referred to as part I, we considered the importance of appropriately scaling the stabilised Stokes operator. It is proved in part I that after diagonal preconditioning, and assuming a quasi-uniform sequence of grids, the eigenvalues of the symmetric indefinite Stokes operator lie in the union of real intervals

$$(-a, -bh) \cup (ch^2, d), \tag{1.1}$$

where h is the characteristic mesh size (which will approach zero under mesh refinement), and a, b, c, and d are positive constants independent of h. The interesting feature of (1.1) is that the negative eigenvalues spread out less rapidly under mesh refinement than the positive eigenvalues. That this is true more generally is proved in this paper, which extends the ideas in part I to cover general block preconditioners. Our results apply to both locally and globally stabilised mixed approximations as well as to methods that are inherently LBB stable. As a special case, we will show that "optimal" preconditioners for the Laplacian subblocks give rise to "optimal" preconditioners for the Stokes problem in the sense that the eigenspectra of the underlying discrete operator are contained in small clusters bounded independently of h. Thus the convergence of PCR will also be independent of h in this case.

We retain the notation of part I throughout this work. For completeness, however, we will review the important definitions from part I in the remainder of this section. The underlying mixed approximation is assumed to involve low-order finite elements so that a *stabilised* discrete Stokes formulation is of the form: find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$\begin{aligned} (\mathrm{grad}\mathbf{u}_h, \mathrm{grad}\mathbf{v}) - (p_h, \mathrm{div}\mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ -(q, \mathrm{div}\mathbf{u}_h) - \beta \mathcal{C}_h(p_h, q) &= 0 \quad \forall q \in P_h. \end{aligned} \tag{1.2}$$

Here, $\mathbf{V}_h$ and P_h are finite element function spaces. $\mathcal{C}_h(\cdot, \cdot)$ is the stabilisation term that is omitted in the conventional (unstabilised) formulation, and $\beta > 0$ is the stabilisation parameter. Expressed in matrix form the system (1.2) is

$$\begin{pmatrix} A & B^t \\ B & -\beta C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ o \end{pmatrix}. \tag{1.3}$$

In (1.3), A represents a block diagonal matrix of discrete Laplacians, $-C$ the stabilisation matrix, and B is the coupling term between velocities and pressure. A is a positive definite $n \times n$ matrix. C is a positive semidefinite $m \times m$ matrix (thus B is $m \times n$). In all practical cases $n > m$.

As discussed in part I, to assess the compatibility or otherwise of the two spaces $\mathbf{V}_h$ and P_h, the Schur complement (from (1.3)) $BA^{-1}B^t + \beta C$ plays a crucial role. In particular, for an LBB stable element ($C = 0$) we know that there exists a constant γ such that

$$\gamma^2 \leq \frac{p^t(BA^{-1}B^t)p}{p^t M_p p} \quad \forall p \in \Re^m - \{0\}. \tag{1.4}$$

Here, as in part I, M_p is the pressure mass matrix.

Note that the boundedness of B implies the existence of a constant Γ such that

$$\frac{p^t(BA^{-1}B^t)p}{p^tM_pp} \leq \Gamma^2 \quad \forall p \in \Re^m - \{0\}. \tag{1.5}$$

For an unstable element, only the upper bound (1.5) on the Schur complement holds. Instability can arise either because the LBB constant γ is zero (corresponding to "pure" spurious pressure modes) or it is dependent on the mesh parameter h. To derive h-independent eigenvalue bounds in the stabilised case, the fundamental property required is that a critical parameter value β_0 exists, independent of h, such that for all parameter values $\beta > \beta_0$

$$\gamma^2 \leq \frac{p^t(BA^{-1}B^t + \beta C)p}{p^tM_pp} \quad \forall p \in \Re^m - \{0\}, \tag{1.6}$$

with γ independent of h and β. This condition is an obvious extension of (1.4) to the stabilised case.

Note that combining (1.6) with the boundedness of C (cf. (2.9)) and B leads to the bound

$$\gamma^2 \leq \frac{p^t(BA^{-1}B^t + \beta C)p}{p^tM_pp} \leq \widetilde{\Gamma}^2. \quad \forall p \in \Re^m - \{0\} \tag{1.7}$$

for all bounded values of $\beta > \beta_0$ and for some constant $\widetilde{\Gamma}$ independent of h. In part I, (1.7) was referred to as the (*uniform-*)*stabilisation condition.* In this paper we aim to determine eigenvalue bounds explicitly in terms of β. As shown in the next section, this can be more conveniently done using (1.6) and (1.5) directly in place of the combined bound (1.7).

The preconditioned conjugate residual algorithm is described in §2 of part I. Convergence results for the method are all that is required here: if r_k denotes the residual of the kth iterate, then

$$\frac{\|r_k\|}{\|r_0\|} \leq \min_{p \in \Pi_k^1} \max_i |p(\mu_i)| = e_k,$$

where Π_k^1 is the set of kth degree real polynomials that satisfy $p(0) = 1$, and $\{\mu_i\}$ are the eigenvalues of the preconditioned Stokes matrix. In [WFS] it is proved that if the eigenvalues μ_i lie in inclusion intervals of the form

$$[-a, -b\alpha] \cup [c\alpha^2, d], \tag{1.8}$$

where a, b, c, and d are positive constants and α is an asymptotically small positive parameter, then the asymptotic PCR convergence rate [V, Eq. 3.24] is

$$\lim_{k \to \infty} e_k^{\frac{1}{k}} = 1 - \alpha^{\frac{3}{2}} \sqrt{bc/ad} + O(\alpha^{\frac{5}{2}}). \tag{1.9}$$

See [WFS] for the proof and further discussion of this estimate. The number of PCR iterations required to reduce the residual by a fixed factor is then $O(\alpha^{-3/2})$. The more "standard" estimate of PCR convergence is based on embedding a spectrum such as

(1.8) in positive and negative intervals that are of equal length. Use of Lebedev's results [L] then gives

$$e_k^{\frac{1}{k}} \leq 2\left(\frac{1-\sqrt{\hat{b}\hat{c}/\hat{a}\hat{d}}}{1+\sqrt{\hat{b}\hat{c}/\hat{a}\hat{d}}}\right)^{\frac{1}{2}}, \tag{1.10}$$

where the spectrum is contained in $[-\hat{a},-\hat{b}]\cup[\hat{c},\hat{d}]$ and $\hat{a}-\hat{b}=\hat{d}-\hat{c}>0$ (see, for example, [CES, Thm. 3.2]). For a spectrum of the form (1.8) this approach gives an estimate of the form

$$\lim_{k\to\infty} e_k^{\frac{1}{k}} \leq 1 - O(\alpha^2). \tag{1.11}$$

The remainder of the paper is set out as follows. In §2 we derive our eigenvalue bounds. In §3 we present the results of numerical experiments that test the sharpness of our eigenvalue calculations, and in §4 we draw some conclusions.

2. Eigenvalue estimates. Our aim is to solve systems with the symmetric coefficient matrix

$$\mathcal{A} = \begin{pmatrix} A & B^t \\ B & -\beta C \end{pmatrix}. \tag{2.1}$$

Given the general positive definite preconditioner

$$\mathcal{M} = \begin{pmatrix} M_A & O \\ O & M_C \end{pmatrix}, \tag{2.2}$$

where M_A and M_C are both symmetric positive definite matrices, the convergence of the PCR algorithm is determined by the values of the optimal minimax polynomials of increasing degree on the eigenvalues of the symmetrically preconditioned matrix

$$\mathcal{M}^{-\frac{1}{2}}\mathcal{A}\mathcal{M}^{-\frac{1}{2}} = \begin{pmatrix} M_A^{-\frac{1}{2}}AM_A^{-\frac{1}{2}} & M_A^{-\frac{1}{2}}B^tM_C^{-\frac{1}{2}} \\ M_C^{-\frac{1}{2}}BM_A^{-\frac{1}{2}} & -\beta M_C^{-\frac{1}{2}}CM_C^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \widetilde{A} & \widetilde{B}^t \\ \widetilde{B} & -\beta\widetilde{C} \end{pmatrix} = \widetilde{\mathcal{A}}. \tag{2.3}$$

Note that this is a slightly different approach from that used in part I; there the preconditioner $\mathcal{M}$ explicitly involved the parameter β.

By defining the congruence transformation of $\widetilde{\mathcal{A}}-\xi I$ given by

$$\begin{pmatrix} I & O \\ \widetilde{B}(\widetilde{A}-\xi I)^{-1} & I \end{pmatrix}\begin{pmatrix} \widetilde{A}-\xi I & O \\ O & -\widetilde{B}(\widetilde{A}-\xi I)^{-1}\widetilde{B}^t-\beta\widetilde{C}-\xi I \end{pmatrix}\begin{pmatrix} I & (\widetilde{A}-\xi I)^{-1}\widetilde{B}^t \\ O & I \end{pmatrix} \tag{2.4}$$

and setting $\xi = 0$, it is clear from Sylvester's law of inertia that the stabilisation condition (1.6) ensures that $\widetilde{\mathcal{A}}$ has a spectrum of n positive and m negative eigenvalues. Retaining the notation of part I we order the eigenvalues of $\widetilde{\mathcal{A}}$ as

$$\mu_{-m} \leq \mu_{-m+1} \leq \cdots \leq \mu_{-1} \leq 0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n, \tag{2.5}$$

the singular values of $\widetilde{B}$ as

$$0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m, \tag{2.6}$$

and the eigenvalues of the preconditioned Laplacian submatrix $\widetilde{A}$ as

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n. \tag{2.7}$$

Also, as in part I, we assume the existence of constants θ, Θ independent of h such that

$$\theta^2 \leq \frac{p^t M_p p}{p^t M_C p} \leq \Theta^2 \quad \forall p \in \Re^m - \{0\}, \tag{2.8}$$

i.e., the diagonal block M_C is required to be spectrally equivalent to the pressure mass matrix M_p. This condition ensures that the correct scaling is enforced between the velocity and pressure fields. The point here is that if the grid refinement is quasi-uniform then a diagonal (or "lumped") mass matrix preconditioner $M_C = \mathrm{diag}(M_p)$ always satisfies (2.8) (with θ "close" to Θ, cf. [W]). When using a continuous P_1 pressure approximation, for example, replacing the mass matrix M_p by its diagonal is very convenient computationally. Furthermore, (2.8) is satisfied in this case with $\theta = \frac{1}{\sqrt{2}}$ and $\Theta = \sqrt{2}$.

We also make use of the boundedness of $\widetilde{C}$, i.e.,

$$\max_{p \in \Re^m - \{0\}} \frac{p^t \widetilde{C} p}{p^t p} = \max_{q \in \Re^m - \{0\}} \frac{q^t C q}{q^t M_C q} \leq \Delta, \tag{2.9}$$

where Δ is a constant independent of h. In particular, $\Delta = 2$ for the two specific stabilised methods considered in the next section.

Our analysis is in two parts. First, we establish estimates of the eigenvalues μ_i in terms of the constants γ, Γ, θ, Θ, Δ, the eigenvalues λ_i, and the singular values σ_i of the preconditioned submatrices. From these we obtain inclusion intervals for the eigenvalues of $\widetilde{\mathcal{A}}$ in terms of the mesh-size parameter h and the constants γ, Γ, θ, Θ, and Δ.

For the unstabilised case, $C = 0$, the following result of Rusten and Winther [RW1, Lem. 2.1] provides some useful eigenvalue bounds.

LEMMA 2.1. *For the case* $C = 0$,

$$\frac{1}{2}\left(\lambda_1 - \sqrt{\lambda_1^2 + 4\sigma_m^2}\right) \leq \mu_{-m}, \tag{2.10}$$

$$\mu_{-1} \leq \frac{1}{2}\left(\lambda_n - \sqrt{\lambda_n^2 + 4\sigma_1^2}\right), \tag{2.11}$$

$$\lambda_1 \leq \mu_1, \tag{2.12}$$

$$\mu_n \leq \frac{1}{2}\left(\lambda_n + \sqrt{\lambda_n^2 + 4\sigma_m^2}\right). \tag{2.13}$$

In [RW1] a simple (3×3) example is given that shows that (2.10), (2.11), (2.12), and (2.13) are sharp. In the uniformly stabilised case, we modify the proof of [RW1] to obtain the following lemma (see also [RW2]).

LEMMA 2.2. *For the case* $C \neq 0$ *where* (2.9) *holds, the eigenvalues of* $\widetilde{A}$ *satisfy* (2.11), (2.12), (2.13), *and*

$$\frac{1}{2}\left(\lambda_1 - \beta\Delta - \sqrt{(\lambda_1 + \beta\Delta)^2 + 4\sigma_m^2}\right) \leq \mu_{-m}. \tag{2.14}$$

Proof. If μ is an eigenvalue of $\widetilde{A}$ then there are vectors u, p not both zero satisfying

$$\widetilde{A}u + \widetilde{B}^t p = \mu u, \tag{2.15}$$

$$\widetilde{B}u - \beta\widetilde{C}p = \mu p. \tag{2.16}$$

If $\mu > 0$ then $u \neq 0$, since otherwise (2.16) implies $p = 0$ as $\widetilde{C}$ is positive semidefinite. If $\mu < 0$ then $p \neq 0$, since otherwise (2.15) implies $u = 0$ as $\widetilde{A}$ is positive definite.

Taking the scalar product of (2.15) with u and the scalar product of (2.16) with p and subtracting gives

$$u^t\widetilde{A}u + \beta p^t\widetilde{C}p = \mu\, u^t u - \mu\, p^t p,$$

which, using (2.7) and the positive semidefiniteness of $\beta\widetilde{C}$, gives

$$(\lambda_1 - \mu)u^t u \leq -\mu p^t p,$$

from which we deduce (2.12) for positive μ since $u \neq 0$ in this case.

Furthermore, for $\mu > 0$, substituting for p from (2.16) into the scalar product of u with (2.15) gives

$$u^t\widetilde{A}u + \frac{1}{\mu}u^t\widetilde{B}^t\left(I + \frac{\beta}{\mu}\widetilde{C}\right)^{-1}\widetilde{B}u = \mu u^t u,$$

where the stated matrix inverse certainly exists because $\beta, \mu > 0$ and $\widetilde{C}$ is positive semidefinite. Moreover, the maximum eigenvalue of $(I + \frac{\beta}{\mu}\widetilde{C})^{-1}$ is 1, thus

$$\mu u^t\widetilde{A}u + u^t\widetilde{B}^t\widetilde{B}u \geq \mu^2 u^t u,$$

from which it follows that

$$0 \geq \mu^2 - \lambda_n\mu - \sigma_m^2,$$

which gives (2.13).

For $\mu < 0$, $\widetilde{A} - \mu I$ is invertible, so we can take the scalar product of (2.16) with p and substitute for u from (2.15) to obtain

$$p^t\widetilde{B}(\widetilde{A} - \mu I)^{-1}\widetilde{B}^t p + \beta p^t\widetilde{C}p = -\mu p^t p. \tag{2.17}$$

Using (2.6), (2.7), and the positive semidefiniteness of $\widetilde{C}$ gives

$$(\lambda_n - \mu)^{-1}\sigma_1^2 \leq -\mu,$$

which is

$$0 \leq \mu^2 - \lambda_n\mu - \sigma_1^2,$$

which yields (2.11). To derive (2.14) we use (2.9) in (2.17) to obtain

$$(\lambda_1 - \mu)^{-1}\sigma_m^2 + \beta\Delta \geq -\mu$$

or

$$0 \geq \mu^2 + (\beta\Delta - \lambda_1)\mu - \sigma_m^2 - \beta\Delta\lambda_1,$$

from which the result easily follows. □

The bound (2.12) was independently proved in part I directly from the congruence transform (2.4). The bounds (2.10), (2.13), and (2.14) are superior to those used in part I, although they are of the same asymptotic order in h as $h \to 0$ for the case of diagonal scaling [WS, Thms. 1 and 2], and in all cases considered in this paper.

The bound (2.11) is pessimistic in practice, however, i.e., when solving a typical Stokes problem, even in the above asymptotic sense. A tighter bound that holds for stable and uniformly stabilised formulations is provided by the following lemma.

LEMMA 2.3.

$$\mu_{-1} \leq \frac{1}{2}\left(\lambda_1 - \sqrt{\lambda_1^2 + 4\gamma^2\theta^2\lambda_1}\right). \tag{2.18}$$

Proof. From (2.17), if $\mu < 0$ is an eigenvalue of $\widetilde{\mathcal{A}}$ then

$$p^t\widetilde{B}\widetilde{A}^{-\frac{1}{2}}(I - \mu\widetilde{A}^{-1})^{-1}\widetilde{A}^{-\frac{1}{2}}\widetilde{B}^t p + \beta p^t\widetilde{C}p = -\mu p^t p,$$

where $p \neq 0$. Because the eigenvalues of $(I - \mu\widetilde{A}^{-1})^{-1}$ are

$$(1 - \mu/\lambda_1)^{-1} \leq (1 - \mu/\lambda_2)^{-1} \leq \cdots \leq (1 - \mu/\lambda_n)^{-1},$$

we have

$$(1 - \mu/\lambda_1)^{-1}p^t\widetilde{B}\widetilde{A}^{-1}\widetilde{B}^t p + \beta p^t\widetilde{C}p \leq -\mu p^t p,$$

and since $0 \leq (1 - \mu/\lambda_1)^{-1} \leq 1$ it follows that

$$(1 - \mu/\lambda_1)^{-1}\left(p^t\widetilde{B}\widetilde{A}^{-1}\widetilde{B}^t p + \beta p^t\widetilde{C}p\right) \leq -\mu p^t p.$$

Using (2.3) to express this in terms of the blocks of the original unpreconditioned Stokes matrix (1.3) gives

$$(1 - \mu/\lambda_1)^{-1}p^t M_C^{-\frac{1}{2}}\left(BA^{-1}B^t + \beta C\right)M_C^{-\frac{1}{2}}p \leq -\mu p^t p.$$

Now using the uniform stabilisation condition (1.6) (or (1.4) when $C = 0$) implies

$$\gamma^2(1 - \mu/\lambda_1)^{-1}p^t M_C^{-\frac{1}{2}}M_p M_C^{-\frac{1}{2}}p \leq -\mu p^t p.$$

Employing (2.8) further implies

$$\gamma^2\theta^2(1 - \mu/\lambda_1)^{-1}p^t p \leq -\mu p^t p.$$

Since $p \neq 0$, this gives

$$0 \leq \mu^2 - \lambda_1\mu - \lambda_1\gamma^2\theta^2,$$

from which (2.18) easily follows. □

For the case of diagonal scaling, this result is proved in part I using an argument that is essentially similar except in that it is motivated by applying the Sylvester Law of Inertia to (2.4) and noting that any $\xi < 0$ for which $-\widetilde{B}(\widetilde{A}-\xi I)^{-1}\widetilde{B}^t-\beta\widetilde{C}-\xi I$ is negative definite is an upper bound on μ_{-1}.

We now estimate σ_m in terms of λ_n.

LEMMA 2.4.

$$\sigma_m \leq \Gamma\Theta\sqrt{\lambda_n}. \tag{2.19}$$

Proof. For all p we have

$$\begin{aligned} p^t\widetilde{B}\widetilde{B}^t p &= p^t M_C^{-\frac{1}{2}} B M_A^{-1} B^t M_C^{-\frac{1}{2}} p \\ &\leq \lambda_n\ p^t M_C^{-\frac{1}{2}} B A^{-1} B^t M_C^{-\frac{1}{2}} p \end{aligned}$$

from (2.7). Using (1.5) gives

$$\begin{aligned} p^t\widetilde{B}\widetilde{B}^t p &\leq \lambda_n\ \Gamma^2\ p^t M_C^{-\frac{1}{2}} M_p M_C^{-\frac{1}{2}} p \\ &\leq \lambda_n\ \Gamma^2\ \Theta^2\ p^t p, \end{aligned}$$

where we have further used (2.8). We have thus proved

$$\sigma_m^2 \leq \lambda_n\ \Gamma^2\ \Theta^2$$

and hence (2.19). □

Note that for stable elements satisfying (1.4) we can derive

$$\gamma\ \theta\ \sqrt{\lambda_1} \leq \sigma_1 \tag{2.20}$$

using a similar argument to that used in Lemma 2.4. This can be used in conjunction with (2.11). For an unstable element such as Q_1–P_0, however, we have $\sigma_1 = 0$ corresponding to the "pure" spurious pressure mode. Thus, the bound (2.11) is trivial even for stabilised formulations for such elements.

Combining the best estimates from Lemmas 2.1, 2.2, 2.3, and 2.4 gives the following theorem.

THEOREM 2.1. *For a stable or stabilised discrete Stokes problem* (1.2) *on a quasi-uniform sequence of grids, assuming* (1.5) *and uniform stability in the sense of* (1.4) *or* (1.6), *and also that* (2.8) *and* (2.9) *hold, the eigenvalues of the preconditioned matrix* (2.3) *lie in the union of intervals*

$$\begin{aligned} &\left[\frac{1}{2}\left(\lambda_1 - \beta\Delta - \sqrt{(\lambda_1+\beta\Delta)^2 + 4\Gamma^2\Theta^2\lambda_n}\right), \frac{1}{2}\left(\lambda_1 - \sqrt{\lambda_1^2 + 4\gamma^2\theta^2\lambda_1}\right)\right] \\ &\cup \left[\lambda_1, \frac{1}{2}\left(\lambda_n + \sqrt{\lambda_n^2 + 4\Gamma^2\Theta^2\lambda_n}\right)\right]. \end{aligned} \tag{2.21}$$

($\Delta = 0$ *in the stable case.*)

To convert the eigenvalue bounds (2.21) into estimates in terms of the mesh size parameter h (which will approach zero under mesh refinement), we will assume that

$$g(h) \leq \frac{u^t A u}{u^t M_A u} \leq 1, \tag{2.22}$$

or equivalently, $g(h) \leq \lambda_1$ and $\lambda_n \leq 1$. Note that in (2.22) we have tacitly assumed that the largest eigenvalue of the preconditioned Laplacian is unity, which means that a suitable scaling is imposed when solving systems involving the operator M_A. The special case of a spectrally equivalent preconditioner $M_A \approx A$, where the function $g(h)$ remains bounded away from zero as $h \to 0$, is discussed separately. Combining (2.21) and (2.22) with the PCR convergence estimate (1.9), leads to the following theorem.

THEOREM 2.2. *For a stable or stabilised discrete Stokes problem* (1.2) *on a quasi-uniform sequence of grids, assuming* (1.5) *and uniform stability in the sense of* (1.4) *or* (1.6), *and also that* (2.8), (2.9), *and* (2.22) *hold with* $g(h) \to 0$ *as* $h \to 0$, *then the eigenvalues of the preconditioned matrix* (2.3) *lie in the union of intervals*

$$\left[-\beta\Delta/2 - \sqrt{\beta^2\Delta^2/4 + \Gamma^2\Theta^2} + O(g(h)), -\gamma\theta\sqrt{g(h)} + O(g(h))\right] \cup \left[g(h)\,,\, 1/2 + \sqrt{1/4 + \Gamma^2\Theta^2}\right], \tag{2.23}$$

and hence the asymptotic convergence rate of the PCR *iteration satisfies*

$$\lim_{k\to\infty} e_k^{\frac{1}{k}} = 1 - g(h)^{\frac{3}{4}} \sqrt{\frac{4\gamma\theta}{\left(\beta\Delta + \sqrt{\beta^2\Delta^2 + 4\Gamma^2\Theta^2}\right)\left(1 + \sqrt{1 + 4\Gamma^2\Theta^2}\right)}} + O(g(h)^{\frac{5}{4}}). \tag{2.24}$$

Examples. For simplicity, we use a, b, c, and d below to denote positive constants that are independent of h but different in each different example.

1. *Diagonal scaling.* $g(h) = O(h^2)$, so (2.23) is

$$\left[-a + O(h^2)\,,\, -bh + O(h^2)\right] \cup \left[ch^2\,,\, d\right],$$

which is proved in part I. The PCR convergence estimate (2.24) becomes

$$\lim_{k\to\infty} e_k^{\frac{1}{k}} = 1 - O(h^{\frac{3}{2}}),$$

so $O(h^{-3/2})$ PCR iterations are required for convergence. The standard estimate (1.11) would imply that no more than $O(h^{-2})$ PCR iterations are required for convergence.

2. *Modified incomplete factorisation.* With an appropriate row sum modification $\lambda_1 = 1$ and $\lambda_n = O(h^{-1})$ [G]. Thus, by trivial scaling to satisfy (2.22), we have that $g(h) = O(h)$ and so (2.23) is

$$\left[-a + O(h)\,,\, -b\sqrt{h} + O(h)\right] \cup \left[ch\,,\, d\right]$$

and (2.24) becomes

$$\lim_{k\to\infty} e_k^{\frac{1}{k}} = 1 - O(h^{\frac{3}{4}}).$$

Hence $O(h^{-3/4})$ PCR iterations are required for convergence. Under the standard estimate (1.11) this would be $O(h^{-1})$.

3. *Hierarchical basis preconditioning.* In this case, the spectral condition number of the preconditioned Laplacian in two dimensions is $O(\log h^{-1})^2$ [Y]. Thus, by an appropriate but trivial scaling to satisfy (2.22), $g(h) = O(1/(\log h)^2)$, and (2.23) is

$$\left[-a + O(1/(\log h)^2)\,,\, -b/|\log h| + O(1/(\log h)^2)\right] \cup \left[c/(\log h)^2\,,\, d\right],$$

in which case (2.24) becomes

$$\lim_{k\to\infty} e_k^{\frac{1}{k}} = 1 - O(|\log h|^{-\frac{3}{2}}),$$

and so $O(|\log h|^{3/2})$ PCR iterations are required for convergence (or $O(|\log h|^2)$ if (1.11) is used).

A key feature of (2.23) in the stabilised case is that the only bound involving β is that which limits the magnitude of the negative eigenvalues of the system. The positive eigenvalues lie in an interval that is independent of β. In contrast, scaling M_C by β, i.e., using a preconditioner

$$\mathcal{M} = \begin{pmatrix} M_A & O \\ O & \beta M_C \end{pmatrix}, \tag{2.25}$$

leads to bounding estimates for μ_{-m}, μ_{-1} and μ_n, which as $\beta \to \infty$ scale like $\frac{1}{\sqrt{\beta}}$ (cf. [WS]). In the case of a "small" stabilisation parameter the unscaled choice (2.2) is clearly superior to (2.25). An important case where a small parameter arises is when using a penalty method to fix the hydrostatic pressure in conjunction with an LBB stable method. In this case, taking $C = M_p$ in (2.1), β represents a "penalty" parameter and must be small (typically $O(10^{-7})$) to reduce consistency error. In such cases it is very important to use an "unscaled" preconditioner of the form (2.2).

The estimate (2.24) leads to an obvious characterisation of the "optimal" value of β, which is stated here formally.

Remark 2.1. Under the assumptions of Theorems 2.1 and 2.2, the "optimal" value of the stabilisation parameter β, giving the best rate of convergence of the PCR iteration associated with the system (2.3), is the value $\beta = \beta_0$ associated with the stability bound (1.6).

To complete our analysis we consider the important case of a spectrally equivalent preconditioner $M_A \approx A$. Directly from Theorem 2.1 and the standard convergence estimate (1.10) we have the following corollary.

COROLLARY 2.1. *If a preconditioner M_A is used that is spectrally equivalent to the (vector-)Laplacian A (so that λ_1 and λ_n are bounded independently of h), then the eigenvalues of the preconditioned Stokes operator are also independent of h, and thus the convergence rate of* PCR *is independent of the mesh.*

In the stabilised case, the bounds (2.21) immediately apply with eigenvalues λ_1 bounded below, and λ_n bounded above independently of h. Although (2.21) holds (with $\Delta = 0$) in the case of an LBB stable element, if an "exact" Poisson preconditioner is used, the following more precise estimate of the positive eigenvalues can be established.

THEOREM 2.3. *Given an unstabilised Stokes problem, that is* (1.3) *with $C = 0$, and using a preconditioner* (2.2) *with $M_A = A$ and M_C such that* (2.8) *holds, then assuming quasi-uniformity of the grid and* LBB *stability in the sense of* (1.4) *and* (1.5), *the eigenvalues of the preconditioned matrix* (2.3) *lie in the union of three intervals*

$$[-b, -a] \cup [1, 1] \cup [1 + a, 1 + b] \tag{2.26}$$

with constants

$$\begin{aligned} a &= -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\gamma^2\theta^2}, \\ b &= -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\Gamma^2\Theta^2} \end{aligned} \tag{2.27}$$

independent of the grid parameter h.

Proof. Substituting $M_A = A$ into (2.3) leads to the eigenproblem

$$\begin{pmatrix} I & \widetilde{B}^t \\ \widetilde{B} & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} I & A^{-\frac{1}{2}} B^t M_C^{-\frac{1}{2}} \\ M_C^{-\frac{1}{2}} B A^{-\frac{1}{2}} & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \mu \begin{pmatrix} u \\ p \end{pmatrix}. \tag{2.28}$$

The $n+m$ eigensolutions can be split into two categories. The first possibility is that of $\mu = 1$, wherein the eigenvectors satisfy the reduced system

$$\begin{pmatrix} O & \widetilde{B}^t \\ \widetilde{B} & -I \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.29}$$

Recalling that B is of full rank for an LBB stable method (cf. [WS]), the first equation of (2.29) implies that $p = 0$. The second equation then implies that the velocity eigenmodes satisfy the constraint $\widetilde{B}u = 0$, so that (2.28) has a repeated eigenvalue of unity with multiplicity $n-m$ corresponding to the dimension of the space of discretely divergence-free velocities (i.e., the kernel of the matrix operator $\widetilde{B}$).

The second possibility is that $\mu \neq 1$, in which case u can be eliminated from (2.28) to give the characteristic eigenproblem:

$$\widetilde{B}\widetilde{B}^t p = \mu(\mu - 1)p = \sigma_j^2 p, \qquad j = 1, \ldots, m. \tag{2.30}$$

Solving the quadratic equation in (2.30) implies that every singular value $\sigma_j, j = 1, \ldots, m$, generates two distinct eigenvalues of (2.28), namely

$$\frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\sigma_j^2} < 0 \quad \text{and} \quad \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\sigma_j^2} > 1. \tag{2.31}$$

Now from (2.19) and (2.20), since $\lambda_1 = 1 = \lambda_n$ we have

$$\gamma\theta \leq \sigma_1 \quad \text{and} \quad \sigma_m \leq \Gamma\Theta. \tag{2.32}$$

Substituting (2.32) into (2.31) then leads to the values a and b in (2.27). □

An obvious feature of (2.31) is that except for the cluster of unit eigenvalues, the spectrum is symmetric about the point $x = \frac{1}{2}$. In this case the PCR convergence estimate (1.10) is directly applicable. (The multiple eigenvalue at 1 accounts for only one PCR iteration.) In general, since we can construct a smooth map from a spectrally equivalent preconditioner $M_A \approx A$, and since the eigenvalues are continuous functions of the entries, this symmetry in the spectrum must be preserved to an extent whenever a spectrally equivalent preconditioner is applied to an LBB stable approximation.

What is intriguing in the stabilised case is that the gap between the two rightmost clusters "fills in" with unstable $O(h)$ eigenvalues associated with the Schur complement (1.4). Thus although uniform stabilisation guarantees eigenvalue bounds that are independent of h, the symmetry of the stable clusters (2.26) is not possible if (1.4) is not satisfied.

3. Numerical examples. In this section, results of some numerical experiments analogous to those in §4 of part I are presented. The test problem is the "leaky" lid driven cavity problem, as described in part I (see also [P]). All computations were done using Matlab-4.0 on a SUN Sparcstation 10. The stable case is discussed first.

TABLE 1
Eigenvalues of $\widetilde{\mathcal{A}}$ with $M_A = A$ and $M_C = M_p$ for the P_1 iso P_2 element.

Grid	μ_{-m}	μ_{-1}	μ_1	μ_n
2×4	-0.2604	-0.0421	1.0000	1.2604
4×8	-0.3359	-0.0475	1.0000	1.3359
8×16	-0.3590	-0.0473	1.0000	1.3590

TABLE 2
Eigenvalues of $\widetilde{\mathcal{A}}$ with $M_A = A$ and $M_C = \mathrm{diag}(M_p)$ for the P_1 iso P_2 element.

Grid	μ_{-m}	μ_{-1}	μ_1	μ_n
2×4	-0.3420	-0.0284	1.0000	1.3420
4×8	-0.4587	-0.0405	1.0000	1.4587
8×16	-0.5644	-0.0407	1.0000	1.5644

3.1. Using LBB stable methods. We start with the case of the stable P_1 *iso* P_2 method (cf. [WS]). In Table 1 the extremal eigenvalues in the case of a block diagonal preconditioner (2.2) with $M_A = A$ and $M_C = M_p$ are given for a sequence of uniformly refined grids of triangular elements. Using a simpler preconditioner, so that M_p is approximated by its diagonal, gave the results in Table 2.

These results can be seen to be consistent with our estimate (2.26)–(2.27). Using the "exact" preconditioner with $M_C = M_p$ (so that $\theta = \Theta = 1$), the bounds (2.27) are tight. Thus, it can be deduced from Table 1 that the LBB constant γ^2 is approximately 0.05, and similarly that Γ^2 is close to 0.5 in this case. In contrast, looking at the results in Table 2 more closely and applying the standard scaled mass matrix bounds $\theta^2 = \frac{1}{2}$ and $\Theta^2 = 2$ for the P_1 approximation [W], the bounds (2.27) appear to be pessimistic in more general cases.

Two important general features of the preconditioned system (2.3) not shown by these tables are the symmetry of the clusters about the point $x = \frac{1}{2}$ and the clustering of the positive eigenvalues around unity. In the case of an exact Poisson preconditioner there is a multiple eigenvalue of unity corresponding to the dimension of the space of discretely divergence free velocities (independent of the choice of M_C). A plot of the eigenvalue distribution corresponding to the 8×16 grid in Table 1 is given in Fig. 1 to illustrate this point.

It is interesting to compare the performance of the block PCR algorithm associated with (2.3) with the obvious alternative of conjugate gradient iteration applied to the preconditioned Schur complement $M^{-1/2}BA^{-1}B^tM^{-1/2}$. This is done in Tables 3 and 4.

In the first two columns of Table 3, PCR iteration counts corresponding to the spectral bounds in Tables 1 and 2 are compared. The convergence criterion used for terminating an iteration was a 10^{-6} reduction in the L^2–norm of the residual. As expected, the number of PCR iterations tends to a constant as $h \to 0$. The results suggest that the approximation of the pressure mass matrix by its diagonal is a false economy in this case. The other two columns in Table 3 give the iteration counts when the standard CG algorithm is applied to the analogous preconditioned Schur complement systems, with a convergence criterion of a 10^{-6} reduction in the norm of the preconditioned Schur complement residual. This gave a solution that was approximately two digits more accurate than in the PCR case. Note that the two

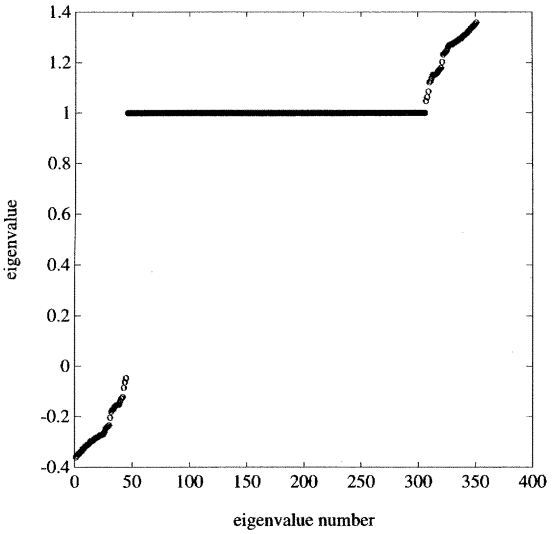

FIG. 1. *P_1 iso P_2 eigenvalue plot 8×16 grid.*

TABLE 3
PCR/PCG iterations: P_1 iso P_2 element.

Grid	PCR $M_C = M_p$	PCR $M_C = D_{M_p}$	PCG $M = M_p$	PCG $M = D_{M_p}$
4×8	21	27	14	15
8×16	23	31	15	22
16×32	19	33	15	23
32×64	19	31	15	23

methods can be implemented with comparable work per iteration in this case, so that the raw iteration counts give a realistic measure of the relative operation counts.

A more practical strategy is to use a block preconditioner based on a spectrally equivalent approximation of the Laplacian. For example, multigrid preconditioners based on a fixed number of V-cycles are known to be optimally efficient in this respect. We present a set of representative results in Table 4. The element used in this case was Q_1 *iso* Q_2, i.e., the quadrilateral analogue of the element used above. The preconditioners A_{MG1} and A_{MG4} represent a multigrid preconditioner based on one and four V-cycles of an "optimally" damped Jacobi iteration, respectively. Iteration and total flop counts using the diagonally scaled PCR method presented in part I and the block preconditioner based on a modified incomplete Cholesky factorisation as discussed in §2 are also included for comparison. Note that the cost of the incomplete factorisation is not included in the flop counts given in the table; preconditioning is via sparse upper and lower triangular matrix solves in this case.

Of course, using a multigrid approximation of the Laplacian is also possible in the Schur complement case; see, for example, [Ve], [BP1], and [BP2]. Using the Verfürth algorithm shows that there are two difficulties that are absent in our approach. The first problem is that the algorithm does not necessarily converge if the (fixed) number of multigrid cycles used to solve the "inner system" is not sufficiently large. In the Q_1 *iso* Q_2 case above the A_{MG1} mutigrid solver led to a divergent method. The

TABLE 4
PCR iterations (megaflops): Q_1 iso Q_2 element.

Grid	$M_A = A_{MG1}$ $M_C = M_p$	$M_A = A_{MG4}$ $M_C = M_p$	$M_A = A_{MIC}$ $M_C = M_p$	$M_A = D_A$ $M_C = D_{M_p}$
4×8	30 (0.44)	26 (0.98)	28 (0.19)	41 (0.23)
8×16	31 (1.97)	24 (3.83)	38 (1.22)	94 (2.25)
16×32	30 (8.10)	24 (15.73)	53 (7.69)	206 (20.63)
32×64	30 (36.33)	24 (66.28)	78 (54.89)	427 (175.04)

second difficulty is that the "wrong" system is being solved, since the consistency error is tied to the inner system solution accuracy. To produce converged solutions of accuracy comparable to those in the PCR case, at least eight multigrid cycles per inner iteration were required.

Comparing the overall merits of the PCG and PCR approaches shows that the PCG approach is clearly the method of choice if an "exact" Laplacian solver is employed. If a multigrid preconditioner of the Laplacian is available, on the other hand, then our experience suggests that the PCR algorithm is likely to be considerably more efficient in almost every case.

3.2. Using stabilised methods. Two classes of stabilisation methods are distinguished in part I. We consider the same two representative methods here. To keep the discussion brief, we will not present a detailed study of the optimal choice of β in either case. For further analysis of this issue see [S].

The first example we consider is the "globally" stabilised P_1–P_1 triangular element with a continuous linear pressure. As discussed in part I, the choice $\beta = 0.025$ is of particular interest. The crucial point is that this value generates the system that would result using an analogous subdivision of P_1–P_1 mini-elements, after elimination of the velocity bubble degrees of freedom [P]. The calculated eigenvalue bounds in the case of an exact mass matrix preconditioner are given in Table 5, and the distribution of eigenvalues on the 8×16 grid is given in Fig. 2.

The results in Table 5 are consistent with the bounds (2.21). In particular, the grid independence of the bounds as h tends to zero is again evident. Comparing Fig. 2 with Fig. 1 shows the interesting feature that the symmetry of the clusters about the point $x = \frac{1}{2}$ has been lost. This can be viewed as being the price to pay for reducing the size of the system by performing static condensation on the mini-element bubble terms.

The second example we consider is the "locally" stabilised Q_1–P_0 quadrilateral element. The method is studied in detail in [S], wherein it is shown that the "optimal" parameter value β_0 is 0.058 for the sequence of uniform grids and the stabilisation used here. The computed eigenvalue bounds in this case are tabulated in Table 6.

As expected, when the block PCR method is applied to either of the stabilised methods above with spectrally equivalent approximations to A and M_p, the number of iterations tends to a constant as h tends to zero. The contraction rate is actually similar in all four mixed approximations considered above. This confirms that the parameter values are indeed close to optimal in the two stabilised cases.

4. Conclusions. In this sequence of two papers we have investigated the use of certain preconditioned iterative solution methods for stabilised and unstabilised finite element approximations of the Stokes problem. In particular, we have considered the use of a single (nonnested) preconditioned conjugate residual iteration. We

TABLE 5
Eigenvalues of $\widetilde{\mathcal{A}}$ with $M_A = A$ and $M_C = M_p$ for the globally stabilised P_1–P_1 element with $\beta = 0.025$.

Grid	μ_{-m}	μ_{-1}	μ_1	μ_n
2×4	−0.3917	−0.0763	1.0000	1.2642
4×8	−0.3927	−0.0736	1.0000	1.3357
8×16	−0.3927	−0.0723	1.0000	1.3587

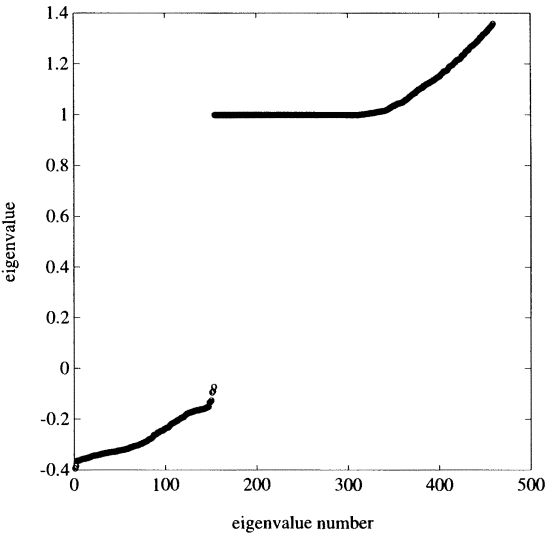

FIG. 2. *P_1–P_1 $\beta = 0.025$ eigenvalue plot 8×16 grid.*

TABLE 6
Eigenvalues of $\widetilde{\mathcal{A}}$ with $M_A = A$ and $M_C = M_p$ for the locally stabilised Q_1–P_0 element with $\beta = 0.058$.

Grid	μ_{-m}	μ_{-1}	μ_1	μ_n
2×4	−0.6597	−0.2203	1.0000	1.5399
4×8	−0.6989	−0.2089	1.0000	1.6023
8×16	−0.7021	−0.1960	1.0000	1.6145

have established estimates of the positive and negative eigenvalues of the symmetric and indefinite Stokes systems on which the iterative convergence rate depends and demonstrated the accuracy of these estimates in computations. The most important outcome of this work is that in almost all cases the negative eigenvalues of the preconditioned Stokes coefficient matrix are in a precise way more clustered than the positive eigenvalues.

In this paper (part II), we have considered the use of block preconditioners based on preconditioners for the (vector-)Laplacian and the pressure "mass matrix." More precise eigenvalue estimates are derived, and certain symmetries that may be of use in alternative preconditioned iterative methods are proved. We demonstrate that if a spectrally equivalent preconditioner for the Laplacian (such as one based on multigrid)

is used, then the number of iterations required to achieve any particular reduction in the residual is independent of the mesh size.

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