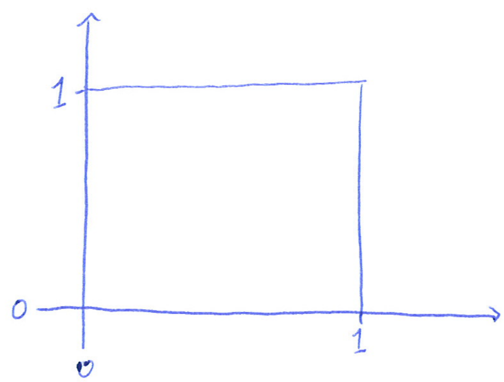
Book: Iterative Methods for Sparse Linear Systems (second edition)　　September 10, 2013
Yousef Saad
↳ available online

Projection Methods
- Krylov subspace methods
→ coming soon

Sparse Matrices arising from discretization of PDEs:

Poisson
Convection-diffusion $\Big\}$ FD



unit square $[0,1] \times [0,1]$

$u = u(x,y)$

$-\nabla^2 u = f$

(or $-\Delta u = f$

$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f$)

on $\Omega = (0,1) \times (0,1)$

$u = g$ on $\partial\Omega$

↳ $\left.\begin{array}{l} u(0,y) \\ u(1,y) \\ u(x,0) \\ u(x,1) \end{array}\right\}$ given Dirichlet

or Neumann $\frac{\partial u}{\partial n}$ given
↳ affects nullity

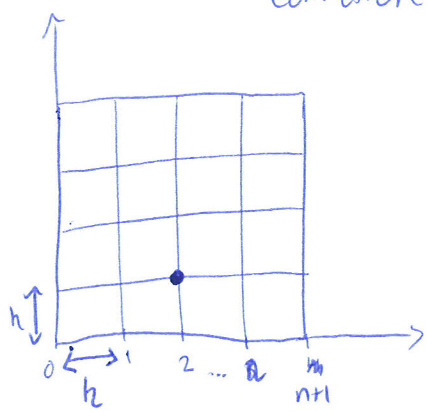
Convection - diffusion

$$-\nabla^2 u + (\sigma, \tau)\nabla u = f$$

gradient : $\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$

$$(\sigma, \tau)\cdot\nabla u = \sigma\frac{\partial u}{\partial x} + \tau\frac{\partial u}{\partial y}$$

↑ convection



$u_{ij} \approx u(ih, jh)$

on unit square $h = \frac{1}{n+1}$

$1 \le i, j \le n$

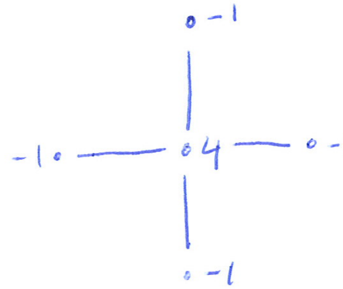
BC: $\left.\begin{matrix} u_{0,j} \\ u_{n+1,j} \\ u_{i,0} \\ u_{i,n+1} \end{matrix}\right\}$ known

$$p(x+h) = p(x) + hp'(x) + \frac{h^2}{2}p''(x) + \dots + \frac{h^n}{n!}p^n(x) + \dots$$

$$+ \quad p(x-h) = p(x) - hp'(x) + \frac{h^2}{2}p''(x) + \dots$$

$$\boxed{p''(x) = \frac{p(x+h) - 2p(x) + p(x-h)}{h^2} + O(h^2)}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x,y)=(ih,jh)} \simeq \frac{u_{i+1} - 2u_{ij} + u_{i-1,j}}{h^2}$$



n^2 unknowns

System:

$-\nabla^2 u = f$

$$\left(\begin{array}{ccc|ccc|c} 4 & -1 & & -1 & & 0 & \\ -1 & 4 & -1 & & 0 & -1 & \\ & -1 & \ddots & & & & \\ \hline -1 & & & \ddots & & & \\ & 0 & & & \ddots & & \\ 0 & & -1 & & & \ddots & \\ \hline & & & & & & -1 \\ & & & & -1 & & 4 \end{array}\right) \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ \vdots \\ u_{n1} \\ u_{12} \\ \vdots \\ u_{n2} \\ \vdots \\ u_{nn} \end{pmatrix} = h^2 \begin{pmatrix} f_{11} \\ f_{21} \\ \vdots \\ \vdots \\ u_{nn} \end{pmatrix} + \begin{pmatrix} \vdots \\ BC \\ \vdots \end{pmatrix}$$

where $f_{ij} = f(ih, jh)$

$Au = f \qquad A = \mathrm{tri}[B\,C\,D]$

$B = D = -I_n$

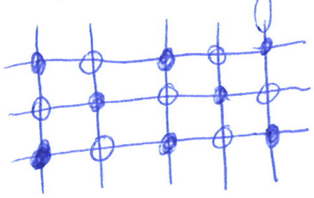
$C = \mathrm{tri}[-1, 4, -1]$

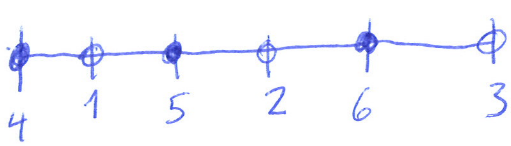
$n = 10$

$(i,j) = (7,6) \rightarrow 57$ (index)

Natural lexicographic ordering

Red Black ordering:





Natural lexiographic

1D - tridiagonal

2D - block tridiagonal

$$\longrightarrow \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = h^2 \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} + \underbrace{\begin{pmatrix} \blacksquare \\ 0 \\ \vdots \\ 0 \\ \blacksquare \end{pmatrix}}_{BC}$$

red-black

1D

$$\begin{pmatrix} B & C \\ D & E \end{pmatrix} \rightarrow \left(\begin{array}{ccc|ccc} 2 & & & -1 & -1 & \\ & 2 & & & \ddots & \ddots \\ & & \ddots & & & -1 \\ & & & & & -1 \\ \hline -1 & & & \ddots & & \\ -1 & \ddots & & & \ddots & \\ & \ddots & \ddots & & & \\ & -1 & -1 & & & 2 \end{array}\right) \begin{pmatrix} u_B \\ u_R \end{pmatrix} = h^2 \begin{pmatrix} f_B \\ f_R \end{pmatrix}$$

2D

$$\left(\begin{array}{c|c} \text{diagonal} & * \\ \hline * & \text{diagonal} \end{array}\right)$$

Schur complement: $E - DB^{-1}C$

$$\begin{pmatrix} B & C \\ D & E \end{pmatrix} \begin{pmatrix} u^{(b)} \\ u^{(r)} \end{pmatrix} = \begin{pmatrix} f^{(b)} \\ f^{(r)} \end{pmatrix}$$

row 2 → row 2 − DB^{-1} row 1:

$$\underbrace{(E - DB^{-1}C)}_{\text{sparse!}} u^{(r)} = f^{(r)} - DB^{-1} f^{(b)}$$

← you can now split up the problem & solve for $u^{(b)}, u^{(r)}$ in parallel.

typically, if you reduce a system → you densify it.

- multicoloring.

matlab: delsq

adding the convective term

$$-\nabla^2 u + (v, \tau)\nabla u = f$$

$$\uparrow$$

$$\approx \frac{u_{j+1} - u_{j-1}}{2} \quad \text{centered}$$

or

$$\approx \frac{u_{j+1} - u_j}{h} \quad \text{back/fwd.}$$

doesn't affect sparsity, but our system is no longer symmetric!