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$$A_i - \sigma_i I = Q_i R_i$$
$$A_{i+1} = R_i Q_i + \sigma_i I$$

Projected Eigensolvers:

Galerkin: $Au - \lambda u \perp K$

$$(Au - \lambda u, v) = 0 \qquad \forall\, v \in K$$

Suppose that $\underbrace{\{v_1, \ldots, v_m\}}_{V}$ is an orthonormal basis for K.

Then,

$$u = y_1 v_1 + y_2 v_2 + \ldots + y_m v_m = Vy$$
$$(AVy - \lambda Vy, v_j) = 0 \qquad j = 1, \ldots, m$$
$$V^T AVy = \lambda y$$

Recall

$$Q^T A Q = \begin{cases} T & \text{- Lanczos} \\ H & \text{- Arnoldi} \end{cases}$$

$$Q_k = \begin{array}{cc} & k \\ n & \begin{bmatrix} \quad & \quad \\ & \end{bmatrix} \end{array}$$

$$\underbrace{Q_k^T A Q_k}_{\substack{T_k \\ H_k}} \in \mathbb{R}^{kk}$$

Ritz values:

eigenvalues of T_k or H_k

↳ approx eig of A

$$\left.\begin{aligned} AV_m &= V_m H_m + h_{m+1,m} V_{m+1} e_m^T \\ V_m^T A V_m &= H_m \end{aligned}\right\} \text{Arnoldi}$$

$$H_m y_i^{(m)} = \lambda_i^{(m)} y_i^{(m)}$$

$$\left.\begin{array}{ll} \lambda_i^{(m)} & \text{Ritz values} \\ u_i^{(m)} = V_m y_i^{(m)} & \text{Ritz vectors} \end{array}\right\} \text{approximations for eigenpairs.}$$

recall $(Q^T A Q)(Q^T x) = \lambda (Q^T x)$

↖ Rayleigh-Ritz.

$$A V_m y_i^{(m)} = V_m \underbrace{H_m y_i^{(m)}}_{\lambda_i^{(m)} y_i^{(m)}} + h_{m+1,m} V_{m+1} e_m^T y_i^{(m)}$$

$$A V_m y_i^{(m)} - \lambda_i^{(m)} V_m y_i^{(m)} = h_{m+1,m} e_m^T y_i^{(m)} V_{m+1}$$

$$\|(A - \lambda_i^{(m)} I) u_i^{(m)}\|_2 = h_{m+1,m} |e_m^T y_i^{(m)}|$$

cheap, computable way to estimate residual

<u>Lanczos:</u>

$$T = V^T A V = [V_m \, V_u]^T A [V_m \, V_u] = \begin{pmatrix} V_m^T A V_m & V_m^T A V_u \\ V_u^T A V_m & V_u^T A V_u \end{pmatrix}$$

where
$V = [V_m \, V_u]$

$$= \begin{pmatrix} T_m & T_{mu} \\ T_{mu}^T & T_{uu} \end{pmatrix}$$

<u>Thm:</u> the minimum of

$$\|A V_m - V_m R\|_2$$

over all **mxm** symmetric matrices R is attained by $R = T_m$.

$$\|A V_m - V_m R\|_2^2 = \lambda_{max}\left((A V_m - V_m R)^T (A V_m - V_m R)\right)$$

recall $\|B\|_2 = \max_{x \neq 0} \frac{\|Bx\|_2}{\|x\|_2} = \sigma(B) = \max_i |\lambda_i(B^T B)|$

write $R = T + Z$

show that $Z = 0$ minimizes this expression.

drop subscripts:

$$(AV - V(T+Z))^T (AV - V(T+Z))$$

$$= (AV - VT)^T (AV - VT) - \overbrace{(AV - VT)^T V Z}^{0 \text{ since } V^T A V = T} - \overbrace{(VZ)^T (AV - VT)}^{0} + \overbrace{(VZ)^T (VZ)}^{Z^T Z = SPSD}$$

Since $Z^T Z$ SPSD,
it can only increase the eigenvalues., therefore, take $Z=0$
Also $(VZ)^T(AV-VT) = Z^T(V^TAV-T)$
$= 0$

$\boxed{\therefore T \text{ is a minimizer}}$

List of accomplishments

- algorithm: Arnoldi + Lanczos
- approx: Ritz values + vectors
- optimality result: best in least squares sense (generalizes R.Q.)

$x_q = p_q(A)x_0 = \sum_{i=1}^{n} p_q(\lambda_i)\gamma_i u_i$ recall $A^k u_i = \lambda_i^k u_i$

↳ polynomial

$\{(\lambda_i, u_i)\}_{i=1}^{n}$ eigenpairs

$x_0 = \sum_{i=1}^{n} \gamma_i u_i$

$x_q = \sum_{i=1}^{n} p_q(\lambda_i)\gamma_i u_i = p_q(\lambda_1)\gamma_1 u_1 + \sum_{i=2}^{n} p_q(\lambda_i)\gamma_i u_i$

Suppose we want the first eigenpair:

$p_q(\lambda_1)$ large

$p_q(\lambda_i)$ small $i>1$

We don't know the eigenvalues, or the polynomial.
We do have Ritz values.

So: $p(t) = (t-\theta_2)(t-\theta_3)\dots(t-\theta_k)$

is small near $\theta_2, \theta_3 \dots \theta_k$

is zero at $\theta_2, \theta_3 \dots \theta_k$

↳ hopefully small near $\lambda_2, \lambda_3 \dots \lambda_k$

restarting:

- Can be done simply by taking the desired Ritz vector so far, reset $m=1$, take it and use it as an initial guess for a fresh Arnoldi

ARPACK