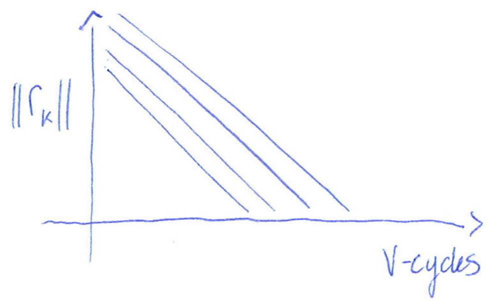
October 24, 2013

Multigrid - a great preconditioner

For Simple problems:



Convergence is independent of the mesh:

smoothing factor → determines the slope of the convergence graph

↳ the amount of reduction (worst case) of error for non-smooth modes

eg. Poisson in 1D: (damped Jacobi)

$$\lambda_k(T_\omega) = 1 - 2\omega \sin^2\left(\frac{k\pi}{2N}\right)$$

$$\lambda_1 = 1 - 2\omega \sin^2\left(\frac{\pi h}{2}\right)$$ bad... (for any ω, $\lambda_1 \simeq 1$ → not going to reduce by much)

However

$$\frac{N}{2} \leq k \leq N-1$$

$$\omega = \frac{2}{3}$$

$$|\lambda_k(T_\omega)| < \frac{1}{3}$$ smoothing factor

optimal ω reduces only high frequency ($\frac{N}{2} \leq k \leq N-1$) modes

↳ the other half taken care of by moving down the V cycle.

- Multigrid as a preconditioner

$$Ax = b \longrightarrow M^{-1}Ax = M^{-1}b$$

M^{-1}: operation of a V-cycle

- Multigrid is delicate

eg: conv-diff.

$$-\Delta u + (\sigma, \tau)\nabla u = f$$

For the first derivative

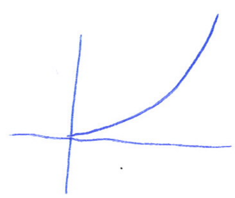
$$\partial_x u_{ij} \simeq \frac{u_{i+1,j} - u_{i-1,j}}{2}$$

Stable only for

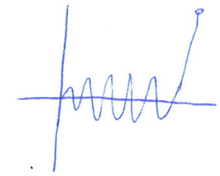
$$\left.\begin{aligned} \beta &= \frac{\sigma h}{2} \\ \gamma &= \frac{\tau h}{2} \end{aligned}\right\} \text{ Mesh Reynolds numbers}$$

$$|\beta|, |\gamma| < 1$$

Centered



true



$|\beta|, |\gamma| > 1$

* important to keep in mind as you move to a coarser mesh

<u>Algebraic Multigrid:</u>

$$Ae \simeq 0$$

$$a_{ii} e_i \approx - \sum_{i \neq j} a_{ij} e_j$$

$\boxed{\text{QMR}}$

Quasi Minimal Residual

$$AQ_k = Q_{k+1} T_{k+1} \qquad \text{Lanczos} \qquad (\text{Symm})$$

$$AQ_k = Q_{k+1} H_{k+1,k} \qquad \text{Arnoldi} \qquad (\text{non sym})$$

- Lanczos (on non-symmetric matrix)
- Solve $\min \|b - Ax_k\|$ on Krylov subspace

$$\equiv \min_z \| Q_{k+1} (\rho e_1 - T_{k+1,k} z) \|$$

pretend Q_{k+1} is orthogonal -> we ignore it

so we instead solve

$$\min \| \rho e_1 - T_{k+1,k} z \|_2$$

(if it is almost symmetric, you are not giving up much).

What if A is skew-symmetric

$$A^T = -A$$

- diagonal is zero

What method do we run?

Orthoganilization procedure

↳ try Arnoldi ... what does $H_{k+1,k}$ look like?

recall: $Q_k^T A Q_k = H_{kk}$

↳ look at transpose

$$Q_k^T A^T Q_k = H_{kk}^T$$

skew symmetric A

$$-Q_k^T A Q_k = H_{kk}^T$$
$$= -H_{kk}$$

So H_{kk} is tridiagonal & skew-symmetric, so

$$H_{kk} = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$$