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A NOTE ON PRECONDITIONING FOR INDEFINITE LINEAR SYSTEMS*

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Abstract. Preconditioners are often conceived as approximate inverses. For nonsingular indefinite matrices of saddle-point (or KKT) form, we show how preconditioners incorporating an exact Schur complement lead to preconditioned matrices with exactly two or exactly three distinct eigenvalues. Thus approximations of the Schur complement lead to preconditioners which can be very effective even though they are in no sense approximate inverses.

Key words. indefinite matrices, preconditioning

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In many areas there arise matrix systems of the form

$$\mathcal{A}u = \begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1}$$

where $A \in \Re^{n\times n}$ and $B, C \in \Re^{m\times n}$ with $n \geq m$ (see, for example, [3]). When $\mathcal{A}$ arises from a constrained variational or optimization problem it is usual that $B = C$. There are also many problems where additionally A is symmetric, so that $\mathcal{A}$ is symmetric (see, for example, [2], [4], [9], [11]). Whether symmetric or not, the matrix $\mathcal{A}$ is generally indefinite (i.e., it has eigenvalues with both positive and negative real parts).

Iterative solutions of systems of the form (1) can be achieved by any of a number of methods. In particular, Krylov subspace methods such as MINRES [10] or GMRES [12] are applicable in the symmetric and nonsymmetric cases, respectively. It is often advantageous (and in many situations necessary) to employ a preconditioner, $\mathcal{P}$, with such iterative methods. The role of $\mathcal{P}$ is to reduce the number of iterations required for convergence while not increasing significantly the amount of computation required at each iteration.

Intuitively, if $\mathcal{P}$ can be chosen so that $\mathcal{P}^{-1}$ is an inexpensive approximate inverse of $\mathcal{A}$, then this might make a good preconditioner; however, it is not necessary for a good preconditioner to have that $\mathcal{P}^{-1}$ be an approximate inverse of $\mathcal{A}$. A sufficient condition for a good preconditioner is that the preconditioned matrix $\mathcal{T} = \mathcal{P}^{-1}\mathcal{A}$ has a low-degree minimum polynomial. This condition is more usually expressed in terms of $\mathcal{T}$ having only a few distinct eigenvalues: in this form we must insist that $\mathcal{T}$ is not degenerate (derogatory) or at least that its Jordan canonical form has Jordan blocks of only small dimension.

In this note we show how preconditioners can be derived for systems of the form (1) based upon an "exact" preconditioner which yields a preconditioned matrix with exactly three or exactly two distinct eigenvalues (in fact, precisely with minimum

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polynomial of exact degree 3 or 2). This observation is based on the Schur complement.

PROPOSITION 1. *If*

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix}$$

is preconditioned by

$$\mathcal{P} = \begin{bmatrix} A & 0 \\ 0 & CA^{-1}B^T \end{bmatrix}, \tag{2}$$

then the preconditioned matrix $\mathcal{T} = \mathcal{P}^{-1}\mathcal{A}$ *satisfies*

$$\mathcal{T}(\mathcal{T} - I)(\mathcal{T}^2 - \mathcal{T} - I) = 0.$$

Proof. Simple calculation yields

$$\mathcal{T} = \mathcal{P}^{-1}\mathcal{A} = \begin{bmatrix} I & A^{-1}B^T \\ (CA^{-1}B^T)^{-1}C & 0 \end{bmatrix}$$

and

$$\left(\mathcal{T} - \frac{1}{2}I\right)^2 = \begin{bmatrix} \frac{1}{4}I + A^{-1}B^T(CA^{-1}B^T)^{-1}C & 0 \\ 0 & \frac{5}{4}I \end{bmatrix}.$$

But $A^{-1}B^T(CA^{-1}B^T)^{-1}C$ is a projection so that

$$\left[\left(\mathcal{T} - \frac{1}{2}I\right)^2 - \frac{1}{4}I\right]^2 = \left[\left(\mathcal{T} - \frac{1}{2}I\right)^2 - \frac{1}{4}I\right],$$

which simplifies to

$$\mathcal{T}(\mathcal{T} - I)(\mathcal{T}^2 - \mathcal{T} - I) = 0. \tag{3}$$

REMARK 1. *Since* (3) *can be factorized into distinct linear factors (over* $\Re$*), it follows that* $\mathcal{T}$ *is diagonalizable and has at most the four distinct eigenvalues*

$$0, \quad 1, \quad \frac{1}{2} \pm \frac{\sqrt{5}}{2}. \tag{4}$$

If $\mathcal{T}$ *is nonsingular then it has the three nonzero eigenvalues.*

REMARK 2. *Proposition* 1 *specifically addresses* left *preconditioning, but by simple similarity transformations involving* $\mathcal{P}$ *or factorizations of* $\mathcal{P}$*, the result of the proposition applies equally for* right *preconditioning,* $\mathcal{T} = \mathcal{A}\mathcal{P}^{-1}$*, or in general, any centered preconditioning* $\mathcal{T} = \mathcal{P}_1^{-1}\mathcal{A}\mathcal{P}_2^{-1}$*, where* $\mathcal{P}_1\mathcal{P}_2 = \mathcal{P}$.

REMARK 3. *It directly follows from Proposition* 1 *that for any vector* r*, the Krylov subspace*

$$\mathrm{span}\{r, \mathcal{T}r, \mathcal{T}^2r, \mathcal{T}^3r, \ldots\}$$

is of dimension at most 3 *if* $\mathcal{T}$ *is nonsingular (or* 4 *if* $\mathcal{T}$ *is singular). Thus, in particular, any Krylov subspace iterative method with an optimality or Galerkin property, for* *example, such as any minimum residual method (see* [6]*), will terminate in at most three iterations with the solution to a linear system of the form* (1) *if the preconditioner* (2) *is used.*

REMARK 4. *If consideration of symmetry is not important, then an argument similar to the above shows that the choice*

$$\mathcal{P} = \begin{bmatrix} A & B^T \\ 0 & CA^{-1}B^T \end{bmatrix} \tag{5}$$

yields a preconditioned system with exactly the two eigenvalues ± 1*. Use of* (5) *rather than* (2) *requires only one more multiplication of a vector by* B^T *per iteration. If* $CA^{-1}B^T$ *is replaced by* $-CA^{-1}B^T$ *in* (5)*, then the preconditioned matrix has only a single eigenvalue of* 1*, but it is* not *in this case diagonalizable: it is similarly elementary to show that the minimum polynomial is* $(\mathcal{P}^{-1}\mathcal{A} - I)^2$*. Since this is again quadratic, there seems nothing to gain by choice of sign.*

We comment that in the context of general symmetric indefinite factorization, Gill et al. [7] have proposed a preconditioner which also gives a preconditioned matrix with eigenvalues ± 1. In the notation of this paper (and in the nonsymmetric case), their preconditioner is

$$\mathcal{P} = \begin{bmatrix} A & B^T \\ C & 2CA^{-1}B^T \end{bmatrix}.$$

REMARK 5. *Proposition* 1 *is of practical use when inexpensive approximations of* A *and of the Schur complement* $CA^{-1}B^T$ *exist. Known examples arise in discretizations of the Stokes problem of incompressible fluid dynamics* [13]*, and indeed in many saddle-point approximations where* inf-sup *or Babuška–Brezzi* [1] *stability holds. In such cases, the eigenvalues lie in three distinct regions; the precise clustering around the nonzero values in* (4) *depends on the quality of the approximations.*

REMARK 6. *If* A^{-1} *is readily approximated then this approximation could be used in the Schur complement as well as in the* (1,1) *block of* $\mathcal{P}$*. (See, for example,* [8]*, where semicirculant approximations which allow the use of fast Fourier transforms are used). Also inner-outer iteration could be effective for this preconditioner (see, for example,* [5]*).*

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