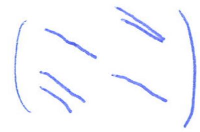
September 12, 2013

Schur Complement:

$$\begin{pmatrix} B & C \\ D & E \end{pmatrix} \begin{pmatrix} u^{(b)} \\ u^{(r)} \end{pmatrix} = \begin{pmatrix} f^{(b)} \\ f^{(r)} \end{pmatrix}$$



$$(*)\quad Bu^{(b)} + Cu^{(r)} = f^{(b)}$$

$$(**)\quad Du^{(b)} + Eu^{(r)} = f^{(r)}$$

$$(**) - D^{-1}B(*) : \underbrace{(E - DB^{-1}C)}_{\text{Schur Complement}} u^{(r)} = f^{(r)} - DB^{-1}f^{(b)}$$

$$\underbrace{-\Delta u + (\sigma, \tau)\nabla u = f}$$

discretize

$(\sigma, \tau) = (0, 0)$: Poisson

$-\Delta_h u$: symmetric positive definite

nonsymmetric

eigenvalues of $-\Delta_h$ n large

$$\lambda_{ij} = 4 - 2\cos\frac{\pi i}{n+1} - 2\cos\frac{\pi j}{n+1} \qquad 1 \le i, j \le n$$

$$0 < \lambda_{ij} < 8$$

Smallest: $\lambda_{11} = 4 - 4\cos\frac{\pi}{n+1}$

largest: $\lambda_{nn} = 4 - 4\cos\frac{\pi n}{n+1}$

$\lessapprox 8$

Condition number:

$$\kappa_2(A) = \frac{\text{maximal singular value}}{\text{minimal singular value}}$$

If A is SPD:

$$\kappa_2(A) = \frac{\max \text{ eig}}{\min \text{ eig}}$$

if A square & non-singular
$$\kappa_2(A) = \|A\| \, \|A^{-1}\|$$

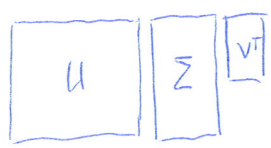
$m \geq n$

$$\mathbb{R}^{m \times n} \ni A = U \Sigma V^T \qquad \Sigma = \begin{pmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{pmatrix} \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$$

orthonormal columns

<u>One way:</u>

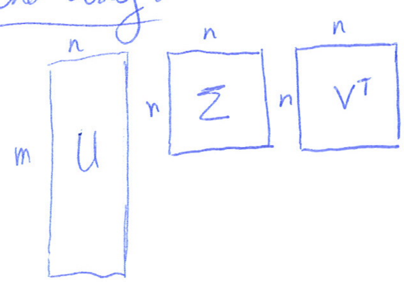
$U \in \mathbb{R}^{m \times m}$
$\Sigma \in \mathbb{R}^{m \times n}$
$V \in \mathbb{R}^{n \times n}$



U, V orthogonal (unitary)

$U^T U = I_m$
$V^T V = I_n$

<u>Second way:</u>



$Ax = b$

$\tilde{x}$: computed sol$^{\underline{n}}$

what is the error? - can we bound it?

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq ?$$

absolute error : $e = x - \tilde{x}$ not computable

residual : $r = b - A\tilde{x}$ computable

Connection : $Ae = r$ $\quad \|b\| \le \|A\| \|x\|$

$e = A^{-1} r$ $\quad \|e\| \le \|A^{-1}\| \|r\|$

$$\frac{\|e\|}{\|x\|} \le \frac{\|A^{-1}\| \, \|r\| \, \|A\|}{\|b\|}$$

$$\underbrace{\left(\|A\| \cdot \|A^{-1}\|\right)}_{K(A)} \left(\frac{\|r\|}{\|b\|}\right)$$

relative residual

- even if relative residual small, if condition # large → can have large error

Laplacian:

largest eig: ≈ 8

Smallest eig: $4 - 4\cos\frac{\pi}{n+1}$

$\cos \approx 1 - \frac{x^2}{2}$, $x \ll 1$

$$\lambda_{min}(A) = 4 - 4\cos\frac{\pi}{n+1} = O\left(\frac{1}{n^2}\right) \equiv O(h^2)$$

refine mesh → better sol^{n}, but worse conditioning

h smaller means:

+ : more accurate discretization

− : larger + more ill-conditioned matrix

~~bad~~ condition number → how many accurate digits can you get (machine # represented w/ 16 values, so condition # of 10^{12} → may only get ~4 accurate digits)

$$\min \ \frac{1}{2}x^TAx - c^Tx$$ Quadratic Problem with

$$\text{s.t. } Bx = d$$ equality constraints.

$$\left.\begin{array}{l} \phi(x,\lambda) = \frac{1}{2}x^TAx - c^Tx + \lambda^T(Bx-d) \\ \nabla\phi = 0 \end{array}\right\} \longrightarrow \overbrace{\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}}^{K}\begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

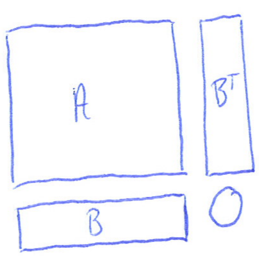
saddle-point system (KKT)

Arises frequently also in fluid dynamics

$$\begin{cases} -\Delta u + \nabla p = f \\ \nabla \cdot u = 0 \end{cases}$$ Stokes

$$\begin{cases} -\Delta u + u\nabla u + \nabla p = f \\ \nabla \cdot u = 0 \end{cases}$$ Navier-Stokes

electromagnetics!



$A \in \mathbb{R}^{n\times n}$

$B \in \mathbb{R}^{m\times n}$ $\quad m < n$

A might be singular! (Maxwell..., $\nabla\times\nabla\times$)

Suppose A is symmetric, positive semi-definite
$(x^TAx \geq 0 \quad \forall x)$

(recall: if $x^TAx > 0 \quad \forall x \neq 0 \Rightarrow$ pos. def.)

We will also often assume A is SPD.

Thm:

K is nonsingular iff

rank $(B) = m$, $\text{null}(A) \cap \text{null}(B) = \{0\}$

(ie. if $Az = 0$, $Bz = 0 \Rightarrow z = 0$).

Satisfied automatically if A is SPD.

Sol$^{\underline{n}}$ methods:

* Schur complement
* Null-space.

Note: a matrix is positive definite iff $\lambda_i(A) > 0$
positive semi-definite $\lambda_i(A) \geq 0$

Definiteness of K: ? -if A is positive semidefinite, B full rank, what can we say about the definiteness of K?

$$K = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

$$x = [x_A, x_B]^T$$

$$[x_A^T \; x_B^T] \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = [x_A^T \; x_B^T] \begin{pmatrix} Ax_A + B^T x_B \\ Bx_A \end{pmatrix}$$

$$= x_A^T (Ax_A + B^T x_B) + x_B^T B x_A$$

$$= x_A^T A x_A + x_A^T B^T x_B + x_B^T B x_A$$

$$= x_A^T A x_A + 2 x_A^T B^T x_B$$

if $x_A = 0 \Rightarrow 0$, so it is not PD, indefinite matrix!

Can also solve an eigenvalue problem.

exercise: complete the argument.

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$Bx = \lambda y \qquad Ax + \frac{1}{\lambda} B^T B x = \lambda x$$

$$y = \frac{1}{\lambda} Bx$$

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

Suppose A is SPD

$$\overbrace{BA^{-1}B^T}^{S}\lambda = -d + BA^{-1}c \qquad \text{range space method.}$$

+: smaller system

+: S is SPD.

−: need A^{-1}

−: take inverse of a sparse matrix → dense system.