October 8, 2013

Arnoldi:

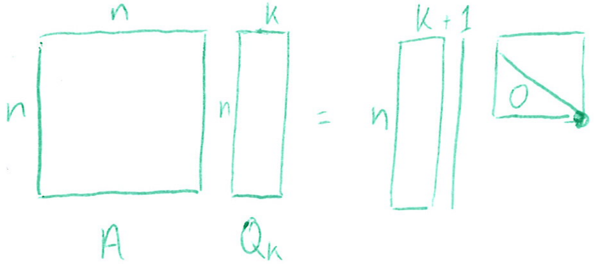
$$AQ_k = Q_{k+1} H_{k+1,k}$$

$$\boxed{Q_k^T A Q_k = H_{k,k}}$$ "truncated similarity transformation"

upper Hessenberg:

$$\begin{matrix} x & x & x & x \\ x & x & x & x \\ & x & x & x \\ & & x & x \\ \hline & & & \boxed{x} \end{matrix} \quad H_{k,k} \Big\} H_{k+1,k}$$

why is this correct?



$\hookrightarrow$ Q_k is orthogonal

$$Q_k^T Q \doteq I_k$$

$$Q_k Q_k^T \neq I_n$$

projection matrix

Krylov subspace solvers:

1. orthogonal projection: $(K = L)$

$$Q_k^T(b - Ax_k) = 0$$

$$Q_k^T b = Q_k^T A x_k$$

we know

$$x_k \in x_0 + K^k(A; r_0) \Rightarrow x_k = x_0 + Q_k z$$

$$\Rightarrow Q_k^T b = \underbrace{Q_k^T A Q_k}_{H_{kk}} z + Q_k^T A x_0$$

$$\Rightarrow Q_k^T r_0 = H_{kk} z$$

recall: $q_1 = \frac{r_0}{\|r_0\|} \Rightarrow \|q_1\| = 1$ and $K^k(A; r_0) = \mathrm{span}\{r_0, Ar_0, \dots, A^{k-1} r_0\}$

so

$$\begin{pmatrix} - q_1^T - \\ - q_2^T - \\ \vdots \\ - q_k^T - \end{pmatrix} \begin{pmatrix} | \\ r_0 \\ | \end{pmatrix} = \begin{pmatrix} \frac{r_0^T r_0}{\|r_0\|} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \beta e_1 \quad \text{where } \beta = \|r_0\|$$

FOM: Full Orthogonalization Method. : $\boxed{H_{KK} z = \rho e_1}$

Algorithm:

- Arnoldi $\longrightarrow \{q_i\}, H_{K,K}$
- $x_r = x_0 + H_{KK}^{-1} \rho e_1$ where $\rho = \|r_0\|$

If A is SPD

$H_{KK} \longrightarrow T_{KK}$ Lanczos

$x_k = x_0 + T_{KK}^{-1} \rho e_1$ · a fancy implementation of this gives CG

At least two nice ways to derive CG:

1. optimization-like

$x_{k+1} = x_k + \alpha_k p_k$

where $p_k^T A p_j = 0 \quad j \neq k$; A-conjugate direction

$\min \phi(x)$ is equivalent to $\min \|e_k\|_A$ over same space

$$\|e_k\|_A^2 = e_k^T A e_k$$

$$= (x - x_k)^T A (x - x_k)$$

$$= \underbrace{x^T A x}_{\text{some fixed number}} \underbrace{- 2 x_k \underbrace{Ax}_{b} + x_k^T A x_k}_{2\phi(x_k)}$$

2. Linear algebra-type way:

$T_{KK} = L_k D_k L_k^T$

and find recurrence relations among the $\{k\}$-related and $\{k+1\}$-related quantities

② Minimum Residual Methods

$\min \|b - A x_k\|_2$

$x_k = x_0 + Q_k z$

$\|b - Ax_k\|_2 = \|b - A(x_0 + Q_k z)\|_2$

$= \|r_0 - AQ_k z\|_2$ Least Squares Problem

$\min \|\rho e_1 - H_{k+1,k} z\|$

use $AQ_k = Q_{k+1} H_{k+1,k}$

and orthogonality of Q_k

- Important property of orthogonal transformations: preserve norm

$b - Ax_k = b - A(x_0 + Q_k z)$

$= r_0 - AQ_k z$

$= \rho q_1 - Q_{k+1} H_{k+1,k} z$

$= Q_{k+1}(\rho e_1 - H_{k+1,k} z)$

Subtlety! → can now multiply by Q_{k+1}^T

This is Arnoldi method $\xrightarrow{\text{fancy implementation}}$ GMRES

$\xrightarrow{\text{if } A \text{ sym (not nec. PD)}}$ MINRES

Algorithm:

Arnoldi: → $AQ_k = Q_{k+1} H_{k+1,k}$

$\min_z \|\rho e_1 - H_{k+1,k} z\|_2$

$x_k = x_0 + Q_k z$

Caveats:

- for A nonsymmetric: long recurrence relations, accumulating cost & storage
  - ↳ Remedy: restarted GMRES
    - after k iterations, refer to r_k as r_0 and start from scratch
    - effective if we have a memory restriction
    - <u>but</u>: we lose optimality
      - Nonsym systems: either optimality or modest memory requirements, but not both