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Operator Preconditioning

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Abstract—Operator preconditioning offers a general recipe for constructing preconditioners for discrete linear operators that have arisen from a Galerkin approach. The key idea is to employ matching Galerkin discretizations of operators of complementary mapping properties. If these can be found, the resulting preconditioners will be robust with respect to the choice of the bases for trial and test spaces. I survey the application of operator preconditioning to finite elements and boundary elements.

Keywords—Operator preconditioning, Galerkin discretization, Duality, Finite elements, Boundary elements.

1. INTRODUCTION

Discrete linear operator equation is a pompous term for a linear system of equations. Nevertheless, showing off sophistication is not my reason for using this phrase. Rather, it forcefully conveys that there is a continuous operator equation posed on infinite-dimensional spaces that underlies the linear system of equations. This seems to be a moot point, for instance in the case of boundary value problems for partial differential equations, but awareness of this connection is key to devising efficient solution strategies for the linear systems.

This is important, because direct elimination often incurs prohibitive computational costs when applied to large discrete operator equations. Then, the only viable options are iterative solution methods [1] whose efficiency hinges on powerful *preconditioners*. The idea of operator preconditioning can furnish those (to some extent).

The reasoning behind operator preconditioning is intriguingly simple. Given a continuous bijective linear operator $A : V \mapsto W$ on function spaces V and W, and another isomorphism $B : W \mapsto V$, then BA will provide an endomorphism of V. The convenient fact about such endomorphism is that their discretization often gives rise to well-conditioned matrices. A standard example is mass matrices in a finite-element context, hence, the consideration to use B as a "preconditioner" for A.

Yet, we will always face a discrete approximation $A_h : V_h \mapsto W_h$ of A, connecting finite-dimensional spaces V_h and W_h. This raises the issue of what kind of discretization B_h of B can still provide a preconditioner B_h for A_h. The next section will give a general answer for variational problems. We shall see that it takes really judicious choices of B_h to make operator preconditioning work.

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The idea of operator preconditioning has probably been put forth many times in special contexts. For instance, in [2, Section 7] it was mentioned as an option for preconditioning saddle point problems, see Section 3 for details. A matrix-oriented presentation is available in [3]. A systematic discussion for elliptic pseudodifferential operators in Sobolev scales is given in [4, Section 3], see also [5]. Subsequently, a generalization was given in [6]. In these works, the focus was on boundary integral equations.

In this paper, I will try and gather some of the scattered applications of operator preconditioning. Further, a unifying theoretical framework will be presented.

2. ABSTRACT THEORY

On two reflexive Banach spaces V, W we consider two continuous sesquilinear forms $a \in L(V \times V, \mathbb{C})$ and $b \in L(W \times W, \mathbb{C})$. Let $V_h \subset V$ and $W_h \subset W$ be finite-dimensional subspaces, on which the sesquilinear forms fulfill the inf-sup-conditions

$$\sup_{v_h \in V_h} \frac{|a(u_h, v_h)|}{\|v_h\|_V} \geq c_A \|u_h\|_V, \qquad \forall u_h \in V_h, \tag{2.1}$$

$$\sup_{w_h \in W_h} \frac{|b(q_h, w_h)|}{\|w_h\|_W} \geq c_B \|q_h\|_W, \qquad \forall q_h \in W_h. \tag{2.2}$$

Further, there is a stable pairing connecting the spaces V_h and W_h: we assume the existence of a continuous sesquilinear form $d \in L(V \times W, \mathbb{C})$ that satisfies another inf-sup-condition

$$\sup_{w_h \in W_h} \frac{|d(v_h, w_h)|}{\|w_h\|_W} \geq c_D \|v_h\|_V, \qquad \forall v_h \in V_h. \tag{2.3}$$

Picking bases $\{b_1, \dots, b_N\}$, $N := \dim V_h$, of V_h and $\{q_1, \dots, q_M\}$, $M := \dim W_h$, of W_h, we can introduce the Galerkin-matrices

$$\mathbf{A} := (a(b_i, b_j))_{i,j=1}^{N}, \qquad \mathbf{D} := (d(b_i, q_j))_{i,j=1}^{N,M}, \qquad \mathbf{B} := (b(q_i, q_j))_{i,j=1}^{M}.$$

THEOREM 2.1. *(See [7].) If, besides (2.1)–(2.3),* $\dim V_h = \dim W_h$, *then*

$$\kappa\left(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-\top}\mathbf{A}\right) \leq \frac{\|a\|\|b\|\|d\|^2}{c_A c_B c_D^2},$$

where $\kappa(\cdot)$ *stands for the spectral condition number of a square matrix.*

PROOF. Denote by $A_h : V_h \mapsto V_h'$, $B_h : W_h \mapsto W_h'$, and $D_h : V_h \mapsto W_h'$ the bounded linear operators associated with the sesquilinear forms a, b, and d. Writing $D_h^* : W_h \mapsto V_h'$ for the adjoint operator of D_h, we immediately conclude

$$\begin{aligned} \|A_h\|_{V_h \mapsto V_h'} &= \|a\|, & \|A_h^{-1}\|_{V_h' \mapsto V_h} &\leq c_A^{-1}, \\ \|B_h\|_{W_h \mapsto W_h'} &= \|b\|, & \|B_h^{-1}\|_{W_h' \mapsto W_h} &\leq c_B^{-1}, \\ \|D_h\|_{V_h \mapsto W_h'} = \|D_h^*\|_{W_h \mapsto V_h'} &= \|d\|, & \|D_h^{-1}\|_{W_h' \mapsto V_h} = \|D_h^{-*}\|_{V_h' \mapsto W_h} &\leq c_D^{-1}. \\ \Rightarrow \quad \|D_h^{-1} B_h D_h^{-*} A_h\| &\leq c_D^{-2}\|a\|\|b\|, & \|A_h^{-1} D_h^* B_h^{-1} D_h\| &\leq \|d\|^2 c_A^{-1} c_b^{-1}. \end{aligned}$$

Recall that the Galerkin matrix corresponding to D_h^* is $\mathbf{D}^\top$. Thus, equipping $\mathbb{C}^N$ with a norm $\|\cdot\|_V$ inherited from the space V_h via the coefficient isomorphism w.r.t. the basis $\{b_1, \dots, b_N\}$, we find

$$\left|\lambda_{\max}\left(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-\top}\mathbf{A}\right)\right| \le \left\|\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-\top}\mathbf{A}\right\| \le c_D^{-2}\|a\|\|b\|,$$
$$\left|\lambda_{\max}\left(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-\top}\mathbf{A}\right)^{-1}\right| \le \left\|\mathbf{A}^{-1}\mathbf{D}^\top\mathbf{B}\mathbf{D}\right\| \le \|d\|^2 c_A^{-1} c_b^{-1}. \qquad \blacksquare$$

REMARK. The bound of Theorem 2.1 is completely independent of the choice of bases for V_h and W_h. The choice of Galerkin spaces V_h and W_h only enters through the constants c_A, c_B, and c_D.

3. FINITE-ELEMENT APPLICATIONS

Now, the role of the space V of Section 2 is played by a Hilbert space H with inner product $(\cdot,\cdot)_H$. As before, a is a bounded sequilinear form on H that satisfies (2.1) on a finite-dimensional subspace $H_h \subset H$.

Specializing the generic setting in Section 2, we choose W as the dual space H', itself a Hilbert space [8]. A suitable finite-dimensional subspace $W_h \subset H'$ with $\dim W_h = \dim H_h$ is furnished by the polar set of the orthogonal complement of H_h in H,

$$W_h := \left\{\varphi \in H' : \langle \varphi, v\rangle_{H'\times H} = 0,\ \forall\, v \in H_h^\perp\right\}.$$

The role of the sesquilinear form $b \in L(W \times W, \mathbb{C})$ will be played by the inner product of H'. In other words, the associated operator $B : W \mapsto W' = H$ boils down to the inverse of the isometric Riesz-isomorphism $R : H \mapsto H'$ [8],

$$\langle Rw, BRv\rangle_{H'\times H} = (Rw, Rv)_{H'} = (w, v)_H = \langle Rw, v\rangle_{H'\times H}, \qquad \forall\, w, v \in H. \tag{3.1}$$

It is immediate that $\|b\| = 1$ and (2.2) holds with $c_B = 1$.

Finally, the sesquilinear pairing $d \in L(V \times W, \mathbb{C})$ will agree with the natural duality pairing $\langle\cdot,\cdot\rangle_{H'\times H}$. Given any basis $\{b_1, \dots, b_N\}$, $N := \dim H_h$, of H_h, we can find $\beta_1, \dots, \beta_N \in W_h \subset H'$ such that $\langle \beta_i, b_j\rangle_{H'\times H} = \delta_{ij}$. This gives a basis for W_h, for which the Galerkin matrix associated with $d(\cdot,\cdot)$ becomes the identity matrix.

$$d(b_i, \beta_j) = \langle \beta_j, b_i\rangle_{H'\times H} = \delta_{ij}, \qquad i, j = 1, \dots, N.$$

Further, the Galerkin matrix $\mathbf{B}$ for $b(\cdot,\cdot)$ with respect to these bases is the inverse of the Riesz matrix $\mathbf{R} := ((b_i, b_j)_H)_{i,j=1}^N$. To see this, note that $RH_h = W_h$ and

$$Rb_i = \sum_{j=1}^N (b_i, b_j)_H\, \beta_j, \qquad i = 1, \dots, N.$$

Similarly, by the definition of the dual basis,

$$B\beta_i = \sum_{j=1}^N (\mathbf{B})_{ji} b_j \Rightarrow (\mathbf{B})_{ji} = \langle \beta_j, B\beta_i\rangle_{H'\times H}, \qquad i, j = 1, \dots, N.$$

Owing to (3.1) this means $\mathbf{B} = \mathbf{R}^{-1}$. The bottom line is that we can conclude from Theorem 2.1 that

$$\kappa\left(\mathbf{R}^{-1}\mathbf{A}\right) \le \|a\| {c_A}^{-1}. \tag{3.2}$$

Usually, the exact evaluation of the action of $\mathbf{R}^{-1}$ will not be feasible, but I would like to point out that $\mathbf{R}^{-1}$ can be replaced by any good preconditioner for the symmetric positive definite matrix $\mathbf{R}$. Such preconditioners might already be known, see the examples given in the remainder of this section: the problem is converted into one already solved.

Q: How does a mathematician heat water in a pot?
A: He simply puts the pot onto a stove.
Q: How does a mathematician heat water in kettle?
A: He pours it into a pot and, thus, has reduced the problem to one already solved.

Folk joke.

3.1. Saddle Point Problems [2]

Consider the mixed variational formulation of second-order elliptic boundary value problems with Dirichlet boundary conditions [9, Ch. 3]. In this case we face $H = \boldsymbol{H}(\mathrm{div};\Omega) \times L^2(\Omega)$. We take for granted a stable finite-element discretization employing pairs of conforming finite-element spaces that satisfy uniform inf-sup-conditions. This guarantees (2.1) independent of the finite-element mesh.

According to the above considerations, the Galerkin matrix induced by the $\boldsymbol{H}(\mathrm{div};\Omega) \times L^2(\Omega)$-inner product can serve as a preconditioner for the indefinite saddle point matrix. Obviously, this matrix is block-diagonal and one block agrees with a well-conditioned mass matrix. The other block may be replaced by one application of a V-cycle of the multigrid method presented in [2,10]. Numerical studies of the efficacy of this preconditioner for accelerating the MINRES method are reported in [2].

Another example is the variational formulation of the Stokes problem [11, III, Section 5], where $H = H_0^1(\Omega) \times L_0^2(\Omega)$. The indefinite matrix arising from a stable conforming finite-element method can be preconditioned by the block-diagonal s.p.d. matrix related to the inner product of H. In turns, this can be replaced by a computationally inexpensive approximation using standard multigrid methods in $H^1(\Omega)$ [12].

3.2. Complex Variational Problems

We target the sesquilinear form

$$a(u,v) := (\alpha \operatorname{grad} u, \operatorname{grad} v)_{L^2(\Omega)} + i\,(\sigma u, v)_{L^2(\Omega)}\,, \qquad u, v \in H_0^1(\Omega),$$

$\alpha, \sigma \in L^\infty(\Omega)$ uniformly positive. Such problems arise, e.g., in models for low-frequency time-harmonic electromagnetic fields [13]. To apply the theory put forth in the beginning of this section, we choose the space $H_{\mathbb{C}} = H_0^1(\Omega)$ of complex-valued functions with inner product

$$(u,v)_H := (\alpha \operatorname{grad} u, \operatorname{grad} v)_{L^2(\Omega)} + (\sigma u, v)_{L^2(\Omega)}\,. \tag{3.3}$$

Since

$$|a(\boldsymbol{u},\boldsymbol{u})| \geq \frac{1}{\sqrt{2}} \left(\left\| \alpha^{1/2} \operatorname{grad} u \right\|_{L^2(\Omega)}^2 + \left\| \beta^{1/2} u \right\|_{L^2(\Omega)}^2 \right), \tag{3.4}$$

we conclude that $c_A \geq (1/2)\sqrt{2}$ and $\|a\| \leq 1$, see (2.1).

What follows is motivated by the observation that a complex linear system of equations $\mathbf{A}x = b$ can be converted into the equivalent real system (subscripts tag real/imaginary parts)

$$\begin{pmatrix} \mathbf{A}_R & -\mathbf{A}_I \\ -\mathbf{A}_I & -\mathbf{A}_R \end{pmatrix} \begin{pmatrix} x_R \\ x_I \end{pmatrix} = \begin{pmatrix} b_R \\ -b_I \end{pmatrix}.$$

Thus, when splitting a variational problem for the sesquilinear form $a(\cdot,\cdot)$ into real and imaginary parts, we end up with a saddle point problem on the real Hilbert space $H := H_0^1(\Omega) \times H_0^1(\Omega)$ (still equipped with inner product (3.3)) related to the bilinear form

$$\tilde{a}\left(\begin{pmatrix} u_R \\ u_I \end{pmatrix}, \begin{pmatrix} v_R \\ v_I \end{pmatrix}\right) := (\alpha \operatorname{grad} u_R, \operatorname{grad} v_R)_{L^2(\Omega)} - (\sigma u_I, v_R)_{L^2(\Omega)} \\ - (\sigma u_R, v_I)_{L^2(\Omega)} - (\alpha \operatorname{grad} u_I, \operatorname{grad} v_I)_{L^2(\Omega)} .$$

As H and $H_{\mathbb{C}}$ are isometrically isomorphic, it goes without saying that $\tilde{a}$ inherits stability and continuity constants from a.

Given an $H_0^1(\Omega)$-conforming finite-element space with a fixed basis, we may write $\mathbf{R} \in \mathbb{R}^{N,N}$ for the Galerkin matrix associated with the inner product (3.3) on $H_0^1(\Omega)$, and $\tilde{\mathbf{A}}$ for the stiffness matrix arising from $\tilde{a}$. Then estimate (3.2) can instantly be applied and yields

$$\kappa\left(\begin{pmatrix} \mathbf{R}^{-1} & \\ & \mathbf{R}^{-1} \end{pmatrix} \tilde{\mathbf{A}}\right) \leq \frac{1}{2}\sqrt{2}.$$

Using MINRES and replacing $\mathbf{R}^{-1}$ by a multigrid cycle will yield a robust iterative solver for the complex variational problem.

REMARK. The investigations could also have been conducted on the algebraic level by showing that for the generalized eigenvalue problem

$$\begin{pmatrix} \mathbf{A}_R & -\mathbf{A}_I \\ -\mathbf{A}_I & -\mathbf{A}_R \end{pmatrix} \begin{pmatrix} x_R \\ x_I \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{A}_R + \mathbf{A}_I & 0 \\ 0 & \mathbf{A}_R + \mathbf{A}_I \end{pmatrix}$$

the moduli of the eigenvalues will always lie in $[(1/2)\sqrt{2}, 1]$. This is true for any positive semidefinite $\mathbf{A}_R$ and positive definite $\mathbf{A}_I$.

4. BOUNDARY ELEMENT APPLICATIONS

The weak forms of boundary integral equations of the first kind on a Lipschitz-surface $\Gamma := \Omega$, $\Omega \subset \mathbb{R}^3$, naturally involve sesquilinear forms defined on trace spaces. Prominent examples are the single-layer boundary integral equation [14, Ch. 3] with

$$a(u,v) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} u(\boldsymbol{x})\bar{v}(\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{x},\boldsymbol{y}), \qquad u,v \in H^{-1/2}\Gamma \tag{4.1}$$

and the electric field boundary integral equation [15] posed on $H^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, for which

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y})\bar{\boldsymbol{v}}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{curl}_\Gamma\, \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{curl}_\Gamma\, \bar{\boldsymbol{v}}(\boldsymbol{x})\right) \mathrm{d}S(\boldsymbol{y},\boldsymbol{x}). \tag{4.2}$$

Let us assume that piecewise constants on a surface mesh (space V_h) are used for a conforming boundary element Galerkin discretization of a variational problem involving the sesquilinear form from (4.1). An approach to operator preconditioning for the resulting discrete boundary integral equation, cf. [4], is suggested by

- the duality $(H^{-1/2}(\Gamma))' \cong H^{1/2}(\Gamma)$ with respect to $L^2(\Gamma)$ as pivot space,
- and the availability of a continuous, bijective (slightly modified [4]) hypersingular boundary integral operator $B : H^{1/2}(\Gamma) \mapsto (H^{1/2}(\Gamma))'$.

The real challenge is to choose a subspace $W_h \subset H^{1/2}(\Gamma)$ such that $\dim W_h = \dim V_h$ and (2.3) holds for $d(v,w) := (v,w)_{L^2(\Gamma)}$. A viable construction is given in [16] and the key idea is to use a pair of *dual meshes*. Meshes are called dual to each other, if the vertex-edge and edge-cell incidence matrices of one agree with the cell-edge and edge-vertex incidence matrices of the

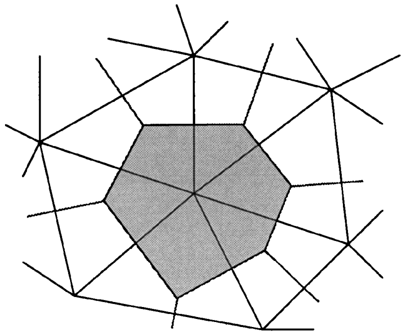

Figure 1. Circumcentric (orthogonal) dual mesh.

other [17]. Geometric realizations are furnished by circumcentric and barycentric constructions, see Figure 1.

Then, given a triangular mesh $\mathcal{M}_h$ of Γ, V_h will contain the piecewise constants on its barycentric dual, whereas W_h is the space of piecewise linear continuous functions on $\mathcal{M}_h$. Evidently, $\dim V_h = \dim W_h$, and continuity ensures $W_h \subset H^{1/2}(\Gamma)$.

In order to show an h-uniform estimate (2.3), we assume a shape-regular and quasi-uniform family of meshes $\{\mathcal{M}_h\}_h$, h standing for their meshwidths. Simple local arguments show that

$$\exists C > 0 : \sup_{v_h \in V_h} \frac{(v_h, w_h)_{L^2(\Gamma)}}{\|v_h\|_{L^2(\Gamma)}} \geq C \|w_h\|_{L^2(\Gamma)}, \qquad \forall w_h \in W_h. \tag{4.3}$$

Here and in the sequel, C stands for a generic positive constant independent of h. Thanks to (4.3),

$$P_h u \in W_h : (P_h u, v_h)_{L^2(\Gamma)} = (u, v_h)_{L^2(\Gamma)}, \qquad \forall v_h \in V_h, \quad u \in L^2(\Gamma),$$

defines a projector $P_h : L^2(\Gamma) \mapsto W_h$. From (4.3) we infer that

$$\begin{aligned} \|P_h u - u\|_{L^2(\Gamma)} &\leq \|P_h u - Q_h u\|_{L^2(\Gamma)} + \|u - Q_h u\|_{L^2(\Gamma)} \\ &\leq C \sup_{v_h \in V_h} \frac{(u - Q_h u, v_h)_{L^2(\Gamma)}}{\|v_h\|_{L^2(\Gamma)}} + \|u - Q_h u\|_{L^2(\Gamma)} \leq C \|u - Q_h u\|_{L^2(\Gamma)}, \end{aligned} \tag{4.4}$$

where $Q_h : L^2(\Gamma) \mapsto W_h$ denotes the $L^2(\Gamma)$-orthogonal projection. Moreover Q_h satisfies [18]

$$\|Q_h u\|_{L^2(\Gamma)} \leq \|u\|_{L^2(\Gamma)}, \qquad \forall u \in L^2(\Gamma), \qquad \|Q_h u\|_{H^1(\Gamma)} \leq C \|u\|_{H^1(\Gamma)}, \qquad \forall u \in H^1(\Gamma),$$

and, by interpolation between $L^2(\Gamma)$ and $H^1(\Gamma)$ we conclude

$$\|Q_h u\|_{H^{1/2}(\Gamma)} \leq C \|u\|_{H^{1/2}(\Gamma)}, \qquad \forall v \in H^{1/2}(\Gamma). \tag{4.5}$$

Similarly, interpolation confirms the error estimate

$$\|v - Q_h v\|_{L^2(\Gamma)} \leq C h^{1/2} \|u\|_{H^{1/2}(\Gamma)}. \tag{4.6}$$

Merging (4.4)–(4.6) and using an inverse inequality yields

$$\begin{aligned} \|P_h u\|_{H^{1/2}(\Gamma)} &\leq \|P_h u - Q_h u\|_{H^{1/2}(\Gamma)} + \|Q_h u\|_{H^{1/2}(\Gamma)} \\ &\leq C \left(h^{-1/2} \|Q_h (P_h u - u)\|_{L^2(\Gamma)} + \|u\|_{H^{1/2}(\Gamma)} \right) \leq C \|u\|_{H^{1/2}(\Gamma)}. \end{aligned} \tag{4.7}$$

Eventually, this estimate combined with the definition of P_h involves

$$\begin{aligned} \sup_{w_h \in W_h} \frac{(w_h, v_h)_{L^2(\Gamma)}}{\|w_h\|_{H^{1/2}(\Gamma)}} &\geq \sup_{u \in H^{1/2}(\Gamma)} \frac{(P_h u, v_h)_{L^2(\Gamma)}}{\|P_h u\|_{H^{1/2}(\Gamma)}} \\ &\geq C \sup_{u \in H^{1/2}(\Gamma)} \frac{(u, v_h)_{L^2(\Gamma)}}{\|u\|_{H^{1/2}(\Gamma)}} = \|v_h\|_{H^{-1/2}(\Gamma)}, \end{aligned} \tag{4.8}$$

which amounts to (2.3) for the pairing $d(v, w) := (v, w)_{L^2(\Gamma)}$.

In the case of the sesquilinear form (4.2) the space V_h of $H^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$-conforming boundary elements is provided by surface edge elements [19,20]. The *Hodge-duality* of $H^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ with itself

$$\sup_{\boldsymbol{v} \in H^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)} \frac{\displaystyle\int_\Gamma (\boldsymbol{n} \times \boldsymbol{v}) \cdot \boldsymbol{w} \, \mathrm{dS}}{\|\boldsymbol{v}\|_{-1/2, \mathrm{curl}_\Gamma}} \approx \|\boldsymbol{w}\|_{-1/2, \mathrm{curl}_\Gamma}, \qquad \forall \boldsymbol{w} \in H^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \tag{4.9}$$

leads us to opt for the skew-symmetric pairing $d(\boldsymbol{v}, \boldsymbol{w}) := \int_\Gamma (\boldsymbol{n} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}} \, \mathrm{dS}$. Now, it is tempting to take the cue from (4.9) and choose $W_h = V_h$. However, in [21] it was shown that even on a family of quasi-uniform and shape-regular meshes

$$\exists \, N_h \subset V_h, \quad c > 0: \quad \forall \, \boldsymbol{u}_h \in N_h: \qquad \sup_{\boldsymbol{v}_h \in V_h} \frac{d(\boldsymbol{u}_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{-1/2, \mathrm{curl}_\Gamma}} \leq c h^{1/2} \|\boldsymbol{u}_h\|_{-1/2, \mathrm{curl}_\Gamma}.$$

This rules out h-uniform stability of the pairing of V_h with itself in the discrete setting. Again, a viable choice for W_h is offered by $H^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$-conforming edge elements on dual meshes, as has been demonstrated by Buffa and Christiansen in [22].

5. ASSESSMENT AND CONCLUSION

Operator preconditioning is a very general recipe for constructing preconditioners for the operators arising from discrete boundary value problems and integral equations. It is simple to apply, but may not be particularly efficient, because, in case the bound of Theorem 2.1 is large, the operator preconditioning policy offers no hint how to improve the preconditioner. Hence, operator preconditioning may often achieve the much-vaunted mesh independence of the preconditioner, but it may not perform satisfactorily on a given mesh: the joke's recommendation on how to heat water in a kettle is not exactly the smartest way to do it.

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