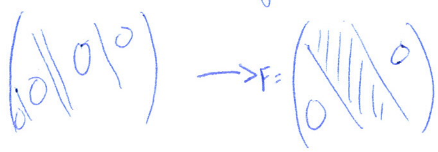
Sept 17, 2013

$Ax = b$

iterative:

direct: $A = LU$ (lu)

$= FF^T$ Cholesky (if A is SPD)



- fill in factorization

↳ can somewhat fix by orderings

- multicolor
- RCM
- AMD
- ⋮

} if you can use direct solver, then go ahead.

but if memory is an issue (eg. 3D)

- iterative methods: mat-vec prod based
- another good reason for iterative:
  - if we have a good initial guess.

OR

- if we want to solve inexactly

Stationary methods:

$A = M - N$ $\qquad A = \begin{pmatrix} \ddots & & F \\ & D & \\ E & & \ddots \end{pmatrix} = D + E + F$

$MX = Nx + b$

↓ x_0 (initial) given

$Mx_{k+1} = Nx_k + b$, $k = 0, 1, \ldots$

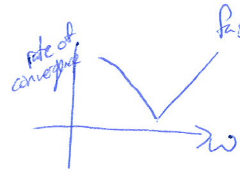
$M = D$ Jacobi $\xrightarrow{O(n^2)\text{ it}}$ poor convergence but parallelizable

$M = D + E$ Gauss-Seidel $\xrightarrow{O(n^2)\text{ it}}$ better than Jacobi, but still bad, good smoother (MG)

$M = \frac{1}{\omega}(D + \omega E)$ SOR $\xrightarrow{O(n)\text{ it.}}$ involves a parameter, but could be significantly faster

↳ $\omega < 1$: under-relaxed

$\omega > 1$: over-relaxed



$$Mx = Nx + b$$

$$Mx_{k+1} = Nx_k + b$$

$$M\underbrace{(x_* - x_{k+1})}_{e_{k+1}} = N\underbrace{(x - x_k)}_{e_k} \longrightarrow e_{k+1} = \underbrace{M^{-1}N}_{T\text{ : iteration matrix}} e_k$$

$$e_k = T^k e_0$$

arbitrary

$$\|e_k\| \le \|T^k\| \, \|e_0\|$$

$$\le \|T\|^k \, \|e_0\|$$

want $\|T\| < 1$

Convergence iff $\rho(T) < 1$

where

$$\rho(T) = \max_i |\lambda_i(T)|$$

spectral radius

The smaller $\rho(T)$ the better,

$$\|e_k\| \lessapprox \rho^k(T) \|e_0\|$$

$$10^{-m} \approx \frac{\|e_k\|}{\|e_0\|} \lessapprox \rho^k \implies -m \approx k \log_{10} \rho$$

$$\boxed{-\log_{10} \rho \approx \frac{m}{k}}$$

if $\rho \approx 1 \to \log_{10} \rho \approx 0$, fixed m, $k \to$ large

$\rho \approx 0 \to -\log_{10} \rho$ large, $k \to$ small

$-\ln \rho$: asymptotic rate of convergence (convergence is linear).

Correction form:

$$x_{k+1} = x_k + M^{-1} r_k$$

where $r_k = b - Ax_k$

$$M x_{k+1} = M x_k + b - A x_k$$
$$= b + (M - A) x_k$$
$$= b + N x_k$$

$$M^{-1} A x = M^{-1} b$$
$$\tilde{A} x = \tilde{b}$$
$$r = M^{-1}(b - Ax)$$
$$= \tilde{b} - \tilde{A} x.$$

So, can w.l.o.g. talk about

$$x_{k+1} = x_k + r_k$$

where r_k is associated w/ $\tilde{A} x = \tilde{b}$

$x_{k+1} = x_k + M^{-1} r_k$ for $Ax = b$ $\Longleftrightarrow$ $x_{k+1} = x_k + r_k$ for $\tilde{A} x = \tilde{b}$ where $\tilde{A} = M^{-1} A$, $\tilde{b} = M^{-1} b$

$$x_{k+1} = x_k + r_k$$

$$b - A(x_{k+1} = x_k + r_k)$$

$$\Downarrow$$

$$b - A x_{k+1} = b - A x_k - A r_k$$

$$r_{k+1} = r_k - A r_k$$
$$= (I - A) r_k$$

$$r_k = (I - A)^k r_0 = p_k(A) r_0.$$

$$x_{k+1} = x_k + r_k$$
$$= (x_{k-1} + r_{k-1}) + r_k$$
$$= \cdots = x_0 + \sum_{i=0}^{k} r_i \quad \leftarrow r_i = p_i(A) r_0, \quad x_k = x_0 + q_{k-1}(A) r_0$$

If the inverse is a polynomial in A, maybe we can solve $Ax - b = 0$ in n iterations exactly?

$r_0 = b - Ax_0$ initial residual

$x_k = x_0 + q_{k-1}(A) r_0$

Let $K^k(A; r_0) := \text{span}\{r_0, Ar_0, \dots, A^{k-1} r_0\}$: Krylov Subspace (k-dim).

$x_{k+1} = x_k + r_k$

$A = M - N$

$M = I$

$T = M^{-1}N = I - M^{-1}A$

$\rho(I - A) < 1$

$x_{k+1} = x_k + \alpha r_k$, $M^{-1} = \alpha I$

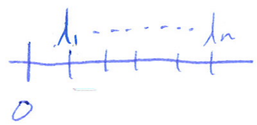
Choose α s.t.

$\rho(I - \alpha A) \to \min$

Suppose A is SPD

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$

eigenvalues $1 - \alpha \lambda_i$



want

$|1 - \alpha\lambda_1|, |1 - \alpha\lambda_n| < 1$ Show that α is optimal if $1 - \alpha\lambda_n = -(1 - \alpha\lambda_1)$

$$\alpha_{opt} = \frac{2}{\lambda_1 + \lambda_n}$$