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Power Method

Inverse Power method

Linear Convergence, asymptotic conv. constant:

want $\left|\frac{\lambda_2}{\lambda_1}\right|^k << 1$ error "kind of" after k iterations

Power method with

$(A-\alpha I)^{-1}$:

linear convergence

$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$

eigenvalues of Power method

$\frac{1}{\lambda_1-\alpha}, \frac{1}{\lambda_2-\alpha}, \dots$

Suppose $\alpha \approx \lambda_1$

$\frac{1}{\lambda_1-\alpha} >> \frac{1}{\lambda_i-\alpha}$

then ratio is given by

$$\frac{\left|\frac{1}{\lambda_2-\alpha}\right|}{\left|\frac{1}{\lambda_1-\alpha}\right|} = \left|\frac{\lambda_1-\alpha}{\lambda_2-\alpha}\right|$$

Choosing α

- we know

$\rho(A) \leq \|A\|$

Rayleigh Quotient

$$(A - \alpha_i I)^{-1}$$

$$\alpha_i = v_i^{\top} A v_i$$

Summary:

Power : linear, $\left|\frac{\lambda_2}{\lambda_1}\right|$, mat-vec prod itns

Inv Power: linear, $\left|\frac{\lambda_1 - \alpha}{\lambda_2 - \alpha}\right|$, solve system (factor $A - \alpha I = LU$ back/for solve

RQ : Cubic solve (and factor) in every iteration

What if we want more eigenvalues?

- leading p eigenvalues

Z_0 is $n \times p$

$Y_{i+1} = A Z_i \qquad i = 0, 1, \ldots$

factor $Y_{i+1} = Z_{i+1} R_{i+1}$ QR factorization.

Orthogonal iteration. $\to$ converge to the dominant p-eigenpairs

convergence will depend on

$$\left|\frac{\lambda_{p+1}}{\lambda_p}\right|$$

QR iteration: given A_0

repeat : $(i = 0, 1 \ldots)$

factor $A_i = Q_i R_i$ (QR decomposition)

$A_{i+1} = R_i Q_i$

Converge to a matrix T, Schur form (assuming eigenvalues of diff magnitudes, real, ..)

- $A = Q T Q^*$, T triangular

Q unitary.

- can read eigenvalues off diagonal of T note

$$A_{i+1} = R_i Q_i$$
$$= Q_i^T \underbrace{(Q_i R_i)}_{A_i} Q_i = Q_i^T A_i Q_i \qquad \text{orthogonally similar}$$

$$A_i = Z_i^T A Z_i$$

$$A Z_i = Z_{i+1} R_{i+1}$$

$$A_i = \underbrace{Z_i^T Z_{i+1}}_{Q} \underbrace{R_{i+1}}_{R}$$

$$RQ = R_{i+1} Z_i^T Z_{i+1}$$
$$= Z_{i+1}^T A \cancel{Z_i Z_i^T} Z_{i+1}$$
$$= A_{i+1}$$

Shifts:

α_i : near an eig of A

factor $A_i - \alpha_i I = Q_i R_i$

$$A_{i+1} = R_i Q_i + \alpha_i I$$
$$= Q_i^T Q_i R_i Q_i + \alpha_i Q_i^T Q_i$$
$$= Q_i^T \underbrace{(Q_i R_i + \alpha_i I)}_{A_i} Q_i$$

again, orthogonal similarity.

If α_i is an exact eigenvalue of A:

$A_i - \alpha_i I$ is singular

but

$A_i - \alpha_i I = Q_i R_i$

so

one of the diagonal elts of R is zero, say $R_i(n,n)$

then

$R_i Q_i \rightarrow$ last row $= 0$

A_{i+1} will have α_i in its (n,n) element

So, the (n,n) entry will converge to eigenvalue

$$\begin{pmatrix} \boxed{\tilde{A}} & \tilde{a} \\ 0 & \alpha_i \end{pmatrix}$$ continue to work on this

Caveats:

- complex eigenvalues?
  - ↳ double shifts (real matrices w/ complex eigenvalues come in complex conjugates)
- Cost
  - order of n^3 in every iteration
  - even if we have only one iteration per eigenvalue $O(n^4)$