October 29, 2013

Bi-orthogonalization

symmetric:
- short recurrences
- optimality

$<$ CG (SPD), MINRES (symm), SYMMLQ

non-symmetric
- long recurrences
- optimality

} GMRES

- longish recurrences
- quasi-optimality

} Restarted GMRES

- short recurrences
- no optimality

} bi-orthogonalization

$$AV_k = V_{k+1} T_{k+1,k}$$

$$W_k^T (b - Ax_k) = 0 \qquad w_i^T v_j = 0$$

$i \neq j$ bi-orthogonal.

$$W_k^T A V_k y = W_k^T b$$

$$T_k y = \|r\|_2 e_1$$ – Bi-Lanczos, two-sided Lanczos

Suppose we have two Krylov subspaces

$$\mathcal{K}_m(A, v_1) = \text{span}\{v_1, Av_1, \ldots, A^{m-1} v_1\}$$

$$\mathcal{K}_m(A^T, w_1) = \text{span}\{w_1, A^T w_1, \ldots, (A^T)^{m-1} w_1\}$$

recall: $Q_k^T A Q_k = T_k$

$$A(q_1 \ldots) = (q_1 q_2 \ldots) \begin{pmatrix} \alpha_1 & \beta_2 & & \\ \beta_2 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \\ & & & \alpha_k \end{pmatrix}$$

and we forced orthogonality
↳ find q_j, $\{\alpha_j\}$, $\{\beta_j\}$ using orthogonality.

Now, use bi-orthogonality to find elements of T_k, $\{v_j\}$, $\{w_j\}$

Core of bi-Lanczos

For $j=1:m$

$\alpha_j = (Av_j, w_j)$

$\hat{v}_{j+1} = Av_j - \alpha_j v_j - \beta_j v_{j-1}$

$\hat{w}_{j+1} = A^T w_j - \alpha_j w_j - \delta_j w_{j-1}$

end

where $(w_j, v_i) = \delta_{ij}$

$$AV_m = V_m T_m + \delta_{m+1} v_{m+1} e_m^T$$

$$A^T W_m = W_m T_m^T + \beta_{m+1} W_{m+1} e_m^T$$

$$W_m^T A V_m = T_m$$

BiCG:

$x_m = x_0 + V_m T_m^{-1}(\rho e_1)$ $\qquad \rho = \|r_0\|_2$

$T_m = L_m U_m$

define

$P_m = V_m U_m^{-1}$

$P_m^* = W_m L_m^{-T}$

What is

$$\begin{aligned}(P_m^*)^T A P_m &= D_m \\ &= L_m^{-1} \underbrace{W_m^T A V_m}_{} U_m^{-1} \\ &\quad L_m^{-1} T_m U_m^{-1} \\ &= L_m^{-1} L_m U_m U_m^{-1} \\ &= I\end{aligned}$$

short recurrences

recall for CG: A-conjugacy

$$p_j^T A p_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$P_k^T A P_k = D_k = I$$

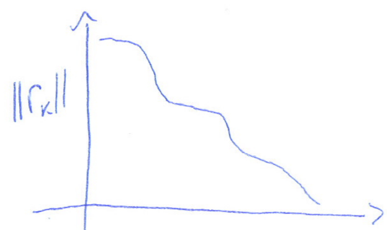
QMR:

$$\min \| V_m^T (\rho e_1 - T_k y) \|$$

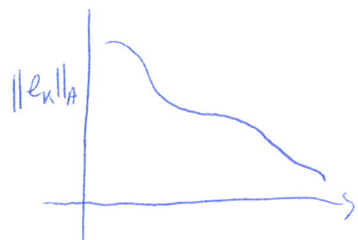
↳ pretend V_m are orthogonal ($\{V_m\}$ from biLanczos)

$$\min_y \| \rho e_1 - T_k y \|_2$$

Convergence behavior



GMRES

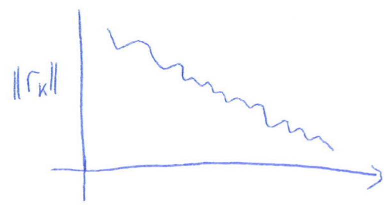


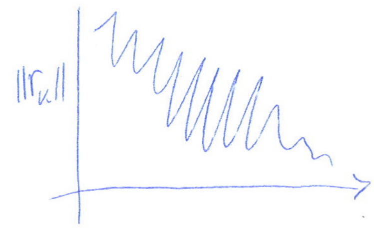
CG

erratic convergence behavior

BiCG

but





BiCG-STAB: smoother / gentler oscillations

BiCG + QMR do require A^T.

↳ Transpose free variants: CGS
BiCG-Stab
TFQMR

Transpose Free versions are based on the observation that many of the parameters can be expressed as, say

$$\alpha_j = \underline{(\phi_j(A) r_0, \phi_j(A^T) r_0^*)} = (\phi_j^2(A) r_0, r_0^*)$$

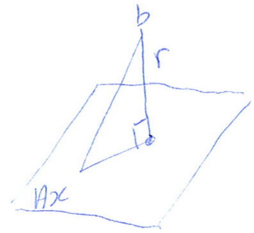
...

LSQR:

$\min \|b - Ax\|_2$

3 ways to solve:

1. Normal eq^{n}s $\quad A^TAx = A^Tb$
2. QR
3. SVD



Augmented form

$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$r = b - Ax$

bi-diagonalization:

$A = UBV^T$

$A^TA = VB^T\underbrace{U^TU}_{I}BV^T = V\underbrace{B^TB}_{\text{tridiagonal}}V^T$

Similarity transformation, the eigenvalues of $T = B^TB$ = eigenvalues of A^TA

Take T - iterate until finding eig.

For sparse case

$AV = UB$

$$A(v_1, v_2, \dots) = (u_1\ u_2 \dots) \begin{pmatrix} \alpha_1 & \beta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \beta_{k-1} \\ & & & \alpha_k \end{pmatrix}$$

do this column by column

$$\begin{cases} AV_k = U_{k+1}B_k \\ A^TU_{k+1} = V_kB_k^T + \alpha_{k+1}v_{k+1}e_k^T \end{cases}$$

$A^TA \rightsquigarrow B_k^TB_k$

$$\begin{pmatrix} U_k^T & \\ & V_k^T \end{pmatrix} \begin{pmatrix} I & A \\ A^T & O \end{pmatrix} \begin{pmatrix} U_k & \\ & V_k \end{pmatrix} \approx \underbrace{\begin{pmatrix} I & B_k \\ B_k^T & O \end{pmatrix}}$$

solve
least-squares problem
on smaller space