Question 2

Code

```
function A2Q2()
clc

NN = [64,128,256,512,1024,2048,4096];
kk = [NN/2;NN];

for JJ = 1:size(kk,1)
    for II = 1:length(NN)

        % Defining descrete Laplacian
        N = NN(II);
        A = Laplacian(N);

        % Defining eigenvalues of descrete Laplacian
        Eigen = @(ii,jj,N) 4-2*(cos(ii*pi/(N+1))+cos(jj*pi/(N+1)));
        SmallestEigA = [Eigen(1,1,N);Eigen(2,1,N);Eigen(1,2,N)];
        LargestEigA = [Eigen(N,N,N);Eigen(N-1,N,N);Eigen(N,N-1,N)];

        b = randn(N^2,1);

        % Calling Lanczos code
        k = kk(JJ,II);
        [T] = lancz(A, b, k);

        % Calculating eigenvalues of T
        OPTS.maxit = 1e6;
        OPTS.tol = 1e-10;

        EIG = eig(full(T));
        EIG = sort(EIG);
        SmallestEigT = EIG(1:3);
        LargestEigT =  EIG(end-2:end);

        % Sort eigenvalues
        SSEA = sort(SmallestEigA);
        SLEA = sort(LargestEigA);
        SSET = sort(SmallestEigT);
        SLET = sort(LargestEigT);

        % Defining table data
        data = [SSEA,SSET,abs(SSEA-SSET)./abs(SSEA),SLEA,SLET,abs(SLEA-SLET)./abs(SLEA)];

        % Set up some options
        tblOpts = {'header',{'Smallest Eig A','Smallest Eig T',...
            'inf-norm rel','Largest Eig A','Largest Eig T'...
            ,'inf-norm rel'},'format',{'%1.4e','%1.4e','%1.4f'...
            ,'%1.6f','%1.6f','%1.4e'},'align','center','delim','|',...
            'printRow',true};
```

```
        for ii = 1:size(data,1);
            table(['Table of Eigenvalues for n = ',num2str(NN(II)^2),...
                ' and k = ',num2str(k)],data(1:ii,:),tblOpts{:}...
                ,'finalRow',ii == size(data,1));
        end

    end
    fprintf('\n\n\n\n\n')
end

    function [A] = Laplacian(n)
        % Creating discretised Laplacian

        e = ones(n,1);

        % Creating sparse diagonal matrices
        I = spdiags(e,0,n,n);
        I1 =spdiags(e,1,n,n);
        I2 = spdiags(e,-1,n,n);

        % Creating 1D Convection-Diffusion matricies
        A1D = 2*I -1*I1 - 1*I2;

        % Creating 2D Convection-Diffusion matrix
        A = kron(I,A1D)+kron(A1D,I);

    end



    function [T,Q] = lancz(A, b, k)
        %function [T,Q] = lancz(A, b, k)
        %
        % Function the performs the Lanczos process
        %
        % Input:
        %       A - Symmetic matrix
        %       b - initial guess
        %       A - number of steps in the Lanczos algorithm
        %
        % Output:
        %       T - Symmetic Hessenberg matrix (Tridiagonal)
        %       Q - (OPTIONAL) orthogonal basis

        n = length(b);
        qprev = sparse(n,1);
        q = b / norm(b);
        beta = [];
        alpha = [];
```

```
        if nargout == 2
            Q = [];
        end

        for i = 1:k
            v = A*q;
            alpha(i) = q' * v;

            if i == 1
                v = v - alpha(i)*q;
            else
                v = v - beta(i-1)*qprev - alpha(i)*q;
            end
            beta(i) = norm(v);
            qprev = q;

            if nargout == 2
                Q = [Q,q];
            end

            if (abs(beta(i)) < 1e-10)
                break
            end
            q = v / beta(i);
        end
        beta = beta(:);
        T = spdiags([beta alpha(:) [0;beta(1:end-1)]],[-1:1],i,i);
    end


end
```

Case 1: $\mathbf{k} = \frac{\sqrt{n}}{2}$

Table of Eigenvalues for n = 4096 and k = 32

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
4.6711e-03	2.5640e-02	4.4891	7.988328	7.838486	1.8758e-02
1.1672e-02	6.8586e-02	4.8760	7.988328	7.920954	8.4341e-03
1.1672e-02	1.5992e-01	12.7009	7.995329	7.977258	2.2602e-03

Table of Eigenvalues for n = 16384 and k = 64

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
1.1861e-03	4.7371e-03	2.9938	7.997035	7.960365	4.5855e-03
2.9649e-03	1.7237e-02	4.8137	7.997035	7.980712	2.0412e-03
2.9649e-03	3.9478e-02	12.3147	7.998814	7.994351	5.5796e-04

Table of Eigenvalues for n = 65536 and k = 128

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
2.9885e-04	8.4862e-04	1.8396	7.999253	7.990894	1.0450e-03
7.4711e-04	4.9382e-03	5.6097	7.999253	7.995929	4.1557e-04
7.4711e-04	9.5807e-03	11.8236	7.999701	7.999109	7.4068e-05

Table of Eigenvalues for n = 262144 and k = 256

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
7.5006e-05	2.6490e-04	2.5317	7.999812	7.997717	2.6189e-04
1.8751e-04	1.0481e-03	4.5893	7.999812	7.998970	1.0537e-04
1.8751e-04	2.4785e-03	12.2180	7.999925	7.999614	3.8845e-05

Table of Eigenvalues for n = 1048576 and k = 512

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
1.8788e-05	5.4249e-05	1.8874	7.999953	7.999404	6.8617e-05
4.6970e-05	2.9117e-04	5.1991	7.999953	7.999755	2.4799e-05
4.6970e-05	6.3521e-04	12.5238	7.999981	7.999950	3.9057e-06

Table of Eigenvalues for n = 4194304 and k = 1024

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel

4.7016e-06	1.1715e-05	1.4917	7.999988	7.999848	1.7482e-05
1.1754e-05	8.0296e-05	5.8313	7.999988	7.999943	5.6411e-06
1.1754e-05	1.5026e-04	11.7833	7.999995	7.999983	1.4910e-06

Table of Eigenvalues for n = 16777216 and k = 2048

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
1.1760e-06	2.5772e-06	1.1915	7.999997	7.999964	4.1215e-06
2.9399e-06	1.4707e-05	4.0025	7.999997	7.999981	1.9719e-06
2.9399e-06	3.5629e-05	11.1189	7.999999	7.999995	5.3818e-07

Case 1: $\mathbf{k} = \sqrt{n}$

Table of Eigenvalues for n = 4096 and k = 64

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
4.6711e-03	4.9254e-03	0.0544	7.988328	7.957860	3.8140e-03
1.1672e-02	1.3954e-02	0.1955	7.988328	7.979281	1.1325e-03
1.1672e-02	4.4271e-02	2.7928	7.995329	7.994390	1.1740e-04

Table of Eigenvalues for n = 16384 and k = 128

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
1.1861e-03	1.6810e-03	0.4172	7.997035	7.989340	9.6218e-04
2.9649e-03	5.3415e-03	0.8016	7.997035	7.993429	4.5088e-04
2.9649e-03	9.9601e-03	2.3593	7.998814	7.997786	1.2849e-04

Table of Eigenvalues for n = 65536 and k = 256

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
2.9885e-04	3.3616e-04	0.1248	7.999253	7.997303	2.4382e-04
7.4711e-04	1.2475e-03	0.6698	7.999253	7.999108	1.8108e-05
7.4711e-04	2.6021e-03	2.4829	7.999701	7.999687	1.7919e-06

Table of Eigenvalues for n = 262144 and k = 512

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
7.5006e-05	8.0829e-05	0.0776	7.999812	7.999275	6.7192e-05
1.8751e-04	2.6584e-04	0.4177	7.999812	7.999648	2.0520e-05

1.8751e-04	5.9906e-04	2.1948	7.999925	7.999841	1.0494e-05

Table of Eigenvalues for n = 1048576 and k = 1024

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
1.8788e-05	4.8293e-05	1.5704	7.999953	7.999872	1.0154e-05
4.6970e-05	8.4837e-05	0.8062	7.999953	7.999943	1.2128e-06
4.6970e-05	1.8231e-04	2.8814	7.999981	7.999978	4.5637e-07

Table of Eigenvalues for n = 4194304 and k = 2048

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
4.7016e-06	9.6027e-06	1.0424	7.999988	7.999959	3.6114e-06
1.1754e-05	1.8401e-05	0.5655	7.999988	7.999978	1.3107e-06
1.1754e-05	4.3829e-05	2.7289	7.999995	7.999994	1.2905e-07

Table of Eigenvalues for n = 16777216 and k = 4096

Smallest Eig A	Smallest Eig T	inf-norm rel	Largest Eig A	Largest Eig T	inf-norm rel
1.1760e-06	3.0872e-06	1.6252	7.999997	7.999989	1.0106e-06
2.9399e-06	5.9548e-06	1.0255	7.999997	7.999995	2.6198e-07
2.9399e-06	1.2085e-05	3.1105	7.999999	7.999998	1.0918e-07

Note: in the tables above I have calculated the relative error between the exact eigenvalue and the Ritz values.

Discussion

From the tables we can see that for case 2 ($k = \sqrt{n}$) we approximate the largest and smallest eigenvalue better than in case 1 (where k is have the value of case 2). However, in both cases we see that the approximation for the largest eigenvalue is better. In fact, we don't get see any digits of accuracy when approximating the smallest eigenvalue. This is not the case when looking at the largest eigenvalue.