Channel Coding Part I

Digital Communication
Chapter 6 & 7

Ola Jetlund

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Introduction
Shannon
Linear Block Codes
Mapping
System model
Encoding
Decoding Linear block codes
Properties
Hamming codes
Convolutional Codes

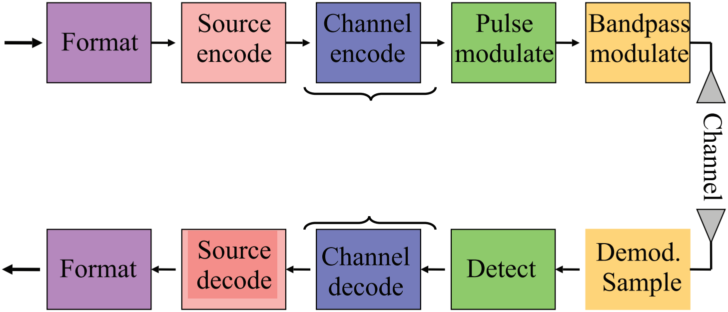
Last Week

- Evaluating the average probability of symbol error for different bandpass modulation schemes
- Comparing different modulation schemes based on their error performances.

This Week (and next week)

- Channel Coding
- Linear Block Codes
- Convolutional Codes

Digital Communication System



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Goals

For a DCS we can define the following goals:

- Maximizing the transmission bit rate
- Minimizing probability of bit error
- Minimizing the required power
- Minimizing required system bandwidth
- Maximizing system utilization
- Minimize system complexity

These are goals that must considered in the design phase.

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Channel Coding

- Transforming signals to improve communication performance
- Waveform coding
- Structured sequences

Error Control Techniques

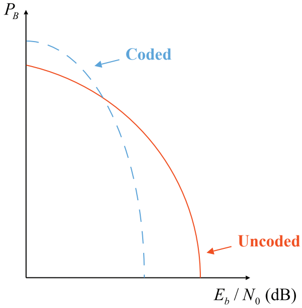
- ARQ - Automatic repeat request
- FEC - Forward Error Correction
- Hybrid ARQ - (ARQ + FEC)

Why error correcting codes?

- Error performance vs. bandwidth
- Power vs. bandwidth
- Data rate vs. bandwidth
- Capacity vs. bandwidth BP

CODING GAIN: Reduction in E_b/N_0 from using a code

$$G = \left(\frac{E_b}{N_0}\right)_u - \left(\frac{E_b}{N_0}\right)_c$$



Practical Channel coding

- Use as few bits as possible to transmit information on a noisy channel
- Decode received information with as few errors as possible

$\Rightarrow$ Utilize the channel capacity[1]:

$$C = W \log_2 \left(1 + \frac{S}{N}\right)$$

Here: capacity for an AWGN channel with:

- W Bandwidth [Hz]
- S Average received signal power
- N Average Noise power

[1] C.E. Shannon, "A mathematical theory of communication," BSTJ, vol. 27, 1948, pp 379-423, 623-657.

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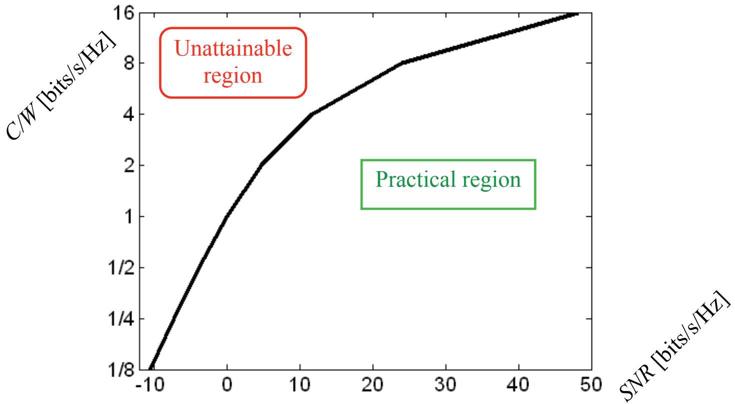
The Shannon theorem:

A limit on transmission data rate R_b

- Transmission with $R_b \leq C$ is possible
  - with an arbitrary small error probability
- For $R_b > C$, transmission cannot achieve an arbitrary small error probability.



The Shannon limit

$$\begin{aligned} C &= W \log_2\left(1+\frac{S}{N}\right) \\ S &= E_b C \\ N &= N_0 W \end{aligned} \quad \Rightarrow \quad \boxed{\frac{C}{W} = \log_2\left(1 + \frac{E_b}{N_0}\frac{C}{W}\right)}$$

As $W \to \infty$ or $\frac{C}{W} \to 0$,

$$\frac{E_b}{N_0} \to \frac{1}{\log_2(e)} = 0.693 \approx -1.6 \text{ [dB]}$$

- No error free transmission (at any rate) for $\frac{E_b}{N_0} < -1.6$ [dB]
- Capicity can be increased by increasing the bandwidth

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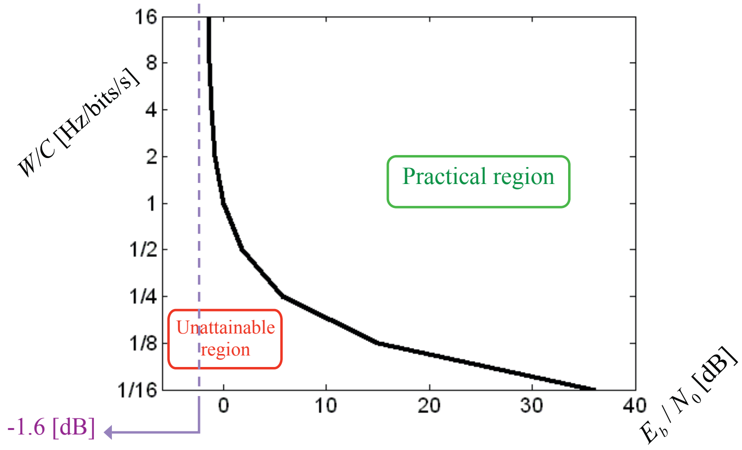
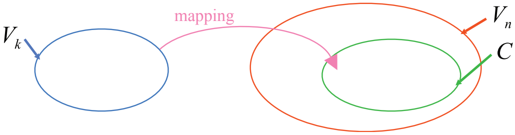
The Shannon limit ...



Linear Block Codes (n, k), I

Coding is a mapping from one space to another:



V_m is a vector space containing all 2^k sequences of length m.

- A set $C \subset V_n$ with cardinality 2^k is called a linear block code if and only if it is a subspace of the vector space V_n.
  - Members of C are called codewords
  - The all-zero word is a codeword
  - any linear combination of a codeword is a codeword

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A note on the binary field

- The set $\{0, 1\}$ under the modulo-2 binary addition and multiplication forms a field.

Addition	Multiplication
$0 \oplus 0 = 0$	$0 \otimes 0 = 0$
$0 \oplus 1 = 1$	$0 \otimes 1 = 0$
$1 \oplus 0 = 1$	$1 \otimes 0 = 0$
$1 \oplus 1 = 0$	$1 \otimes 1 = 1$

- AKA the Galois field: $\mathrm{GF}(2)$
- Example V_3

$$V_3 = \{(000),(001),(010),(011),\\ (100),(101),(110),(111)\}$$

It has $2^k = 2^3 = 8$ members (cardinality)

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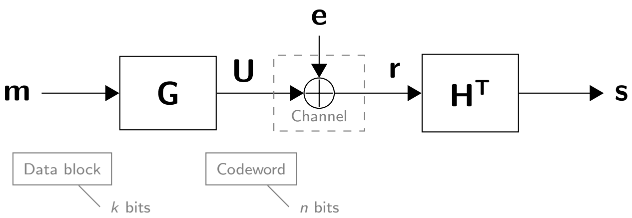
Linear Block Codes (n, k), II



- **m** Message
- **G** Generator matrix
- **U** Codeword
- **e** Error introduced by channel
- **r** Received codeword
- **H** Parity check matrix
- **s** Syndrome (received data)

Note! We consider only binary sequences!

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Convolutional Codes

- The (n, k) linear block code takes k data bit and produces a coded sequence of length n
  - $n - k$ parity bits
  - Code rate:

$$R_c = \frac{k}{n}$$

- Coding:

$$\mathbf{U} = \mathbf{m} \cdot \mathbf{G}$$

  The generator matrix $\mathbf{G}$ is of size $k \times n$ and it can be
  - systematic, or
  - nonsystematic.

Example, nonsystematic $(7,4)$

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{g_1} \\ \mathbf{g_2} \\ \mathbf{g_3} \\ \mathbf{g_4} \end{bmatrix}$$

Given $\mathbf{m} = [1\,0\,0\,0]$, then

$$\mathbf{U} = \mathbf{d} \cdot \mathbf{G} = [1\,1\,0\,0\,0\,1\,0]$$

Note that all rows in $\mathbf{G}$ are codewords.

Remember: A linear combination of two codewords are a codeword

Thus we can manipulate G as follows:

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{g_1} \\ \mathbf{g_2} \\ \mathbf{g_3} \\ \mathbf{g_4} \end{bmatrix}$$

$$\mathbf{g_5} = \mathbf{g_1} + \mathbf{g_2} = [1\,1\,0\,0\,0\,1\,0] + [0\,1\,0\,0\,1\,1\,1] = [1\,0\,0\,0\,1\,0\,1]$$

$$\mathbf{g_6} = \mathbf{g_3} + \mathbf{g_3} = [0\,0\,0\,1\,1\,1\,0] + [0\,0\,1\,1\,1\,1\,0] = [0\,0\,1\,0\,0\,0\,0]$$

Replace $\mathbf{g_1}$ with $\mathbf{g_5}$, $\mathbf{g_4}$ with $\mathbf{g_6}$ to obtain new generator matrix:

$$\mathbf{G}' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{g_5} \\ \mathbf{g_2} \\ \mathbf{g_3} \\ \mathbf{g_6} \end{bmatrix}$$

Swap 3rd and 4th row in $\mathbf{G}'$ to obtain a new generator matrix:

$$\mathbf{G}'' = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Note that

$$\mathbf{G}'' = \left[\ \mathbf{I}_4 \ \middle|\ \mathbf{P}\ \right]$$

Systematic block codes (n, k)

For a systematic code

- the first k bits are the information bits:

$$\mathbf{G} = \left[\ \mathbf{I}_k \ \middle|\ \mathbf{P}\ \right]$$

$\mathbf{I}_k$ is a $k \times k$ identity matrix
$\mathbf{P}$ is a $k \times (n-k)$ matrix

- Thus,

$$\begin{aligned}\mathbf{U} &= (u_1, u_2, \ldots, u_n) \\ &= (\overbrace{m_1, m_2, \ldots, m_k}^{\text{message bits}}, \overbrace{p_1, p_2, \ldots, p_{n-k}}^{\text{parity bits}})\end{aligned}$$

(Syndrom) Decoding I

For any linear block code with generator matrix $\mathbf{G}$ there exists a matrix $\mathbf{H}$ of size $(n-k) \times n$ such that

$$\mathbf{G} \cdot \mathbf{H}^T = \mathbf{0}$$

- $\mathbf{H}$ is called the parity check matrix and its rows are linearly independent.
- For systematic linear block codes:

$$\mathbf{H} = \left[\begin{array}{c|c} \mathbf{P}^T & \mathbf{I}_{n-k} \end{array} \right]$$

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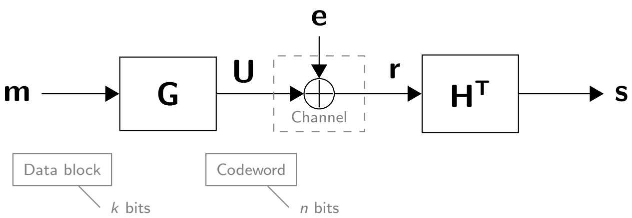
Decoding II



$$\mathbf{s} = \mathbf{r} \cdot \mathbf{H}^T \qquad \mathbf{r} = \mathbf{U} + \mathbf{e} \qquad \mathbf{U} = \mathbf{m} \cdot \mathbf{G} \qquad \mathbf{G} \cdot \mathbf{H}^T = \mathbf{0}$$

The syndrome $\mathbf{s}$:

$$\begin{aligned}\mathbf{s} &= \mathbf{r} \cdot \mathbf{H}^T = (\mathbf{U} + \mathbf{e}) \cdot \mathbf{H}^T = (\mathbf{m} \cdot \mathbf{G} + \mathbf{e}) \cdot \mathbf{H}^T \\ &= \mathbf{m} \cdot \mathbf{G} \cdot \mathbf{H}^T + \mathbf{e} \cdot \mathbf{H}^T = \mathbf{m} \cdot \mathbf{0} + \mathbf{e} \cdot \mathbf{H}^T \\ &= \mathbf{0} + \mathbf{e} \cdot \mathbf{H}^T = \mathbf{e} \cdot \mathbf{H}^T.\end{aligned}$$

Decoding III

$$\mathbf{s} = \mathbf{r} \cdot \mathbf{H}^T = \mathbf{e} \cdot \mathbf{H}^T$$

1. If $\mathbf{s} = \mathbf{0}$ then $\mathbf{r}$ is a legal codeword and the decoded message $\widehat{\mathbf{m}}$ is found from $\widehat{\mathbf{m}}\mathbf{G} = \mathbf{r}$,
2. and if $\mathbf{s} \neq \mathbf{0}$ then $\mathbf{r}$ is a not legal codeword.
   2.1 Find an error vector $\mathbf{e}'$ such that $\mathbf{r} - \mathbf{e}'$ is a legal codeword:

$$\text{Find } \mathbf{e}' \text{ such that } (\mathbf{r} - \mathbf{e}') \cdot \mathbf{H}^T = \mathbf{0}$$

Properties of linear block codes

For a (n, k) linear block code there are 2^k legal codewords: $\mathbf{U}_i,\ i \in 1, 2, \ldots 2^k$.

- Hamming weight: $w(\mathbf{U}_i) =$ the number of non-zero elements in $\mathbf{U}_i$.
- Hamming distance: $d(\mathbf{U}_i, \mathbf{U}_j) = w(\mathbf{U}_i \oplus \mathbf{U}_j)$
- Minimum distance:

$$d_{\min} = \min_{i \neq j} d(\mathbf{U}_i, \mathbf{U}_j) = \min_i w(\mathbf{U}_i)$$

A (n, k) linear block code can then

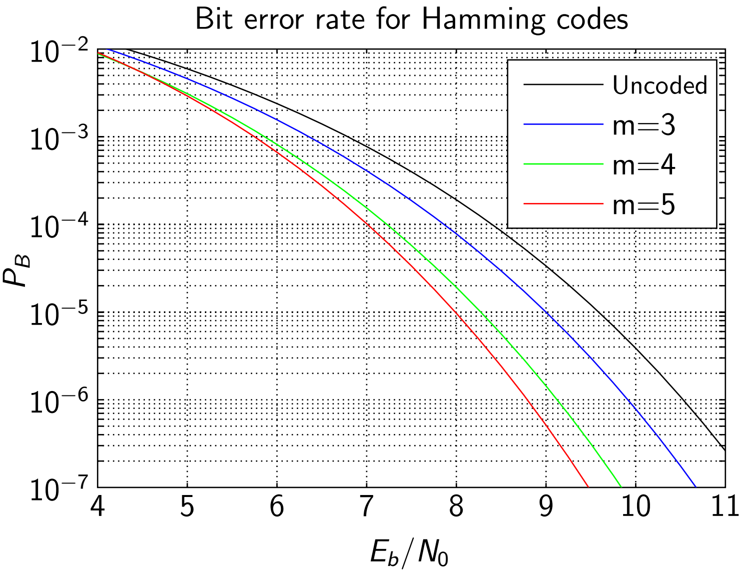
- Detect $e = d_{\min} - 1$ errors
- Correct $t = \lfloor \frac{d_{\min} - 1}{2} \rfloor$ errors

Example: Systematic Hamming code (m)

- Number of bits in codeword: $n = 2^m - 1$
- Number of information bits: $k = 2^m - m - 1$
- Number of parity bits: $n - k = m$
- Code rate $\frac{k}{n} = \frac{2^m - m - 1}{2^m - 1} = 1 - \frac{m}{2^m - 1}$

For Hamming codes the columns in **H** represents **all** binary vectors of length 2^{n-k} (except the all-zero codeword.

m	n	k
3	7	4
4	15	11
5	31	26

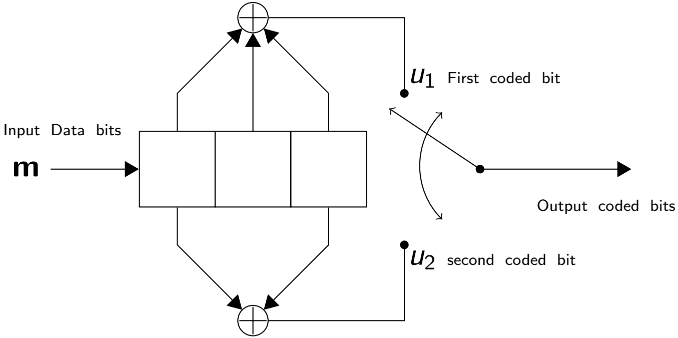

Bit error rate for Hamming codes
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
P_B
4
5
6
7
8
9
10
11
E_b/N_0
Uncoded
m=3
m=4
m=5

Convolution versus block encoding

- Linear block codes
  - Rate $R_c = \frac{k}{n}$
  - n is the length of the codewords
  - k is the length of the information sequence coded (or mapped) to one codeword
- Convolutional codes
  - Rate $R_c = \frac{k}{n}$
  - n does not define a block or a codeword
  - k usually set to 1
  - K constraint length - counting the number of memory elements (which is $K-1$)

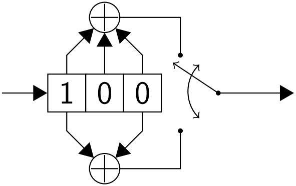
A convolutional code encodes the entire stream of data into a single codeword.

Example: Rate $R_c = \frac{1}{2}$

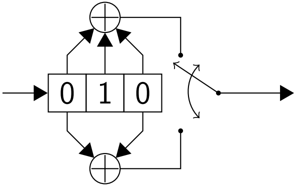


Here, $K = 3$

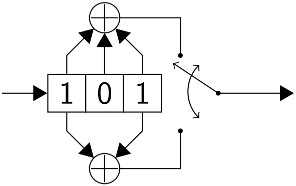
Let $\mathbf{m} = (101)$ and find the output.



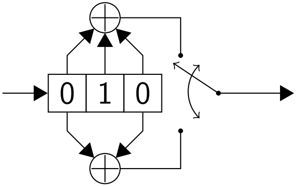
time t_1: $(u_1, u_2) = (1, 1)$



time t_2: $(u_1, u_2) = (1, 0)$



time t_3: $(u_1, u_2) = (0, 0)$



time t_4: $(u_1, u_2) = (1, 0)$

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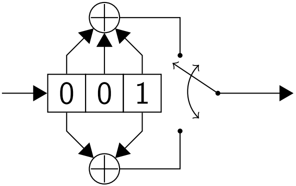
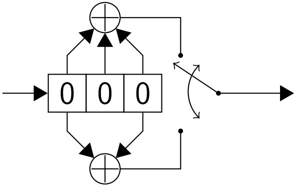




time t_5: $(u_1, u_2) = (1, 1)$ time t_6: $(u_1, u_2) = (0, 0)$

$$\mathbf{m} = (101) \longrightarrow \boxed{\text{Encoder}} \longrightarrow \mathbf{U} = (11\ 10\ 00\ 10\ 11)$$

Description of convolutional codes

A Tree diagrams (not very common)

B State Diagram

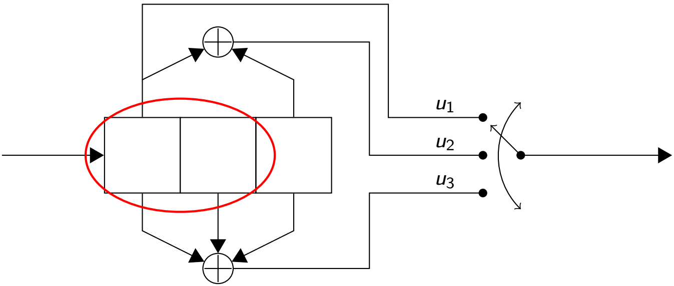
- Often used to find the exact error correcting properties of a code

C The trellis diagram

- Is often used to visualize the decoding process

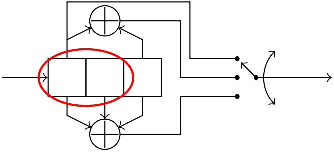
Another larger will illustrate descriptions B and C.

Example: $R_c = \frac{1}{3}$



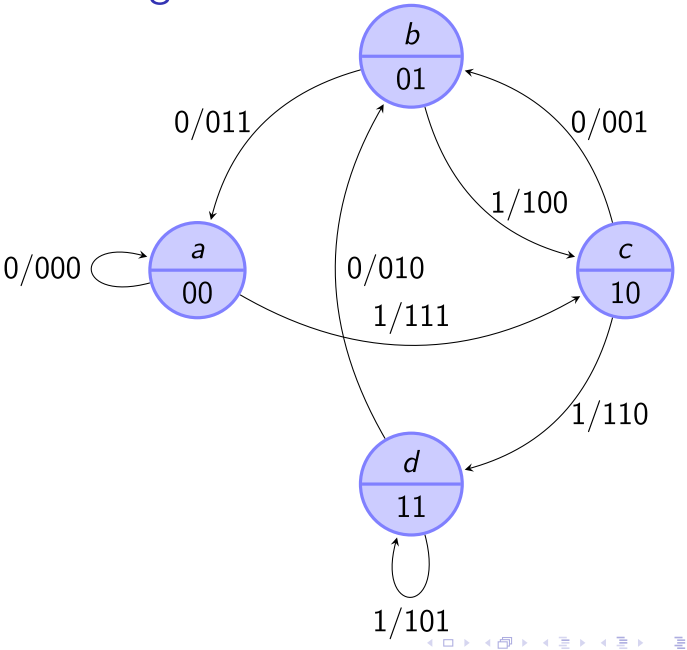
The State of a convolutional code are the $K-1$ first bits in the encoder

Inputs, states, and outputs

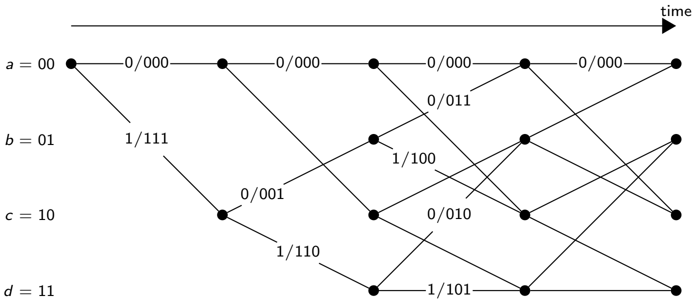


Input	Current State	Next State	Output	
0	0 0	0 0	0 0 0	"0"
1	0 0	1 0	1 1 1	"7"
0	0 1	0 0	0 1 1	"3"
1	0 1	1 0	1 0 0	"4"
0	1 0	0 1	0 0 1	"1"
1	1 0	1 1	1 1 0	"6"
0	1 1	0 1	0 1 0	"2"
1	1 1	1 1	1 0 1	"5"

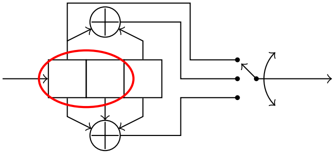
State Diagram

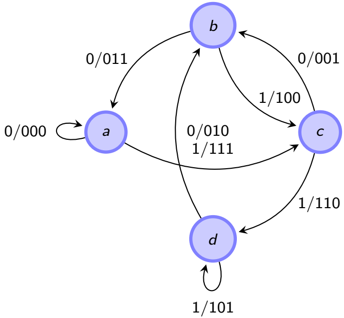


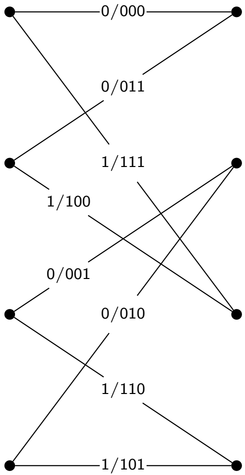
Trellis Diagram



Convolutional codes (block -, state -, and trellis diagram







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