Mathematical Paradoxes

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Abstract

Paradoxes have been an abundant source of contention in mathematics. In this project we explore the origins and resolutions to some such paradoxes. These include but are not exclusive to the Banach-Tarski Paradox, Skolem's Paradox, and Russell's Paradox. Russell's Paradox is a result of naive set theory that was resolved by adopting our current system of axiomatic set theory. This foundation is itself questioned by the Banach-Tarski Paradox, since its unintuitive result follows from the Axiom of Choice, which is used to arrive at various important results in mathematics. Axiomatisation as a source of truth is also interrogated in Skolem as we look at different models for set theory which have led many to believe that set theoretic concepts may be relative. The aim of these investigations is to understand the disconnection between intuition and formalism while shedding light on how this imbalance can be informative.

Declaration

We declare that this thesis was composed by ourselves and that the work contained therein is our own, except where explicitly stated otherwise in the text.

(*Manjari Agrawal, Charlie Alston, Daniel Jekov and Jack Watts*)

Antony, thank you for all your help and guidance.

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Chapter 1

Introduction

> I am the wisest man alive, for I know one thing, and that is that I know nothing.
>
> *Plato*

The study of paradoxes has posed many questions throughout history that have baffled great mathematicians, philosophers and scientists alike. They challenge the way in which we see the world, giving us contradictory and often counter intuitive results which lead us to reevaluate our mathematical reasoning.

Before we proceed, let us define what a paradox is, as it is a term often misused. It is derived through Latin from the Greek word ‘paradoxon’ which directly translates to ‘contrary(opinion)’. The Oxford Dictionary states two definitions [1]:

i. a statement or proposition which, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory;

ii. a seemingly absurd or self-contradictory statement or proposition which when investigated or explained may prove to be well founded or true.

Within our analysis of a series of paradoxes we hope to explore the philosophical and mathematical implications of the troubling results they pose. By the end of this project, we hope to shed light on some important paradoxes within mathematics and philosophy and offer insight into the constructive consequences of their results and role in mathematical study.

It is also true that paradoxes threaten the very foundations of mathematics. Throughout history, several have shook the academic community and sparked a redefinition of the fundamental laws that we operate with. Perhaps most famously was in relation to Russell’s Paradox which highlighted a contradiction in set theory and mathematical logic, areas which many consider to be the source of all mathematics. This paradox is the reason that we use an axiomatised system of set theory. Otherwise, we encounter contradictions whose existence we cannot ignore if we wish to explore mathematical concepts meaningfully. We shall discuss Russell's paradox in more detail in chapter 5. We shall also discuss the theory of types in that chapter. Type theory is a another system that can be used as a foundation for mathematical logic. It was formulated in an attempt to avoid paradoxes in logic and mathematics such as Russell's Paradox. Just as we use modern set theory to understand and evaluate concepts in mathematics, we can use the theory of types as our foundation. We shall explore it in depth in chapter 5 and look at how it can be used to resolve certain paradoxes.

Chapter 4 concerns what is known as Skolem's Paradox. Like Russell's, Skolem's Paradox elicited questions about the validity of set theory by considering two apparently incompatible theorems. The theorems in question were Downward Löwenheim-Skolem Theorem, which concerns countable first order axiomatisations, and Cantors theorem, which tells us there are more sets than natural numbers. The chapter will explain how the 'paradox' is not an actual antinomy, but rather a bizarre reality that exists between model and set theories. As a result it has raised interesting questions about relativity surrounding set theoretic concepts.

Chapter 3 will discuss a result first discovered by Stefan Banach and Alfred Tarski in 1924 known as The Banach-Tarski Paradox. It is a remarkable result in the field of set-geometric theory, which initially casts doubt on our perceptions of the properties of space and sets, as well as the understanding of the physical world around us. The paradox states that, in its most basic form, it is feasible to split a solid sphere into a finite number of parts and then reassemble those pieces into two solid spheres that are the same size as the original. We will later see how this paradox is heavily confined to two very important mathematical concepts, the idea of unmeasurable sets and the controversial Axiom of Choice. This seemingly illogical outcome has significant implications for the Axiom of Choice, opening a huge discussion on whether it should be part of our axiomatic system, as well as questioning the nature of infinity and the foundations of mathematical reasoning.

In investigating these paradoxes, as well as others, we are forced to analyse the long standing feud between mathematical formalists and intuitionists. Furthermore, we are also able to recognise that mathematical paradoxes can be and have been a great source of learning and development.

Chapter 2

Classification and Examples

> There is no clear-cut distinction between example and theory.
>
> *Michael Atiyah*

2.1 A Few Brief Examples

In order to aid understanding of what they are, we shall explore a few examples of paradoxes. Firstly, we look at the concept of Sierpinski's Gasket, which is a shape with infinite perimeter and zero area. The idea that such a thing can exist is of a paradoxical nature itself and shall be explained now.
Consider that we construct an equilateral triangle of side length n, and thus, perimeter, $P = 3n$, and area, $A = n^2\sqrt{3}/4$.

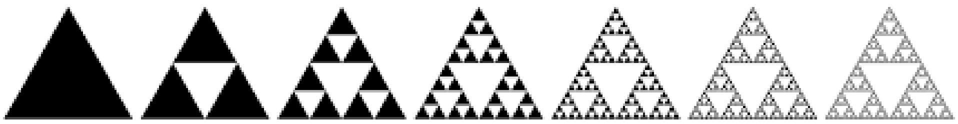

Figure 2.1: The Construction of Sierpinski's Gasket [2]

Now we imagine that, as in the second image in 2.1, we partition our triangle into four new identical equilateral triangles and remove the middle one. It can be seen that now, the area has been reduced by a quarter and the perimeter increased by a half. $P = 3n \times 3/2 = 9n/2$, $A = n^2\sqrt{3}/4 \times 1/2 = n^2\sqrt{3}/8$. Now within each of the remaining three triangles, continue to repeat this process. As the number of iterations of this process approaches infinity, we see that:

$$P = \lim_{\alpha\to\infty}\left(3n \times \left(\frac{3}{2}\right)^{\alpha}\right) = \infty,$$
$$A = \lim_{\beta\to\infty}\left(\frac{n^2\sqrt{3}}{4} \times \left(\frac{1}{2}\right)^{\beta}\right) = 0;$$

and so we have elegantly constructed our shape of infinite perimeter and zero area using only elementary operations. This is a powerful result, and the fact that such a shape can be constructed from a Euclidean shape by inserting straight lines and removing subsections of it at each stage is counter-intuitive. Throughout our education we may have noticed that there is a link between area and perimeter, but we would usually associate this relationship as being positive, and often proportional. This example gives a simple introduction to the types of problems that we will be exploring and the sort of thinking that is required to understand the concepts we will introduce.

We then have paradoxes that a high school student could feasibly understand, such as Gabriel's horn. Employing logarithms, calculus and volumes of revolution we can construct a shape of infinite surface area yet finite volume.

Gabriel's Horn, discovered by Evangelista Torricelli in 1643, is a shape that is constructed by rotating the region which is bounded by the x-axis and the graph $y = 1/x$ where x $\in [1, \infty)$ and looks like so:

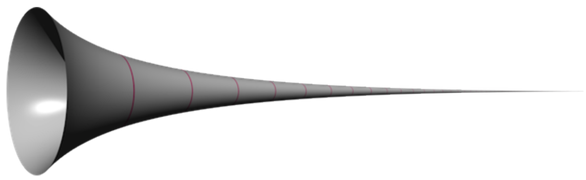

Figure 2.2: Diagram of Gabriel's Horn [2]

We recall the definitions of the volume and surface area of a shape produced by a volume of revolution and apply them in this case. Firstly for volume:

$$\begin{aligned} V &= \lim_{\gamma \to \infty} \left(\pi \int_1^\gamma \frac{1}{x^2}\, dx \right) \\ &= \pi \lim_{\gamma \to \infty} \left(1 - \frac{1}{\gamma} \right) \\ &= \pi. \end{aligned}$$

And now for surface area:

$$\begin{aligned} SA &= 2\pi \int_1^\gamma \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2} \right)^2}\, dx \\ &= 2\pi \int_1^\gamma \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx. \end{aligned}$$

Which is not any elementary integral to compute, but we may observe that it is true for all x that:

$\sqrt{1+\frac{1}{x^4}} \geq 1$ and $\frac{1}{x} > 0$, in order to show that:

$$\begin{aligned} SA &= 2\pi \int_1^{\gamma} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \\ &\geq 2\pi \int_1^{\gamma} \frac{1}{x}\,dx \end{aligned}$$

Which implies that:

$$SA \geq 2\pi \ln \gamma.$$

But, $\lim_{\gamma\to\infty} 2\pi \ln \gamma = \infty$, and so, it follows that Gabriel's horn indeed has infinite surface area, but only a finite volume, π.

The methods that allow us to calculate this result are found in calculus. Fundamental to the theory of calculus is the method of differentiation via finding a derivative. The way in which the derivative is intuitively explained to students is that it is 'the instantaneous rate of change'. The phrase is perpetuated by teachers and lecturers, yet when the students sit down and mull over the actual words they may have the same realisation that the definition in itself seems to describe something paradoxical. Change cannot be instantaneous surely? Of course the word that makes this definition mathematically rigorous is infinitesimal, the fact that 'h' within our definition is not actually zero at any point, it just approaches it. However, it is understandable that students may find this notion counter-intuitive. This kind of paradoxical thinking is integral to good mathematical practice, an idea that we develop further in chapter 6.

Then with just a small introduction to undergraduate set theory we have Russell's Paradox, which shall be discussed shortly. The overriding message is that these paradoxes are things that we encounter and make use of every single day of our lives, despite their implausible results they can have constructive consequences.

Even in everyday life, completely disjoint from the world of mathematics, we have sayings and phrases which are paradoxical such as "less is more", which even appear in popular culture like Trigger's broom (in the same vain as Theseus' Ship) from *Only Fools and Horses*.

2.2 Classification of Paradoxes

Throughout the history of paradoxes, there have been many attempts to classify them. The most accepted of these is the work of W. V. Quince [3]. He categorised them into three sets; veridical, falsidical and antimony.

Veridical paradoxes are described as being "truth telling". Though our initial reaction may be to dismiss it as being false, it can indeed be shown to be true. An example of this is the St. Petersberg Paradox which was first thought of by Nicholaus Bernoulli in 1713. It derives its name from the St. Petersberg game and has the following rules [4]:

A fair coin is flipped until the first instance that it comes up heads. Then the player receives £2^f, where we have that f is the number of flips that have occured. We ask the question, how much should you bet on this game?

When asked this question people may naively say a modest sum. It could come as a great surprise then to them that the mathematical 'answer' to that question is that you should bet an infinite amount of money, or at least as much money as you can possibly muster. This fact comes from probability theory and the idea of expected value. Recall the definition of this in the discrete case: $\mathbb{E}(X) = \sum_i x_i p(x_i)$, with x_i the event, or in this case the value, and $p(x_i)$ is the probability of the event, or in this case the probability of the value occurring.

The probability of you flipping a head on the first turn would result in you getting £2, occurring with a probability of 1/2. The probability of you flipping a tail then a head is 1/4, with a payout of £4, and so on. We can now calculate the expected value.

$$\begin{aligned}\mathbb{E}(X) = \frac{1}{2}\cdot 2 \ + \frac{1}{4}\cdot 4 \ + \frac{1}{8}\cdot 8 \ + \ldots &= 1+1+1+\ldots \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \cdot 2^n \\ &= \infty.\end{aligned}$$

The fact that the result of this paradox is so far away from people's intuition is what defines this as a veridical paradox and it highlights just how bad our brains are at predicting non-linear functions.

Falsidical paradoxes, are unsurprisingly just that. These seem at first to be nonsensical and despite the moderate attempts of 'proof' it does not take much thought to realise that they are nonsensical. However they can be just as interesting to consider. Indeed, dealing with absurdity is something which mathematicians do every day. 'Reductio ad absurdum' i.e proof by contradiction, is a method that we use frequently. The method essentially aims to make two things that are mutually exclusive be true by making assumptions at the start of the proof. Perhaps the most famous example of these are the paradoxes of Zeno.

There are many that fall under his name but the most popular are about motion, specifically Achilles and the tortoise [5]. The paradox, which can take many forms each illustrating the same idea, is one of the oldest paradoxes that we know but it is still relevant today. Suppose we have Achilles chasing a tortoise, which starts $0.9m$ ahead of him. Achilles is travelling at $1m/s$ and the tortoise $0.1m/s$, and so we can conclude that Achilles should catch up to the tortoise after 1 second, at a distance of $1m$ from where he started.

However. we could consider his motion in a different way by dissecting it into sections. Before Achilles may catch the tortoise, he clearly must first get to the point from which the tortoise started. Then in the time it took him to get to that point, the tortoise would have moved a little bit further on. We could label this stage one. Then Achilles must travel to the point that the tortoise reached at the end of stage one - by which point, again, the tortoise will have made further progress. We can label this stage two. This process continues in an iterative fashion for k stages as k approaches infinity. So, Achilles has to do an infinite number of these 'catch' up stages, each of finite length, before he reaches the tortoise, so we conclude that he never actually catches the tortoise. This is our paradox.

Clearly, this is unfounded. It is a similar argument to the following: hold you hands apart from each other and hold the left one stationary, then move your right hand to 'clap' the left one. However you could consider that motion as your right hand getting half the distance to your left, and then three quarters, and then seventh eighths. Then by this logic your hands would never actually 'clap'. Well of course they will as we operate in the real world and the idea that they will not is clearly incorrect. This is the same for Achilles and the tortoise. There are obvious logical failings in the arguments (solved by the theory of limits) and so this paradox can be thought as being false, or incorrect.

A paradox which does not fit into either of these cases is called **antimony**.

Arguably the most interesting of these is the paradox surrounding the sum of the natural numbers. At the face of it, this appears to not be a difficult or controversial question. An analyst would argue that $1+2+3+\dots$ is a divergent series. Indeed we may express the partial sum of the series to be:

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2} \implies 1+2+3+\cdots = \lim_{m\to\infty}\left(\sum_{i=1}^{m} i\right) = \lim_{m\to\infty}\left(\frac{m(m+1)}{2}\right) = \infty,$$

and so it would appear to be sensible to deduce that the sum of the natural numbers is infinite.

However we may also progress by considering the series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \quad x < 1.$$

Then if we take the derivative of both sides we get:

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}.$$

Then we let $x = -1$, which gives:

$$1 - 2 + 3 - 4 + \ldots = \frac{1}{(1 - -1)^2} = \frac{1}{4}$$

We now introduce the **Euler-Riemann Zeta Function**. It is usually referred to as the **Riemann Zeta Function**, when its argument is complex, and the **Euler Zeta Function** when it is real. It was first discovered by Euler c. 1749 when he was trying to resolve the famous Basel Problem, which is this function evaluated at 2 and remarkably has a value of $\pi^2/6$. Riemann utilised the formula over 100 years after it was discovered and it is central to the infamous **Riemann Hypothesis**; for which the prize for a solution is \$1 000 000, courtesy of the Clay Mathematics Institute. It is defined by:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \ldots$$

Now if we focus on only real inputs, we see:

$$2^{-z}\zeta(z) = 2^{-z} = 2^{-z} + 4^{-z} + 6^{-z} + 8^{-z} + \ldots,$$

and from this we can take:

$$\begin{aligned}(1 - 2 \cdot 2^{-z})\zeta(z) &= 1 + 2^{-z} + 3^{-z} + 4^{-z} + 5^{-z} + 6^{-z} + \ldots \\ &\quad -2\big(\quad 2^{-z} \qquad + 4^{-z} \qquad + 6^{-z} + \ldots\big) \\ &= 1 - 2^{-z} + 3^{-z} - 4^{-z} + \ldots\end{aligned}$$

Now if we set $s = -1$, we get that:

$$\begin{aligned}(1 - 2 \cdot 2^1)\zeta(-1) &= 1 - 2^1 + 3^1 - 4^1 + \ldots \\ -3\zeta(-1) &= 1 - 2 + 3 - 4 + \ldots\end{aligned}$$

But from earlier we showed that $1 - 2 + 3 - 4 + \ldots = \frac{1}{4}$, and so

$$-3\,\zeta(-1) = \frac{1}{4} \implies \zeta(-1) = -\frac{1}{12}.$$

Following from the definition of the Euler-Riemann Zeta Function, we have that

$$1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}.$$

This is our paradox. All of our mathematical intuition would tell us that this result is entirely wrong, yet this fact is utilised in many areas of physics, including in string theory. In fact, it is a pivotal part of proving that there are

26 dimensions in Bosonic String Theory[6, p. 22]. So, is it really false? It has legitimate and important real-life applications and of major significance to the academic community, but it is dependant on this seemingly absurd formula. It is because of this that we may argue this does not fall solely into the category of falsidical paradoxes.

Chapter 3

Banach-Tarski Paradox

> Mathematics is the most beautiful and most powerful creation of the human spirit.
>
> *Stefan Banach*

The Banach-Tarski Paradox is a result in mathematics that shows it is possible to partition a solid three-dimensional object, such as a sphere, into a finite number of pieces, and then reassemble those pieces to form two identical copies of the original object. This is completely counter-intuitive to our understanding of the preservation of volume by rigid motions; a **rigid motion** of a space is a transformation that does not change the (Euclidean) distance between two points. Rigid motions can include translations (moving an object without rotating or changing its shape), rotations (turning an object around a fixed point), and reflections (flipping an object across a mirror line). A combination of any of these operations is also considered a rigid motion, so it is seemingly impossible to double the volume of an object by splitting it into pieces and rearranging them rigidly.

This is a consequence of the existence of sets, most of the sets constructed in the proof of Banach-Tarski have no volume to preserve. As a result, the pieces that the sphere has been decomposed into have no well-defined volume. The existence of such unmeasurable sets is a direct consequence of the **Axiom of Choice**. We will now consider the idea of measure and the Axiom of Choice.

The axioms of mathematics serve as the fundamental building blocks and starting point for all mathematical reasoning and proof. Without these axioms, there would be no coherent mathematical system, and mathematical concepts and theorems would have no foundation. These axioms were devised to unite all branches of Mathematics. The set of Axioms that we accept in Mathematics today are called the **Zermelo-Fraenkel Axioms**. Most of these axioms are reasonable and intuitive. However, the most controversial one is the Axiom of Choice.

The Axiom of Choice 1 ([7], p. 5)**.** *For any nonempty collection of disjoint nonempty sets, there is a set containing exactly one element from each set in the* *collection.*

Formally put, it states:

The Axiom of Choice 2 ([8]). *Let I be a set. A collection indexed by I is a collection of sets $\{S_i\}_{i\in I}$. In other words, the collection contains one set for each element of I. A choice function is a function,*

$$f : I \to \bigcup_{i\in I} S_i, \tag{3.1}$$

such that $f(i) \in S_i$ for all $i \in I$. The Axiom of Choice states that for any indexed collection of nonempty sets, there exists a choice function.

The axiom is intuitive enough, however it is completely independent of all the other Zermelo-Fraenkel axioms. You can neither prove nor disprove the Axiom of Choice based on these axioms.

3.1 Measure

In mathematics, size is typically described using the concept of **measure**. A measure is a real-valued function that assigns a non-negative value to sets, indicating their size or magnitude - it is a way of assigning a numerical value, or measure, to the size of sets of real numbers and it formalizes the notion of length in 1-D, area in 2-D and volume in 3-D. The most common example of a measure is the **Lebesgue measure**, which is defined on the Euclidean space and assigns a volume to sets in 3-D space.

Definition 1. *The Length Function on the set of intervals in $\mathbb{R}$ is the function $l : \mathbb{I} \to [0, \infty]$ defined for all intervals $I \subseteq \mathbb{R}$ with endpoints $a, b \in \mathbb{R}^*$ by:*

$$l(I) = \begin{cases} |b-a| & \textit{if } I \textit{ is bounded.} \\ \infty & \textit{if } I \textit{ is unbounded.} \end{cases}$$

The Lebesgue measure μ , defined as $\mu : \omega \to \mathbb{R}$,takes some interval and returns its 'measure', which is some positive real number. This measure is defined as the length of the interval I, $\mu(I) = l(I)$.

Given a set A of real numbers, its Lebesgue measure, $\mu(A)$, should it be defined, holds the following properties:

- Non-Negativity: $\mu(A) \geq 0$ for all $A \in \mathbb{R}$.
- Countable Additivity: Suppose $A_1, A_2 \ldots$ are disjoint subsets of $\mathbb{R}$. Then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{n=1}^{\infty} \mu(A_i)$.
- Translation Invariance: For some number $x \in R$, define $A + x := \{x + y \mid y \in A\}$. It follows that $\mu(A + x) = \mu(A)$.
- Monotonicity: If $A_1 \subseteq A$, then $\mu(A_1) \leq \mu(A) \leq \infty$.

All Lebesgue measurable sets satisfy these properties, however not all subsets of $\mathbb{R}$ are measurable, as we have already mentioned. Perhaps one of the most famous examples of unmeasureable sets are **Vitali sets**, which are a subset of the real numbers and were first introduced in mathematics by Giuseppe Vitali in 1905. They can be formally defined as:

Definition 2 ([9], p. 2)**.** *A subset $V \subseteq [0,1]$ is called a Vitali set if V contains a single point from each coset of $\mathbb{Q}$ in $\mathbb{R}$.*

A coset of $\mathbb{Q}$ in $\mathbb{R}$ is defined as any set of the form $x + \mathbb{Q} = \{x + q \mid q \in \mathbb{Q}\}$. We can generate an uncountable amount of Vitali sets, and they exist as a result of the Axiom of Choice. The Vitali Theorem asserts the existence of such sets.

Theorem 1 ([9], p. 3)**.** *If $V \subseteq [0,1]$ is a Vitali set, then V is not Lebesgue measurable.*

Proof. ([10])

Let, $x \sim y$ iff $x - y \in \mathbb{Q}$ define an equivalence relation on $\mathbb{R}$.

$\mathbb{R}$ is divided into equivalence classes, $[x]_\sim = \{x + q : q \in \mathbb{Q}\}$, by this relation. Each equivalence class contains a point in the interval $[0,1]$. Let V be a set that contains only one point from each equivalence class, denoted by $V \subseteq [0,1]$. (V exists by the Axiom of Choice).

Then if $x \in [0,1]$, there must exist a $y \in V$ such that $x \in [y]$. Then, for some $q_i \in \cap [-1,1]$, $x - y = q_i$ hence we have $x \in V + q_i \equiv V_i$. So,

$$V_i \equiv V + q_i, \forall q_i \in \mathbb{Q} \cap [-1,1] .$$

Then, notice that

$$[0,1] \subseteq \bigcup_{i \in \mathbb{N}} V_i \subseteq [-1,2] .$$

We can also see that,

$$V_i \cap V_j = \varnothing, \text{ if } i \neq j.$$

As if $y, z \in V$ and $y \neq z$ we would have $y + q_i = z + q_j$, this is impossible due to our choice of V.

Suppose V is measurable, then all $V_i, i \in \mathbb{N}$ and $\bigcup_{i \in \mathbb{N}} V_i$ are.

From $[0,1] \subseteq \bigcup_{i \in \mathbb{N}} V_i \subseteq [-1,2]$, and the monotonicity property of the Lebesgue measure λ, we get

$$1 = \lambda([0,1]) \leq \lambda\left(\bigcup_{i \in \mathbb{N}} V_i\right) \leq \lambda([-1,2]) = 3.$$

Now using the translation invariance property of the Lebesgue measure, we have

$$0 < \lambda(V_i) = \lambda(V) .$$

So by using the fact that V_i, for $i \in \mathbb{N}$, are pairwise disjoint and that the Lebesgue measure is σ-additive, we get

$$\lambda\left(\bigcup_{i\in\mathbb{N}} V_i\right) = \Sigma_{i\in\mathbb{N}}\lambda\left(V_i\right) = \infty.$$

This contradicts $1 = \lambda([0,1]) \leq \lambda\left(\bigcup_{i\in\mathbb{N}} V_i\right) \leq \lambda([-1,2]) = 3$.

Hence, V cannot be measurable. □

As shown, the construction of the Vitali set is dependent on the Axiom of Choice. It is a significant instance of an unmeasurable set which paved the way for the construction of the Banach-Tarski Paradox by showing that such unmeasurable sets exist.

So why is this possible? Logically, there doesn't seem to be a problem with this axiom as it's highly intuitive. If you have a collection of non-empty sets, you can clearly take an element from each set and construct a new set. However, the problem lies in the fact that this newly formed arbitrary set has no structure. The Axiom of Choice states that a choice function always exists, however it is not always clear how that choice function should be defined. This is contrary to the other Axioms, which tell us exactly how to construct a set.

We will now explore some notions in group theory before going on to construct and prove most of the results behind the Banach-Tarski Paradox. We begin by looking at the idea of **Free groups**, which will play a central role in the final proof of the paradox as will be seen below.

3.2 Free Groups

Before going any further, let's actually define what it means for a group to be paradoxical in Mathematics. A group is called **paradoxical** if it can be partitioned into a finite number of subsets, each of which can be rearranged in a different way to create two disjoint copies of the original group.

A set X is considered paradoxical with respect to a group G if it can be divided into two different subsets in such a way that when G acts on those subsets, the resulting sets appear to be identical to the original set X. Formally put, we can define it as,

Definition 3 ([11], p. 4)**.** *Let G be a group acting on a set X. A non-empty subset $E \subseteq X$ is G-paradoxical (or paradoxical with respect to G) if there are pairwise disjoint subsets $A_1, ...A_m, B_1, ..., B_n$ of E and elements $g_1, ...g_m, h_1, ...h_n \in G$, such that,*

$$E = \bigcup_{i=1}^{m} g_i(A_i) = \bigcup_{i=1}^{m} h_j(B_j),$$

then X is paradoxical with respect to G or X is G-paradoxical.

Note that an action of a group G on a set X is paradoxical if X is paradoxical. The group itself is called paradoxical if the action of the group on itself by left multiplication is paradoxical.

Definition 3.2.1 ([12], p. 6)**.** *A free group with generating set S, comprises all finite length reduced words (any written product of group elements and their inverses) formed using letters from $\{\sigma, \sigma^{-1} : \sigma \in S\}$, which includes the empty word represented by e. A reduced word is one that does not contain any instances of $\sigma\sigma^{-1}$ or $\sigma^{-1}\sigma$.*

The free group on n letters, $a_1, \dots a_n$, is the set of all possible words (such as $a_1^2 a_{n-1}^{-1} a_n$) that can be generated and simplified with the relation $a_i a_i^{-1} = e$, where the group operation is concatenation. Moving forward, we need to understand free groups with 2 generators, say a and b. Let us denote this group by F_2.

Theorem 2 ([13], p. 2)**.** *A group G is paradoxical if and only if there exists a free action (every element other than the identity in the group has no fixed points) on a set X which is paradoxical.*

Proof. ([13], p. 2) Define a paradoxical decomposition of G as,

$$G = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j.$$

Using the Axiom of Choice we can select a subset M of X, containing only one element from each orbit of G. We end up with $\bigcup_{g \in G} gM$, which is a disjoint partition of M. Then if $gx = hy$ for some $g, h \in G$ and $x, y \in X$, we get that $x = y$ based on the choice of M. Due to this action being free, we get $g = h$.

We now define,

$$\widehat{A_j} = \bigcup_{g \in A_i} gM \text{ and } \widehat{B_j} = \bigcup_{g \in B_j} gM.$$

We observe that these sets remain disjoint. However, we also have,

$$X = \bigcup_{i=1}^{n} g_i \widehat{A_i} = \bigcup_{j=1}^{m} h_j \widehat{B_j}.$$

Now consider an orbit $\mathcal{O}$ of a point X, defined as $Gx = \{g \cdot x \mid g \in G\}$. From this the action of G on $\mathcal{O}$ is an action on the cosets space by some subgroup $H < G$. Therefore, the paradoxical decomposition of G is implied by the paradoxical decomposition of $\mathcal{O}$. $\square$

In particular, this means that paradoxical decomposition is possible for all free actions of $\mathbb{F}_2$.[13, p. 3]

Corollary 1 ([12], p. 4)**.** *If G has a paradoxical subgroup, then G is also paradoxical.*

Proof. If G has a paradoxical subgroup, it follows that this subgroup acts on G without non-trivial fixed points as a result of Theorem 2. $\square$

This is a very important result. If we now show that F_2 is indeed paradoxical, we can transfer its properties because of this Corollary.

Theorem 3 ([11], p. 5)**.** *A free group of rank 2, F_2, is F_2-paradoxical, where F_2 acts on itself by left multiplication.*

Proof. we can partition F_2 as follows:

$$F_2 = \{e\} \cup S(a) \cup S\left(a^{-1}\right) \cup S(b) \cup S\left(b^{-1}\right),$$

where the subsets are pairwise disjoint and $S(a), S\left(a^{-1}\right), S(b), S\left(b^{-1}\right)$ are the set of words starting with a, a^{-1}, b, b^{-1} respectively.

Now, observe,

$$\begin{aligned} F_2 &= S(a) \cup aS\left(a^{-1}\right) \\ &= S(b) \cup bS\left(b^{-1}\right). \end{aligned}$$

The reason for this is that $aS(a^{-1})$ includes all words that do not start with a, as we only allow reduced words. Using the same reasoning, $bS(b^{-1})$ comprises all words that do not begin with b and as a result, we observe that F_2 is paradoxical. □

Now that we have found a paradoxical-decomposition of F_2, we will try to find a group of 3-D rotations ($SO3$), which is a free group with 2 generators. Since we know that F_2 has paradoxical actions, finding a group which is isomorphic to F_2 in $SO3$ will allow us to use the paradoxical decomposition of said group to manipulate a shape in 3-D space. This is possible because of the result which we will state below.

Corollary 2. *If a group G has a paradoxical subgroup F_2, then G itself is paradoxical.*

This is a direct result of Corollary 1 and Theorem 2.

3.3 A Free Group of Rotations of $\mathbb{R}^3$

The next goal is to identify a free group in $\mathbb{R}^3$ that consists of rotations around the origin with 2 generators.

We will follow the steps as shown in [14, p. 3]. Denote by A a rotation by θ about the x-axis, and by B a rotation by θ about the z-axis where $\theta = \cos^{-1}\left(\frac{1}{3}\right)$.

The matrix of A and B can be calculated as follows.

$$A = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix}, \qquad B = \frac{1}{3}\begin{pmatrix} 1 & -2\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 1 \end{pmatrix}, \quad B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Now, we can define ð as the group generated by A and B. We chose $\theta = \cos^{-1}\left(\frac{1}{3}\right)$ as this is an irrational multiple of π, ensuring that the orbit of these rotations has no repetition (the orbit of a rotation is the set of all points in a space that can be reached by applying the rotation to a given point).

Lemma 1 ([14], p. 3)**.** *Let $a_1 a_2 \dots a_n$ be a word with $a_i \in \{A, A^{-1}, B, B^{-1}\}$ in the reduced form with length n. Then,*

$$\begin{aligned} &a_1 a_2 \cdots a_n(0,1,0) = \frac{1}{3^n}(a\sqrt{2}, b, c\sqrt{2}), \\ &a, b, c \in \mathbb{Z}. \end{aligned}$$

Proof. For $n = 0$, for the empty word ϕ, $\phi(0,1,0) = (0,1,0)$. So the base case holds.

Assume the lemma holds for all words of length $n-1$.

Let ϕ be a word of length $n-1$, $n \geq 1$.

By inductive hypothesis, assume,

$$\phi(0,1,0) = \frac{1}{3^{n-1}}(a\sqrt{2}, b, c\sqrt{2}).$$

Then, using the matrices from above, we have the following calculations,

$$\begin{aligned} A\phi(0,1,0) &= \frac{1}{3^n}(3a\sqrt{2}, b-4c, (2b+c)\sqrt{2}), \\ A^{-1}\phi(0,1,0) &= \frac{1}{3^n}(3a\sqrt{2}, b+4c, (-2b+c)\sqrt{2}), \\ B\phi(0,1,0) &= \frac{1}{3^n}((a-2b)\sqrt{2}, 4a+b, 3c\sqrt{2}), \\ B^{-1}\phi(0,1,0) &= \frac{1}{3^n}((a+2b)\sqrt{2}, -4a+b, 3c\sqrt{2}). \end{aligned}$$

The above expressions are all in the required form. By mathematical induction, the lemma holds for all words ϕ of length n. □

Theorem 4 ([14], p. 3)**.** ð *is a free group.*

Proof. Assuming ð is a free group, there should be no non-trivial identity in ð by Theorem 2. Suppose to the contrary that there exists some non-trivial identity ρ in ð. Thus $\rho(0,1,0) = (0,1,0)$. By Lemma 1, $\rho(0,1,0)$ must be of the form $\frac{1}{3^n}(a\sqrt{2}, b, c\sqrt{2})$, so we must have that $a = c = 0$ and $b = 3^n$ where $n > 0$. Therefore $a \equiv b \equiv c \equiv 0 \pmod 3$. It can be shown by induction and listing all possible results of applying A and B modulo 3 that this is not possible. □

The concept of paradoxical decomposition in Euclidean space is founded on the existence of free subgroups within the group of rotations in three-dimensional Euclidean space. All free actions of a free group with 2 generators leads to paradoxical actions. This coupled with Corollary 2 can help us very strong statements about $SO3$. Now that we have seen that a free group with 2 generators exists in $SO3$ (As a result of Theorem 4), we can use Corollary 2 to get the following proposition.

Proposition 1. *The group $SO3$ of rotations in $\mathbb{R}^3$ has a free subgroup of rank 2, hence $SO3$ is paradoxical.*

We now have the tools to discuss another significant Paradox which paved the way for the Banach-Tarski Paradox. Hausdorff's Paradox states the following:

Theorem 5 ([12], p. 7)**.** *There exists a countable set $D \subset S^2$ such that $S^2 \backslash D$ is $SO3$-paradoxical.*

We can prove this theorem by finding a subgroup of $SO3$ free on 2 generators making it paradoxical on itself, as follows by Corollary 1. The next step is to identify a subset D of S^2 that will enable the paradoxical property to be transferred to $S^2 \backslash D$ through Theorem 2.

Defining D to be the set of nontrivial fixed points of the countable subgroup F_2 of $SO3$ when it acts on the sphere S^2. Each element of F_2 has two nontrivial fixed points on S^2, which are the points where the axis of rotation intersects the sphere. Since F_2 is countable, the set D is also countable. After removing this countable set, Theorem 2 implies that the resulting set, $S^2 \backslash D$, is paradoxical under the action of $SO3$.

3.4 Equidecomposability

The Banach-Tarski Paradox involves the decomposition of a unit ball into finitely many pieces, rearranging each of these pieces into equal unit balls. So far, we have managed to show through Hausdorff's Paradox that a Sphere with a numbered set of points removed is $SO3$ paradoxical. By Theorem 2, we can deduce paradoxicity on a group which is being acted on if and only if there exists a free action on that group. We cannot achieve this by only using the group of rotations in 3-D, as there are infinitely many points that remain fixed on the unit ball under rotation. In order to deal with this problem, we need to introduce translations. We will now introduce the notion of **equidecomposablity**, which will give us some strong results and help to re-define the formal statement of the Banach-Tarski Paradox.

Definition 3.4.1 ([15], p. 4)**.** *Two subsets $A, B \subseteq R_n$ are called equidecomposable if they can be partitioned into the same finite number of pieces, which can be matched such that each pair of corresponding pieces are congruent.*

Or more formally defined:

Definition 3.4.2 ([11], p. 25)**.** *Suppose G acts on X and $A, B \subseteq X$. Then A and B are G equidecomposable (sometimes called finitely G-equidecomposable or piece-wise G-congruent) if A and B can each be partitioned into the same finite number of respectively G-congruent pieces. Formally,*

$$A = \bigcup_{i=1}^{m}(A_i), \;\; B = \bigcup_{i=1}^{m}(B_i),$$

with $A_i \cap A_j = \emptyset = B_i \cap B_j$ if $i \leq j \leq n$, and there are $g_1 \cdots g_n \in G$ such that, for each $i \leq n, g_i(A_i) = B_i$.

Note that equidecomposibility is an equivalence relation ($A \sim B$).

We now have the following propositions.

Proposition 2 ([13], p. 6)**.** *Let G be a group that acts on a set X. If F is paradoxical and F is equidecomposable to E, then E is paradoxical.*

Proof. Define a paradoxical decomposition of E, F,

$$F = \bigcup_{i=1}^{n}(g_i)(A_i) = \bigcup_{i=1}^{m}(h_j)(B_j).$$

E is equidecomposable to both $\bigcup_{i=1}^{n}(g_i)(A_i)$ and $\bigcup_{i=1}^{m}(h_j)(B_j)$ by transitivity. Thus E is paradoxical. □

Proposition 3. *([13], p. 6) If D is a countable subset of S^2 then $S^2 \backslash D$ and S^2 are $SO3$-equidecomposable.*

Proof. ([13], p. 6)

Using the fact that D is countable, we can find a line L which does not intersect with D. Define Λ as a collection of all $\alpha \in [0, 2\pi]$ where there exists some natural number n and a point x in D such that both x and $\rho(x)$ are in D. We define ρ as the rotation about L by an angle $n\alpha$. Observe that Λ is countable, due to this we can find $\theta \in [0, 2\pi) \setminus \Lambda$. Now let ρ be a rotation by θ around L, hence $\rho^n(D) \cap \rho^k(D) = \emptyset$ for all natural numbers $k \neq n$. So for $D' = \bigcup_{n \geq 0} \rho^n(D)$, we get that,

$$S^2 = D' \cup (S^2 \setminus D') \sim \rho(D') \cup (S^2 \setminus D') = S^2 \setminus D,$$

□

proving the claim.

Hausdorff's paradox and this proposition give way to the following corollary.

Corollary 3. *S^2 is $SO3$-paradoxical.*

The above statement is a consequence of two observations. Firstly, that $S^2 \backslash D$ is paradoxical under the action of $SO3$, and secondly, that S^2 can be partitioned into subsets that can be rearranged using $SO3$ transformations to form $S^2 \backslash D$. It follows that S^2 is also paradoxical under the action of $SO3$.

Now, we can finally assemble all of these propositions to prove the Banach-Tarski Paradox.

Theorem 6. *The closed unit ball in $\mathbb{R}^3$ centred at the origin with it's centre removed, $\mathbb{B} \backslash \{0\}$, is $SO3$-paradoxical.*

Proof. ([13], p. 7) Using Corollary 3, we can find $A_1, \dots, A_n$, $B_1, \dots, B_n \subset S^2$ and $g_1, \dots, g_n, h_1, \dots, h_m \in SO3$ which satisfy Definition 3. Now we define,

$$\widehat{A}_i = \{tx : t \in (0, 1], x \in A_i\} \text{ and } \widehat{B}_j = \{tx : t \in (0, 1], x \in B_j\}.$$

Hence, $\widehat{A}_1, \dots, \widehat{A}_n$, $\widehat{B}_1, \dots, \widehat{B}_n \subset \mathbb{B} \backslash \{0\}$ are pairwise disjoint and $\mathbb{B} = \bigcup_{1 \leq j \leq n} g_i \widehat{A}_i = \bigcup_{1 \leq j \leq m} h_j \widehat{A}_j$. Therefore, $\mathbb{B} \backslash \{0\}$ is paradoxical. □

Theorem 7. *(The Banach-Tarski Paradox).* $\mathbb{B}$ *is* G_3*-paradoxical. (Group of isometries in* $\mathbb{R}^3$*).*

Proof. It has been shown that $\mathbb{B}\backslash\{0\}$ is paradoxical, all that's left to show is that $\mathbb{B}\backslash\{0\}$ is equidcompasbale to $\mathbb{B}$. Then, by proposition 2 it follows that $\mathbb{B}$ is also $SO3$ paradoxical. This implies G_3 paradoxicity since $SO3$ is paradoxical and a subgroup of G_3.

([15], p. 7)We consider a small circle that is entirely contained in $\mathbb{B}$ and passes through the point 0. Let ρ be a 1-radian rotation of this circle. As a result, the points $0, \rho 0, \rho^2 0, \rho^3 0, \cdots$ are distinct, hence if we apply ρ to these points, we end up with the same set except that 0 is missing. From this we get a two-piece equidecomposition of $\mathbb{B}$ with $\mathbb{B}\backslash\{0\}$, with one piece being $0, \rho 0, \rho^2 0, \rho^3 0, \cdots$ going to $\rho 0, \rho^2 0, \rho^3 0, \cdots$ under the rotation ρ. The other piece of the equidecomposition being the rest of the ball going to itself. Then, by proposition 2, we have the statement. □

The Banach-Tarski Paradox is a profound and intriguing result that reveals the complex and unexpected behavior of sets in 3-D space. Its proof relies on advanced mathematical concepts and techniques, including group theory and measure theory, which highlights the deep connections between different branches of mathematics. By challenging our intuition about volume and measure, the paradox reminds us of the limitations of our understanding and the importance of rigour and abstraction within mathematical reasoning.

Moreover, the Banach-Tarski Paradox has significant implications for the foundations of mathematics and the nature of infinity. It shows that seemingly basic concepts such as volume and measure are not as well-behaved as expected, and some sets defy our intuitive understanding of physical space. The paradox also raises questions about the role of intuition and logic in mathematical reasoning, highlighting the power and limitations of set theory and axiomatic systems.

Despite its unsettling nature, the Banach-Tarski Paradox showcases the remarkable power and flexibility of mathematical thinking. By using creative and innovative approaches, mathematicians can unlock new insights and solutions to long-standing problems. Ultimately, this paradox underscores the richness and depth of mathematical knowledge and the endless possibilities for discovery and exploration.

Chapter 4

Skolem's Paradox

> Later generations will regard set theory as a disease from which one has recovered.
>
> *Henri Poincaré*

Skolem's Paradox describes an initially troubling result that comes from comparing two important theorems from model theory and set theory. Discovered by Norwegian mathematician Thoraf Skolem, he noticed what appeared to be a mathematical contradiction between his own **Löwenheim-Skolem Theorem**, concerning the cardinality of models, and **Cantor's Theorem** on cardinality of power sets [16].

The issue is as follows. Löwenheim-Skolem Theorem tells us that first-order axiomatisations of set theory of infinite model, have a model that is countable. Cantor's Theorem proves the existence of uncountable sets. How then can a model who's domain is only countable satisfy the claim that uncountable sets exist? Surely there can be no element of a countable model that is uncountable as then how would the model be countable in the first place? In this chapter we will explain how this is not, in fact, a contradiction in set theory. I will first outline the appropriate prior knowledge of model theory required to understand the details of the issue. We will then explain both Löwenheim-Skolem Theorem and Cantor's Theorem so we might understand how the problem arose. Finally, we will show that there is in fact no issue at all, and that the intrigue is just a rather peculiar result of model and set theory.

4.1 Model Theory

In order to explain Skolem's Paradox we need a basic understanding of model theory, which we will use this section to delineate.

Model Theory is a field, often placed under the umbrella of mathematical logic, which studies the relationship between formal languages and their interpretations [17]. A model consists of a language used to satisfy given statements, this language is called a **signature**. This signature holds all the symbols used to formalise these statements. Mathematical statements hold no inherent truth or meaning without a way to interpret them [18], a model tells us how to interpret these symbols and thus determine whether a statement is true or false.

More formally, given some mathematical statement S, meaning a collection of symbols (i.e. quantifiers, constants, etc.), and given an interpretation $\mathcal{I}$, we say that $\mathcal{I}$ is a model of S if it satisfies the statement's claim, that is, if under the interpretation $\mathcal{I}$ S is true. This can be expressed symbolically as:

$$\mathcal{I} \models S.$$

Given a set of statements A which define a class of interpretations that model A, we say that A is a set of axioms [18]. For example consider the statements:

$$\begin{gathered}
\forall a, b, c \in X : (a \cdot b) \cdot c = a \cdot (b \cdot c), \\
\exists i \in X : \forall a \in X, i \cdot a = a \cdot i = a, \\
\forall a \in X \exists b \in X : a \cdot b = b \cdot a = i, \\
\forall a, d \in X : a \cdot d = d \cdot a.
\end{gathered}$$

if we take $\cdot$ to be $+$, i to be 0, and d to be $-a$, you will most likely recognise these as the axioms for abelian groups. The interpretations that model abelian groups are thus of signature $0, +, -$.

Models are made of what we call **structures** [19, p. 5]. The kind of structures Löwenheim-Skolem Theorem is concerned with are known as **first-order structures**, structures that are built using first order logic. Skolem's Paradox breaks down for higher-order structures such as systems built using second-order logic [16]. Structures are given by some set which is its domain, as well as an interpretation for each of the elements of the prescribed signature. Thus the domain will consist of all the interpreted constant, function and relation symbols of the given language.

Definition 4.1.1 ([19], p. 5)**.** *We say $\mathcal{M}$ is a $\gamma-$structure if it consists of some set M and interprets the constant symbols C_γ, function symbols F_γ and relation symbols R_γ of some signature γ as follows:*

i. $\{f_i \in F_\gamma\}$ *corresponds to some i placed function $f : M^{n_i} \to M$ for some $n_i \geq 1$.*

ii. $\{R_j \in R_\gamma\}$ *corresponds to some j placed relation $R \subset M^{m_j}$ for some $m_j \geq 1$.*

iii. $\{c_k \in C_\gamma\}$*, which correspond to some constant $x \in M$.*

Here M is the domain, or **universe** [19, p. 6], of $\mathcal{M}$ (we will denote the universe of some model $\mathcal{A}$ by its normal typeset A). Function and relation symbols are both of some arity (the number of arguments they take), here n_i and m_j respectively. When symbols are used with respect to a certain model we may like to indicate this with a subscript [17, p. 20]. For instance, given $\mathcal{M}$ is a model of signature:

$$\mathcal{L} = \{\leq, +, \cdot\},$$

we might denote each element:

$$\leq_{\mathcal{M}}, +_{\mathcal{M}}, \cdot_{\mathcal{M}}.$$

So given a signature, a structure is made when interpretations are assigned to its symbols. Let us think back to abelian groups now. Let us denote an abelian group by G, we have from above that its signature is then given by 0_G, $+_G$, $-_G$. Under the interpretation function we understand 0 to be the additive identity and $+_G$ and $-_G$ are functions $G \times G \to G$ for addition and subtraction.

Thus model theory gives us a way to interpret mathematical statements. Note model theory is not exclusive to mathematics, models can be created to interpret anything [18]. However, the interest in such a field of study is in the properties of models.

4.1.1 Sub-Structures

Note for the rest of this chapter we will use structure and model interchangeably.

Definition 4.1.2 ([19], p. 19)**.** *Given two γ structures $\mathcal{M}$ and $\mathcal{N}$, we say that $\mathcal{N}$ is a substructure of $\mathcal{M}$, denoted $\mathcal{N} \subseteq \mathcal{M}$, if there is an injective map $e : \mathcal{N} \to \mathcal{M}$ such that all the interpretations of the language γ are preserved. Precisely it is the map:*

i. $e(f^{\mathcal{M}}(a_1, ..., a_{n_f})) = f^{\mathcal{N}}(e(a_1), ..., e(a_{n_f}))$ *for all function symbols f in γ and $a_1, ..., a_{n_f}$ in $\mathcal{M}$.*

ii. $(a_1, ..., a_{n_R}) \in R^{\mathcal{M}} \iff (e(a_1), ..., e(a_{n_R})) \in R^{\mathcal{N}}$ *for all relation symbols R in γ and $a_1, ..., a_{n_R}$ in $\mathcal{M}$.*

iii. $e(c^{\mathcal{M}}) = c^{\mathcal{N}}$ *for all constant symbols c in γ.*

This preservation of interpretations is known as an **embedding** [20, p. 18].

Definition 4.1.3 ([19], p. 15)**.** *Two $\gamma-$structures are said to be* elementary equivalent*, written $\mathcal{M} \equiv \mathcal{N}$, if for every $\gamma-$sentence ϕ ,*

$$\mathcal{M} \models \phi \iff \mathcal{N} \models \phi.$$

Given an embedding in γ $e : \mathcal{M} \to \mathcal{N}$, we say that it is **elementary** if:

$$\mathcal{M} \models \phi(a_1, ..., a_n) \iff \mathcal{N} \models \phi(e(a_1), ..., e(a_n)).$$

Thus given $\mathcal{N} \subseteq \mathcal{M}$, say that that $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$ if the preservation map is elementary. This is written $\mathcal{N} \preceq \mathcal{M}$. In this case we say that $\mathcal{M}$ is an **elementary extension** of $\mathcal{N}$

4.1.2 Model Theoretic Satisfaction

In order to understand the mathematics of the Löwenheim-Skolem Theorem, I will illustrate a quick example of a model theoretic statement and it's satisfaction. This will help understand the notation used which is more akin to logic than set theory.

Given a formula, ϕ, we may introduce it as $\phi(\bar{x})$, where $\bar{x}$ is a vector of variables, meaning the free variables of ϕ all lie in $\bar{x}$ [17, p. 24]. So given $\bar{a} = a_1, ..., a_n$, $\phi(\bar{a})$ is the formula we get by replacing the free variable terms x_i with a_i. As we have seen above we use the $\models$ symbol to denote that a structure models some given statements. So given a formula $\phi(\bar{a})$, if an interpretation $\mathcal{I}$ satisfies it we may write:

$$\mathcal{I} \models \phi(\bar{a})$$

Consider the structure $\mathcal{I} = \{1, +, <\}$. let ϕ be the formula:

$$\phi \quad = \quad \forall x(x < (y + 1)).$$

Thus $\mathcal{I} \models \phi$. A value upon which a statement is satisfied is called a witness [17]. For instance given an existential statement of the form $\exists x\phi(x)$, we say that y witnesses ϕ if $\phi(y)$ is true.

With all these model theoretic tools at our disposal we can now understand Löwenheim-Skolem Theorem.

4.2 Löwenheim-Skolem Theorem

Löwenheim-Skolem Theorem describes a group of theorems initially penned by German mathematician Leopald Löwenheim and further developed by the aforementioned Thoraf Skolem. It is a fundamental result of model theory concerning the cardinality of models. We understand the cardinality of a model in the same way in which we would a set i.e. the cardinality of a model is the number of elements it contains (the cardinality of its domain). Essential in the proof of Löwenheim-Skolem Theorem is what is known as the **Tarski-Vaught test**. It is a criteria that ensures that a subtructure is an elementary substructure.

Theorem 8 (Tarski-Vaught Test [21], p. 3)**.** *let $\mathcal{M}$, $\mathcal{N}$ be two first order structures with in the language γ. Suppose $\mathcal{M} \subseteq \mathcal{N}$. Then $\mathcal{M} \preceq \mathcal{N}$ if and only if for every $\gamma-$formula $\phi(x, \bar{y})$, and $\bar{a}$ in $\mathcal{M}$, if $\mathcal{N} \models \exists x\phi(x, \bar{a})$, then there is b in $\mathcal{M}$ such that $\mathcal{N} \models \phi(b, \bar{a})$.*

Proof. We first prove the forward claim. Fix $\bar{a} \in \mathcal{M}$ and some formula $\phi(x, \bar{a})$. If $\mathcal{N} \models \exists x\phi(x, \bar{a})$, as $\mathcal{M} \preceq \mathcal{N}$, we have that $\mathcal{M} \models \exists x\phi(x, \bar{a})$. Thus if we pick $b \in \mathcal{M}$ such that $\mathcal{M} \models \phi(b, \bar{a})$, we have $\mathcal{N} \models \phi(b, \bar{a})$ by elementarity.
Now for the backward claim. The case is trivial for formulae with no logical connectives. We consider cases in which formulae contain only existential quantifiers as formulas are logically equivalent to formulas without universal quantifiers. Fix $\bar{a} \in \mathcal{M}$ and assume $\forall b \in \mathcal{M}$, we have that $\mathcal{M} \models \phi(b, \bar{a}) \iff \mathcal{N} \models \phi(b, \bar{a})$. We

want to show $\mathcal{M} \models \exists x \phi(x, \bar{a}) \iff \mathcal{M} \models \exists x \phi(x, \bar{a})$. The forward claim is immediate by our assumption; the backward claim Follows from the Tarski-vaught test. $\square$

Here $\gamma-$formula refers to some formula in the language of γ.

What I have been referring to as Löwenheim Skolem Theorem is in fact an amalgamation of two theorems that describe different cases. These are the **upward** and **downward** Löwenheim-Skolem Theorems [16]. The upwards case tells us that we can find elementary extensions of larger infinite cardinality than the original model, whereas the downwards case tells us that we can find elementary submodels of lower infinite cardinality than the original model. Only the downward version is integral to understanding Skolem's Paradox and as such I will omit the proof for the upward case, however I will include the theorem for completeness.

Theorem 9 (Upward Löwenheim-Skolem Theorem [19], p. 25)**.** *Let γ be a first order language, $\mathcal{M}$ an infinite $\gamma-$structure and κ a cardinal such that $|\gamma| \leq \kappa$ and $|\mathcal{M}| \leq \kappa$. Then $\mathcal{M}$ has an elementary extension of cardinality κ.*

Theorem 10 (Downward Lowenheim-Skolem Theorem [19])**.** *Let $\mathcal{M}$ be a $\gamma-$structure and let κ be any cardinal such that $|\gamma| \leq \kappa \leq |M|$. Then $\mathcal{M}$ necessarily has a elementary substructure $\mathcal{N}$ such that $|\mathcal{N}| = \kappa$. Further, if $X \subseteq \mathcal{M}$ and $|X| \leq \kappa$, then $X \subseteq \mathcal{N}$.*

Proof. Assume $|X| = \kappa$. We generate our substructure by recursively defining sets X_i, $i \in \mathbb{N}$, where $X = X_0 \subset X_1 \subseteq ...X_i \subseteq ...$ and for each $\gamma-$formula $\phi(x_0, ..., x_n)$, each $j \leq n$ and each $b_0, ..., b_n$ in X_i such that:

$$\mathcal{M} \models \exists x_j \phi(b_0, ..., b_n).$$

We have $v \in X_{i+1}$ such that:

$$\mathcal{M} \models (b_0, ...b_{j-1}, v, b_{j+1}, ..., b_n)$$

.

We know that $|\gamma| \leq \kappa$, thus there are at most κ γ-formulas as each formula is just a finite selection of symbols from γ, hence we need only add κ elements from $\mathcal{M}$ to each X_i. So, without out loss of generality, let $|X_i| = \kappa$. Let the universe of $\mathcal{N}$ be the union of our recursively defined X_i's, that is $N = \bigcup\{X_i | i \in \mathbb{N}\}$. We have then that $|N| = \kappa$. N is closed under functions from $\mathcal{M}$ and contains all its constants, thus we can define a substructure $\mathcal{N} \subseteq \mathcal{M}$. By the Tarski-Vaught test $\mathcal{N} \preceq \mathcal{M}$. $\square$

Downward Löwenheim-Skolem tells us that given a model of infinite cardinality we can always find an elementary submodel that is countable. Thus given a first-order theory with a model of infinite domain, we can find a model that interprets the theory that is surely countable.

4.3 Cantor's Theorem

Cantor was first to posit the idea of **uncountability**[22] and it is by now a universally accepted fact within maths there are different types of countability. Given a set we say that it is countable if its cardinal number is finite, of course if a set has a finite number of elements we can count them all, or if there exists a bijective function between its elements and the set of natural numbers $\mathbb{N}$. For instance the set of all even numbers is countable, as we can construct an enumerating function from the natural numbers to the even ones by simply multiplying each element of $\mathbb{N}$ by 2:

$$f : \mathbb{N} \to 2\mathbb{N},$$

$$f(x) = 2x, \forall x \in \mathbb{N}.$$

The rational numbers $\mathbb{Q}$ is also a notable infinite set that is countable.

Problems arrive, however, when we consider sets like the reals $\mathbb{R}$, or the set of irrational numbers, $\mathbb{R}\backslash\mathbb{Q}$. If we try to find some bijection between these sets and $\mathbb{N}$ we can not, hence they are uncountable. There are many proofs showing that some sets are uncountable but the first came from Cantor where In 1874 he proved that the set $\mathbb{R}$ was uncountable via what is known as Cantor's diagonal method [22]. I will outline now perhaps Cantor's most important theorem surrounding countability.

Given some set A it's power set, denoted $\mathcal{P}(A)$, is defined as the set of all subsets of A [23]. For instance, let $A = \{1, 2\}$, its power set is given by:

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

.

Clearly $|A| < |\mathcal{P}(A)|$. Cantor wondered if this was always the case and further more how it would apply to sets of infinite cardinality.

Theorem 11 (Cantor's Theorem [23], p. 1)**.** *Given a set A, there is no surjective function from A to its power set $\mathcal{P}(A)$. That is $|A| < |\mathcal{P}(A)|$.*

Proof. Suppose that there exists a surjective function $f : A \to \mathcal{P}(A)$. Consider the set:

$$S = \{a \in A : A \notin f(a)\} \subseteq A.$$

Clearly $S \in \mathcal{P}(A)$, as f surjective $\exists x \in A$ such that $f(x) = S$ hence $x \in S$ and $x \notin S$. This is a contradiction and thus no such surjection exists. □

In fact, given some set A there are $2^{|A|}$ elements in its power set. Take $A = \mathbb{N}$, its power set has $2^{|\mathbb{N}|}$ elements which by Cantor's Theorem we know is strictly greater than $|\mathbb{N}|$. So to Cantor's surprise he discovered that there are different sizes of infinity. Things like the natural numbers and integers are relatively small. The fact they can be put into one to one correspondence with $\mathbb{N}$ means they are **countably infinite**. Whereas sets such as the real numbers or the power set of $\mathbb{N}$, or indeed the power set of any infinite set, are **uncountably infinite**, as no such bijection exists. Cantor even came up with a way to order these different types of infinity through what is known as the **Aleph number** $\aleph$ [24, p. 30]. $\aleph_0$ is the cardinality of $\mathbb{N}$, the reals have cardinality $\aleph_1$, think of this as each element in the reals has $\aleph_0$ elements preceding it (when looking at 1, 1.1, 1.11, ... we can expand this an infinite amount of times). Similarly sets of size $\aleph_2$ have $\aleph_1$ elements preceding each element. Therefore we can use this notion of $\aleph$ to order different kinds of infinity:

$$\aleph_0 < \aleph_1 < ... < \aleph_n$$

for some non-negative integer n.

4.4 The Appearance of a Paradox

Skolem's result understandably came as quite a shock when paired with Cantor's Theorem. It is worth noting that, although regarded as the father of set theory, Cantor worked under a much more informal theory in which a set wasn't even defined, sometimes called **naive** set theory [25]. Regardless, the same result can be proved using any standard first order axiomatisation of set theory [16], which are precisely the kind of structures that the Löwenheim-Skolem Theorems are concerned with.

Let us work within the universe of Zermelo-Fraenkel set theory (ZFC). How can it be then that the same axioms which prove the existence of uncountable sets, have a model which is countable? Surely in order for a model to satisfy such a claim it must have an uncountable number of elements to 'give' the set. Indeed it is not difficult to find countable models of ZFC. Kurt Gödel developed his own model known as the constructable universe [26], where the sets are structured in a hierarchy which are built up from the empty set using the power set and union operators. Timothy Bays [27, p. 5] uses a method that suggests constructing endless arbitrary models via substitution. To outline the conflict specifically between downward Löwenheim-Skolem Theorem and Cantor's Theorem, we will introduce some notation. Let $\mathcal{N}$ be a model for ZFC. By downward Löwenheim-Skolem Theorem, ZFC necessarily has a countable elementary submodel $\mathcal{M}$. Within ZFC we can prove the statement:

$$x \textit{ is uncountable.}$$

Or more formally the formula:

$$\Phi(x) \equiv \exists x(\forall f(f : \mathbb{N} \to x \implies \text{“}f \text{ is not surjective”})).$$

As $\mathcal{M}$ models ZFC it must believe its statements are true:

$$\mathcal{M} \models x \textit{ is uncountable},$$

thus:

$$\mathcal{M} \models \Phi(x).$$

Now if we take x to be the power set of $\mathbb{N}$ this is precisely the fact that this set cannot be enumerated by the natural numbers i.e. $\mathcal{P}(\mathbf{N})$ is uncountable, yet how can $\mathcal{M}$ satisfy this if its domain is countable. To understand the solution to the paradox we must consider the different ways things are interpreted **within** and **without** the model. Within here meaning we consider our models domain, the objects it knows that exist. As well as this, when we are looking from within elements of the model are understood in the way in which the model interprets them. By without we mean an analysis of the objects of our model from outside it, hence our frame of reference may be more robust. In this context without means our domain is the entire set theoretic universe. Indeed when looking from without our model we understand each element in the way that we usually would, quantifiers like '$\in$' are interpreted as the real membership relation. However, from the perspective of our model $\mathcal{M}$, the membership relation is understood with respect to its own interpretation $\in_{\mathcal{M}}$ [16], that being what ever meaning the interpretation function prescribes it. In fact, there is no reason why the meanings of $\in$ and $\in_{\mathcal{M}}$ need align at all. Given a model we could specify that $\in_{\mathcal{M}}$ takes on the meaning:

$$m_1 \in m_2 \iff 0 \leq m_1 + m_2 \leq 20, m_1 \geq 8 \text{ and } 0 \leq m_2 \leq 15$$

which is not at all similar to the real membership relation. Indeed it could be that $\mathcal{M} \models m_1 \in m_2$ where m_1 and m_2 are not even sets [27, p. 9]!

This fact of the variability of the model theoretic interpretation of its quantifiers means there really is no reason to expect that the set theoretic and model theoretic interpretation of Φ are alike at all. Similarly, the sets upon which we range over when determining cardinality can be entirely different when viewed from within or without our model [16]. The number of objects that our model $\mathcal{M}$ sees is substantially smaller than from our set theoretic view. We have that $\mathcal{M}$ satisfies the existence of the real numbers. However, as our model is countable, there are necessarily $2^{\aleph_0}$ reals that our model does not see. Clearly our model interprets the set $\mathbb{R}$ wildly different to the genuinely uncountable set we know. Bays describes a similar example looking at submodels of Von Nuemann universe V [27, p. 5]. Let $\mathcal{S}$ be an elementary submodel of the Von Nuemann universe V_κ (κ is an inaccessible cardinal), we have that $V_\kappa \models ZFC$, hence $\mathcal{S} \models ZFC$. Despite $\mathcal{S}$ being countable it contains the uncountable element $(\aleph_1)^V$.

The set we are ranging over under the model theoretic interpretation is $\{x|\mathcal{M} \models x \in \mathbb{R}\}$, which is countable (as it is all the elements x within our model that are real), the problem is we are conflating it with the uncountable set $\{x|x \in \mathbb{R}\}$ [27]. We must then distinguish between our model theoretic and set theoretic interpretations. Formula Φ at surface level is understood in the usual sense, that is that its quantifiers take their typical interpretations and range over the set theoretic universe. However, from the perspective of our model we would have to replace each element with its model theoretic counterpart, so Φ becomes:

$$\Phi(x) \equiv \forall_{\mathcal{M}} N(\exists_{\mathcal{M}} x(\forall_{\mathcal{M}} f(f : N \to_{\mathcal{M}} x \implies \text{“}f \text{ is not surjective”}_{\mathcal{M}}).$$

Here the existential and universal quantifier and membership relation can have wildly different meanings to what we would expect. Hence, a rather vague solution to the paradox can be chalked up to an equivocation between these sets, and between the quantifiers $\in, \forall, \exists$, and $\in_{\mathcal{M}}, \forall_{\mathcal{M}}, \exists_{\mathcal{M}}$. From each perspective the statement that "*x is uncountable*", will very likely have different truth values from within and without the model.

This indeed is where the solution lies. However to avoid confusion with the syntactic alignment between statements in ZFC and their model theoretic interpretations, let us look at cases where they are understood in the same way.

4.4.1 Transitive Models

A **transitive model** of set theory is one in which each element of the model is equivalent with its set theoretic counterpart.

Definition 4.4.1 (Transitive model [27], p.17)**.** *A model $\mathcal{M}$ is transitive if:*
1) Every elements of $\mathcal{M}$ is a set.
2) Every member of an element is also an element of $\mathcal{M}$.

Meaning we have that $1_{\mathcal{M}} = 1$, $2_{\mathcal{M}} = 2$, ..., $\mathbb{R}_{\mathcal{M}} = \mathbb{R}$. Further $\in_{\mathcal{M}} = \in$. These rare occurrences make Skolem's result all the more puzzling. As previously mentioned, the explanation comes down to internal and external views.
If we now take $\mathcal{N}$ to be a countable transitive model of ZFC (we can do this by **Lowenheim-Skolem-Mostowski Theorem** [28, p. 9]), given n_1 and n_2 in N (the domain of $\mathcal{N}$) then $n_1 \in n_2$ if and only if $\mathcal{N} \models n_1 \in n_2$, thus membership in $\mathcal{N}$ really is the real membership relation. Since $\mathcal{N}$ models ZFC we can make the claim:

$$\mathcal{N} \models \exists a(\neg \exists b(b \text{ is an enumeration of } a)).$$

Thus there must be some object x in N such that:

$$\mathcal{N} \models \neg \exists b(b \text{ is an enumeration of } x).$$

However as $\mathcal{N}$ is a countable transitive model, all its elements are countable, thus there must be some enumeration of x. The problem here now is determining where the model theoretic view differs from the set theoretic view. Indeed there is such an enumerating function, call if f. As $\mathcal{N}$ is transitive, the above expressions are identical regardless of our perspective. If $\mathcal{N}$ is countable how then can it satisfy this uncountability statement? The problem here lies with when we add the existential quantifier $\exists$. The explanation of this fact is similar to above, within the model such a bijection does not exist. Our model can only range over elements of its domain and when we come across '$\exists$' and begin to look for such a function $\mathcal{N}$ does not recognise its existence. Whereas when we step back and look at the set theoretic universe we can find plenty of bijections like this. So the genuinely countable element x within our model appears uncountable to $\mathcal{N}$, so the model still satisfies the statement. To put it as Button [28], our model is a brain in a vat and is oblivious to things outwith it.

Thus, it is not surprising that the relatively tiny model $\mathcal{N}$ is unaware of functions that we may find in the set theoretic universe. If we consider $\mathcal{U}$ to be the set theoretic universe, we would say that $\mathcal{P}(\mathbb{N})$ ($\mathcal{P}(\mathbb{N})$ as understood by our model) is countable with respect to $\mathcal{U}$ but not $\mathcal{N}$ - this is because our quantifiers are restricted to each domain. Here $\mathcal{U}$ contains an enumeration which $\mathcal{N}$ does not. This explains the Skolem Paradox, and brings to light the relativity of the notion of countability.

4.5 Aftermath

Skolem himself never considered this fact a paradox [16], however he was concerned with the idea of using set theory as the foundation of mathematics. He believed that axiomatic set theory lead to the relativity of set theoretic concepts [29]. To argue his case, Skolem put forth his Löwenheim-Skolem Theorem and its consequent paradox to bolster his claim. He said about Löwenheim-Skolem Theorem [29, p. 130]:

> *"It's most important application is the critique of the set-theoretical concepts, and most especially that of higher infinite powers".*

We will concern ourselves for the rest of this chapter on the idea of **absolute** countability. By absolute we mean sets that are countable from an absolute point of view, meaning regardless of the model, an object is always countable. It would be natural then to consider the real numbers uncountability to be absolute. However, many proponents of Skolem, and himself, thought this idea was nonsense, and that the Löwenheim-Skolem theorems proved the relativity of countability. After all we have seen that the notion of countability can change relative to our model.

We will label believers of this view **Skolemite's**. Let us briefly look over Skolem's views and similar contemporary Skolemite's. In 1922, Skolem released a paper heavily criticising Zermelo's axiomatisation of set theory [29]. Skolem believed antinomies were still present within this conception and even claimed he had found up to eight errors. Skolem went as far as to prove Löwenheim-Skolem Theorem without any use of set theoretic tools as a means to display its redundancy [29]. In doing so Skolem showed that this relativity is inherent within axiomatic systems, regardless of the axiomatisation, as the applicability of Löwenheim-Skolem Theorem is so general we can guarantee a countable model. This relativism Skolem explained similarly to the solution of the paradox, that concepts are relative as quantifiers only range over the model rather than the (much larger) set theoretic universe; thus different truths correlate to different models. For Skolem, part of the problem lay between intuition and formalism, he claimed there was an intuitionistic aspect to axiomatic set theory and that the notion of uncountability was circular [29].

Many responded to Skolem's views saying that perhaps it is not fair to call specific models *models* of set theory if they do not understand set theoretic notions in the intended manner. They thought if it was possible to vary things, like the membership relation, so that accepted set theoretic truths are completely wrong then how could we accept these models as legitimate models of ZFC, they are **non-standard** [29]. Consider Myhills perspective [29, p. 144]:

"Indeed there is evidently only one standard model of set theory, because the predicate letter epsilon, the only one which appears, is already preassigned on interpretation."

Myhill here argues for some sort of canonical interpretation, one in which the membership relation is *preassigned*. He said that the intuitionist notion of membership is what makes set-theory itself. He argued that if we abandoned intuition, and stick to a purely formal understanding, although we may have the same underlying structure it's meaning may be *"grotesquely different"* [29, p. 144] to what we expect. This canonical interpretation is one where there exist no bijections between the natural numbers and its power set, where the reals are still absolutely uncountable. However this specification of some intended interpretation is not specified within the axioms of ZFC and thus who is really to say under which interpretation set theory is supposed to be understood? This was Skolem's argument.

We can then ask, what do modern Skolemites make of this result? Resnik [30] outlines what he calls the **strong Skolemite** perspective. He describes the strong Skolemite as holding the belief that no set theory is equipped to produce genuine uncountable sets, and that in fact all sets are countable from an absolute perspective [30]. This idea seems to rest upon the notion of a kind of hierarchy of sets. That is a progression of sets where despite a set containing an uncountable element relative to itself, if we jump further up the hierarchy we can find some enumerating function. This claim can be taken seriously due to the fact that many set theories can be shown to only exhibit countable elements when consistently extended [30].

The strong Skolemite may try to use this fact to extend some system containing $\mathbb{R}$, to a countable one, and thus claiming $\mathbb{R}$ is countable by assumption that the new system is countable. This 'proof', however, is severely lacking, and completely sidesteps the discovery of some enumerating function. Another argument is given by Hao Wang [30]. Wang offers a hierarchical system based on a theory of orders. Within this system we can order systems in terms of order. So systems of order 0 we will denote Σ_0. This idea of orders is explained by Resnik:

" $\Sigma_{\alpha+1}$ contains all the objects of order α plus those sets of objects of order α which are determined by predicates whose bound variables range over no objects of any order greater than α."

Each system contains uncountable objects relative to that given system of some arbitrary order α. However, all elements within this systems are countable in a system of order $\alpha + 2$. Here Wang manages to avoid absolute notions of countability as objects are either **countable**$_\alpha$ or **uncountable**$_\alpha$. The problem with Wang's system is how it tackles uncountable sets. Consider the real numbers, within Wang's system it cannot tell us whether this set is countable or uncountable as no Σ_α contains the reals, but rather subsets of the reals of order α. The strong Skolemite thesis, Resnik claims, is thus flimsy or illusive, lacking the rigour to substantiate their claims. Skolemite's are still yet to produce a bijection between $\mathbb{N}$ and some previously thought uncountable set.

Skolem's Paradox is a product of misunderstanding. Given our explanation indeed the issue is not very paradoxical at all. Despite this difference in schools of thought, the general consensus within the mathematical community is that Skolem's Paradox poses no threat to mathematics or its consistency. It's exclusivity to first-order axiomatisations mean the result is easily avoidable if one is not convinced of its passivity. The feared antinomy appears to be merely an *"unexpected feature of formal systems"*, to quote van Heijenoort [16], which has blossomed into an important discourse on the absoluteness of set theoretic concepts. We come across the same battle between intuitionism and formalism, can axiomatic systems really satisfy all our mathematical questions? Does it function as a completely sound basis for mathematics? As mathematicians we strive for consistency in our logic, however Skolem's Paradox demonstrates that axiomatisation may still leave something to be desired.

Chapter 5

Russell's Paradox and Type Theory

> Mathematics takes us still further from what is human, into the region of absolute necessity, to which not only the world, but every possible world, must conform.
>
> *Bertrand Russell*

5.1 Russell's Paradox and Similar Paradoxes

Russell's Paradox is one of the most famous paradoxes in set theory. It is a result of naive set theory that assumes an unrestricted **Comprehension Axiom**. The axiom simply states that we can form a set from any formula (property) and free variable [31], so we are able to form sets by grouping everything that has a certain property. In mathematical notation we have that:

$$\exists A \forall x (x \in A \equiv \phi(x)).$$

This leads to problems. Russell uses the following set to illustrate this [31]:

$$R = \{x : x \notin x\}.$$

The Russell set leads immediately to a contradiction - in that we see that if $R \in R$ then $R \notin R$ and if $R \notin R$ then $R \in R$.

There have been various attempts at avoiding the paradox, the most commonly used being Zermelo set theory that supposes the **Separation** or **Subset Axiom**. Russell himself came up with the theory of types in order to get around this and similar paradoxes. He talks about a few paradoxes that are avoided by adopting the theory of types in *Principia Mathematica*. He recognises that these paradoxes are characteristically self-referential and that is what gives rise to contradictions [32, p. 61].

To see this we can consider the 'liar paradox', talked about famously by Epimenides in ancient Greece. If someone says "I am lying", considering the truth value of that statement itself leads to contradictions. The self-reference arises when we assume the truth/falsehood being referenced in the sentence is of the same type as the truth/falsehood of the sentence itself. This is basically what the theory of types aims to do. According to Russell and Whitehead, the truth value itself is ambiguous so we need different orders of truths and falsehoods in order to not run into contradictions.

Apart from this classical paradox, he also mentions the 'Burali-Forti paradox' in mathematics [32, p. 60]. This paradox is a variation on the idea that the collection of all sets cannot be a set itself. The paradox concerns ordinals, which are well-ordered, transitive sets. The set of all ordinals up to and including an ordinal is itself an ordinal. Now, when we consider the set of all ordinals, Ord, we encounter a problem since it itself is a well-ordered, transitive set, and thus, an ordinal. However, then Ord is not the set of all ordinals since it does not contain itself. In set theory, as we have that the collection of all sets cannot be a set for similar reasons, here we have that the collection of all ordinals is not a set and thus, cannot itself be an ordinal. The self-reference here is apparent since the moment we talk about a set of all ordinals we are creating a new ordinal.

5.2 Type Theory

Russell came up with the theory of types first in his work Principles of Mathematics [33], but he worked further on it and published his ramified theory of types with Whitehead in *Principia Mathematica* [32]. In order to delve deeper into how the theory of types can resolve various paradoxes, we will need to understand the underlying concepts.

As mentioned above the theory aimed at making statements involving self-reference meaningless. This is referred to as the **vicious circle principle** in the account of the theory [32, p. 37]. The idea is that when something talks about "all" objects then that something cannot be a part of that "all", since otherwise we get caught up in a vicious circle. So, collections whose existence would mean they are part of themselves are illegitimate totalities, and any reference to such collections is a fallacy. Thus, collections such as the set of all ordinals can be shown to be meaningless.

To start understanding the theory of types, firstly, we are going to consider propositional functions. Propositional functions are just functions that take in a value to produce a proposition, both mathematical and non-mathematical. In order to make meaningful propositions there needs to be restriction on the values the variable can take. Letting x be the variable, ϕx is defined to be a single value the function can take out of all its values. While $\phi\hat{x}$ denotes all the values the function can take, i.e. it represents the function itself [32, p. 40]. First, observe that the value of the function exists separately to the function. The function does not need to be determined in order for us to be able to understand a proposition. However, in order to determine a function we need to assume the values that it can take such that if we are given an object we can know if it is a value of the function or not. Thus, the function cannot take values that can only be defined in terms of the function itself because of the vicious circle principle. If this was the case, then the values of the function would contain the function itself, but then this would form an illegitimate totality.

Another way of putting it is that if a function can take a value that is defined in terms of itself then the values of the function presuppose the function. However, the function itself presupposes its values. Then, $\phi(\phi\hat{x})$ becomes meaningless [32, p. 40]. We also want that two functions that take, say a as argument, cannot take as argument the other function. Consider the statement "x is mortal". We see that then '$\phi\hat{x}$ is mortal' does not denote anything definite because $\phi\hat{x}$ is ambiguous. Furthermore, '$(x).\phi x$ is a man' is meaningless because the argument is a proposition, but in order for the statement to say something substantial about the proposition, we need to be able to say something about the constituents of the proposition which we are not able to do in this case [32, p. 48] (here, $(x).\phi x$ means that ϕx in all cases or always). Though there are functions that can only take functions as arguments. $(x).\phi x$ is a function in $\phi\hat{x}$ and by the vicious circle principle, we might be skeptical about it. However, why we cannot get rid of it is because it denotes something definite since there are values that $\phi\hat{x}$ takes which it is dependent on. When talking in terms of "all" or "some" we can only talk in terms of a function.

Another idea that Russell considers in the book is the vagueness of the notions of truth and falsehood. We can see that there are different types of truths or falsehoods by considering **elementary judgments** and **general judgments**. An elementary judgement is about something definite and precise. Such judgments can be about relations between two or three objects or the quality of an object. While a general judgement would be about "all" or "some" objects. Such a judgement would look something like asserting a proposition ψx for all x given that ϕx also holds [32, p. 45]. Elementary statements then will have elementary truth and if ϕx is elementary then $(x).\phi x$ is something that would have second-truth or second-falsehood. Similarly, $(\exists x).\phi x$ would have second order truth value. So, saying $(x).\phi x$ has second-truth is the same as saying '$(x).(\phi x$ has elementary truth)' has second-order truth [32, p. 45]. The idea is that the sense in which ϕx is true is not the same as the sense in which $(x).\phi x$ is true. This is because ϕx denotes a single value is true while $(x).\phi x$ denotes a number of values and they all have to be true in order for it to be true.

Now, we have the tools to talk about the hierarchy of functions. To classify functions we can start with an **individual**, say a, and then classify first-order functions as those that can take a as an argument; and higher order functions which can take those functions as arguments. However, when we talk about all functions that can take a as an argument or "a-functions" we are in essence talking about an illegitimate totality [32, p. 49]. Consider a function in two variables, $f(\phi\hat{z}, x)$, that takes in a function and a variable. The following statement '$(\phi).f(\phi\hat{z}, x)$' is just a function in x. However, $(\phi).f(\phi\hat{z}, x)$ represents the totality of all possible functions $\phi\hat{z}$ to which x is an argument and that itself cannot be part of the totality. This function that can take x as an argument should be part of the set of all functions that can take x as an argument but it cannot, since it itself represents that set. So, $(\phi).f(\phi\hat{z}, x)$ cannot represent all functions that take x as an argument since that is an illegitimate totality and thus, meaningless. Having established this we can now form a hierarchy that avoids this problem.

To form the hierarchy meaningfully and concisely, we want to understand that certain propositions can be derived from others, and thus only thinking about proposition **matrices** that represent all propositions of a type, is enough. Specifically, propositions with **apparent variables** can be derived from simpler propositions. A proposition has an apparent variable when it talks about all or some of the values that the variable can take [32, p. 50]. So, for instance a sentence, let's say represented by $(x, y).\phi(x, y)$, has two apparent variables and no actual variables. This proposition was derived from $\phi(x, y)$ by first making x an apparent variable and then making y an apparent variable. Thus, we can obtain the original proposition from a proposition matrix that has only real variables and no apparent variables. Given this understanding of propositions and variables we can now consider the first type of functions.

First-order functions are those that only take individuals as an argument. That is, they can take as argument only objects that are not propositions or functions. The matrices that can be used to represent such functions and to derive more functions of this type are $\phi(x), \psi(x, y), ..$ and so on. We can form other first-order functions by making some variables apparent like $(x, y).\phi(x, y, z)$. The functions are either considering a finite set of individuals or a totality of individuals, which means there is no involvement of an illegitimate totality. We denote "any first-order function" by $\phi!\hat{x}$ and any value of that function by $\phi!x$ [32, p. 51].

Given these set of functions, in order to talk about them meaningfully we can introduce a new set of functions that take first-order functions and individuals as arguments, but no apparent variables. We call them second-order functions. Before introducing matrices for such functions which would be similar to what we have already seen, we should look at $\phi!x$. This is itself a function. But more interestingly, it is itself a second-order function since it is a function of two variables, namely $\phi!\hat{z}$ and x [32, p. 52]. As well as that there are no apparent variables since ϕ is itself variable. Similarly, $\phi!a$ is a function of only $\phi!\hat{z}$. This is because a is a definite individual and $\phi!a$ takes $\phi!\hat{z}$ as argument to give out values of first-order functions at a.

Now, we can immediately see matrices for second-order functions:
$f(\phi!\hat{z}), g(\phi!\hat{x}, \psi!\hat{z}), ..$ We can form more functions by converting some variables to apparent variables as we did in the case of first-order functions. Together they all form second-order functions. We can also extend our notation to this new order of functions. $f(\hat{\phi}!\hat{z})$ is a function with one variable which is a first order function while $f(\phi!\hat{z})$ is a value of such a function. An example we just saw of

such a function would be $\phi!a$ for some definite individual, a. $f(\hat{\phi}!\hat{z}, x)$ would be a function that takes a first-order function as an argument as well as an individual, the simplest example would be $\phi!x$ [32, p. 52]. Further notation for second-order functions with different combination of arguments would follow this pattern.

We have established a system that lets us classify functions in a way that does not lead to illegitimate totalities. As we have formed the first two classes of functions we can go on to form higher orders of functions in a similar way. So, the order of a particular function is going to be $n + 1$ when the highest order of its variables (arguments or apparent variables) is n. We can also define a predicative function of one variable. A predicative function is a function whose order is the least order it can take, given the order of the argument. So, if the order of the argument for a function is n and the function has order $n+1$, then it is a predicative function [32, p. 53]. Predicative functions of more variables can be defined in a similar way where the order of the function is inferred from the argument which has the highest order. Thus, predicative functions and apparent variables are enough to construct all functions.

We have thus, constructed a hierarchy of functions, motivated by a number of troubling inconsistencies in Mathematics and Logic due to a naive understanding of set theory. However, we are missing an axiom that Russell proposed in *Principia Mathematica*, which was an attempt to make certain concepts of Logic and Mathematics still accessible given the hierarchy we have just gone through in depth.

5.2.1 Axiom of Reducibility

The axiom of reducibility is a way of still being able to talk about, say for instance, "all a-functions". With the theory of types above, when we make a statement about such functions we are considering functions of only the n^{th} order, but never about all functions of all orders. The axiom is a way to be able to do that in a non-paradoxical sense, since intuitively we wish to talk about such totalities without running into inconsistencies. In similar notation to Russell and Whitehead, the axiom is stated as follows[32, p. 56]:

$$\vdash (\exists \psi)(\phi(x) \equiv_x \psi!x).$$

Here $\vdash$ prefixes the statement in order to clarify that it is an assertion; a statement that is complete and believed to be true within the realm of type theory (an axiom or theorem) [34]. The statement itself says that given any function of any order $\phi\hat{x}$, there exists a predicative function which is formally equivalent to the original function. Formal equivalence holds between two functions when if one of the functions is true with a certain argument, then the other is also true with that argument - and if one of them is false then the other is false as well. More informally, we have that given a collection of objects satisfied by a statement, there exists a predicate that only takes that collection of objects as arguments and no others [32, p. 56].

The motivation behind the axiom is illustrated in a few ways by Russell. A general example would be when we consider making a statement about someone that can be also made about another finite number of people. All these people are bound to share a predicate because each one of them has to have a predicate unique to herself. The authors, for instance, consider the exact moment of each of their births [32, p. 56]. Since no other person shares those, we can consider a disjunction (union) of all these unique predicates to arrive at a predicate that is only common to that set of people. Finally, that predicate would be equivalent to the original statement since it identifies only that group of people.

A more interesting property that the axiom of reducibility lets us define is **identity**. The fact that two objects are identical is not immediate from the theory of types when considered without the axiom. We would want to say that x and y are identical if $\forall\phi, \phi(x) \implies \phi(y)$ [32, p. 57]. But this is not possible to do since '$\forall\phi$' is not a valid totality that we can talk about. In order for the statement to have meaning we must restrict '$\forall\phi$' to just predicates, or second-order functions, or any functions of only one order. But if we want x and y to be identical, we would want more than the fact that all predicates of x belong to y. With the axiom of reducibility we are able to resolve this problem [32, p. 57]. Assume all the predicates of x belong to y, or both satisfy a single order of functions. Now, consider all the objects identical to x. The axiom says that they all have a predicate in common. x has this predicate since it is identical with itself. Then y has this predicate because of our assumption. Thus, y is identical with x.

The theory of types is supposed to be an account that does not assume the existence of classes. However, the axiom of reducibility is something that is a trivial consequence if we assume the existence of classes [32, p. 58]. This is because the arguments to a given function, say $\phi\hat{x}$, form a class. Then, being a member of the this class is a predicate that applies to all its arguments. That is, if α is the class of all arguments to $\phi\hat{x}$, then $x \in \alpha$ is a predicative function of x.

Russell and Whitehead clarified that they chose to incorporate the axiom instead of the existence of classes because it is a weaker assumption and it is preferred to make the weakest assumption required [32, p. 58]. Regardless, Russell considers the theory of classes in depth in his book in order to reconcile mathematical notions with type theory. He explains how classes could arise given the axiom of reducibility, even though he does not consider them to be **definable** objects. It is worth exploring some of what he says about classes in *Principia Mathematica* here because these concepts are necessary for understanding and resolving mathematical paradoxes, and of course, Russell's Paradox.

5.2.2 Theory of Classes

The idea of a class is much needed and comes about naturally in mathematics. For instance, we want a function to determine a class by grouping its possible arguments. However, when we want to take a more philosophical approach, it is not as natural to incorporate classes into our understanding. Russell tries to do so in order to solve both semantical and mathematical paradoxes through his theory.

The theory of classes that Russell tries to form given the Axiom of Reducibility and his hierarchy of functions is based on the treatment of **intensional** and **extensional** functions.

Extensional functions functions of functions, such that given any formally equivalent arguments to the functions their truth value does not change [32, p. 73]. Let $f(\phi\hat{x})$ be a function, then it is an extensional function of $\phi\hat{x}$ if it is equivalent to (has the same truth-value as) $f(\psi\hat{x})$, given that $\phi\hat{x}$ is formally equivalent to $\phi\hat{x}$. An intensional function is just defined as a function of a function that is not extensional. Russell gives various examples to illustrate what are intensional and extensional functions. For instance 'x is a man $\implies$ x is mortal' is an extensional function of 'x is a man', since we can substitute any formally equivalent sentence into the function and it will still have the same truth value. 'x is a featherless biped $\implies$ x is mortal' is equivalent to the original statement [32, p. 73].

However, when we consider someone's beliefs, for instance, we cannot say the same. 'A believes that 'x is a man' $\implies$ x is mortal', is an intensional function because 'A' does not have to believe all featherless bipeds are men [32, p. 73]. So, substituting that equivalent statement into the original function might just change its truth value.

Now that we have an intuitive understanding of such functions what we see is that extensional functions are the concern of mathematical logic while intensional functions are products of non-mathematical statements. The notion of classes becomes relevant when we want to consider the group of arguments satisfying an extensional function. Instead of considering a function to be the argument to the extensional function, it is more convenient to think of it as having for its argument the class determined by the argument function. Russell calls this idea of class an extension. Functions have the same extension when they are formally equivalent and the extension itself is the class determined by all formally equivalent functions that are arguments to an extensional function [32, p. 74]. We want to think of the argument to the extensional function as this class because it is representative of the commonality between all formally equivalent functions. Instead of considering each of those functions separately to give rise to multiple equivalent statements, we want to consider a single function that is informative. This is not something that we can do however, with intensional functions. The consideration of classes is immediate when talking about extensional functions, but in order to talk about classes in general we need a different strategy.

The idea is to try and write all functions as extensional functions and eliminate the need to differentiate between extensional and intensional functions. This can be done by deriving an extensional function from any function of a predicative function. The derived function needs to have certain properties. It needs to be able to take as argument any function $\phi\hat{z}$ that takes the same type of arguments as $\psi!x$, where $\psi!z$ is the predicative function, which is the argument to the original function [32, p. 74]. Furthermore, if the original function of a predicative function, $f(\psi!z)$, is extensional then it needs to be equivalent to the derived extensional function [32, p. 74]. Now, we can define the derived function.

The derived function for $f(\psi!z)$ is "there is a predicative function which is formally equivalent to $\phi\hat{z}$ and satisfies f" [32, p. 74]. We want to check that this definition of the derived function satisfies the conditions that we stipulated above. Let's say $\phi\hat{z}$ is a predicative function which is the argument to f and the derived function, which is written as $f\{\hat{z}(\phi z)\}$. When $f(\phi\hat{z})$ is extensional, the derived function is only true when $\phi\hat{z}$ is true [32, p. 75]. So, if f is extensional then it is equivalent to its derived function, given they both take predicative functions as arguments. On the other hand, if $f(\phi\hat{z})$ is an intensional function, then the derived function is true when $f(\phi\hat{z})$ is true, but it might also be true when $f(\phi\hat{z})$ is not true [32, p. 75]. This is because if $f(\phi\hat{z})$ is false, then there can still be a predicative function formally equivalent to $\phi\hat{z}$ that satisfies f, since f is intensional. However, we just want the derived function to be extensional.

The derived function is able to take any type of function, $\phi\hat{x}$ as its argument as long as f can take any predicative function $\psi!z$ as its argument [32, p. 75]. This is immediate from the hypothesis that $\psi!z$ and $\phi\hat{x}$ are formally equivalent and that $\psi!z$ and $\phi\hat{x}$ can have different orders. This is because formal equivalence does not require them to be of the same order as long as they take arguments of the same type. Russell calls this a **systematic ambiguity** that occurs due to the use of negation, disjunction, the universal quantifier, and the existential quantifier in constructing functions. This also leads to an ability to interpret f with different types of arguments [32, p. 75]. When for instance $\phi\hat{z}$ and $\psi!z$ are of different orders but formally equivalent, we see that $f(\phi\hat{z})$ still makes sense. Given that $\phi\hat{z}$ is formally equivalent to $\psi!z$, if $f(\phi\hat{z})$ and $f(\psi!z)$ are also equivalent in such a case, we have that $f\{\hat{z}(\phi z)\}$ is equivalent to $f(\phi\hat{z})$. Of course, the premise of finding a $\psi!z$ that is formally equivalent to $\phi\hat{z}$ is redundant, given the Axiom of Reducibility. We are thus able to see the equivalence of any extensional function and its derived function, since their truth values align given any argument.

We have thus, arrived at a point where we can define classes which do what we want them to do. The class is represented by $\hat{z}(\phi z)$ and it contains the arguments that satisfy $\phi\hat{z}$. Since this is the class that we want for our extensional function to take as argument instead of $\phi\hat{z}$ itself, we write our derived extensional function as $f\{\hat{z}(\phi z)\}$. We defined this function above in words, but we can think of it in a type theoretic notation as follows for the rest of this section (this is not the exact notation used by the authors since we wanted to make it easier to read);

$$f\{\hat{z}(\phi z)\} = (\exists\psi)((\phi x \equiv_x \psi!x) \text{ and } f\{\psi!\hat{z}\}).$$

This says exactly what we said above, that we can find a predicative function $\psi!\hat{z}$ equivalent to $\phi\hat{x}$ such that f is satisfied by the predicative function. Given this definition and the Axiom of Reducibility we can check that this definition does indeed satisfy the usual properties of classes that we need and for which, we have done all this work. We firstly need that all propositional functions determine a class. Secondly, two functions determine the same classes if and only if they are formally equivalent. Third of all, we want the existence of classes of classes. And lastly, the most important property required to avoid our paradoxes, we want that a class cannot be a member of itself [32, p. 76].

Each propositional function determines a class because of the Axiom of Reducibility. For any $\phi\hat{x}$, we know there is a formally equivalent predicative function, $\psi!\hat{z}$. This predicative function satisfies some function f and it follows that $\phi\hat{x}$ also satisfies that function. Thus, we can always form our derived function $f\{\hat{z}(\phi z)\}$, which is a proposition about our class.

Say we have two equal classes, $\hat{z}(\phi z)$and$\hat{z}(\psi z)$, then they both satisfy some function f. By the definition of $f\{\hat{z}(\phi z)\}$, we have that $\phi x \equiv_x \chi!x$ for some predicative function χ and $\psi x \equiv_x \theta!x$ for some predicative function θ, where both predicative functions are equal to the classes. This is because if one of the classes satisfies some function f, then one of the predicative functions has to satisfy f. Thus, $\chi!\hat{z} = \theta!\hat{z}$. Which is the same thing as saying $\phi x \equiv_x \psi x$, since both are equivalent to the same predicative function. So, saying the two classes are equal is equivalent to saying that $\phi x \equiv_x \psi x$.

The last two requirements necessitate the notion of membership that we have not discussed in the context of type theory. We want '$x \in \hat{z}(\phi z)$' to have meaning. Since '$x \in \hat{z}(\phi z)$' or 'x is a member of the class $\hat{z}(\phi z)$' is a function in $\hat{z}(\phi z)$, it is a derived function and consequently needs to be derived from the definition of $f\psi!\hat{z} = x \in \psi!\hat{z}$.

We define it as $x \in \psi!\hat{z} = \psi!x$ [32, p. 78]. Now, we get $x \in \hat{z}(\phi z)$ from the definition of the derived function. '$x \in \hat{z}(\phi z)$' implies that '$x \in \psi!\hat{z}$' as per the definition of a derived function, which in turn implies $\psi!x$ from our definition of membership, and finally that implies ϕx since it is formally equivalent to $\psi!x$ [32, p. 78]. Alternatively, we also have that for all $\phi\hat{x}$ we have a predicative function, $\psi!\hat{z}$ formally equivalent to it. $\phi x \implies \psi!x$ which in turn implies that $x \in \psi!\hat{z}$ and $x \in \hat{z}(\phi z)$ [32, p. 79]. So, x is a member of $\hat{z}(\phi z)$ if and only if it satisfies ϕ:

$$\vdash (x \in \hat{z}(\phi z) \equiv \phi x).$$

A class of classes is defined as one would expect. So, given a function of a function F, the class of classes would be all the values $\hat{z}(\phi z)$ which satisfy f, where f is the argument to F. Stating $\hat{z}(\phi z)$ as α, this class of classes can then be written as $\hat{\alpha}(f\alpha)$ and the corresponding derived function is $F\{\hat{\alpha}(f\alpha)\}$. The usual definitions and properties follow and therefore the existence of classes of classes is easily available.

The fact that a class cannot have itself as its member is immediate from the definition of membership. $\hat{z}(\phi z) \in \hat{z}(\phi z)$ would mean that there is a predicative function $\psi!\hat{z}$ equivalent to $\phi\hat{z}$ such that $\psi!(\psi!\hat{z})$, since $\psi!\hat{z} \in \psi!\hat{z} = \psi!(\psi!\hat{z})$. This is of course meaningless from our discussion of types of arguments that a function can take. This also leads us to discard the notion of the class of all classes. The existence of the class of all classes would lead to self-membership. So, the collection of all classes cannot itself be a class and must be a different type of object, which is of course an idea we require in mathematics. All the other consequences required mentioned above follow from definitions discussed and especially from the Axiom of Reducibility.

Having defined classes given the Axiom of Reducibility, we want to end with how these ideas can let us talk about some totalities that we might want, but have until now been unable to due to them being illegitimate. a-functions define a-classes, that is, all functions that take 'a' as an argument define the classes that have 'a' as a member. Is the collection of all a-classes a legitimate totality? Yes it is, since by the Axiom of Reducibility each a-function is equivalent to a predicative a-function. Thus, all a-classes are just defined by all predicative a-functions, which is of course a legitimate totality since we are only considering a collection of a single order of functions [32, p. 76]. So, the vicious-circle principle is restricted to a great extent by the Axiom of Reducibility and our definition of classes. Since these are concepts that we want to talk about, especially in mathematics, it seemed to Russell that extending type theory in this direction was justified.

5.2.3 Resolving Russell's Paradox

We have discussed the theory of types in detail as well as the theory of classes as explained in *Principia Mathematica*. Thus, we have the tools to resolve Russell's Paradox, the very thing that led to Russell developing this theory. The Russell set (we shall switch from type theoretic language to set theoretic language, and refer to classes as sets) is formed of sets that are not members of themselves. However as we have just seen, a set cannot be a member of itself since it leads to us having to consider a function that takes itself as argument. So, given this we also should not be able to talk about a set of sets that do not contain themselves, because such a phrase would be meaningless.

How was this paradox resolved by the axiomatic set theory that has been the foundation of mathematics for years? Zermello-Frankael set theory with Choice or ZFC relies on certain axioms that restrict our ability to form sets. To be able to avoid Russell's Paradox we do not require all the axioms. Actually, we can just resolve it from the axioms of Zermello set theory. We can also solve the problem of the set of all sets as a consequence, as we did with type theory. We need the Axiom of Extensionality and the Subset Axiom (or the Separation Axiom). Extensionality basically defines equality of sets. It says

$$X = Y \Leftrightarrow (\forall x \in X \Leftrightarrow x \in Y).$$

The Subset Axiom states that

$$\forall X \text{ and } \phi, \exists Z, \text{ such that } x \in Z \implies (x \in X \text{ and } \phi(x)).$$

Firstly, the Russell Set if defined as the set of all sets which are not members of themselves leads to the obvious contradiction, that $R \in R \Leftrightarrow R \notin R$. This set's existence is immediately refuted by the Extensionality Axiom. As well as that, from the Subset Axiom, we know that we cannot form a set such as the following $\{x : x \notin x\}$. So, the revised Russell Set can now be written as $S = \{x \in X : x \notin x\}$ where X is just another set that exists from our axioms and we also have that $\phi(x) = x \notin x$. Now, we do not run into any problems since there is no way

S is a member of itself since we are only considering members of S which do not contain themselves.

However, if we consider X to be the set of all sets V, then we again run into the same problem of the Russell Set [35, p. 4]. This is why we cannot call the collection of all sets a set. This conclusion was reached in a similar manner in type theory where we first observed that our definition of sets cannot allow a set to be a member of itself. Then we moved on to say that the class of all classes would imply self-membership, which of course is meaningless in our theory. Nevertheless, Russell wanted to give a theory that did not assume sets existed. From the beginning, we could not discard the Russell set without going into the hierarchy of functions or the Axiom of Reducibility. The definition of sets is more complicated since their existence is not assumed, though the way they are handled with respect to the paradox is very similar.

Type theory is just one way of resolving the paradox. Our current system of axiomatic set theory is another. The wider use of one is merely a result of historical factors and should not be considered a reflection of the other's lack of advantages. Russell's ramified theory of types that we have considered in this paper was developed and modified by a number of mathematicians in the twentieth century. These type theories just as various set theories have their advantages and disadvantages. Type theories have been particularly useful to the development of computer science. The **Simple Theory of Types** introduced by Alonzo Church, considers two objects, terms and types. Given these objects and some typing rules we can avoid Russell's Paradox. This Simple Type Theory can be easily used to define simple data types like disjoint unions and Cartesian products, and thus, has been widely used when designing programming [36].

When we recognise the advantages of all these theories we start recognising that paradoxes are useful tools. If their existence hinders us from conceptualising mathematics meaningfully, we can find ways to resolve them. In resolving them we are bound to explore a number of ways to do so and this exploration can be pivotal to developing our fields. Russell's Paradox led to a lot of controversy, but what mattered in the end was the process of resolving it. It led to formulations and a greater understanding of set theory, type theory, model theory and even category theory. This acceleration in progress given the paradox is definitely something we as mathematicians are able to appreciate.

Chapter 6

Further Discussion

> Either mathematics is too big for the human mind or the human mind is more than a machine.
>
> *Kurt Gödel*

6.1 Are They Actually Useful?

Paradoxes can sometimes be deemed to be thought experiments from which we can derive no tangible benefit, but to mathematicians as a whole they are of great use. They can be used as a vessel to describe and explain otherwise unintuitive concepts, perhaps most successfully in the case of David Hilbert's Grand Hotel Paradox on infinite sets. It is a great demonstrations of a theoretically complicated mathematical idea that can really enlighten one's understanding of the topic.

You could not criticise someone for thinking that infinity is just a number like any other and extrapolating this to make statements about it like $\infty/\infty = 1$, $\infty + \infty = 2\infty$, or that $1 + \infty > \infty$. Of course we know these statements are all incorrect. Similarly, many could reasonably conclude that there are more natural numbers than even numbers, and more multiples of 2 than multiples of 4, for which they would be incorrect about again.

Hilbert's Hotel offers a way in which you can understand the idea of the cardinality of the natural numbers intuitively, which resonates with mathematicians and non-mathematicians alike [37, p. 73] Assume there exists a hotel with a (countably) infinite number of rooms, and each room is occupied. Now suppose that someone enters the lobby of the hotel and asks for a room. Not wanting to turn the man away, the hotel manager constructs a plan so that he can accommodate the new customer. He asks every guest to move to the next room, so the guest in room 1 moves to room 2, room 2 to room 3 and so forth so the guest in room n moves to room $n + 1$. Now there is space for the new customer to enter the newly vacated room 1. Despite the infinite number of complaints that will now arrive on the desk of the manager, he has resolved the issue.

Should a bus full of forty new guests arrive, he follows a similar procedure by asking the guest in room n to move to room $n + 40$.

Next, an infinitely long bus arrives in the seemingly infinitely long car park, containing an infinite number of people wanting a room. The manager must now fit all of these new customers in, but realises that he cannot just ask the current occupants of the room to move along a constant number of rooms. Instead he asks them to move to the room number that is exactly double of the one they were currently in. In this case the current occupants would now occupy all the even numbered rooms and the new arrivals could take all the odd numbered rooms. This is a nice way to show that '∞', '$\infty + c$' and '2∞' all have the same 'size', or cardinality, namely of $\aleph_0$, due to the fact we can construct a bijective function between each of the sets, a concept discussed in chapter 4. The point here being that, despite the absurd premise and apparent disconnection from reality, this paradox is an engaging way to illustrate mathematical concepts, which have very real implications, without the dense jargon.

6.2 Paradoxical Thinking

A Harvard University researcher, Albert Rothenburg was one of the first researchers to investigate the general use of paradoxical thought in his work 'The Janusian Process in Scientific Creativity'. Janusian process is defined by the author to be the action of "actively conceiving and using multiple opposites or antitheses simultaneously" [38, p. 1] and continues to describe that:

> "a basis for the creative function of the process is the importance of paradox in scientific inquiry. Concepts are posited as simultaneously true and not-true, and contradictions are conceived as operating side by side."

Rothenburg closely followed the careers of many famous scientists and interviewed twenty-two Nobel laureates from areas of chemistry, physics and medicine and noted that a characteristic. He noted a characteristic that was common amongst them in their successful research careers was the ability to consider multiple conditions that contradicted each other and derive results [38]. The paradoxes themselves are not the proof or general idea used to show their discoveries to the world. They however, can be the defining start point where the researchers actually made a noteworthy observation from which they could explore things further. Consider the example of Einstein's twin paradox [39] which was instrumental in the development of his Theory of Special Relativity. So instrumental in fact that the thought experiment was even referenced in his paper.

Indeed the concept of quantum superposition is itself contradictory and a paradox, as within quantum mechanics a particle may be in more than one state *à la* Schrödinger's cat. The benefit of paradoxical thinking extends to all fields of research and just having an open and curious mind has enabled so many of these famous discoveries that are useful to us today. The paradoxical ideas of quantum superposition are integral to the quantum computers that we use today.

In fact, one could argue that it is paradoxical itself for there to be unprovable statements in mathematics. This was proved via the revered **Gödel's Incompleteness Theorems**, which was a shattering blow to the mathematical community. For decades from the late 1800s, factions of mathematicians argued over the very soul of maths. A group known as the **formalists**, headed by David Hilbert, believed that a rigorous system based on Cantor's set theory could mathematically prove, and resolve any issues in the field [40]. However the paradox explored in chapter 5, Russell's Paradox, left questions unanswered.

Hilbert wished to answer three main questions, which are printed below along with alternative phrasing:

1. Is mathematics complete? (Can every true statement be proven?)
2. Is mathematics consistent? (Is mathematics free of contradictions?)
3. Is mathematics decidable? (Is there a system which can determine if a statement is derived from the axioms?)

Hilbert believed the answer to all three questions to be yes. During a speech he gave in the 1930s in Königsberg, in response to apathetic feelings around the answer to these questions, he proudly opined [41, p. 154]:

> *"Instead of this silly ignorance, on the contrary let our fate be: We must know, we will know."*

This was later printed as the epitaph on his grave in Götingen.

Soon after, in a crushing blow to the formalists and Hilbert personally, Gödel demonstrated that irrespective of which axiomatic system you operate in, there will always be true statements which cannot be proved [42].

Many view mathematics to be the subject of rigorous proof. Students spend years of their university lives being professionally trained in the art of proving statements but unfortunately we cannot construct a mathematical system in which all statements are provable. To many contemporary mathematicians this was understandably an alarming realisation, there truly was something romantic and hopeful about the prospect that we could one day achieve this. It is even possible some mathematicians found that their life's work towards a problem may have been in vain. Indeed part of the allure of mathematics is its impartiality, the exactness of provably right and wrong statements, but now there was this chasm in the middle of undecidability or impossibility.

A whole project could have been written on Hilbert and Gödel and the battle for the foundation of mathematics, or indeed just the Incompleteness Theorems, but they do link nicely to the topic of paradoxes. The proof used ideas similar to paradoxes like the liar paradox, briefly mentioned at the beginning of chapter 5. Through a process known as **Gödel numbering**, Gödel was able to give a number to symbols used in mathematics, beginning with the first twelve shown below:

Constant sign	Gödel number	Usual Meaning
~	1	not
∨	2	or
⊃	3	if...then...
∃	4	there is an...
=	5	equals
0	6	zero
s	7	the successor of
(	8	punctuation mark
)	9	punctuation mark
,	10	punctuation mark
+	11	plus
×	12	times

Figure 6.1: First twelve Gödel numbers [43]

Using this, he could now construct desired axioms or formulas as he wished. For example, '$0 = 0$', with 0 represented by the Gödel number 6 and the '$=$' by 5 [43]. Then in the order that they appear in the formula, they are used as the exponents of the primes, which are put in ascending order and we take the product. Consequently $0 = 0$ becomes $2^6 \times 3^5 \times 5^6 = 243\,000\,000$, and so clearly when these statements become more complicated the values that represents them are going to be enormous in magnitude [43]. Hence from the **Fundamental Theorem of Arithmetic** we can deduce that each of these formulas that we make will be unique. Then as a proof is essentially just a construction of these statements combined together. He constructed a proof in a similar way, say of the first step of the proof was the statement '$0 = 0$', then the sequence for the proof would be $2^{243\,000\,000} \times \ldots$ and so on. Due to the size of these numbers, Gödel began labelling them with letters, like 'a'.

The details are beyond the scope of this paper. However, the idea was to construct a sentence that would be labelled 'g' that said 'there is no proof for the statement with Gödel number, g'. Now consider that if the statement is false and there indeed is a proof, then you have just proven that there is no proof. Alternatively, the statement could be true at which point again you have shown that the mathematical system has unprovably true statements in it [44]. In **Gödel's Second Incompleteness Theorem**, he continued to show that no consistent axiomatic system can prove its own consistency.

This was all made possible due to the nature of self-referential paradoxes and is another example of how knowledge of paradoxes and how they work led to yet another brilliant discovery which fundamentally changed mathematics from that point onwards.

Mathematician Alan Turing was the man to answer the third of Hilbert's questions, again showing that his intuition had been incorrect [45]. I will not delve in to details of how he proved this but it mimics ideas similar to that of Gödel utilising paradoxical thinking. Although the answer to each of Hilbert's three questions was no, the ideas developed in the process of answering these questions advanced our development and understanding of the world and our mathematical systems. Indeed, Turing's discovery was pivotal to the first development of computers which has led to the rapid technological development of the last century [46].

It would appear then that despite paradoxes within mathematics giving us apparently unintuitive results, they in fact can act as a catalyst for development in our way of thinking. Old mathematical paradigms are challenged in the face of a paradox and massive upheaval of accepted truths may be the result of the solution. The ability to manipulate ideas and think in a way that imitates paradoxical ideas has been a skill that has improved our arsenal when tackling problems as mathematicians.

Chapter 7

Conclusion

> In mathematics, the art of proposing a question must be held of higher value than solving it.
>
> *Georg Cantor*

Each of the paradoxes which we have discussed have exposed a disconnection between our intuition and the results of our foundations of mathematics. Russell's Paradox was a result of naive intuition and its resolution through our axiomatisation of set theory gave us the tools to do mathematics meaningfully. Consider the Axiom of Choice. The Axiom of Choice, even though a source of contention among mathematicians, can be used to prove a number of results in various areas of mathematics. These results are not just useful in mathematics but even in fields like physics. One of the first results we talk about in linear algebra is that every vector space has a basis. This result is actually equivalent to the Axiom of Choice. The Axiom of Choice is required to show that infinite dimensional spaces have a basis[35]. Consequences of the axiom, such as this, seem like reasons to accept it.

Even within model theory we have statements that are equivalent to the axiom and used widely. For instance the **Löwenheim-Skolem-Tarski Theorem**. It says that a first-order sentence having a model of infinite cardinality κ also has a model of any infinite cardinality μ such that $\mu \leq \kappa$ [47]. This was proved equivalent to Axiom of Choice by Tarski.

These consequences are countered by the Banach-Tarski paradox since our immediate reaction is to reject such a result. But peculiarly, this paradox's existence is something most mathematicians are willing to accept in order to be able to use the Axiom of Choice in areas where it gives us especially useful consequences. So even though we want mathematical concepts to align with our intuitions we are never able to completely achieve this since in order to do so, we have to restrict ourselves to a great extent. This might be something that hinders our ability to progress in the same manner that we have been doing until now. Perhaps we do not require our intuitions to inform mathematics at all times.

Given our discussion of Skolem's Paradox we know there are various set theories we can work under. There is no one way of actually talking about mathematical concepts. We can understand and use them relative to the system we are working under. Maybe mathematics would be better used if we can explore its many avenues with an open mind, because as discussed, paradoxes can be a great source of information. Skolem's Paradox might look contradictory at a glance but it is a widely accepted result in model theory; not because it is intuitive but because it relies on a reasoning we cannot counter substantially. Thus, our work here is more a reflection of the relevance of paradoxes in mathematics rather than of them being holes in its foundations.

Mathematical paradoxes force us to reconsider how we view the field. Although not explicit, there is often a tacit intuition attached to mathematical concepts, and for good reason. Intuition is helpful at solidifying complex mathematical ideas and linking them to our real world experiences. It bridges the gap from abstract algebraic nomenclature to what the maths 'tells' us. However we must be careful not to forget the structures that we are considering, despite being integral to our understanding of the world these concepts exist outwith reality, rooted solely in logic. If we impart our own experience into our mathematical understanding perhaps we are taking a biased interpretation and as we have seen, absolving ourselves of an expected outcome can often lead to a deeper understanding.

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