# Propositional Logic

CSE2315, Chapter 1-2

# Class Review

- Well Formed Formula (wff)
- Order of Precedence
- How to evaluate truth value in the Truth Table
- Tautology & Contradiction
- Tautological equivalence
  - Commutative/Associative/etc $(A \land B \Leftrightarrow B \land A)$ / $(A \land B) \land C \Leftrightarrow A \land (B \land C)$
- De Morgan's Laws - $(A \lor B)' \Leftrightarrow A' \land B'$
- Pseudocode / Algorithm
- Tautology Test Algorithm

# Propositional Logic

- Statements are sometimes called *propositions*
- The wffs also called *propositional wffs*, because the wffs represent statements
- In this section, we learn tools how to derive conclusions from formal logic based on given statements
- The formal system that uses propositional wffs:
  - Propositional logic
  - Statement logic
  - Or, propositional calculus

# Propositional Logic

- Definition of Argument:
  - An argument is a sequence of statements in which the conjunction of the initial statements (called the premises/hypotheses) is said to imply the final statement (called the conclusion).
- An argument can be presented in symbolic form as

$$(P_1 \land P_2 \land \ldots \land P_n) \rightarrow Q$$

  - where $P_1$, $P_2$, ..., $P_n$ are given statements, called hypotheses
  - and Q is the conclusion.

# Valid Argument

- What is a valid argument?
  - When can Q be *logically deduced from* $P_1, P_2, \ldots, P_n$ ?
  - When is Q a *logical conclusion from* $P_1, P_2, \ldots, P_n$ ?
  - When does $P_1, P_2, \ldots, P_n$ *logically imply* Q ?
  - When does Q *follow logically from* $P_1, P_2, \ldots, P_n$ ?
- Informal answer: Whenever the truth of hypotheses leads to the conclusion
- We need to focus on the relationship of the conclusion to the hypotheses and not just any knowledge we might have about the conclusion Q.

# Valid Argument

- Example 1 (valid argument):
- Two hypotheses:
  - $P_1$: If George Washington was the first president of the United States, then John Adams was the first vice president.
  - $P_2$ : George Washington was the first president of the United States.
- Conclusion:
  - Q: John Adams was the first vice president.
- Symbolic representation
  - $(A \rightarrow B) \land A \rightarrow B$

    P1 P2 Q

| A | B | A→B | (A→B) Λ A | (A→B) Λ A→B |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | T |

# Valid Argument

- Example 2 (not a valid argument):
  - $P_1$: George Washington was the first president of the United States.
  - $P_2$ : Thomas Jefferson wrote the Declaration of Independence.

    And the conclusion
  - Q: Every day has 24 hours.
  - This wff $P_1 \wedge P_2 \rightarrow Q$ is not "intrinsically true".
- Definition of valid argument:
  - The propositional wff $P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow Q$ is a valid argument when it is a tautology
- How to arrive at a valid argument?
  - Using a proof sequence

# Proof Sequence

- Definition of Proof Sequence
  - A proof sequence is a sequence of wffs in which each wff is either a hypothesis or the result of applying one of the formal system’s derivation rules to earlier wffs in the sequence.

# Derivation Rules

- Derivation Rules:
  - To test whether $P_1 \land P_2 \land \ldots \land P_n \rightarrow Q$ is tautology
  - We can use truth table or the algorithm, TautologyTest
  - Instead, we can use derivation rules which manipulates wffs in a truth preserving manner
  - Equivalence Rules & Inference Rules
- Equivalence Rules
  - Allows individual wffs to be rewritten
  - Truth preserving rules
- Inference Rules
  - Allows new wffs to be derived
  - Work only in one direction

# Equivalence Rules

- These rules state that certain pairs of wffs are equivalent, hence one can be substituted for the other with no change to truth values.
- The set of equivalence rules are summarized here:
  - Let P, Q, and R be wffs

| Expression | Equivalent to | Abbreviation for rule |
|---|---|---|
| $P \vee Q$<br>$P \wedge Q$ | $Q \vee P$<br>$Q \wedge P$ | Commutative - comm |
| $(P \vee Q) \vee R$<br>$(P \wedge Q) \wedge R$ | $P \vee (Q \vee R)$<br>$P \wedge (Q \wedge R)$ | Associative-<br>ass |
| $(P \vee Q)'$<br>$(P \wedge Q)'$ | $P' \wedge Q'$<br>$P' \vee Q'$ | De-Morgan's Laws<br>De-Morgan |
| $P \rightarrow Q$ | $P' \vee Q$ | implication - imp |
| $P$ | $(P')'$ | Double Negation- dn |
| $P \leftrightarrow Q$ | $(P \rightarrow Q) \wedge (Q \rightarrow P)$ | Equivalence - equ |

# Inference Rules

- Inference rules allow us to add to the proof sequence a new wff that matches the last part of the rule pattern, if one or more wffs that match the first part of the rule already exist in the proof sequence.

| From (first part) | Can Derive (last part) | Abbreviation for rule |
|---|---|---|
| $P, P \to Q$ | $Q$ | Modus Ponens- mp |
| $P \to Q, Q'$ | $P'$ | Modus Tollens- mt |
| $P, \ Q$ | $P \land Q$ | Conjunction-con |
| $P \land Q$ | $P, Q$ | Simplification- sim |
| $P$ | $P \lor Q$ | Addition- add |

- Note: Inference rules ***do not*** work in both directions, unlike equivalence rules.

# Examples of the rules

- Example for using equivalence rule in a proof sequence:
  - Simplify $(A' \vee B') \vee C$

| | |
|---|---|
| 1. $(A' \vee B') \vee C$ | |
| 2. $(A \wedge B)' \vee C$ | 1, De Morgan |
| 3. $(A \wedge B) \rightarrow C$ | 2, imp |

- Example of using inference rule
  - If it is bright and sunny today (P), then I will wear my sunglasses. (Q) $(P \rightarrow Q)$

<u>Modus Ponens</u>

It is bright and sunny today. (P) Therefore, I will wear my sunglasses. (Q)

<u>Modus Tollens</u>

I will not wear my sunglasses. (Q') Therefore, it is not (bright and sunny) today. → Therefore, it is not bright or not sunny today. (P')

# Examples

- Ex. 12) Suppose that $A \to (B \wedge C)$ and A are two hypotheses of an argument. The following is a proof sequence:

  1. $A \to (B \wedge C)$ hyp
  2. $A$ hyp
  3. $B \wedge C$ 1,2, mp

- Ex. 13) Suppose that $(A \to B) \vee C$ and A are two hypotheses of an argument. The following is a proof sequence:

  1. $(A \to B) \vee C$ hyp
  2. $A$ hyp

# Deduction Method

- To prove an argument of the form

$$P_1 \land P_2 \land \ldots \land P_n \to R \to Q$$

- Deduction method allows for the use of R as an additional hypothesis and thus prove

$$P_1 \land P_2 \land \ldots \land P_n \land R \to Q$$

- Prove $(A \to B) \land (B \to C) \to (A \to C)$

Using deduction method, prove $(A \to B) \land (B \to C) \land A \to C$

1. $A \to B$ hyp
2. $B \to C$ hyp
3. $A$ hyp
4. $B$ 1,3 mp
5. $C$ 2,4 mp

- The above is called the rule of Hypothetical Syllogism or hs in short. (table 1.14)
- Many such other rules can be derived from existing rules which thus provide easier and faster proofs.

# Additional Inference Rules

- These additional rules can be derived by the previous rules.

| From | Can Derive | Name / Abbreviation |
|---|---|---|
| $P \to Q, Q \to R$ | $P \to R$ | Hypothetical syllogism- hs |
| $P \lor Q, P'$ | $Q$ | Disjunctive syllogism- ds |
| $P \to Q$ | $Q' \to P'$ | Contraposition- cont |
| $Q' \to P'$ | $P \to Q$ | Contraposition- cont |
| $P$ | $P \land P$ | Self-reference - self |
| $P \lor P$ | $P$ | Self-reference - self |
| $(P \land Q) \to R$ | $P \to (Q \to R)$ | Exportation - exp |
| $P, P'$ | $Q$ | Inconsistency - inc |
| $P \land (Q \lor R)$ | $(P \land Q) \lor (P \land R)$ | Distributive - dist |
| $P \lor (Q \land R)$ | $(P \lor Q) \land (P \lor R)$ | Distributive - dist |

# Proofs of Inference Rules

- Prove that $(P \to Q) \to (Q' \to P')$ is a valid argument (called Contraposition – con).
  - $(P \to Q) \wedge Q' \to P'$ deduction method
  - Modus Tollens (mt)

- Prove $P \wedge P' \to Q$ (called Inconsistency - inc)

| | | |
|---|---|---|
| 1. | $P$ | hyp |
| 2. | $P'$ | hyp |
| 3. | $P \vee Q$ | 1, add |
| 4. | $Q \vee P$ | 3, comm |
| 5. | $(Q')' \vee P$ | 4, dn |
| 6. | $Q' \to P$ | 5, imp |
| 7. | $(Q')'$ | 2, 6, mt |
| 8. | $Q$ | 7, dn |

# Proofs using Propositional Logic

- Prove the argument

  $A \wedge (B \to C) \wedge [(A \wedge B) \to (D \vee C')] \wedge B \to D$

  - First, write down all the hypotheses.

  1. $A$
  2. $B \to C$
  3. $(A \wedge B) \to (D \vee C')$
  4. $B$

  Use the inference and equivalence rules to get at the conclusion D.

  5. $C$ — 2,4, mp
  6. $A \wedge B$ — 1,4, con
  7. $D \vee C'$ — 3,6, mp
  8. $C' \vee D$ — 7, comm
  9. $C \to D$ — 8, imp

     and finally

  10. $D$ — 5,9 mp

  The idea is to keep focused on the result and sometimes it is very easy to go down a longer path than necessary.

# More Proofs

- $(A \wedge B)' \wedge (C' \wedge A)' \wedge (C \wedge B')' \rightarrow A'$ is a valid argument

| | | |
|---|---|---|
| 1. | $(A \wedge B)'$ | hyp |
| 2. | $(C' \wedge A)'$ | hyp |
| 3. | $(C \wedge B')'$ | hyp |
| 4. | $A' \vee B'$ | 1, De Morgan |
| 5. | $B' \vee A'$ | 4, comm |
| 6. | $B \rightarrow A'$ | 5, imp |
| 7. | $(C')' \vee A'$ | 2, De Morgan |
| 8. | $C' \rightarrow A'$ | 7, imp |
| 9. | $C' \vee (B')'$ | 3, De Morgan |
| 10. | $(B')' \vee C'$ | 9, comm |
| 11. | $B' \rightarrow C'$ | 10, imp |
| 12. | $B' \rightarrow A'$ | 8, 11, hs |
| 13. | $(B \rightarrow A') \wedge (B' \rightarrow A')$ | 6, 12, con |

# Not done yet!!

- At this point, we have now to prove that

$(B \to A') \wedge (B' \to A') \to A'$

- Proof sequence

1. $B \to A'$ hyp
2. $B' \to A'$ hyp
3. $A \to B'$ 1, cont
4. $A \to A'$ 3, 2, hs
5. $A' \vee A'$ 4, imp
6. $A'$ 6, self

# Verbal Arguments

- Russia was a superior power, and either France was not strong or Napoleon made an error. Napoleon did not make an error, but if the army did not fail, then France was strong. Hence the army failed and Russia was a superior power.
- Converting it to a propositional form using letters A, B, C and D

| | |
|---|---|
| A: Russia was a superior power | |
| B: France was strong | B′: France was not strong |
| C: Napoleon made an error | C′: Napoleon did not make an error |
| D: The army failed | D′: The army did not fail |

- Combining the statements using logic

| | |
|---|---|
| $(A \wedge (B' \vee C))$ | hypothesis |
| $C'$ | hypothesis |
| $(D' \to B)$ | hypothesis |
| $(D \wedge A)$ | conclusion |

- Combining them, the propositional form is

$$(A \wedge (B' \vee C)) \wedge C' \wedge (D' \to B) \to (D \wedge A)$$

# Proving Verbal Argument

- Prove $(A \wedge (B' \vee C)) \wedge C' \wedge (D' \to B) \to (D \wedge A)$
- Proof sequence

| | | |
|---|---|---|
| 1. | $A \wedge (B' \vee C)$ | hyp |
| 2. | $C'$ | hyp |
| 3. | $D' \to B$ | hyp |
| 4. | $A$ | 1, sim |
| 5. | $B' \vee C$ | 1, sim |
| 6. | $C \vee B'$ | 5, comm |
| 7. | $B'$ | 2, 6, ds |
| 8. | $B' \to (D')'$ | 3, cont |
| 9. | $(D')'$ | 7, 8, mp |
| 10. | $D$ | 9, dn |
| 11. | $D \wedge A$ | 4, 10 , con |

# Class Exercise

- Prove the following arguments
    - $(A' \rightarrow B') \wedge (A \rightarrow C) \rightarrow (B \rightarrow C)$

- If the program is efficient, it executes quickly. Either the program is efficient, or it has a bug. However, the program does not  execute quickly. Therefore it has a bug. (use letters E, Q, B)

- The crop is good, but there is not enough water. If there is a lot of rain or not a lot of sun, then there is enough water. Therefore the crop is good and there is a lot of sun. (use letters C, W, R, S)