# Sets

CSE2315, Chapter 4-1

# Set Theory

- A set is a collection of distinct objects called elements.
- Sets: Powerful tool in computer science to solve real world problems.
- Traditionally, sets are represented by capital letters, and elements by lower case letters.
  - Set: A, B, C
  - Elements: *a,b,c*
- The symbol $\in$ means "belongs to" and is used to represent the fact that an element belongs to a particular set. Hence, $a \in A$ means that element *a* belongs to set A.
- $b \notin A$ implies that *b* is not an element of A.
- Braces {} are used to indicate a set.

- A = {2, 4, 6, 8, 10}    $3 \notin A$ and $2 \in A$

# Set Theory

- Ordering is not imposed on the set elements and listing elements twice or more is redundant.
- Two sets are equal if and only if they contain the same elements.
  - A = { 1,2,3,4}  B = {4,3,2,1}
  - C = { 1,1,2,2,3,3}  D = {1,2,3}
- Hence, $A = B$ means

  $$(\forall x)[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$
- Two types of set representation
  - List up all the elements of a set
  - Describe a property that characterizes the set elements
- Finite and infinite set: described by number of elements in a set
- Members of infinite sets cannot be listed, but a pattern for listing elements could be indicated.
- e.g. $S = \{x \mid x \text{ is a positive even integer}\}$ or using predicate notation.

  $S = \{x \mid P(x)\}$ means $(\forall x)[(x \in S \rightarrow P(x)) \wedge (P(x) \rightarrow x \in S)]$ where $P$ is the unary predicate.
- Hence, every element of $S$ has the property $P$ and everything that has a property $P$ is an element of $S$.

# Set Theory Examples

- Describe each of the following sets by listing the elements:
- $\{x \mid x \text{ is a month with exactly thirty days}\}$
  - {April, June, September, November}
- $\{x \mid x \text{ is an integer and } 4 < x < 9\}$
  - $\{5, 6, 7, 8\}$
- What is the predicate for each of the following sets?
- $\{1, 4, 9, 16\}$
  - $\{x \mid x \text{ is one of the first four perfect squares}\}$
- $\{2, 3, 5, 7, 11, 13, 17, \ldots\}$
  - $\{x \mid x \text{ is a prime number}\}$

# Set Theory Basics

- A set that has no elements is called a null or empty set and is represented by $\varnothing$ or $\{\}$.
  - Note that $\varnothing$ is different from $\{\varnothing\}$. The latter is a set with 1 element, which is the empty set.
- Some notations used for convenience of defining sets
  - N = set of all nonnegative integers (note that $0 \in N$)
  - Z = set of all integers
  - Q = set of all rational numbers
  - R = set of all real numbers
  - C = set of all complex numbers
- Using the above notations and predicate symbols, one can describe sets quite easily

# Examples

- Using the above notations and predicate symbols, one can describe sets quite easily
- A = $\{x \mid (\exists y)[(y \in \{0,1,2\})$ and $(x = y^2)]\}$
  - Hence, A = {0,1, 4}
- B = $\{x \mid x \in$ N and $(\exists\, y)(y \in$ N and $x \leq y)\}$
- C = $\{x \mid x \in$ N and $(\forall y)(y \in \text{N} \rightarrow x \leq y)\}$
- D = $\{x \mid x \in$ N and $(\forall y)(y \in \{2, 3, 4, 5\} \rightarrow x \geq y)\}$
- E = $\{x \mid (\exists y)(\exists z)(y \in \{1,2\}$ and $z \in \{2,3\}$ and $x = z - y)\}$

# Open and Closed Interval

$$\{x \in R \mid -2 < x < 3\}$$

- Denotes the set containing all real numbers between -2 and 3. This is an open interval, meaning that the endpoints -2 and 3 are not included.
  - By all real numbers, we mean everything such as 1.05, -3/4, and every other real number within that interval.

$$\{x \in R \mid -2 \leq x \leq 3\}$$

- Similar set but on a closed interval
  - It includes all the numbers in the open interval described above, plus the endpoints.

# Relationship between Sets

- Say S is the set of all people, M is the set of all male humans, and C is the set of all computer science students.
- M and C are both subsets of S, because all elements of M and C are also elements of S.
- M is not a subset of C, however, since there are elements of M that are not in C (specifically, all males who are not studying computer science).
- For sets S and M, M is a subset of S if, and only if, every element in M is also an element of S.
  - Symbolically: $M \subseteq S \Leftrightarrow (\forall x)$, if $x \in M$, then $x \in S$.
- If $M \subseteq S$ and $M \neq S$, then there is at least one element of S that is not an element of M, then M is a proper subset of S.
  - Symbolically, denoted by $M \subset S$

# Relationship between Sets (contd.)

- A superset is the opposite of subset. If M is a subset of S, then S is a superset of M.
  - Symbolically, denoted $S \supseteq M$.
- Likewise, if M is a proper subset of S, then S is a proper superset of M.
  - Symbolically, denoted $S \supset M$.
- Cardinality of a set is simply the number of elements within the set.
- The cardinality of S is denoted by $|S|$.
- By the above definition of subset, it is clear that set M must have fewer members than S, which yields the following symbolic representation:

$$S \supset M \Rightarrow |M| < |S|$$

# Class Exercise

- A = $\{x \mid x \in \mathrm{N} \text{ and } x \geq 5\} \Rightarrow \{5, 6, 7, 8, 9, \ldots\ldots\}$
- B = $\{10, 12, 16, 20\}$
- C = $\{x \mid (\exists y)(y \in \mathrm{N} \text{ and } x = 2y)\} \Rightarrow \{0, 2, 4, 6, 8, 10, \ldots\ldots\}$

| | |
|---|---|
| $B \subseteq C$ | $B \subset A$ |
| $A \subseteq C$ | $26 \in C$ |
| $\{11, 12, 13\} \subseteq A$ | $\{11, 12, 13\} \subset C$ |
| $\{12\} \in B$ | $\{12\} \subseteq B$ |
| $\{x \mid x \in \mathrm{N} \text{ and } x < 20\} \not\subset B$ | $5 \subseteq A$ |
| $\{\emptyset\} \subseteq B$ ? | $\emptyset \in A$ ? |

# Set of Sets

- For the following sets, prove $A \subset B$.
- $A = \{ x \mid x \in R \text{ such that } x^2 - 4x + 3 = 0\}$
  - $A = \{1, 3\}$
- $B = \{ x \mid x \in N \text{ and } 1 \leq x \leq 4\}$
  - $B = \{1, 2, 3, 4\}$
- All elements of A exist in B, hence $A \subset B$.
- From every set, many subsets can be generated. A set whose elements are all such subsets is called the power set.
- For a set S, $\wp(S)$ is termed as the power set.
- For a set $S = \{1, 2, 3\}$ ; $\wp(S) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- For a set with $n$ elements, the power set has $2^n$ elements.

# Set Operations

- An ordered pair of elements is written as (x,y) and is different from (y,x).
  - Two ordered pairs (a,b) and (c,d) are equal if and only if a = c and b = d.
  - If S = {2,3}, the ordered pairs of this set are (2,2), (2,3), (3,2), (3,3).
- Binary and unary operators
  - Binary acts on two elements, say $x$-$y$ or $y$-$x$.
  - Unary acts on a single element, say negation of an element $x$ is $-x$.
  - Example: +, - and * are all binary operators on Z.
- $\circ$ is a 'binary operation' on a set S:
- if for every ordered pair $(x,y)$ of elements of S, $x \circ y$ exists, and is unique, and is a member of S.
  - The fact that $x \circ y$ exists, and is unique is the same meaning as that the binary operation $\circ$ is well-defined.
  - The fact that $x \circ y$ always belongs to S means that S is closed under the operation $\circ$.
  - The operator $\circ$ is just a placeholder for a real operator like +, -, * etc.

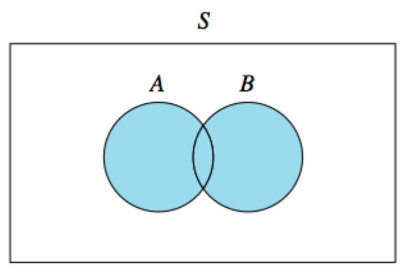
# Binary or Unary Operations

- Which of the followings are not binary operations on the given sets? Why?

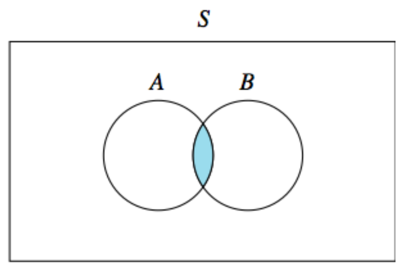
- $x \circ y = x/y$; S = set of all nonnegative integers
  - S is not closed under division ($x/0$ does not exist).
- $x \circ y = x/y$; S = set of all positive rational numbers
- $x \circ y = x^y$; S = R
  - $0^0$ is not defined
- $x \circ y$ = maximum of $x$ and $y$; S = N
- $x \circ y = x + y$; S = set of Fibonacci numbers
  - S is not closed under addition since, 1+3 = 4 and 4 is not a Fibonacci number

# Operation on Sets

- New sets can be formed in a variety of ways, and can be described using both set builder notation and Venn diagrams.
- Let A, B $\in \wp(S)$.
- The **union** of set A and B, denoted by A $\cup$ B is the set that contains all elements in either set A or set B, i.e. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- The **intersection** of set A and B, denoted by A $\cap$ B contains all elements that are common to both sets i.e. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$




$A \cup B$

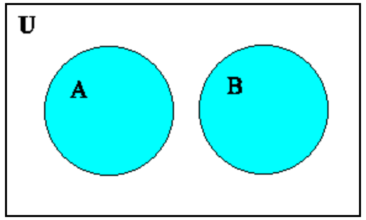



$A \cap B$

- If A = { 1,3,5,7,9} and B = {3,7,9,10,15} ;
  $A \cup B = \{1, 3, 5, 7, 9, 10, 15\}$ and $A \cap B = \{3, 7, 9\}$.

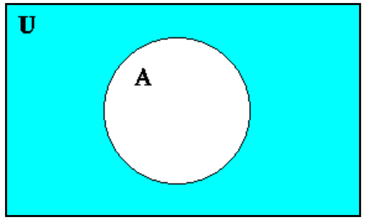
# Disjoint, Universal and Difference Sets

- Given set A and set B, if $A \cap B = \varnothing$, then A and B are disjoint sets. In other words, there are no elements in A that are also in B.




- For a set $A \in \wp(S)$, the complement of set A, denoted as $\sim A$ or $A'$, is the set of all elements that are not in A.

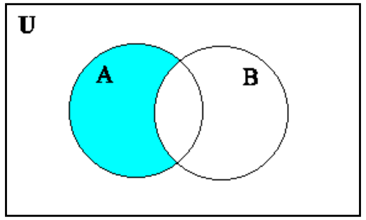
$$A' = \{x \mid x \in S \text{ and } x \notin A\}$$




- The difference of A-B is the set of elements in A that are not in B. This is also known as the complement of B relative to A.

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Note $A - B = A \cap B' \neq B - A$



# Class Exercises

- Let $A = \{1, 2, 3, 5, 10\}$
  $B = \{2, 4, 7, 8, 9\}$
  $C = \{5, 8, 10\}$
  be subsets of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Find
- $A \cup B$
- $A - C$
- $B' \cap (A \cup C)$
- $A \cap B \cap C$
- $(A \cup B) \cap C'$

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# Cartesian Product

- If A and B are subsets of S, then the **cartesian product (cross product)** of A and B denoted symbolically by $A \times B$ is defined by

$$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$$

- The Cartesian product of 2 sets is the set of all combinations of ordered pairs that can be produced from the elements of both sets. Example, given 2 sets A and B, where A = {a, b, c} and B = {1, 2, 3}, the Cartesian product of A and B can be represented as

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$$

- Is $A \times B = B \times A$?
- Cross-product of a set with itself is represented as $A \times A$ or $A^2$
- Ordered pair
  - Example, the coordinates of points on a graph are ordered pairs, where the first value must be the *x* coordinate, and the second value must be the y coordinate.
- $A^n$ represents the set of all ordered *n*-tuples $(x_1, x_2, \ldots, x_n)$ of elements of A.

# Class Exercise

- Let A={1,2} and B={3,4}
  - Find A x B
  - Find B x A
  - Find $A^2$
  - Find $A^3$

# Basic Set Identities

- Given sets A, B, and C, and a universal set S and a null/empty set $\emptyset$, the following properties hold:

- Commutative property (cp)

  $A \cup B = B \cup A$ $\quad$ $A \cap B = B \cap A$

- Associative property (ap)

  $A \cup (B \cup C) = (A \cup B) \cup C$ $\quad$ $A \cap (B \cap C) = (A \cap B) \cap C$

- Distributive properties (dp)

  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- Identity properties (ip)

  $\emptyset \cup A = A \cup \emptyset = A$ $\quad$ $S \cap A = A \cap S = A$

- Complement properties (comp)

  $A \cup A' = S$ $\quad$ $A \cap A' = \emptyset$

# More set identities

- Double Complement Law

  $(A')' = A$

- Idempotent Laws

  $A \cup A = A$ $\quad\quad$ $A \cap A = A$

- Absorption properties

  $A \cup (A \cap B) = A$ $\quad\quad$ $A \cap (A \cup B) = A$

- Alternate Set Difference Representation

  $A - B = A \cap B'$

- Inclusion in Union

  $A \subseteq A \cup B$ $\quad\quad$ $B \subseteq A \cup B$

- Inclusion in Intersection

  $A \cap B \subseteq A$ $\quad\quad$ $A \cap B \subseteq B$

- Transitive Property of Subsets

  if $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$

# Exercise

- Use the set identities to prove

$$[A \cup (B \cap C)] \cap ([A' \cup (B \cap C)] \cap (B \cap C)') = \emptyset$$

- Proof:

| | |
|---|---|
| $[A \cup (B \cap C)] \cap ([A' \cup (B \cap C)] \cap (B \cap C)') =$ | |
| $([A \cup (B \cap C)] \cap [A' \cup (B \cap C)]) \cap (B \cap C)'$ | using ap |
| $([(B \cap C) \cup A] \cap [(B \cap C) \cup A']) \cap (B \cap C)'$ | using cp twice |
| $[(B \cap C) \cup (A \cap A')] \cap (B \cap C)'$ | using dp |
| $[(B \cap C) \cup \emptyset] \cap (B \cap C)'$ | using comp |
| $(B \cap C) \cap (B \cap C)'$ | using ip |
| $\emptyset$ | using comp |