Predicate Logic

CSE2315, CHAPTER 1-4

Predicate Logic

- Similar to propositional logic for solving arguments, build from quantifiers, predicates and logical connectives.
- A valid argument for predicate logic need not be a tautology.
- The meaning and the structure of the quantifiers and predicates determines the interpretation and the validity of the arguments
- Basic approach to prove arguments:
  - Strip off quantifiers
  - Manipulate the unquantified wffs
  - Reinsert the quantifiers

Four new inference rules

Two rules to strip the quantifiers	Two rules to reinsert the quantifiers

- Note to remember: $P(x)$ could be $(\forall y)\ (\forall z)\ Q(x,y,z)$

Inference Rules

From	Can Derive	Name / Abbreviation	Restrictions on Use
$(\forall x)P(x)$	$P(t)$ where t is a variable or constant symbol	Universal Instantiation- ui	If t is a variable, it must not fall within the scope of a quantifier for t ex) $(\forall x)(\exists y)P(x,y)$ to $(\exists y)P(\underline{y},y)$
$(\exists x)P(x)$	$P(a)$ where a is a constant symbol not previously used in a proof sequence	Existential Instantiation- ei	Must be the first rule used that introduces a
$P(x)$	$(\forall x)P(x)$	Universal Generalization- ug	$P(x)$ has not been deduced from any hypotheses in which x is a free variable nor has $P(x)$ been deduced by ei from any wff in which x is a free variable
$P(x)$ or $P(a)$	$(\exists x)P(x)$	Existential Generalization- eg	To go from $P(a)$ to $(\exists x)P(x)$, x must not appear in $P(a)$ ex) $P(a,y)$ to $(\exists y)P(\underline{y},y)$

Examples: Proofs using Predicate Logic (ui)

- Prove the following argument:
  - All flowers are plants. Sunflower is a flower. Therefore, sunflower is a plant.
  - $P(x)$ is "x is a flower"
  - a is a constant symbol (Sunflower)
  - $Q(x)$ is "x is a plant"
- The argument is $(\forall x)[P(x) \rightarrow Q(x)] \wedge P(a) \rightarrow Q(a)$
- The proof sequence is as follows:

1. $(\forall x)[P(x) \rightarrow Q(x)]$ hyp
2. $P(a)$ hyp
3. $P(a) \rightarrow Q(a)$ 1, ui
4. $Q(a)$ 2, 3, mp

UI continued…

- (One more ui example) Prove the argument
  - $(\forall x)[P(x) \to Q(x)] \wedge [Q(y)]' \to [P(y)]'$
- Proof sequence:

1. $(\forall x)[P(x) \to Q(x)]$ hyp
2. $[Q(y)]'$ hyp
3. $P(y) \to Q(y)$ 1, ui
4. $[P(y)]'$ 2, 3, mt

Examples: Proofs using Predicate Logic (ei)

- The following would be legitimate steps in a proof sequence

1.	$(\forall x)[P(x) \to Q(x)]$	hyp
2.	$(\exists y)[P(y)]$	hyp
3.	$P(a)$	2, ei
4.	$P(a) \to Q(a)$	1, ui
5.	$Q(a)$	3,4 mp

- What if we switch the order, 3 and 4?

Examples: Proofs using Predicate Logic (eg)

- Prove the argument $(\forall x)P(x) \to (\exists x)P(x)$
- Proof sequence:

1. $(\forall x)P(x)$ hyp
2. $P(x)$ 1, ui
3. $(\exists\, x)P(x)$ 2, eg

Examples: Proofs using Predicate Logic (ug)

- Prove the argument $(\forall x)[P(x) \to Q(x)] \wedge (\forall x)P(x) \to (\forall x)Q(x)$
- Proof sequence:

1.	$(\forall x)[P(x) \to Q(x)]$	hyp
2.	$(\forall x)P(x)$	hyp
3.	$P(x) \to Q(x)$	1, ui
4.	$P(x)$	2, ui : no restriction on ui about reusing a name
5.	$Q(x)$	3, 4, mp
6.	$(\forall x)Q(x)$	5, ug

- Note: step 6 is legitimate since x is not a free variable in any hypotheses nor was ei used before

Restrictions on ug

- Incorrect ug 1

1.	$P(x)$	hyp
2.	$(\forall x)P(x)$	1, incorrect ug; x was free variable in the hypothesis

- Incorrect ug 2

1.	$(\forall x)\ (\exists y)\ Q(x,y)$	hyp
2.	$(\exists y)\ Q(x,y)$	1. ui
3.	$Q(x,a)$	2, ei
4.	$(\forall x)\ Q(x,a)$	3, incorrect ug; $Q(x,a)$ was deduced by ei from the wff in step2, in which x is free variable

Examples: Proofs using Predicate Logic

- Prove the argument

  $(\forall x)[P(x) \land Q(x)] \rightarrow (\forall x)P(x) \land (\forall x)Q(x)$
- Proof sequence:

1. $(\forall x)[P(x) \land Q(x)]$ hyp
2. $P(x) \land Q(x)$ 1, ui
3. $P(x)$ 2, sim
4. $Q(x)$ 2, sim
5. $(\forall x)P(x)$ 3, ug
6. $(\forall x)Q(x)$ 4, ug
7. $(\forall x)P(x) \land (\forall x)Q(x)$ 5, 6, con

Examples: Proofs using Predicate Logic

- Prove the argument

  $(\forall y)[P(x) \rightarrow Q(x,y)] \rightarrow [P(x) \rightarrow (\forall y)Q(x,y)]$

- Using the deduction method, we can derive

  $(\forall y)[P(x) \rightarrow Q(x,y)] \land P(x) \rightarrow (\forall y)Q(x,y)$

- Proof sequence:

1.	$(\forall y)[P(x) \rightarrow Q(x,y)]$	hyp
2.	$P(x)$	hyp
3.	$P(x) \rightarrow Q(x,y)$	1, ui
4.	$Q(x,y)$	2, 3, mp
5.	$(\forall y)Q(x,y)$	4, ug

Temporary hypotheses

- A temporary hypothesis can be inserted into a proof sequence. If T is inserted as a temporary hypothesis and eventually W is deduced from T and other hypotheses, then the wff $T \to W$ has been deduced from other hypotheses and can be reinserted into the proof sequence
- Prove the argument

  $[P(x) \to (\forall y)Q(x,y)] \to (\forall y)[P(x) \to Q(x,y)]$
- Proof sequence:
  1. $P(x) \to (\forall y)Q(x,y)$ hyp
  2. $P(x)$ temporary hypothesis (T)
  3. $(\forall y)Q(x,y)$ 1, 2, mp
  4. $Q(x,y)$ 3, ui (W)
  5. $P(x) \to Q(x,y)$ temp. hyp discharged ($T \to W$)
  6. $(\forall y)[P(x) \to Q(x,y)]$ 5, ug

More Examples

- Prove the sequence $[(\exists x)A(x)]' \Leftrightarrow (\forall x)[A(x)]'$
  - To prove equivalence, implication in each direction should be proved
- Proof sequence for $[(\exists x)A(x)]' \rightarrow (\forall x)[A(x)]'$

1.	$[(\exists x)A(x)]'$	hyp
2.	$A(x)$	temp. hyp
3.	$(\exists x)A(x)$	2, eg
4.	$A(x) \rightarrow (\exists x)A(x)$	temp. hyp discharged
5.	$[A(x)]'$	1, 4, mt
6.	$(\forall x)[A(x)]'$	5, ug

- Proof sequence for $(\forall x)[A(x)]' \rightarrow [(\exists x)A(x)]'$

1.	$(\forall x)[A(x)]'$	hyp
2.	$(\exists x)A(x)$	temp. hyp
3.	$A(a)$	2, ei
4.	$[A(a)]'$	1, ui
5.	$[(\forall x)[A(x)]']'$	3, 4, inc (inconsistency)
6.	$(\exists x)A(x) \rightarrow [(\forall x)[A(x)]']'$	temp. hyp discharged
7.	$[((\forall x)[A(x)]')']'$	1, dn
8.	$[(\exists x)A(x)]'$	6, 7, mt

Proving Verbal Arguments

- Every crocodile is bigger than every alligator. Sam is a crocodile. But there is a snake, and Sam isn't bigger than that snake. Therefore, something is not an alligator.
  - Use C(x): x is a crocodile; A(x): x is an alligator, B(x,y): x is bigger than y, s is a constant (Sam), S(x): x is a Snake
- Hence prove argument

$$(\forall x)(\forall y)[C(x) \wedge A(y) \rightarrow B(x,y)] \wedge C(s) \wedge (\exists x)(S(x) \wedge [B(s,x)]') \rightarrow (\exists x)[A(x)]'$$

1.	$(\forall x)(\forall y)[C(x) \wedge A(y) \rightarrow B(x,y)]$	hyp
2.	$C(s)$	hyp
3.	$(\exists x)(S(x) \wedge [B(s,x)]')$	hyp
4.	$(\forall y)[C(s) \wedge A(y) \rightarrow B(s,y)]$	1, ui
5.	$S(a) \wedge [B(s,a)]'$	3, ei
6.	$C(s) \wedge A(a) \rightarrow B(s,a)$	4, ui
7.	$[B(s,a)]'$	5, sim
8.	$[C(s) \wedge A(a)]'$	6, 7, mt
9.	$[C(s)]' \vee [A(a)]'$	8, De Morgan
10.	$[C(s)] \rightarrow [A(a)]'$	9, imp
11.	$[A(a)]'$	2, 10, mp
12.	$(\exists x)[A(x)]'$	11, eg

Class Exercise

- Prove the argument

  $(\forall x)[(B(x) \vee C(x)) \to A(x)] \to (\forall x)[B(x) \to A(x)]$

Class Exercise

- Every ambassador speaks only to diplomats, and some ambassadors speak to someone. Therefore, there is a diplomat.
- Use A(x): x is an ambassador; S(x,y): x speaks to y; D(x): x is a diplomat

Prove the argument

$(\forall x)\ (\forall y)[(A(x) \wedge S(x,y)) \rightarrow D(y)] \wedge (\exists x)(\exists y)(A(x) \wedge S(x,y)) \rightarrow (\exists x)D(x)$