Induction

CSE2315, Chapter 2-2

Principles of mathematical induction

- First Principle:

P(1) is true
$(\forall k)$P(k) true $\rightarrow$ P(k+1) true
P(n) is true for all positive integers n

- P(1) is true
  - "1" has the property of P
- For any positive integer k, P(k) $\rightarrow$ P(k+1)
  - If any number has the property P, so does the next number
- An example of an implication
  - 2 hypotheses and 1 conclusion
- **Basis (Basis step)**; establishing the truth of P(1)
- **Inductive step**; establishing the truth of P(k) $\rightarrow$ P(k+1)
- When we assume P(k) to be true to prove the inductive step, P(k) is called **inductive assumption (inductive hypothesis)**.

Example 1

- Prove that $1+3+5+\ldots+(2n-1) = n^2$ for any $n \geq 1$ (n, integer)

Example 1 (contd.)

- Prove that $1+3+5+\ldots+(2n-1) = n^2$ for any $n \geq 1$ (n, integer)
- What is P(n)?
- Basis step; P(1) is the equation $1 = 1^2$, which is true.
- Inductive hypothesis; $P(k)$: $1+3+5\ldots+(2k-1) = k^2$ (assumed true)
- Inductive step; $P(k+1)$: $1+3+5\ldots+[2(k+1)-1] = (k+1)^2$
- Again, rewriting the sum on the left side of P($k+1$) reveals how the inductive assumption can be used:

$$
\begin{aligned}
1+3+5\ldots+[2(k+1)-1] &= 1+3+5\ldots+(2k-1)+[2(k+1)-1] \\
&= k^2+[2(k+1)-1] \\
&= k^2+2k+1 \\
&= (k+1)^2
\end{aligned}
$$

- Therefore,

$$1+3+5\ldots+[2(k+1)-1] = (k+1)^2$$

and verifies P($k+1$) and completes the proof.

Example 2

- Prove that $1+2+2^2\ldots+2^n = 2^{n+1}-1$ for any $n \geq 1$.
- What is $\mathrm{P(n)}$?
- Basis; $\mathrm{P}(1)$ is the equation $1+2 = 2^{1+1}-1$ or $3 = 2^2-1$, which is true.
- We take $\mathrm{P}(k)$: $1+2+2^2\ldots+2^k = 2^{k+1}-1$ as the inductive hypothesis and try to establish $\mathrm{P}(k+1)$: $1+2+2^2\ldots+2^{k+1} = 2^{k+1+1}-1$
- Again, rewriting the sum on the left side of $\mathrm{P}(k+1)$ reveals how the inductive assumption can be used:

$$\begin{aligned} 1+2+2^2\ldots+2^{k+1} &= 1+2+2^2\ldots+ 2^k + 2^{k+1} \\ &= 2^{k+1}-1 + 2^{k+1} \quad \text{(by inductive hypothesis)} \\ &= 2(2^{k+1})-1 \\ &= 2^{k+1+1}-1 \end{aligned}$$

- Therefore,

$$1+2+2^2\ldots+2^{k+1} = 2^{k+1+1}-1$$

and verifies $\mathrm{P}(k+1)$ and completes the proof.

Example 3

- Prove that for any positive integer n, the number 2^{2n}-1 is divisible by 3.
- The basis step is to show P(1), that $2^{2(1)}-1 = 4-1 = 3$ is divisible by 3. Clearly this is true.
- P(k): We assume that $2^{2k}-1$ is divisible by 3,which means that

  $2^{2k}-1 = 3m$ for some integer m, or $2^{2k} = 3m+1$.

  P(k+1): We want to show that $2^{2(k+1)}-1$ is divisible by 3.

$$
\begin{aligned}
2^{2(k+1)} -1 &= 2^{2k+2} -1 \\
&= 2^2 \cdot 2^{2k} - 1 \\
&= 2^2(3m+1) - 1 \text{ (by the inductive hypothesis)} \\
&= 12m+4-1 \\
&= 12m+3 \\
&= 3(4m+1) \text{ where } 4m+1 \text{ is an integer}
\end{aligned}
$$

- Thus $2^{2(k+1)}-1$ is divisible by 3.

Principles of mathematical induction

- First Principle:

  $P(1)$ is true

  $(\forall k)P(k)$ true $\rightarrow P(k+1)$ true

  $P(n)$ is true for all positive integers n

- Second Principle:

  $P(1)$ is true

  $P(r)$ true for all r, $1 \leq r \leq k \rightarrow P(k+1)$ true

  $P(n)$ is true for all positive integers n

- Major difference is in the second statement.
  - Use the second principle when assuming $P(k)$ is not enough to prove $P(k+1)$.
  - Assuming $P(r)$ for any r where $1 \leq r \leq k$ gives more ammunition to prove the relation.
  - Use second principle when the case $k+1$ depends on results further back than k.

Example: Second Principle of Induction

- Prove that the amount of postage greater than or equal to 8 cents can be built using only 3-cent and 5-cent stamps.
- Proof using the second principle of Induction
- Hence, we have to prove that $\mathrm{P}(n)$ is true for $n \geq 8$ where $\mathrm{P}(n)$ is the sum of 3 and 5 cent stamps used to make the n cents worth of postage.
  - Basis step: $\mathrm{P}(8) = 3 + 5 = 8$ which is true.
  - In this case we will have to establish additional cases to prove this. Hence, establish two cases $\mathrm{P}(9)$ and $\mathrm{P}(10)$
  - $\mathrm{P}(9) = 3 + 3 + 3 = 9$ and $\mathrm{P}(10) = 5 + 5$
  - Assume $\mathrm{P}(r)$ is true for any r, $8 \leq r \leq k$ and consider $\mathrm{P}(k+1)$
  - We have proved $\mathrm{P}(r)$ is true for $r = 8,9,10$
  - So, let's assume $k+1 \geq 11$
  - If $k+1 \geq 11$ then $(k+1)$-3= k-$2 \geq 8$, hence, by inductive hypothesis, $\mathrm{P}(k$-$2)$ is true.
  - Hence, k-2 can be written as a sum of 3s and 5s. Adding an additional 3 results in $k+1$ as a sum of 3s and 5s.
  - This verifies $\mathrm{P}(k+1)$ is true and hence verifies the proof.