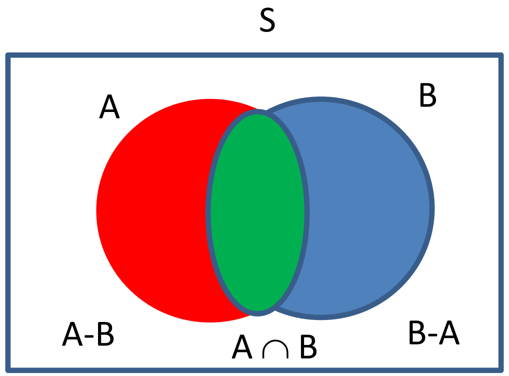
Principle of Inclusion & Exclusion

CSE2315, Chapter 4-3

Principle of Inclusion & Exclusion

- If A and B are subsets of universal set S, then (A-B), (B-A) and ($A \cap B$) are disjoint sets.
- As seen from the figure, (A-B) $\cup$ (B-A) $\cup$ ($A \cap B$) is the same as $A \cup B$.



- For three disjoint sets

$$|(A-B) \cup (B-A) \cup (A \cap B)| = |A-B| + |B-A| + |A \cap B|$$

- We have solved for two finite sets A and B

$|A-B| = |A| - |A \cap B|$ and $|B-A| = |B| - |A \cap B|$

- Hence, using this, we get

$$\begin{aligned}|(A-B) \cup (B-A) \cup (A \cap B)| &= |A| - |A \cap B| + |B-A| + |A \cap B| \\ &= |A| - |A \cap B| + |B| - |A \cap B| + |A \cap B|\end{aligned}$$

- Hence, $|A \cup B| = |A| + |B| - |A \cap B|$

Principle of Inclusion & Exclusion

- The principle of inclusion and exclusion for two sets A and B.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- The name comes from the fact that to calculate the elements in a union, we include the individual elements of A and B but subtract the elements common to A and B so that we don't count them twice.
- This principle can be generalized to *n* sets.

Example: Inclusion and Exclusion Principle

- **<u>Example 1</u>**: How many integers from 1 to 1000 are either multiples of 3 or multiples of 5?
  - Let us assume that A = set of all integers from 1 to 1000 that are multiples of 3.
  - Let us assume that B = set of all integers from 1 to 1000 that are multiples of 5.
  - $A \cup B$ = The set of all integers from 1 to 1000 that are multiples of either 3 or 5.
  - $A \cap B$ = The set of all integers that are both multiples of 3 and 5, which also is the set of integers that are multiples of 15.

Example: Inclusion and Exclusion Principle

- To use the inclusion/exclusion principle to obtain $|A \cup B|$, we need $|A|$, $|B|$ and $|A \cap B|$.
  - From 1 to 1000, every third integer is a multiple of 3, each of this multiple can be represented as 3*p*, for any integer *p* from 1 through 333, Hence $|A| = 333$.
  - Similarly for multiples of 5, each multiple of 5 is of the form 5*q* for some integer *q* from 1 through 200. Hence, we have $|B| = 200$.
  - To determine the number of multiples of 15 from 1 through 1000, each multiple of 15 is of the form 15*r* for some integer *r* from 1 through 66.
  - Hence, $|A \cap B| = 66$.
- From the principle, we have the number of integers either multiples of 3 or multiples of 5 from 1 to 1000 given by

  $|A \cup B| = 333 + 200 - 66 = 467$.

Example: Inclusion/exclusion principle for 3 sets

- **<u>Example 2:</u>** In a class of students undergoing a computer course the following were observed.
  - Out of a total of 50 students: 30 know Pascal, 18 know Fortran, 26 know COBOL, 9 know both Pascal and Fortran, 16 know both Pascal and COBOL, 8 know both Fortran and COBOL, 47 know at least one of the three languages.
- From this we have to determine
  - a. How many students know none of these languages?
  - b. How many students know all three languages?

Example: Inclusion/exclusion principle for 3 sets

a. We know that 47 students know at least one of the three languages in the class of 50. The number of students who do not know any of three languages is given by the difference between the number of students in class and the number of students who know at least one language.
   - Hence, the students who know none of these languages = 50 – 47 = 3.

b. Students know all three languages, so we need to find $|A \cap B \cap C|$.

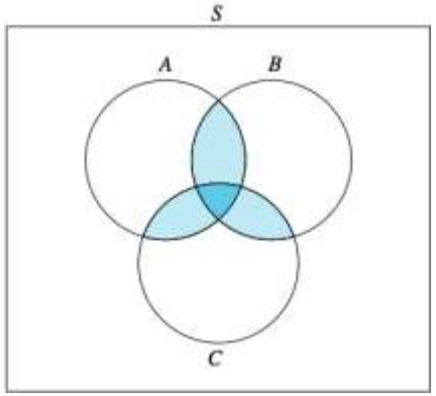
   - A = All the students who know Pascal in class.
   - B = All the students who know COBOL in the class.
   - C = All the students who know FORTRAN in class.

- We have to derive the inclusion/exclusion formula for three sets

  $|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$ (refer to the next line↓)

Example: Inclusion/exclusion principle for 3 sets

- $|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$
  $= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$
  $= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |A \cap B \cap C|)$
  $= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$
- Given in the problem are the following:
  $|A \cap B| = 9$ (both P & F)
  $|A \cap C| =16$ (both P & C)
  $|B \cap C| = 8$ (both F & C)
  $|A \cup B \cup C| = 47$
- Hence, using the above formula, we have
  $47 = 30 + 18 + 26 -8 -9 -16 + |A \cap B \cap C|$
  Hence, $|A \cap B \cap C| = 6$



Class Exercise

- A group of students plan to order pizza. If 13 will eat sausage topping, 10 will eat pepperoni, 12 will eat extra cheese, 4 will eat both sausage and pepperoni, 5 will eat both pepperoni and extra cheese, 7 will eat both sausage and extra cheese, and 3 will eat all three toppings, how many students are in the group?
- Let, A = {students who will eat sausage}

  B = {students who will eat pepperoni}

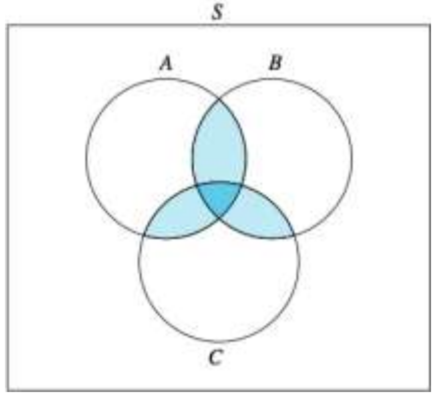
  C = {students who will eat extra cheese}

  Then,

  $|A| = 13$, $|B| = 10$, $|C| = 12$, $|A \cap B| = 4$, $|B \cap C| = 5$, $|A \cap C| = 7$, and

  $|A \cap B \cap C| = 3$

  $|A \cup B \cup C| = ?$



- Hence, using the above formula, we have

  $|A \cup B \cup C| = 13 + 10 + 12 - 4 - 5 - 7 + 3$

  Hence, $|A \cup B \cup C| = 22$

Principle of Inclusion and Exclusion

Given the finite sets $A_1, \ldots\ldots, A_n$, $n \geq 2$, then

$$|A_1 \cup \ldots\ldots \cup A_n| = \sum_{1\leq i\leq n} |A_i| - \sum_{1\leq i\leq j\leq n} |A_i \cap A_j| + \sum_{1\leq i\leq j\leq k\leq n} |A_i \cap A_j \cap A_k| - \ldots . + (-1)^{n+1} |A_1 \cap \ldots \cap A_n|$$

- In this equation above, the term $\sum_{1\leq i\leq j\leq n} |A_i \cap A_j|$ says add together the number of elements in all intersections of the form $A_i \cap A_j$ where $i < j$.
- What are these terms for $n = 3$? This gives $A_1 \cap A_2$ (i=1, j=2), $A_1 \cap A_3$ (i=1, j=3), and $A_2 \cap A_3$ (i=2, j=3). This agrees with the proof of inclusion/exclusion principle for three sets in the previous slide, where A_1 = A, A_2 = B, and A_3 = C.

Pigeonhole Principle

- If more than k items are placed into k bins, then at least one bin has more than one item.

- How many people must be in a room to guarantee that two people have the last name begin with the same initial?

- How many times must a single die be rolled in order to guarantee getting the same value twice?

(From 2-1) Direct Proof: Contraposition

- Example 2: Prove that "If n+1 separate passwords are issued to n students, then some student gets ≥ 2 passwords."
  - The contrapositive is:
    - If every student gets < 2 passwords, then n+1 separate passwords were NOT issued."
  - Suppose every student has < 2 passwords
  - Then, every one of the n students has at most 1 password.
  - The total number of passwords issued is at most n, not n+1.