Quantifiers, Predicates, and Validity

CSE2315, Chapter 1-3

Variable and Statement

- Variable
  - A variable is a symbol that stands for an individual in a collection or set. For example, the variable x may stand for one of the days. We may let x = Monday or x = Tuesday, etc
- Incomplete Statements
  - A sentence containing a variable is called an incomplete statement. An incomplete statement is about the individuals in a definite domain or set. When we replace the variable by the name of an individual in the set we obtain a statement about that individual.
  - Example of an incomplete statement: "x has 30 days."
  - Here, x can be any month and substituting that, we will get a complete statement.

Quantifiers and Predicates

- Quantifiers:
  - Quantifiers are phrases that refer to given quantities, such as “for some” or “for all” or “for every,” indicating how many objects have a certain property.
- Two kinds of quantifiers:
  - Universal and Existential
- Universal Quantifier: represented by $\forall$
  - The symbol is translated as and means “for all”, “given any”, “for each,” or “for every,” and is known as the universal quantifier.
- Existential Quantifier: represented by $\exists$
  - The symbol is translated as and means variously “for some,” “there exists,” “there is a,” or “for at least one”.

Quantifiers and Predicates

- Predicate
- It is the verbal statement that describes the property of a variable. Usually represented by the letter P, the notation P(x) is used to represent some unspecified property or predicate that x may have
  - e.g. P(x) = x has 30 days.
  - P(April) = April has 30 days.
- Combining the quantifier and the predicate, we get a complete statement of the form $(\forall x)$P(x) or $(\exists x)$P(x).
- The collection of objects is called the domain of interpretation.
- Truth value of expressions formed using quantifiers and predicates
  - What is the truth value of $(\forall x)$P(x) where x is all the months and P(x) = x has less than 32 days.
  - Undoubtedly, the above is true since no month has 32 days.

Truth value of the following expressions

- Truth of expression $(\forall x)P(x)$
  1. $P(x)$ is the property that x is yellow, and the domain of interpretation is the collection of all flowers:
  2. $P(x)$ is the property that x is a plant, and the domain of interpretation is the collection of all flowers:
  3. $P(x)$ is the property that x is positive, and the domain of interpretation consists of integers:

  – Can you find one interpretation in which both $(\forall x)P(x)$ is true and $(\exists x)P(x)$ is false?

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- Predicates involving properties of single variables : unary predicates
- Binary, ternary and *n*-ary predicates are also possible.
  – $(\forall x)(\exists y)Q(x, y)$ is a binary predicate. This expression reads as "for every x there exists a y such that $Q(x, y)$."

Instant Excercises

- What is the truth value of each of the following wffs in the interpretation where the domain consists of the integers, $O(x)$ is "x is odd," $L(x)$ is "x<10," and $G(x)$ is "x>9"?
  - A. $(\exists x)\ O(x)$
  - B. $(\forall x)[L(x) \rightarrow O(x)\ ]$
  - C. $(\exists\ x)[L(x) \wedge G(x)]$
  - D. $(\forall x)[L(x) \vee G(x)]$

- $(\forall x)\ (\exists y)\ \mathrm{Q}(x,y)$ & $(\exists y)\ (\forall x)\ \mathrm{Q}(x, y)$?
  - Domain of interpretation is integers
  - $Q(x, y) : x < y$
  - $(\forall x)\ (\exists y)\ \mathrm{Q}(x, y)$ → For any integers (x), there is a larger integer (y)

  - $(\exists y)\ (\forall x)\ \mathrm{Q}(x, y)$ → There is a single integer (y), that is larger than any integer (x)

Interpretation

- Formal definition: An interpretation for an expression involving predicates consists of the following:
  - A collection of objects, called the domain of interpretation, which must include at least one object.
  - An assignment of a property of the objects in the domain to each predicate in the expression.
  - An assignment of a particular object in the domain to each constant symbol in the expression.
- Predicate wffs can be built similar to propositional wffs using logical connectives with predicates and quantifiers.
- Examples of predicate wffs
  - $(\forall x)[P(x) \rightarrow Q(x)]$
  - $(\forall x)\ ((\exists\ y)[P(x, y) \vee Q(x, y)] \rightarrow R(x))$
  - $S(x, y) \wedge R(x, y)$

Scope of a variable in an expression

- Brackets are used wisely to identify the scope of the variable.
  - $(\forall x)\ [\exists(y)[P(x, y) \vee Q(x, y)] \rightarrow R(x)]$
    - Scope of $(\exists y)$ is $P(x, y) \vee Q(x, y)$ while the scope of $(\forall x)$ is the entire expression.
  - $(\forall x)S(x) \vee (\exists y)R(y)$
    - Scope of x is $S(x)$ while the scope of y is $R(y)$.
  - $(\forall x)[P(x, y) \rightarrow (\exists y)\ Q(x, y)]$
    - Scope of variable y is not defined for $P(x, y)$ hence y is called a free variable. Such expressions might not have a truth value at all.
- What is the truth value of the expression
  - $\exists(x)[A(x) \wedge (\forall y)[B(x, y) \rightarrow C(y)]]$ in the interpretation, where
  - $A(x)$ is "$x > 0$" , $B(x, y)$ is "$x > y$" and $C(y)$ is "$y \leq 0$" where the domain of x is positive integers and the domain of y is all integers

True, $x = 1$ is a positive integer and any integer less than x is ≤ 0

Translation: Verbal statements to symbolic form

- "Every person is nice" can be rephrased as "For any thing, if it is a person, then it is nice." So, if $P(x)$ is "x is a person" and $Q(x)$ be "x is nice," the statement can be symbolized as
  - $(\forall x)[P(x) \to Q(x)]$
  - Variations: "All persons are nice" or "Each person is nice"

– How about$(\forall x)[P(x) \land Q(x)]$?

  - Domain: whole world, then not everything in the world is nice person.

- "There is a nice person" can be rewritten as "There exists something that is both a person and nice."
  - $(\exists x)[P(x) \land Q(x)]$
  - Variations: "Some persons are nice" or "There are nice persons."
- How about $(\exists x)[P(x) \to Q(x)]$?
  - This will only be true if there are no persons in the world but that is not the case.
- Hence such a statement is false, so almost always, $\exists$ goes with $\land$ (conjunction) and $\forall$ goes with $\to$ (implication).

Translation

- To translate an English statement into wff, use intermediate English statement
- The word "only" can be tricky depending on its presence in the statement.
  - X loves only Y ⇔ If X loves anything, then that thing is Y.
  - Only X loves Y ⇔ If anything loves Y, then it is X.
  - X only loves Y ⇔ If X does anything to Y, then it is love.
- Example for forming symbolic forms from predicate symbols
  - J(x) is "x is John"; M(x) is "x is Mary"; L(x, y) is "x loves y"
  - John loves only Mary ⇔
    For any thing, if it is John then, if it loves anything, that thing is Mary ⇔
    $(\forall x)[J(x) \rightarrow (\forall y)(L(x, y) \rightarrow M(y))]$
  - Only John loves Mary ⇔
    For any thing, if it is Mary then, if anything loves it, that thing is John ⇔
    $(\forall x)\ [M(x) \rightarrow (\forall\ y)\ (L(y, x) \rightarrow J(y))]$
  - John only loves Mary ⇔
    For any thing, if it is John then, for any other thing, if that thing is Mary, then John loves it ⇔
    $(\forall x)[J(x) \rightarrow (\forall\ y)(M(y) \rightarrow L(x, y))]$

Tips for translation to predicate wff

- Textbook p45
- Look for the key words that signify the type of quantifier
  - For all, for every, for any, for each : universal quantifier
  - For some, there exists : existential quantifier
- English sometimes uses “understood” universal quantifiers
  - “Dogs chase rabbits” is understood to mean, “All dogs chase all rabbits.”
- If you use a universal quantifier, the connective that goes with is almost always “implication”
- If you use an existential quantifier, the connective that goes with is almost always “conjunction”
- Whatever comes after the word “only” is the conclusion of an implication.
- You are most apt to arrive at a correct translation if you follow the order of the English words

Class exercise

- S(x): x is a student; I(x): x is intelligent; M(x): x likes music
- Write wffs that express the following statements:
  - All students are intelligent.
  - Some intelligent students like music.
  - Everyone who likes music is a stupid student.
  - Only intelligent students like music.

Negation of statements

- A(x): Everything is fun
- Negation will be "it is false that everything is fun," i.e. "something is nonfun."
- In symbolic form, $[(\forall x)A(x)]' \Leftrightarrow (\exists x)[A(x)]'$
- Similarly negation of "Something is fun" is "Nothing is fun" or "Everything is boring."
- Hence, $[(\exists x)A(x)]' \Leftrightarrow (\forall x)[A(x)]'$

Class Exercise

- What is the negation of “Everybody loves somebody sometime.”
  - Everybody hates somebody sometime
  - Somebody loves everybody all the time
  - Everybody hates everybody all the time
  - Somebody hates everybody all the time

- What is the negation of the following statements?
- Some pictures are old or faded.

- All people are tall and thin.

- Some students eat only pizza.

- Only students eat pizza.

Validity

- Analogous to a tautology of propositional logic.
- Truth of a predicate wff depends on the interpretation.
- A predicate wff is valid if it is true in all possible interpretations just like a propositional wff is true if it is true for all rows of the truth table.
- A valid predicate wff is intrinsically true.

	Propositional Wffs	Predicate Wffs
Truth values	True or false – depends on the truth value of statement letters	True, false or neither (if the wff has a free variable)
Intrinsic truth	Tautology – true for all truth values of its statements	Valid wff – true for all interpretations
Methodology	Truth table to determine if it is a tautology	No algorithm to determine validity

Validity Examples

- $(\forall x)P(x) \rightarrow (\exists x)P(x)$
  - This is valid because if every object of the domain has a certain property, then there exists an object of the domain that has the same property.
- $(\forall x)P(x) \rightarrow P(a)$
  - Valid – quite obvious since *a* is a member of the domain of x.
- $(\exists x)P(x) \rightarrow (\forall x)P(x)$
  - Not valid since the property cannot be valid for all objects in the domain if it is valid for some objects of that domain. Can use a mathematical context to check as well.
  - Say $P(x)$ = "x is even," then there exists an integer that is even but not every integer is even (domain: integer)
- $(\forall x)[P(x) \vee Q(x)] \rightarrow (\forall x)P(x) \vee (\forall x)Q(x)$
  - Invalid, can prove by mathematical context by taking $P(x) = x$ is even and $Q(x) = x$ is odd (domain: integer).
  - In that case, the hypothesis is true but the conclusion is false because it is not the case that every integer is even or that every integer is odd.

Class Exercise

- What is the truth of the following wffs where the domain consists of integers:
  1. $(\forall x)[L(x) \rightarrow O(x)]$ where $O(x)$ is "x is odd" and $L(x)$ is "$x < 10$"?
  2. $(\exists y)(\forall x)(x + y = 0)$?
  3. $(\exists y)(\exists x)(x^2 = y)$?
  4. $(\forall x)[x < 0 \rightarrow (\exists y)(y > 0 \land x + y = 0)]$?

- Using predicate symbols and appropriate quantifiers, write the symbolic form of the following English statement:
- $D(x)$ is "x is a day" M is "Monday" T is "Tuesday"
- $S(x)$ is "x is sunny" $R(x)$ is "x is rainy"
  1. Some days are sunny and rainy.
  2. It is always a sunny day only if it is a rainy day.
  3. It rained both Monday and Tuesday.
  4. Every day that is rainy is not sunny.