Permutations and Combinations

CSE2315, Chapter 4-4

Permutations and Combinations

- A = {1,2,3}
- (1,2,3), (1,3,2), (2,1,3), (2,3,2), (3,1,2), (3,2,1)
- (1,2,3)

Permutations

- An ordered arrangement of objects is called a permutation.
  - Hence, a permutation of n distinct elements is an ordering of these n elements.
- It is denoted by $P(n,r)$ or ${}_nP_r$.
- Ordering of last four digits of a telephone number if digits are allowed to repeat = $10 \cdot 10 \cdot 10 \cdot 10$ = **10000**
- Ordering of four digits if repetition is not allowed = $10 \cdot 9 \cdot 8 \cdot 7$ = **5040** = $10!/6!$

  where $n! = n*(n-1)*(n-2)*\ldots*3*2*1$ and by definition $0! = 1$
- Hence, mathematically, for $r \le n$, the number of permutations of r distinct objects chosen from n objects is defined by
- $P(n,r) = n*(n-1)*(n-2)*\ldots*(n-r+1) = \dfrac{n*(n-1)*(n-2)*\ldots*(n-r+1)*(n-r)!}{(n-r)!}$

$\Rightarrow P(n,r) = \dfrac{n!}{(n-r)!}$ for $0 \le r \le n$

- Hence, $P(10,4) = 10! / (10-4)! = 10!/6! = 5040$

Permutations: Some special cases

- $P(n,0) = n! / n! = 1$
- This means that there is only one ordered arrangement of 0 objects, called the empty set.

- $P(n,1) = n!/ (n-1)! = n$
- There are *n* ordered arrangements of one object (i.e. *n* ways of selecting one object from *n* objects).

- $P(n,n) = n!/(n-n)! = n!/0! = n!$
- This means that one can arrange *n* distinct objects in *n*! ways, that is nothing but the multiplication principle.

Permutation Examples

1. Ten athletes compete in an Olympic event. Gold, silver and bronze medals are awarded to the first three in the event, respectively. How many ways can the awards be presented?

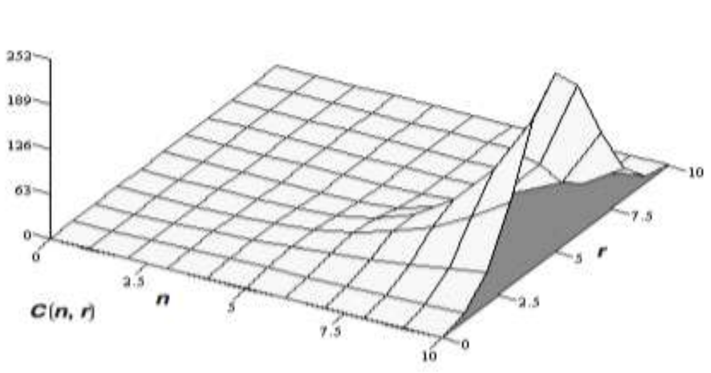
2. How many ways can six people be seated on six chairs?
3. How many permutations of the letters ABCDEF contain the letters DEF together in any order?

4. The professor’s dilemma: how to arrange four books on OS, seven on programming, and three on data structures on a shelf such that books on the same subject must be together?

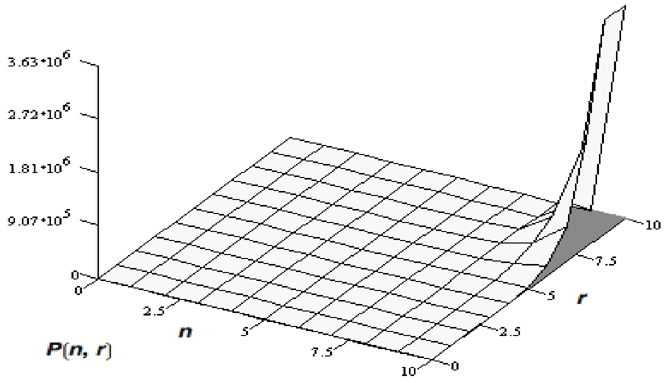
Combinations

- When order in permutations becomes immaterial, i.e. we are just interested in selecting *r* objects from *n* distinct objects, we talk of combinations denoted by

  $C(n,r)$ or ${}_nC_r$
- For each combination, there are $r!$ ways of ordering those r chosen objects
- Hence, from multiplication principle,
- $C(n,r) * r! = P(n,r) \Rightarrow$ $C(n,r) = \dfrac{P(n,r)}{r!} = \dfrac{n!}{(n-r)!r!}$ for $0 \le r \le n$
  - Note: $C(n,r)$ is much smaller than $P(n,r)$ as seen from the graphs below:



(a)



(b)

Combinations: Special Cases

- $C(n,0) = 1$
- Only one way to choose 0 objects from *n* objects - choose the empty set

- $C(n,1) = \text{n}$
- Obvious, since *n* ways to choose one object from *n* objects

- $C(n,n) = 1$
- Only one way to choose *n* objects from *n* objects

Combinations: Examples

- In how many ways can three athletes be declared winners from a group of 10 athletes who compete in an Olympic event?

Combinations: Examples

- How many ways can we select a committee of three from 10?
  - $C(10,3) = 10! / (3! \cdot 7!) = 10 \cdot 9 \cdot 8 / 3 \cdot 2 \cdot 1 = 120$
- How many ways can a committee of two women and three men be selected from a group of five different women and six different men?

- How many five-card poker hands can be dealt from a standard 52-card deck?

- How many five-card poker hands contain cards all of the same suit?

- How many poker hands contain three cards of one denomination and two cards of another denomination?

Eliminating Duplicates

- How many ways can a committee of two be chosen from four men and three women and it must include at least one man.

  Incorrect and impulsive answer = C(4,1)*C(6,1)
  <u>Correct answer</u> = C(7,2) – C(3,2) = C(4,1)*C(6,1) – C(4,2)
  C(7,2) = all committees possible
  C(3,2) = all committees with no men on it
  C(4,2) is the number of committees with two men on it. It has to be subtracted since we are counting it twice in C(4,1)*C(6,1).

- How many distinct permutations can be made from the characters in the word FLORIDA?

  Simple: 7!

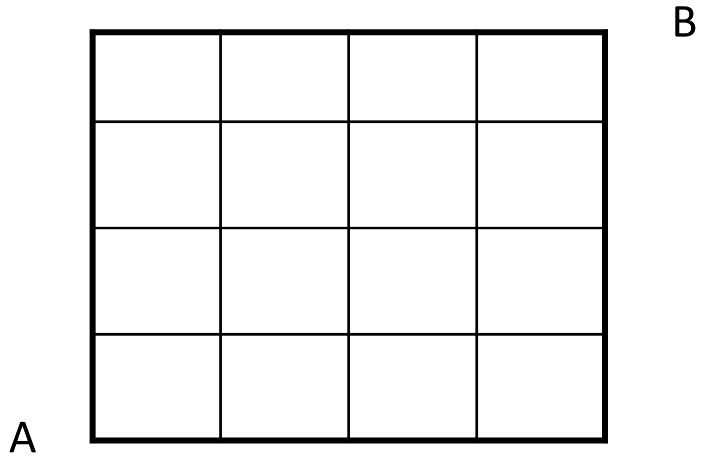
- How many distinct permutations can be made from the characters in the word MISSISSIPPI?

  Since we have more than one S, interchanging the S's at the same position will not result in a distinguishable change. Hence for four S's, 4! possible permutations that look alike.

  Hence total number of permutations = $\dfrac{11!}{4!4!2!}$

Combinations: Examples

- How many routes are there from the lower-left corner of an *n* by *n* square grid to the upper-right corner if we are restricted to traveling only to the right (R) or upward (U)?



Permutations with Repetitions

- Permutations with repetition
- For example, we have 4 numbers, 1,2,3, and 4. We'd like to make a 3-digit numbers.
  - Hundreds digit : 1,2,3,4
  - Tens digit: 1,2,3,4
  - Ones digit: 1,2,3,4
  - 4^3
- Generally, if we choose ***r*** objects out of ***n*** distinct objects with repetition allowed, the permutation is n^r
- Next, combination with repetition allowed

Combinations with Repetitions

- Combinations with repetition
- For example, we have 2 numbers, 1 and 2. We'd like to pick one number three times with repetition allowed (order doesn't matter).
  - (1 1 1), (1 1 2), (1 2 2), (2 2 2)
- Now, try to add 0,1,2 respectively and in that order to the numbers
  - (1,2,3), (1,2,4), (1,3,4), (2,3,4)
- Then the problem turns to combination without repetition allowed
- Initially n=2, r=3, then now C(4,3) ← C(2+3-1,3)
- Therefore, C(n+r-1,r)

Combinations with Repetitions

- Second idea for picking up 3 objects (times) out of 2 distinct objects
- n=2, r=3
- (1 1 1 /) (1 1 / 2) (1 / 2 2) (/ 2 2 2)
- We need to have three (*r*) slots to hold 1,1,1 or 1,1,2 etc
- And we need to have *n*-1 markers (separators) to group them
- This gives *r* + (*n* – 1) slots to fill, and we want to know the number of ways to select r of these.

$$C(r+n-1,r)=\frac{(r+n-1)!}{r!(r+n-1-r)!}=\frac{(r+n-1)!}{r!(n-1)!}$$

Summary of Counting Techniques

TABLE 4.2

You Want to Count the Number of ...	Technique to Try
Subsets of an n-element set	Use formula 2^n.
Outcomes of successive events	Multiply the number of outcomes for each event.
Outcomes of disjoint events	Add the number of outcomes for each event.
Outcomes given specific choices at each step	Draw a decision tree and count the number of paths.
Elements in overlapping sections of related sets	Use principle of inclusion and exclusion formula.
Ordered arrangements of r out of n distinct objects	Use $P(n, r)$ formula.
Ways to select r out of n distinct objects	Use $C(n, r)$ formula.
Ways to select r out of n distinct objects with repetition allowed	Use $C(r + n - 1, r)$ formula.

Class Exercises

- How many permutations of the characters in the word COMPUTER are there? How many of these end in a vowel?
- How many distinct permutations of the characters in ERROR are there?
- In how many ways can you seat 11 men and 8 women in a row if no two women are to sit together?
- A set of four coins is selected from a box containing five dimes and seven quarters.
  - Find the number of sets which has two dimes and two quarters.
  - Find the number of sets composed of all dimes or all quarters.