Odle Math Club

# 2022-2023 Mathcounts Team Selection Test

## Solutions

*prepared by Director Owen Zhang (Class A), with contributions from problem writers*

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The following pages provide solutions to the 2022-2023 Mathcounts Team Selection Test for Odle Middle School. We hope the test was enjoyable!

For all problems, we provide one way of solving the problem that we believe is the most helpful. You may have solved the problem in a variety of other ways.

We would highly appreciate any feedback about the problems, solutions, and logistics of the test! If you have any comments, here are the people in charge of specific portions of this event. For specific problems, the problem writer is listed next to the corresponding solution.

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Tyee Competition Math Club, for the useful LaTeX template.

# 1 Sprint

**Problem 1** *(Owen Zhang).* This is straightforward arithmetic. Adding the numbers gives $\boxed{135800}$. We can also sum each place value individually to get $10+90+700+5000+30000+100000$, which gives the same answer.

**Problem 2** *(Owen Zhang).* Notice that the problem asks for the second smallest positive integer, as the first one is 1, which is trivial. In order for a number to be both a perfect square, perfect cube, and perfect fourth power, all of its prime factors must have an exponent divisible by $2, 3$, and 4. Since $\text{lcm}(2, 3, 4) = 12$, the smallest such number is $X = 2^{12} = 4096$. Since Daniel will be $X$ years old tomorrow, he must be $X - 1$ days old today, or $\boxed{4095}$ days old.

**Problem 3** *(Owen Zhang).* Each hour, the four of them combined can write $1000 + 100 + 10 + 1 = 1111$ problems. Thus, they need $\frac{9999}{1111} = \boxed{9}$ hours to finish the test, as desired.

**Problem 4** *(Gene Yang).* There are $\binom{6}{1}$ ways to choose the pair of socks, $\binom{2}{1}$ ways to choose his water bottle, $\binom{9}{2}$ ways to choose his two pencils, and $\binom{3}{1}$ ways to choose his eraser. Thus, in total, he has

$$\binom{6}{1}\binom{2}{1}\binom{9}{2}\binom{3}{1} = 6(2)(36)(3) = \boxed{1296}$$

ways to pick his items.

**Problem 5** *(Owen Zhang).* Since $33^2 + 44^2 = 55^2$, this is a right triangle, so we can obtain the area by multiplying the two legs and dividing by 2, giving us an area of $\frac{33 \cdot 44}{2} = \boxed{726}$ square units.

**Problem 6** *(Owen Zhang).* Intuitively, we want to use a lot of factors of 2s and 3s to maximize the number of factors. Notice that $\boxed{120}$ is $2^3 \cdot 3 \cdot 5$, and thus has 16 factors. It's easy to check that no other number under 125 has 16 or more factors, so this is our answer.

**Problem 7** *(Gene Yang).* With the 20 frisbees, we can immediately trade to obtain 30 tennis balls. With the 44 golf balls, we can trade to get 18 frisbees, which can then get us 27 more tennis balls. Although we have 2 golf balls leftover, we can not trade anything with them, so our answer is just $\boxed{57}$ tennis balls.

**Problem 8** *(Gene Yang).* This is just a system of equations. Let $h$ be the number of minutes he handwrites, and let $t$ be the number of minutes he types for. Then we have the following system:

$$\begin{cases} 138t + 18h = 337 \\ 23t + 2h = 49 \end{cases}$$

The key observation is that $138 = 23 \cdot 6$. We can now multiply the second equation by 6, and then subtract from the first equation, to get $6h = 43$, so Maximilian should handwrite for $\frac{43}{6}$ minutes or equivalently $\boxed{430}$ seconds.

**Problem 9** *(Owen Zhang).* For simplicity, let's suppose $d = o+1, l = o+2$, and $e = o+3$. Then, we can easily see that $o \leq 3$ gives a sum that is far too large, so we turn our attention to $o = 4$. Now, we see that

$$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{319}{420}$$

is just slightly over our threshold of $\frac{3}{4}$, but using $o = 5$ instead of $o = 4$ decreases the sum by another $\frac{1}{4} - \frac{1}{8}$, giving us a sum that is well under $\frac{3}{4}$. Hence, our smallest solution set is $(o, d, l, e) = (5, 6, 7, 8)$, and thus the product is $\boxed{1680}$.

**Problem 10** *(Owen Zhang).* Although we have three named siblings, notice that we only care about the sum of their ages, so it is irrelevant which name is assigned to which person. Since we have an arithmetic sequence, we must have some common difference, $d$. There are two cases: either $d = 68$, or $d = 34$ (make sure you understand why).

**Claim.** Given $n$ numbers in an arithmetic sequence with common difference $d$, such that $n$ and $d$ are relatively prime, exactly one of the $n$ numbers must be divisible by $n$.

**Proof of Claim.** Let the numbers be $a_0, a_1, a_2, \cdots, a_{n-1}$. Then, the claim is equivalent to showing that $a_i \not\equiv a_j \pmod{n}$ for any $i \neq j$, as the Pigeonhole Principle would imply that at least one of the numbers is $0 \pmod{n}$. Suppose otherwise for the sake of contradiction. Then, $a_i - a_j \equiv 0 \pmod{n}$, so $d(i-j) \equiv 0 \pmod{n}$. But we assumed that $d$ was relatively prime to $n$, so $i - j$ must be a multiple of $n$, which is impossible since $|i - j| < n$. Hence, all the numbers are distinct modulo $n$, and exactly one of them is divisible by $n$ as desired.

Since 68 and 34 are both relatively prime to 3, by the Claim, at least one of the three ages is divisible by 3. Now you may be thinking, how can the ages all be prime then? The answer is that this observation **forces** one of the ages to be 3; in particular the youngest one, since that is the only multiple of 3 that is prime.

Hence, the ages are either $3, 37, 71$ or $3, 71, 139$. One can check that all of these numbers actually work out to be primes, so we desire $M = 213$ and $N = 111$, giving us an answer of $\boxed{102}$.

**Problem 11** *(Owen Zhang).* Holden has enough eggs to make 33 cakes (with 2 eggs leftover), enough sugar to make exactly 32 cakes, and enough flour to make exactly 32 cakes. Thus, he can make at most $\boxed{32}$ cakes, since we can not make a cake if we do not have all the materials.

**Problem 12** *(Owen Zhang).* Notice that $12 + 23 + 34 + 45 + 56 = 170$, so let the other two sides be $a$ and $830 - a$ with $830 - a \geq a$. By the Triangle inequality, we have that $12+23+34+45+56+a > 830-a$, or $170+a > 830-a$, so $2a > 660$, which means $a \geq 331$. Since we want to maximize the difference between $830 - a$ and $a$, which is $830 - 2a$, we want to minimize $a$, so using $a = 331$ gives a difference of $\boxed{168}$, as desired.

**Problem 13** *(Gene Yang).* Notice that all even $n$ satisfy the condition but no odd $n$ do, so the answer is just $\boxed{310}$, as desired.

**Problem 14** *(Gene Yang).* If the height of the resulting cylinder is 9, then the circumference of the circle must be 14, which means the radius is $\frac{7}{\pi}$ giving a volume of $9 \cdot \frac{49}{\pi}$. Similarly the volume when the height of the cylinder is 14 must be $14 \cdot \frac{81}{4\pi}$, giving the ratio between the two as $\boxed{\frac{14}{9}}$.

**Problem 15** *(Owen Zhang).* This is not too difficult to casework, particularly because we can fix Anna's location (and then just multiply by 3, since Anna's location doesn't change the symmetry). Suppose Anna is on the second seat. Then we can casework on the person in the first seat to get a total of 11 possibilities, which we would multiply by 3 to get $\boxed{33}$.

An alternative approach is that the question is asking for $\frac{3}{4}$ the number of derangements of 5, which can be calculated to be $\frac{3}{4} \cdot 44 = 33$.

**Problem 16** *(Nishka Kacheria).* Extend lines $AF$ past $A$ and $BE$ past $B$, labeling the intersection $X$. Similarly, extend lines $BC$ past $C$ and $FD$ past $D$, labeling the intersection $Y$ to form square $FXEY$. Then, notice congruent right triangles $AXE$ and $CYF$ with hypotenuses $AE$ and $FC$. Finally, Pythagorean's Theorem on the edges ($AX = 12$, $XE = 12 + 5 = 17$), gives the solution $12^2 + 17^2 = \boxed{433}$.

**Problem 17** *(Owen Jianwen Zhang).* Notice that the only area in which the snail can not do so is the inner $40 \times 40$ square, which the snail is placed in with probability $\frac{1600}{10000} = \boxed{\frac{4}{25}}$.

**Problem 18** *(Owen Zhang).* This is based off a real life scenario known as "gerrymandering" – the problem in itself is not so difficult, but I thought it would be interesting to put. Notice that the minimum amount of votes required in each committee is 6, and they would need 6 committee points to win – for a total of $\boxed{36}$ votes.

**Problem 19** *(Owen Zhang).* Let $S = \frac{1}{3} + \frac{1}{3^2} + \frac{2}{3^3} + \frac{3}{3^4} + \frac{5}{3^5} + \cdots$. Then, notice that $\frac{1}{3}S = \frac{1}{3^2} + \frac{1}{3^3} + \frac{2}{3^4} + \frac{3}{3^5} + \cdots$, so we can subtract to see that $S - \frac{1}{3}S = \frac{1}{3} + \frac{1}{9}S$, and hence $S = \boxed{\frac{3}{5}}$.

**Problem 20** *(Owen Zhang).* The key observation is that since $\angle DBC = 180 - \angle DEC = \angle AED$, it follows that triangles $ADE$ and $ACB$ are similar. Since $\frac{AD}{AC} = \frac{2}{5}$, we know that $\frac{2}{BC} = \frac{1}{AB} = \frac{2}{5}$, so $BC = 5$ and $AB = \frac{5}{2}$, and the perimeter of the triangle is just $AB + AC + BC = \frac{5}{2} + 5 + 5 = \boxed{\frac{25}{2}}$.

**Problem 21** *(Owen Zhang).* Notice that any triangle with its three vertices drawn on this circle will have three angle measures that are multiples of $\frac{360}{36 \cdot 2} = 5$ degrees, since the inscribed angle is always half of the subtended arc measure. Hence, a kind of cute triangle must have at least one angle measuring exactly 85 degrees. To get an angle of 85 degrees, we need an angle that subtends an arc with exactly 18 points (including the endpoints), since that will produce exactly 17 "gaps" where each gap is a 10 degree arc. There are 36 pairs of points that are exactly 17 "gaps" away, and for each pair, we can choose any of the 18 other points as the third vertex, producing a total of $36 \cdot 18$ triangles. However, the triangles with **two**

85 degree angles are double counted, so we subtract by 36, giving a final of $36 \cdot 17 = \boxed{612}$ triangles.

**Problem 22** *(Gene Yang).* Since 24 has exactly one power of 3 in it, there must be either a 3x magnifying glass or a 6x magnifying glass. If there is a 3x magnifying glass, then there is a factor of 8 remaining, so Ethan either uses $\{2, 4, 3\}$ or $\{2, 2, 2, 3\}$ in some order. If there is a 6x magnifying glass, then there is a factor of 4 remaining, so Gene either uses $\{4, 6\}$ or $\{2, 2, 6\}$. Now, we can add up $3! + \frac{4!}{3!} + 2! + \frac{3!}{2!}$ to get a total of $6 + 4 + 2 + 3 = \boxed{15}$ ways.

**Problem 23** *(Owen Zhang).* To get to the point $(20, 22)$, Peter must get to $(20, 20)$ first, so let's just consider his journey from the origin to $(20, 20)$. The key observation is that he can only move 5 units upward or to the right at a time – otherwise, neither coordinate will be divisible by 5. In other words, we can define a new operation $A$ which is equivalent to moving $U$ five times in a row, and a new operation $B$ which is equivalent to moving $R$ five times in a row. Now the question just asks how many ways we can rearrange 4 A's and 4 B's, which is just $\binom{8}{4} = \boxed{70}$ ways.

**Problem 24** *(Owen Zhang).* Let $O$ be the circumcenter of $ABC$. Then, notice that $[ABDEF] = [AOD]+[DOE]+[EOF]+[FOB]-[AOB]$, and since $[AOB]$ is constant, we just seek to maximize $[AOD]+[DOE]+[EOF]+[FOB]$. Even better, since $AO, DO, EO, FO, BO$ are all constant, we just seek to maximize $\sin(AOD) + \sin(DOE) + \sin(EOF) + \sin(FOB)$. For simplicity, let those angles be rewritten as $w, x, y$, and $z$ respectively. Now, since sine is a concave function over $[0, \pi]$, we have by Jensen's inequality that

$$\frac{\sin(w) + \sin(x) + \sin(y) + \sin(z)}{4} \leq \sin\left(\frac{w + x + y + z}{4}\right) = \sin(30^\circ),$$

so the sum of the sines is at most $4\sin(30^\circ) = 2$, which is clearly achievable by letting $D, E$, and $F$ be equidistant. Now, we can just sum the triangles and subtract $[AOB]$ to get a final maximum of

$$4 \cdot \frac{1}{2} \cdot \left(\frac{2\sqrt{3}}{3}\right)^2 \cdot \sin(30^\circ) - \frac{\sqrt{3}}{3} = \boxed{\frac{4 - \sqrt{3}}{3}}.$$

**Problem 25** *(Owen Zhang).* Suppose we have a valid rolling, and Let the rolls be $a < b < c$. We can use stars and bars on the "gaps" between the rolls to get that there are $\binom{14}{3} = 364$ different sets of rols we could have, which we multiply by 3! to get 2184 possible ordered triplets, for a final answer of $\frac{2184}{20^3} = \boxed{\frac{273}{1000}}$.

**Problem 26** *(Owen Zhang).* Notice that for all $m \geq 7$, $m!$ is divisible by every one-digit number. Thus we just need to consider what numbers $1! + 2! + 3! + 4! + 5! + 6! = 873$ is divisible by. Clearly it is odd, so we immediately eliminate $2, 4, 6$, and 8. We can check that the sum of the digits is 18, so it **is** divisible by 3 and 9. Finally, it is obviously not divisible by 5, and we can check that it is also not divisible by 7. Thus, our answer is just $1+3+9 = \boxed{13}$.

**Problem 27** *(Owen Zhang).* Let $y = x - 15$. Then, we can rewrite the equation as

$$\frac{1}{y + 12} + \frac{1}{y + 4} + \frac{1}{y - 4} + \frac{1}{y - 12} = 0,$$

or

$$\frac{2y}{y^2-144}+\frac{2y}{y^2-16}=0.$$

Letting $z=y^2-80$, we now have

$$\frac{\sqrt{z+80}}{z-64}+\frac{\sqrt{z+80}}{z+64}=0,$$

or

$$\frac{2z\sqrt{z+80}}{z^2-64^2}=0,$$

so we must have $2z\sqrt{z+80}=0$. If $z=0$, then we know that $y=\pm\sqrt{80}$, so $x=15\pm 4\sqrt{5}$. If $\sqrt{z+80}=0$, then we know that $z=-80$, so $y=0$, and thus $x=15$ is a solution. Thus in total we have **3** distinct solutions that sum to **45** and multiply to **2175**, and our answer is $r+s+p=3+45+2175=\boxed{2223}$.

**Problem 28** *(Owen Jianwen Zhang).* Notice that we can rewrite

$$f(x)=(x(x-6))^2+18x+12,$$

so

$$f(a)+f(b)+f(c)=18(a+b+c)+12\cdot 3+\sum_{\text{cyc}}(a(a-6))^2.$$

By the Trivial Inequality, $\sum_{\text{cyc}}(a(a-6))^2\geq 0$, so clearly $f(a)+f(b)+f(c)\geq 252$. Indeed, $a=0, b=c=6$ achieves equality, so $\boxed{252}$ is the desired minimum.

**Problem 29** *(Gene Yang).* Notice that drawing the height $H$ down from $B$ onto $AC$ splits $ABC$ into a 5-12-13 and a 9-12-15 triangle. Let the midpoint of $AC$ be $M$, so that $AM=CM=7$. Now, since triangles $CMD$ and $CHB$ are similar, we must have $\frac{CD}{CB}=\frac{CM}{CH}$ or $\frac{CD}{15}=\frac{7}{9}$. Thus $CD=\frac{35}{3}$, and $BD=15-35/3=10/3$, giving us a final answer of $\boxed{\frac{7}{2}}$.

**Problem 30** *(Owen Zhang).* For each element, we can either put it in none of the subsets, just $A$, just $B$, just $C$, or just $A$ and $C$, which is 5 choices per element. Thus the answer is just $5^5=\boxed{3125}$.

## 2 Target

**Problem 1** *(Owen Zhang).* Each time I roll the dice, it has a $\frac{1}{2}$ chance of returning an even number. Thus, since all rolls are independent, the answer is $\left(\frac{1}{2}\right)^5 = \boxed{\dfrac{1}{32}}$.

**Problem 2** *(Owen Zhang).* Let the three numbers in $S$ be $x, y, z$ with $x \leq y \leq z$. Since there is a unique mode, we either have $x = y, y = z$, or $x = y = z$. Consider the case that $x = y$. Then, the mean is $\frac{2y+z}{3}$, while the median is $y$, but since they are equal this forces $y = z$, so $x = y = z$. Similarly, if $y = z$, then we must have $x = y$ as well. Hence, all three elements have to be the same for the condition to hold, and the answer is just $\boxed{100}$.

**Problem 3** *(Owen Jianwen Zhang).* It is easy to see that after an odd number of moves, Julian must be at an odd number, since on every move you either move from an even number to an odd number or vice versa. The furthest left he can go is $-2023$ and the furthest right he can go is 2023, for a total of $\boxed{2024}$ locations.

**Problem 4** *(Hanting Li).* Notice that the total cost of the letters is $27 \cdot 26 + (1 + 2 + \cdots + 26) = \frac{3}{2} \cdot 27 \cdot 26 = 1053$ cents, which can be paid in 42 quarters and 3 pennies for a total of $\boxed{45}$ coins.

**Problem 5** *(Gene Yang).* The difference in volume between a 2 meter radius sphere and a 1 meter radius sphere is $\frac{4}{3}\pi(2^3 - 1^3) = \frac{28}{3}\pi$. Since air weighs 1.2 kilograms per cubic meter, we can multiply to get $\frac{28}{3}\pi \cdot 1.2$ kilograms, which is $\approx \boxed{35186}$ grams.

**Problem 6** *(Owen Zhang).* Since $\frac{5}{\frac{15}{2}} = \frac{8}{12}$, triangles $ADE$ and $ABC$ are similar. Let $[AGE] = x$. Then, $[AGD] = \frac{2}{3}x$ and $[AFB] = [AGD] \cdot \left(\frac{8}{5}\right)^2 = \frac{64}{25} \cdot \frac{2}{3}x = \frac{128}{75}x$, giving us $[DGBF] = [AFB] - [ADE] = \frac{78}{75}x$, so the desired ratio is just $\boxed{\dfrac{26}{25}}$.

**Problem 7** *(Owen Zhang).* Let $f(n) = 1^1 + 2^2 + \cdots + n^n$. Notice that the sequence $f(n) \pmod 3$ repeats mod 18, and $f(n) \equiv 0 \pmod 3$ only when $n \equiv 0, 4, 7, 14, 15, 17 \pmod{18}$. Thus, to get the 2022nd smallest solution, we just need $337 \cdot 18 = \boxed{6066}$.

**Problem 8** *(Owen Zhang).* Let $n$ be called "special" if $\partial(\partial(n)) = n$ and $\partial(n) \neq n$. Notice that if $n$ is special, then $\partial(n)$ is necessarily special as well, so special numbers come in pairs. Thus, $\partial(871) = 45361$ is one of the other three numbers. Our intuition then drifts to 872, since 872 is one more than 871 **and** $\partial(872) = \partial(871) + 1$. Indeed, we can check that $\partial(872) = 45362$, and $\partial(45362) = 872$, so our other two numbers are 872 and 45362, giving us a sum of $\boxed{91595}$.

*Notes:* This property of a number is actually relatively well-known, and several research papers have been written proving that these are the only four such numbers. Of course, you did not need to know this to solve the problem, as being given 871 simplified it significantly. To learn more, search up the keyterms "factorion" and "amicable factorion." Specifically, the numbers in question here are factorions in a cycle of length 2.