

distributions_math

Waveley Qiu (wq2162)

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Full Posterior Distribution

$$\begin{aligned} Y_i(t+6) &= \beta_{0i} + \beta_{1i}Y_i(t) + \beta_{2i}\Delta_{i1}(t) + \beta_{3i}\Delta_{i2}(t) + \beta_{4i}\Delta_{i3}(t) + \varepsilon_i(t) \\ &= \eta_i(t) \end{aligned}$$

$$Y = (Y_1, Y_2, \dots, Y_n)$$

$$\text{Let } \eta = (\eta_1, \eta_2, \dots, \eta_n), \text{ where } \eta_i = X_i \beta_i^T$$

$$Y_i \sim MVN(\eta_i, \sigma^2 I_{n_i})$$

$$f(Y_i|B, \mu, \Sigma, \sigma^2) = (2\pi)^{-\frac{n_i}{2}} |\sigma^2 I_{n_i}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2} (Y_i - X_i \beta_i^T)^T (\sigma^2 I_{n_i})^{-1} (Y_i - X_i \beta_i^T)\right)$$

$$\begin{aligned} L(Y|B, \beta, \Sigma, \sigma^2) &= \prod_{i=1}^N f(Y_i|B, \beta, \Sigma, \sigma^2) \\ &= \prod_{i=1}^N (2\pi)^{-\frac{n_i}{2}} |\sigma^2 I_{n_i}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2} (Y_i - X_i \beta_i^T)^T (\sigma^2 I_{n_i})^{-1} (Y_i - X_i \beta_i^T)\right) \end{aligned}$$

$$B = (\beta_1^T, \beta_2^T, \dots, \beta_n^T)^T$$

$$\beta_i \sim MVN(\mu, \Sigma)$$

$$f(B|\beta, \Sigma) = \prod_{i=1}^n (2\pi)^{-\frac{5}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2} (\beta_i - \mu)^T \Sigma^{-1} (\beta_i - \mu)\right)$$

$$f(\mu) \propto 1$$

$$f(\Sigma) \propto |\Sigma|^{-(d+1)} \exp\left(-\frac{1}{2} \Sigma^{-1}\right), \text{ where } d = 5$$

$$f(\sigma^2) \propto \frac{1}{\sigma^2}$$

$$\begin{aligned}
&\Rightarrow f(B, \mu, \Sigma, \sigma^2 | Y) \propto L(Y | B, \beta, \Sigma, \sigma^2) f(B, \beta, \Sigma, \sigma^2) \\
&= L(Y | B, \beta, \Sigma, \sigma^2) f(B | \beta, \Sigma) f(\beta) f(\Sigma) f(\sigma^2) \\
&= \prod_{i=1}^N (2\pi)^{\frac{n_i}{2}} |\sigma^2 I_{n_i}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(Y_i - X_i \beta_i)^T (\sigma^2 I_{n_i})^{-1} (Y_i - X_i \beta_i^T)\right) |\Sigma|^{-\frac{n}{2}} \\
&\quad \exp\left(-\frac{1}{2}(\beta_i - \mu)^T \Sigma^{-1} (\beta_i - \mu)\right) |\Sigma|^{-(d+1)} \exp\left(-\frac{1}{2}\Sigma^{-1}\right) \frac{1}{\sigma^2} \\
&= \prod_{i=1}^N |\sigma^2 I_{n_i}|^{-\frac{1}{2}} |\Sigma|^{-(d+\frac{n}{2}+1)} \frac{1}{\sigma^2} \exp\left[-\frac{1}{2}[(Y_i - X_i \beta_i^T)^T (\sigma^2 I_{n_i})^{-1} (Y_i - X_i \beta_i^T) + (\beta_i - \mu)^T \Sigma^{-1} (\beta_i - \mu)]\right] \exp\left(-\frac{1}{2}\text{trace}(\Sigma^{-1})\right) \\
&= \prod_{i=1}^N \sigma^{2n_i(-\frac{1}{2})} |\Sigma^{-1}|^{(d+\frac{n}{2}+1)} \sigma^{-2} \exp\left[-\frac{1}{2}[(Y_i - X_i \beta_i^T)^T (\sigma^2 I_{n_i})^{-1} (Y_i - X_i \beta_i^T) + (\beta_i - \mu)^T \Sigma^{-1} (\beta_i - \mu)]\right] \exp\left(-\frac{1}{2}\text{trace}(\Sigma^{-1})\right) \\
&= \prod_{i=1}^N \sigma^{-n_i} |A|^{d+\frac{n}{2}+1} \sigma^{-2} \exp\left[-\frac{1}{2}[(Y_i - X_i \beta_i^T)^T (\sigma^2 I_{n_i})^{-1} (Y_i - X_i \beta_i^T) + (\beta_i - \mu)^T A (\beta_i - \mu)]\right] \exp\left(-\frac{1}{2}\text{trace}A\right) \\
&= \sigma^{-\sum_{i=1}^N n_i - 2} |A|^{d+\frac{n}{2}+1} \exp\left[-\frac{1}{2} \sum_{i=1}^N [(Y_i - X_i \beta_i^T)^T (\sigma^2 I_{n_i})^{-1} (Y_i - X_i \beta_i^T) + (\beta_i - \mu)^T A (\beta_i - \mu)]\right] \exp\left[-\frac{1}{2}\text{trace}A\right]
\end{aligned}$$

Conditional distribution of σ^2

$$\begin{aligned}
f(\sigma^2 | B, \mu, \Sigma, Y) &\propto f(B, \mu, \Sigma, \sigma^2, Y) \\
&\propto \sigma^{-\sum_{i=1}^N n_i - 2} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^N [(Y_i - X_i \beta_i^T)^T I_{n_i} (Y_i - X_i \beta_i^T)]\right] \\
&= (\sigma^{-2})^{\frac{\sum_{i=1}^N n_i}{2} + 1} \exp\left[-\frac{1}{2} \sum_{i=1}^N [(Y_i - X_i \beta_i^T)^T (Y_i - X_i \beta_i^T)] \sigma^{-2}\right] \\
&= \left(\frac{1}{\sigma^2}\right)^{\frac{\sum_{i=1}^N n_i}{2} + 1} \exp\left[-\frac{1}{2} \sum_{i=1}^N \sum_{t=1}^{n_i} [(y_{it} - x_{it} \beta_i^T)^2] \frac{1}{\sigma^2}\right]
\end{aligned}$$

Let $W = \sigma^2$. It follows:

$$\begin{aligned}
&\left(\frac{1}{\sigma^2}\right)^{\frac{\sum_{i=1}^N n_i}{2} + 1} \exp\left[-\frac{1}{2} \sum_{i=1}^N [(Y_i - X_i \beta_i^T)^T (Y_i - X_i \beta_i^T)] \frac{1}{\sigma^2}\right] \\
&= \left(\frac{1}{W}\right)^{\frac{\sum_{i=1}^N n_i}{2} + 1} \exp\left[-\frac{1}{2} \sum_{i=1}^N [(Y_i - X_i \beta_i^T)^T (Y_i - X_i \beta_i^T)] \frac{1}{W}\right] \\
&\Rightarrow W \sim \text{inverse Gamma}\left(\frac{\sum_{i=1}^N n_i}{2}, \frac{1}{2} \sum_{i=1}^N \sum_{t=1}^{n_i} (y_{it} - x_{it} \beta_i^T)^2\right)
\end{aligned}$$

Conditional Distribution of Big Sigma

$$\begin{aligned}
f(\Sigma^{-1}|B, \mu, \sigma^2, Y) &\propto |A|^{d+\frac{N}{2}+1} \exp \left[-\frac{1}{2} \sum_{i=1}^N (\beta_i - \mu)^T A (\beta_i - \mu) \right] \exp(-\frac{1}{2} \text{trace} A) \\
&= |A|^{d+\frac{N}{2}+1} \exp \left[-\frac{1}{2} \text{trace} \left(A + \sum_{i=1}^N (\beta_i - \mu)^T A (\beta_i - \mu) \right) \right] \\
&= |A|^{d+\frac{N}{2}+1} \exp \left[-\frac{1}{2} \text{trace} \left(A \left(I + \sum_{i=1}^N (\beta_i - \mu)(\beta_i - \mu)^T \right) \right) \right]
\end{aligned}$$

$$V^{-1} = I + \sum_{i=1}^N (\beta_i - \mu)(\beta_i - \mu)^T$$

$$\frac{1}{2}(n^* - p - 1) = d + \frac{N}{2} + 1$$

$$\implies n^* = 3d + N + 2$$

Thus, $A = \Sigma^{-1} \sim \text{Wishart} \left(n^*, I + \sum_{i=1}^N (\beta_i - \mu)(\beta_i - \mu)^T \right)$