Project3

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$$Y_i(t+b) = \underbrace{\beta_{0i} + \beta_{1i} Y_i(t) + \beta_{2i}\cdot\Delta_{i1}(t) + \beta_{3i}\cdot\Delta_{i2}(t) + \beta_{4i}\cdot\Delta_{i3}(t)}_{\text{"}X_i\beta_i^T\text{"}} + \varepsilon_i(t)$$

let $X_i\beta_i^T = \beta_{0i} + \beta_{1i}\cdot Y_i(t) + \beta_{2i}\cdot\Delta_{i1}(t) + \beta_{3i}\cdot\Delta_{i2}(t) + \beta_{4i}\cdot\Delta_{i3}(t)$

$\therefore\ Y_j \sim N(X_i\beta_i^T, \sigma^2)$

$\therefore$ pdf of $Y_i(t)$: $f(Y_{ji} \mid B, \beta, \Sigma, \sigma^2) = \prod_{j=1}^{n_j} \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot \exp\left(-\frac{(Y_{ji}-X_j\beta_i^T)^2}{2\sigma^2}\right)$ "only consider $Y_i(t)$ has a corresponding $Y_i(t-b)$"

$\because\ Y = (Y_1, Y_2, \cdots, Y_N)$

$\therefore\ Y_i \sim MVN(X_i\beta_i^T, \sigma^2 I_{n_i})$

$\therefore\ f(Y_i \mid B, \beta, \Sigma, \sigma^2) = (2\pi)^{-\frac{n_i}{2}}\cdot |\det(\sigma^2 I_{n_i})|^{-\frac{1}{2}}\cdot \exp\left(-\frac{1}{2}(Y_i - X_i\beta_i^T)^T\cdot(\sigma^2 I_{n_i})^{-1}\cdot(Y_i - X_i\beta_i^T)\right)$

$\therefore\ L(Y \mid B, \beta, \Sigma, \sigma^2) = \prod_{i=1}^{N} f(Y_i \mid B, \beta, \Sigma, \sigma^2) = \prod_{i=1}^{N} (2\pi)^{-\frac{n_i}{2}}\cdot |\det(\sigma^2 I_{n_i})|^{-\frac{1}{2}}\cdot \exp\left(-\frac{1}{2}(Y_i - X_i\beta_i^T)^T\cdot(\sigma^2 I_{n_i})^{-1}\cdot(Y_i - X_i\beta_i^T)\right)$

$\because\ B = (\beta_1^T; \beta_2^T, \cdots, \beta_N^T)^T$

$\because\ \beta_i \sim MVN(\mu, \Sigma)$

$\therefore\ f(B \mid \beta, \Sigma) = \prod_{i=1}^{N} (2\pi)^{-\frac{5}{2}}\cdot |\Sigma|^{-\frac{1}{2}}\cdot \exp\left(-\frac{1}{2}(\beta_i-\mu)^T\cdot\Sigma^{-1}(\beta_i-\mu)\right)$

$f(\beta) \propto 1$

$f(\Sigma) \propto |\Sigma|^{-(d+1)}\cdot\exp\left(-\frac{1}{2}tr(\Sigma^{-1})\right)$, where $d=5$

$f(\sigma^2) \propto \frac{1}{\sigma^2}$

$\Rightarrow f(B, \beta, \Sigma, \sigma^2 \mid Y) \propto L(Y \mid B, \beta, \Sigma, \sigma^2)\cdot f(B, \beta, \Sigma, \sigma^2)$

$= L(Y \mid B, \beta, \Sigma, \sigma^2)\cdot f(B \mid \beta, \Sigma)\cdot f(\beta)\cdot f(\Sigma)\cdot f(\sigma^2)$

$= \prod_{i=1}^{N} (2\pi)^{-\frac{n_i}{2}}\cdot |\det(\sigma^2 I_{n_i})|^{-\frac{1}{2}}\cdot\exp\left(-\frac{1}{2}(Y_i - X_i\beta_i^T)^T\cdot(\sigma^2 I_{n_i})^{-1}\cdot(Y_i-X_i\beta_i^T)\right)\cdot(2\pi)^{-\frac{5}{2}}\cdot|\Sigma|^{-\frac{N}{2}}\cdot\exp\left(-\frac{1}{2}(\beta_i-\mu)^T\cdot\Sigma^{-1}(\beta_i-\mu)\right)$
$|\Sigma|^{-(d+1)}\cdot\exp\left(-\frac{1}{2}tr(\Sigma^{-1})\right)\cdot\frac{1}{\sigma^2}$

$= \prod_{i=1}^{N} |\det(\sigma^2 I_{n_i})|^{-\frac{1}{2}}\cdot|\Sigma|^{-(d+\frac{N}{2}+1)}\cdot\frac{1}{\sigma^2}\cdot\exp\left[-\frac{1}{2}\left[(Y_i-X_i\beta_i^T)^T\cdot(\sigma^2 I_{n_i})^{-1}\cdot(Y_i-X_i\beta_i^T) + (\beta_i-\mu)^T\cdot\Sigma^{-1}\cdot(\beta_i-\mu)\right]\right]\cdot\exp\left[-\frac{1}{2}tr(\Sigma^{-1})\right]$ $\because |A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

replace $\Sigma^{-1} = A$

$= \prod_{i=1}^{N} \sigma^{2\cdot n_i\cdot-\frac{1}{2}}\cdot|\Sigma^{-1}|^{d+\frac{N}{2}+1}\cdot\sigma^{-2}\cdot\exp\left[-\frac{1}{2}\left[(Y_i-X_i\beta_i^T)^T\cdot(\sigma^2 I_{n_i})^{-1}\cdot(Y_i-X_i\beta_i^T) + (\beta_i-\mu)^T\cdot\Sigma^{-1}\cdot(\beta_i-\mu)\right]\right]\cdot\exp\left[-\frac{1}{2}tr(\Sigma^{-1})\right]$

$= \prod_{i=1}^{N} \sigma^{-n_i}\cdot|A|^{d+\frac{3}{2}}\cdot\sigma^{-2}\cdot\exp\left[-\frac{1}{2}\left[(Y_i-X_i\beta_i^T)^T\cdot(\sigma^2 I_{n_i})^{-1}\cdot(Y_i-X_i\beta_i^T) + (\beta_i-\mu)^T\cdot A\cdot(\beta_i-\mu)\right]\right]\cdot\exp\left[-\frac{1}{2}tr(A)\right]$

$= \sigma^{-\sum_{i=1}^{N} n_i-2}\cdot|A|^{d+\frac{N}{2}+1}\cdot\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left[(Y_i-X_i\beta_i^T)^T\cdot(\sigma^2 I_{n_i})^{-1}\cdot(Y_i-X_i\beta_i^T) + (\beta_i-\mu)^T A(\beta_i-\mu)\right]\right]\cdot\exp\left[-\frac{1}{2}tr(A)\right]$

① $f(\sigma^2 \mid B, \mu, \Sigma, Y) \propto f(B, \mu, \Sigma, \sigma^2, Y) \propto \sigma^{-\sum n_i-2}\cdot\exp\left[\underbrace{-\frac{1}{2\sigma^2}}_{-\frac{1}{2}\cdot\sigma^{-2}}\cdot\sum_{i=1}^{N}\left[(y_i-x_i\beta_i^T)^T\cdot I\cdot(y_i-x_i\beta_i^T)\right]\right].$

$= (\sigma^{-2})^{\frac{\sum n_i}{2}+1}\cdot\exp\left[-\frac{1}{2}\cdot\sum_{i=1}^{N}\left[(y_i-x_i\beta_i^T)^T\cdot(y_i-x_i\beta_i^T)\right]\cdot\sigma^{-2}\right].$

(let $w = \sigma^2$)

$= (\sigma^{-2})^{\frac{\sum n_i}{2}+1}\cdot\exp\left[-\frac{1}{2}\cdot\sum_{i=1}^{N}\sum_{t=1}^{n_i}(y_{it} - x_{it}\cdot\beta_i^T)^2\cdot\sigma^{-2}\right]$

$= \frac{1}{\sigma^2}^{\frac{\sum n_i}{2}+1}\cdot\exp\left[-\frac{1}{2}\cdot\sum_{i=1}^{N}\sum_{t=1}^{n_i}(y_{it} - x_{it}\cdot\beta_i^T)^2\cdot\frac{1}{\sigma^2}\right]$

$= \frac{1}{w}^{\overbrace{\boxed{\frac{\sum n_i}{2}}}^{}_{\alpha}+1}\cdot\exp\left[-\underbrace{\boxed{\frac{1}{2}\cdot\sum_{i=1}^{N}\sum_{t=1}^{n_i}(y_{it} - x_{it}\cdot\beta_i^T)^2}}_{\beta}\cdot\frac{1}{w}\right]$

$w \sim$ inverse Gamma $\left(\frac{\sum n_i}{2}, \frac{1}{2}\sum_{i=1}^{N}\sum_{t=1}^{n_i}(y_{it}-x_{it}\cdot\beta_i^T)^2\right)$

② $f(\overset{A}{\Sigma^{-1}} \mid B, \mu, \sigma^2, Y) \propto |A|^{d+\frac{N}{2}+1}\cdot\exp\left[-\frac{1}{2}\sum_{i=1}^{N}(\beta_i-\mu)^T\cdot A\cdot(\beta_i-\mu)\right]\cdot\exp\left(-\frac{1}{2}tr(A)\right)$

$= |A|^{d+\frac{N}{2}+1}\cdot\exp\left[-\frac{1}{2}tr\left(A + \sum_{i=1}^{N}(\beta_i-\mu)^T\cdot A\cdot(\beta_i-\mu)\right)\right]$

$= |A|^{d+\frac{N}{2}+1}\cdot\exp\left[-\frac{1}{2}tr\left(A\left(I + \sum_{i=1}^{N}(\beta_i-\mu)(\beta_i-\mu)^T\right)\right)\right]$

$= |A|^{\underbrace{d+\frac{N}{2}+1}_{\frac{1}{2}(n^*-p-1)}}\cdot\exp\left[-\frac{1}{2}\cdot tr\left(A\underbrace{\left(I + \sum_{i=1}^{N}(\beta_i-\mu)(\beta_i-\mu)^T\right)}_{V^{-1}}\right)\right]$

$d + \frac{N}{2} + 1 = \frac{1}{2}(n^* - p - 1)$ $\because p = d$

$2d + N + 1 = n^* - d - 1$

$n^* = 3d + N + 2$

$A \sim \text{Wishart}\left(n^*, I + \sum_{i=1}^{N}(\beta_i-\mu)^T(\beta_i-\mu)\right)$

③ $f(\mu \mid \cdots) \propto \exp\left(-\frac{1}{2}\sum_{i=1}^{N}(\beta_i-\mu)^T\cdot\Sigma^{-1}(\beta_i-\mu)\right)$

$\propto \exp\left(\frac{1}{2}\left(\underbrace{\boxed{\sum_{i=1}^{N}\beta_i^T\Sigma^{-1}\beta_i}}_{R} + \mu^T\underbrace{\boxed{N\Sigma^{-1}}}_{\substack{\text{variance}\\ \Downarrow\\ V}}\mu - \underbrace{\boxed{2\sum_{i=1}^{N}\beta_i^T\Sigma^{-1}}}_{\substack{\text{mean}\\ \Downarrow\\ M=\sum_{i=1}^{n}(\Sigma^{-1}\beta_i)}}\mu\right)\right)$ ①

$\Rightarrow V = N\Sigma^{-1}$

$R = \sum_{i=1}^{N}\beta_i^T\Sigma^{-1}\beta_i$

$M = \sum_{i=1}^{N}(\Sigma^{-1}\beta_i)$

$\Rightarrow$ ① $\propto R + \beta V\beta - 2M\beta \propto (\beta - V^{-1}M)^T V(\beta - V^{-1}M)$

$\mu \sim MVN(V^{-1}M, V^{-1})$

④. $f(\beta_i \mid \cdots) \propto \exp\left(-\frac{1}{2}\left[(Y_i - X_i\beta_i^T)^T(\sigma^2 I_{n_i})^{-1}(Y_i - X_i\beta_i^T) + (\beta_i-\mu)^T\Sigma^{-1}(\beta_i-\mu)\right)\right)$

$= Y_i^T\sigma^{-2}I_{n_i}Y_i^T + \beta_i^T X_i^T\sigma^{-2}I_{n_i}X_i\beta_i - 2Y_i^T\sigma^{-2}I_{n_i}X_i\beta_i + \beta_i^T\Sigma^{-1}\beta_i + \mu^T\Sigma^{-1}\mu - 2\mu^T\Sigma^{-1}\beta_i$

$= \underbrace{Y_i^T\sigma^{-2}I_{n_i}Y_i + \mu^T\Sigma^{-1}\mu}_{R} + \beta_i^T(\underbrace{\Sigma^{-1} + X_i^T\sigma^{-2}I_{n_i}X_i}_{V})\beta_i - 2(\underbrace{Y_i^T\sigma^{-2}I_{n_i}X_i + \mu^T\Sigma^{-1}}_{M})\beta_i$

$= R + \beta_i^T V\beta_i - 2M\beta_i \quad \cdots (1)$

where : $R = Y_i^T\sigma^{-2}I_{n_i}Y_i + \mu^T\Sigma^{-1}\mu$

$V = \Sigma^{-1} + X_i^T\sigma^{-2}I_{n_i}X_i$

$M = \sigma^{-2}X_i^T Y_i + \Sigma^{-1}\mu$

(1) $\propto (\beta_i - V^{-1}M)^T\cdot V(\beta_i - V^{-1}M)$

$\therefore f(\beta_i \mid \cdots) \sim N(V^{-1}M, V^{-1})$