distributions_math

Waveley Qiu (wq2162)

2022-05-04

Full Posterior Distribution

$$
\begin{aligned}
Y_i(t+6) &= \beta_{0i} + \beta_{1i} Y_i(t) + \beta_{2i}\Delta_{i1}(t) + \beta_{3i}\Delta_{i2}(t) + \beta_{4i}\Delta_{i3}(t) + \varepsilon_i(t) \\
&= \eta_i(t)
\end{aligned}
$$

$$
\begin{aligned}
Y &= (Y_1, Y_2, ..., Y_n) \\
\text{Let } \eta &= (\eta_1, \eta_2, ..., \eta_n), \text{ where } \eta_i = X_i \beta_i^T \\
Y_i &\sim MVN(\eta_i, \sigma^2 I_{n_i}) \\
f(Y_i | B, \mu, \Sigma, \sigma^2) &= (2\pi)^{\frac{n_i}{2}} \left|\sigma^2 I_{n_i}\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\left(Y_i - X_i\beta_i^T\right)(\sigma^2 I_{n_i})^{-1}(Y_i - X_i\beta_i^T)\right) \\
L(Y | B, \beta, \Sigma, \sigma^2) &= \prod_{i=1}^{N} f(Y_i | B, \beta, \Sigma, \sigma^2) \\
&= \prod_{i=1}^{N} (2\pi)^{-\frac{n_i}{2}} \left|\sigma^2 I_{n_i}\right|^{-\frac{1}{2}} \exp(-\frac{1}{2}(Y_i - X_i\beta_i^T)^T (\sigma^2 I_{n_i})^{-1} (Y_i - X_i\beta_i^T)^T) \\
B &= (\beta_1^T, \beta_2^T, ..., \beta_n^T)^T \\
\beta_i &\sim MVN(\mu, \Sigma) \\
f(B | \beta, \Sigma) &= \prod_{i=1}^{n} (2\pi)^{-\frac{5}{2}} |\sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\beta_i - \mu)^T \Sigma^{-1} (\beta_i - \mu)\right) \\
f(\mu) &\propto 1 \\
f(\Sigma) &\propto |\Sigma|^{-(d+1)} \exp\left(-\frac{1}{2}\Sigma^{-1}\right), \text{ where } d = 5 \\
f(\sigma^2) &\propto \frac{1}{\sigma^2}
\end{aligned}
$$

$$
\begin{aligned}
&\implies f(B,\mu,\Sigma,\sigma^2|Y) \propto L(Y|B,\beta,\Sigma,\sigma^2)f(B,\beta,\Sigma,\sigma^2)\\
&= L(Y|B,\beta,\Sigma,\sigma^2)f(B|\beta,\Sigma)f(\beta)f(\Sigma)f(\sigma^2)\\
&= \prod_{i=1}^{N}(2\pi)^{\frac{n_i}{2}}\left|\sigma^2 I_{n_i}\right|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}(Y_i - X_i\beta_i)^T(\sigma^2 I_{n_i})^{-1}(Y_i - X_i\beta_i^T)\right)|\Sigma|^{-\frac{n}{2}}\\
&\quad \exp\left(-\frac{1}{2}(\beta_i-\mu)^T\Sigma^{-1}(\beta_i-\mu)\right)|\Sigma|^{-(d+1)}\exp(-\frac{1}{2}\Sigma^{-1})\frac{1}{\sigma^2}\\
&= \prod_{i=1}^{N}\left|\sigma^2 I_{n_i}\right|^{-\frac{1}{2}}|\Sigma|^{-(d+\frac{n}{2}+1)}\frac{1}{\sigma^2}\exp\left[-\frac{1}{2}\left[(Y_i - X_i\beta_i^T)^T(\sigma^2 I_{n_i})^{-1}(Y_i - X_i\beta_i^T) + (\beta_i-\mu)^T\Sigma^{-1}(\beta_i-\mu)\right]\right]\exp(-\frac{1}{2}\text{trace}(\Sigma^{-1}))\\
&= \prod_{i=1}^{N}\sigma^{2n_i(-\frac{1}{2})}|\Sigma^{-1}|^{(d+\frac{N}{2}+1)}\sigma^{-2}\exp\left[-\frac{1}{2}\left[(Y_i - X_i\beta_i^T)^T(\sigma^2 I_{n_i})^{-1}(Y_i - X_i\beta_i^T) + (\beta_i-\mu)^T\Sigma^{-1}(\beta_i-\mu)\right]\right]\exp(-\frac{1}{2}\text{trace}(\Sigma^{-1}))\\
&= \prod_{i=1}^{N}\sigma^{-n_i}|A|^{d+\frac{N}{2}+1}\sigma^{-2}\exp\left[-\frac{1}{2}\left[(Y_i - X_i\beta_i^T)^T(\sigma^2 I_{n_i})^{-1}(Y_i - X_i\beta_i^T) + (\beta_i-\mu)^T A(\beta_i-\mu)\right]\right]\exp(-\frac{1}{2}\text{trace}A)\\
&= \sigma^{-\sum_{i=1}^{N}n_i-2}|A|^{d+\frac{N}{2}+1}\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left[(Y_i - X_i\beta_i^T)^T(\sigma^2 I_{n_i})^{-1}(Y_i - X_i\beta_i^T) + (\beta_i-\mu)^T A(\beta_i-\mu)\right]\right]\exp\left[-\frac{1}{2}\text{trace}A\right]
\end{aligned}
$$

Conditional distribution of sigma^2

$$
\begin{aligned}
f(\sigma^2|B,\mu,\Sigma,Y) &\propto f(B,\mu,\Sigma,\sigma^2,Y)\\
&\propto \sigma^{-\sum_{i=1}^{N}n_i-2}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left[(Y_i - X_i\beta_i^T)^T I_{n_i}(Y_i - X_i\beta_i^T)\right]\right]\\
&= \left(\sigma^{-2}\right)^{\frac{\sum_{i=1}^{N}n_i}{2}+1}\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left[(Y_i - X_i\beta_i^T)^T(Y_i - X_i\beta_i^T)\right]\sigma^{-2}\right]\\
&= \left(\frac{1}{\sigma^2}\right)^{\frac{\sum_{i=1}^{N}n_i}{2}+1}\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\sum_{t=1}^{n_i}\left[(y_{it} - x_{it}\beta_i^T)^2\right]\frac{1}{\sigma^2}\right]
\end{aligned}
$$

Let $W = \sigma^2$. It follows:

$$
\begin{aligned}
&\left(\frac{1}{\sigma^2}\right)^{\frac{\sum_{i=1}^{N}n_i}{2}+1}\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left[(Y_i - X_i\beta_i^T)^T(Y_i - X_i\beta_i^T)\right]\frac{1}{\sigma^2}\right]\\
&= \left(\frac{1}{W}\right)^{\frac{\sum_{i=1}^{N}n_i}{2}+1}\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left[(Y_i - X_i\beta_i^T)^T(Y_i - X_i\beta_i^T)\right]\frac{1}{W}\right]\\
&\implies W \sim \text{inverse Gamma}(\frac{\sum_{i=1}^{N}n_i}{2}, \frac{1}{2}\sum_{i=1}^{N}\sum_{t=1}^{n_i}(y_{it} - x_{it}\beta_i^T)^2)
\end{aligned}
$$

Conditional Distribution of Big Sigma

$$\begin{aligned}
f(\Sigma^{-1}|B,\mu,\sigma^2,Y) &\propto |A|^{d+\frac{N}{2}+1}\exp\left[-\frac{1}{2}\sum_{i=1}^{N}(\beta_i-\mu)^TA(\beta_i-\mu)\right]\exp(-\frac{1}{2}\text{trace}A) \\
&= |A|^{d+\frac{N}{2}+1}\exp\left[-\frac{1}{2}\text{trace}\left(A+\sum_{i=1}^{N}(\beta_i-\mu)^TA(\beta_i-\mu)\right)\right] \\
&= |A|^{d+\frac{N}{2}+1}\exp\left[-\frac{1}{2}\text{trace}\left(A\left(I+\sum_{i=1}^{N}(\beta_i-\mu)(\beta_i-\mu)^T\right)\right)\right] \\
V^{-1} &= I+\sum_{i=1}^{N}(\beta_i-\mu)(\beta_i-\mu)^T \\
\frac{1}{2}(n^*-p-1) &= d+\frac{N}{2}+1 \\
2d+N+1 &= n^*-d-1 \\
&\implies n^*=3d+N+3
\end{aligned}$$

Thus, $A=\Sigma^{-1}\sim\text{Wishart}\left(n^*,\ I+\sum_{i=1}^{N}(\beta_i-\mu)(\beta_i-\mu)^T\right)$