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The Relationship Between Two Statistical Tests for the Comparison of Two Exponential Distributions

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Summary

To compare two exponential distributions with or without censoring, two different statistics are often used; one is the F test proposed by Cox (1953) and the other is based on the efficient score procedure. In this paper, the relationship between these tests is investigated and it is shown that the efficient score test is a large-sample approximation of the F test.

Key words: Exponential distribution; F test; Efficient score.

1. Introduction

The problem of comparing the survival experience of two groups of individuals has been much studied and a number of statistical tests for the hypothesis of equality of the survival distributions have been proposed. When the times to failure/event follow an exponential distribution, Cox (1953) showed that an F test can be used to test the difference between two groups whether or not the data include censored observations. Another parametric procedure frequently presented in various textbooks is the efficient score test (e.g., KALBFLEISCH and PRENTICE, 1980, page 53). In this short note, we demonstrate the relationship between these two test statistics.

2. Methods

Suppose there are n_1 and n_2 individuals in groups 1 and 2, respectively. Let $x_1, x_2, \ldots, x_{d1}$ be the d_1 uncensored observations (i.e. with events) and $x'_{d1+1}, \ldots, x'_{n1}$ be the (n_1-d_1) right-censored observations of group 1. In group 2, let $y_1, y_2, \ldots, y_{d2}$ be the times to events and $y'_{d2+1}, \ldots, y'_{n2}$ be the (n_2-d_2) right-censored observations. Also let

$$T_1=\sum_{i=1}^{d_1} x_i+\sum_{j=d_1+1}^{n_1} x'_j$$

$$T_2=\sum_{i=1}^{d_2} y_i+\sum_{j=d_2+1}^{n_2} y'_j$$

be the total exposure times in groups 1 and 2, respectively. If it is known that the survival times of the two groups follow the exponential distribution with density functions

$$f_i(t) = \varrho_i \exp(-t\varrho_i); \quad t \geqq 0 \quad \text{and} \quad i = 1, 2,$$

then testing the equality of two exponential distributions is equivalent to testing the hypothesis

$$H_0: \varrho_1 = \varrho_2 .$$

Suppose we consider this null hypothesis against the one-tailed alternative

$$H_A: \varrho_1 < \varrho_2$$

then the efficient score procedure (e.g. KALBFLEISCH and PRENTICE, 1980, page 53) yields an asymptotic standard normal statistic

$$z = \frac{d_1 T_2 - d_2 T_1}{\{T_1 T_2 (d_1 + d_2)\}^{\frac{1}{2}}} .$$

This statistic can be rewritten as

$$(1) \qquad z = \frac{d_1 - (d_1 + d_2)\left[\dfrac{T_1}{T_1 + T_2}\right]}{\left\{(d_1 + d_2)\left[\dfrac{T_1}{T_1 + T_2}\right]\left[\dfrac{T_2}{T_1 + T_2}\right]\right\}^{\frac{1}{2}}} = \frac{d_1 - n\Pi}{[n\Pi(1 - \Pi)]^{\frac{1}{2}}}$$

and can be seen as a normal approximation to the conditional binomial distribution $B(n, \Pi)$ of d_1 where

$$n = d_1 + d_2$$

and

$$\Pi = \frac{T_1}{T_1 + T_2} .$$

Within the framework of this conditional distribution, the level of significance (or p-value) for testing H_0 against H_A is given by

$$(2) \qquad p = \frac{1}{2}\left\{\sum_{u=0}^{d_1} \binom{n}{u} \Pi^u (1 - \Pi)^{n-u} + \sum_{v=0}^{d_1 - 1} \binom{n}{v} \Pi^v (1 - \Pi)^{n-v}\right\} .$$

Now we have

(i) If p_1 is such that

$$p_1 = \sum_{u=0}^{d_1} \binom{n}{u} \Pi^u (1 - \Pi)^{n-u}$$

then

$$T_1/T_2 = (\nu_1' F_{\nu_1', \nu_2', 1-p_1})/\nu_2'$$

(see JOHNSON and KOTZ, 1969, pp. 58–59) where n, Π, T_1 and T_2 are as previously defined,

$$\nu_1' = 2\,(d_1+1)$$
$$\nu_2' = 2d_2$$

and $F_{\nu_1', \nu_2', 1-p_1}$ is the lower 100 $(1-p_1)$ percentage point of the F distribution with (ν_1', ν_2') degrees of freedom. This implies that

$$F_1 = \frac{T_1/\nu_1'}{T_2/\nu_2'} = \frac{T_1/(d_1+1)}{T_2/d_2} = F_{\nu_1', \nu_2', 1-p_1}\,. \tag{3}$$

(ii) Similarly, if p_2 is such that

$$p_2 = \sum_{v=0}^{d_1-1} \binom{n}{v} \Pi^v\,(1-\Pi)^{n-v}$$

then

$$F_2 = \frac{T_1/\nu_1^*}{T_2/\nu_2^*} = \frac{T_1/d_1}{T_2/(d_2+1)} = F_{\nu_1^*, \nu_2^*, 1-p_2} \tag{4}$$

where

$$\nu_1^* = 2d_1$$
$$\nu_2^* = 2\,(d_2+1)\,.$$

In other words, from equations (2) to (4), using the conditional binomial distribution $B(n, \Pi)$ of d_1 is equivalent to "averaging" F_1 of equation (3) and F_2 of equation (4). The question is how to average these slightly different F-statistics.

(i) One option to take as degrees of freedom

$$\nu_1 = \nu_1^* = 2d_1$$
$$\nu_2 = \nu_2' = 2d_2$$

corresponding to the F-statistic

$$F = \frac{T_1/d_1}{T_2/d_2}$$

as being used by many (see COX and OAKES, 1984, page 84).

(ii) The other option is to average the degrees of freedom of F_1 and F_2, i.e.

$$\begin{aligned} \nu_1 &= (\nu_1' + \nu_1^*)/2 \\ &= 2d_1 + 1 \\ \nu_2 &= (\nu_2' + \nu_2^*)/2 \\ &= 2d_2 + 1 \end{aligned}$$

corresponding to the F-statistic

$$F_c = \frac{T_1/(d_1 + .5)}{T_2/(d_2 + .5)}$$

as originally suggested by Cox (1953). There are empirical evidence to show that F performs slightly better than F_c with small samples.

Finally, equation (1) shows that the efficient score test is a large-sample approximation of this procedure when $(d_1 + d_2)$ is large.

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