Online version of the theorem of Thue

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Abstract

A sequence S is *nonrepetitive* if no two adjacent blocks of S are the same. In 1906 Thue proved that there exist arbitrarily long nonrepetitive sequences over 3 symbols. We consider the online variant of this result in which a nonrepetitive sequence is constructed during a play between two players: Bob is choosing a position in a sequence and Alice is inserting a symbol on that position taken from a fixed set A. The goal of Bob is to force Alice to create a repetition, and if he succeeds, then the game stops. The goal of Alice is naturally to avoid that and thereby to construct a nonrepetitive sequence of any given length.

We prove that Alice has a strategy to play arbitrarily long provided the size of the set A is at least 12. This is the online version of the Theorem of Thue. The proof is based on nonrepetitive colorings of outerplanar graphs. On the other hand, one can prove that even over 4 symbols Alice has no chance to play for too long. The minimum size of the set of symbols needed for the online version of Thue's theorem remains unknown.

1 Introduction

A *repetition* of *size* $t \geq 1$ in a sequence $S = s_1 s_2 \ldots s_n$ is a subsequence of the form $s_{i+1} s_{i+2} \ldots s_{i+2t}$ satisfying $s_{i+j} = s_{i+t+j}$ for all $j = 1, 2, \ldots, t$. A sequence S is *nonrepetitive* if there is no repetition (of any size) in S. A celebrated theorem of Thue [15] (cf. [4]) from 1906 asserts that there are arbitrarily long nonrepetitive sequences over the set of just 3 symbols. This result has lots of applications and generalizations (cf. [1], [5], [8], [12], [14]). Recently, some game theoretic variants has been introduced leading to new challenges in the area [13]. A basic idea is that a nonrepetitive sequence is created by two players, Alice and Bob, say, but only Alice cares of avoiding repetitions. For instance, they may be picking alternately symbols from a fixed set A and appending them at the end of the existing sequence. Whenever a repetition occurs, its second part is being erased immediately. It is proved in [9] that Alice can still create arbitrarily long nonrepetitive sequences (no matter what Bob is doing) provided the size of A is at least 8.

In this paper we introduce another Thue type game, which we call the *online Thue game*. In one round of the game Bob chooses a position in the existing sequence S, which is specified by a number $i \in \{0, 1, \ldots, n\}$, and Alice is picking a symbol $x \in A$ which is inserted right after s_i, thereby giving a new sequence $S' = s_1 \ldots s_i x s_{i+1} \ldots s_n$, with $i = 0$ meaning that x is placed at the beginning of S. The goal of Bob is to force Alice to create a repetition, while Alice will try to avoid that for as long as possible. For instance, if $A = \{a, b, c\}$ and $S = acbc$, then Bob catches Alice in one move by choosing $i = 1$. Indeed, picking any $x \in A$ results in a repetition in S': **aa**cbc, a**bcbc**, a**cc**bc. Actually, it is easy to check that Alice has no chance for a longer play over 3 symbols. A bit more effort is needed to check that the same is true when A is of size 4, and one may start thinking that there is no finite bound at all. However we shall prove the following theorem.

Theorem 1 *There is a strategy guaranteeing Alice arbitrarily long play in the online Thue game on the set of* 12 *symbols.*

The proof is based on a former result of Kündegen and Pelsmajer [11], and Barát and Varjú [3], on nonrepetitive coloring of outerplanar graphs. A vertex coloring of a graph G is *nonrepetitive* if no repetition can be found along any simple path of G.

Theorem 2 ([3], [11]) *Every outerplanar graph has a nonrepetitive coloring using* 12 *colors.*

The proof of Theorem 1, in a slightly different setting, is given in the next section. The last section contains discussion and some open problems.

2   Proof of the main result

We start with introducing an equivalent setting for the online Thue game that will be more convenient for our purposes. First notice that this game can be played on the real line in such a way that Bob is choosing points of the line and Alice is coloring them using A as the set of colors. Then after n rounds we have a sequence of points $B = b_1 b_2 \ldots b_n$, $b_i \in \mathbb{R}$, written in increasing order $b_1 < b_2 < \ldots < b_n$, and the corresponding sequence of colors $S = c(b_1)c(b_2)\ldots c(b_n)$, $c(b_i) \in A$, assigned to the points b_i by Alice. The goal of Alice is to avoid repetitions in S.

Proof of Theorem 1. Without loss of generality we may assume that Bob starts with 0, and whenever he wants to extend the sequence B to the left or to the right, he always chooses the closest integer point. So, in the second move he picks either -1 or 1. In consequence the extreme points b_1 and b_n of B are always integers. Furthermore we assume that when Bob decides to insert a point between b_i and b_{i+1}, then the new point lies exactly in the middle of these two. So, if he has chosen 1 in the second round, then he may now choose $-1, 1/2$, or 2. The set of points accessible in this way is just the set of *dyadic rationals* D which consists of all numbers of the form $a/2^k$, where a is any integer, and k is any nonnegative integer. We assume that $(a, 2^k) = 1$, and we call the exponent k the *depth* of $a/2^k$. Denote by D_k the set of all dyadic rationals of depth k. Notice that D_0 is just the set of all integers.

Now, for each integer $k \geq 0$ consider a graph P_k whose vertex set is the union $D_0 \cup D_1 \cup \ldots \cup D_k$, with two points joined by an edge if and only if there is no other point of the union

between them on the line. Clearly P_k is a bi-infinite path. Consider finally a graph G on the vertex set D whose set of edges is the union of sets of edges of all paths P_k. It is not hard to see that G is an outerplanar graph. Indeed, we may draw the edges of each path P_k as half-circles of diameter $1/2^k$ lying above the line. Then no two edges of G can cross, and clearly all vertices of G touch the outer face of this embedding.

A simple (but crucial) observation is that any sequence $B = b_1 b_2 \ldots b_n$ that can be formed by Bob during the game is a path in G. Actually, B is a path in a subgraph G_n of G which is the union of paths $P_0, P_1, \ldots, P_n$. Indeed, assume inductively that this holds for n and consider a sequence B' from the next round. Suppose first that a new point x is placed at the beginning of B, that is $B' = xB$. Then, accordingly to the rules, x must be the closest integer to b_1, which itself is an integer. Hence xb_1 is an edge in P_0, and consequently B' is a path in G_{n+1}. The same holds for $B' = Bx$. Now, suppose x has been inserted between b_i and b_{i+1}. By inductive assumption these two points lie consecutively on some path P_j, $0 \le j \le n$. Hence, b_i, x, b_{i+1} lie consecutively on path P_{j+1}. In other words, the edge $b_i b_{i+1}$ has been subdivided, and therefore B' is a path in G_{n+1}.

To complete the proof we need only to demonstrate that graph G has a nonrepetitive coloring using 12 colors. But this follows easily from Theorem 2 by invoking the compactness principle. ∎

3 Remarks and open problems

First notice that we have actually proved a stronger result asserting that Alice has a strategy to play *infinitely* long, not just to play *arbitrarily* long. The strategy seems nonconstructive, as it is based on nonrepetitive coloring of an infinite graph obtained via compactness principle. However, looking into details of the proof of Theorem 2, one has an impression that it can be adapted to infinite graphs so that an explicit formula for the nonrepetitive coloring of G is perhaps possible.

Notice also that in fact we need a weaker coloring in which only forward-directed paths are nonrepetitive. We suspect that this can be achieved with smaller number of colors, perhaps 9 are sufficient.

Another observation leads to a major open problem of the area of nonrepetitive colorings of graphs. Notice that graph G from the proof of Theorem 1 has *page number* 1. What about nonrepetitive coloring of graphs with higher page number?

Conjecture 3 *There is a finite constant N such that every graph of page number 2 has a nonrepetitive coloring using N colors.*

This innocently looking question was propounded some years ago by Idziak. However, now we know that it is strong enough to imply the following statement.

Conjecture 4 *There is a finite constant M such that every planar graph has a nonrepetitive coloring using M colors.*

The best result to date [6] gives a logarithmic upper bound (in the number of vertices of a graph). Perhaps it would be easier to verify directed versions of the above problems.

Finally, one may consider online versions of other theorems in combinatorics on words. Particularly interesting seems the case of *abelian repetitions*, that is subsequences of the form $r_1 \ldots r_t r_{\sigma(1)} \ldots r_{\sigma(t)}$, where σ is any permutation of the set $\{1, 2, \ldots, t\}$. Answering a question of Erdős [7], Keränen [10] proved that there exist arbitrarily long sequences over 4 symbols avoiding abelian repetitions of any size. Is the online version of this result possible?

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