This work is a field guide for Machine Learning, and Numerical Linear Algebra. By field guide we mean a book intended to help guide the reader while actively engaged in algorithm development. These pages have variously served as my personal notes and software capability demonstrations. It is the authors hope that the will be useful or interesting to mathematicians making the transition from theory to application. A certain amount of mathematical sophistication is assumed at points. We use concepts from measure theory and functional analysis without comment.

I am grateful to my parents and family for their patience and *love*. Without them this work would never have come into existence (literally).

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Pittsburgh, PA Bruce Campbell
2013

Contents

This work addresses

> *the application and implementation of techniques in Machine Learning, Statistical Pattern Recognition, & Spectral Graph Theory*

to *problems in real world data analysis.*

0.1 Learning Theory and Functional Analysis

Supervised learning in its most abstract setting requires finding a function $f(x)$ given instances $(x_i, f(x_i))$. Typical assumptions are that x_i is an iid sample from some unknown distribution. A loss function is a random variable

$$L : Ran(f) \times Ran(f) \rightarrow \mathbb{R}^+$$

defining the cost of misclassification. The risk associated with a candidate function f' is defined to be the expectation of the loss over the sample space Ω,

$$R(f') = \int L(f(\omega), f'(\omega)) d\omega \tag{0.1.1}$$

. Statistical learning theory is concerned with assessing the approximations to f given by minimizing the empirical loss associated with a sample $(x_i, f(x_i))$.

The notion of a loss function goes back to the roots of modern probability theory and economics. The St. Petersburg paradox is an example of a random variable $S : \mathbb{N} \rightarrow \mathbb{R}^=$ with infinite expectation limited utility. Let $W(k)$ be the winnings after k plays of from a game with outcome S that pays 2^{i-1} with probability $p_i = 1/2^i$. $\lim_{k \longrightarrow \infty} W(k)/k = E(S) = \sum_{i=1}^{\infty} p_i 2^{i-1} = \infty$ The implication for a decision theory based only expected value is that a rational player would pay an infinite amount of money to play this game. Bernoulli introduced the notion of expected utility which takes into account the fact that a payout of 2^i may not have twice the utility of a payout of 2^{i+1} when i gets large. The utility U is a random variable on the sample space representing preferences of an agent. Loss represents the aversion of an agent to the outcomes of the sample space,

$$L(\omega) + U(\omega) = \alpha \; \forall \; \omega \in \Omega$$

where α is constant. Expected loss $R(f')$ is the risk associated with choosing the approximation f'. Restricting the class of functions to consider when minimizing the risk for a candidate approximation to f is a key aspect of classifier design.

Gaussian processes provide a class of models and learning algorithms for real world problems that have a long history and are well characterized. Learning algorithms are cast as minimization problems $min_(H)R()$ in a Hilbert space $\mathcal{H}$ with a dot product that encapsulates a model and sample data. Bayesian methods are often employed for estimation and inference with Gaussian processes. They allow an intuitive approach to incorporating prior knowledge in classification problems and the ability to obtain confidence intervals for predictions. Many common regression and classification algorithms can be cast as minimization problems in a Reproducing Kernel Hilbert Space (RKHS).

0.2 Learning With Kernels

[12], [25], [36], [39], [38], [41], [47], [49], [40]

Kernel learning is a paradigm for classification and regression where prior belief is expressed in the construction of a similarity matrix of distanced between points in a feature space Ω by embedding via a non linear map ϕ in a higher [often infinite] dimensional Hilbert space using the kernel as an inner product.

$$\begin{aligned} K(x,x') &=< \phi(x), \phi(x') > \\ K &\succeq 0 \\ SPD \Rightarrow \sum_{x\in\Omega}\sum_{x'\in\Omega} f(x)K(x,x')f(x') &\geq 0 \forall f \ \in \ell^2(\Omega) \end{aligned}$$

Recall that infinitely divisible probability distributions aries as the sum of *iid* random variables. Infinitely divisible kernels have the representation

$$\begin{aligned} K &= K^{\frac{1}{n}} \dots K^{\frac{1}{n}} \\ K &= e^{\beta H} \\ e^{\beta H} &= \lim_{n\to\infty}(1+\frac{\beta H}{n})^n \end{aligned}$$

We construct a mutli-resolution representation of the data with exponentiated kernels. The sequence of kernels $K(\beta)$ represents a one parameter group associated with a diffusion on the graph of the data. A $\beta \to 0\infty$ the kernel moves from the identity to one that represents the clusters in the off diagonal components. The local structure of Ω is preserved in H while the global geometry of the data set is progressively revealed in $K(\beta)$ as we push the diffusion forward with the one parameter group. We can construct exponentiated kernels over direct products of sets $\Omega_1 \bigotimes \Omega_2$ that will allow for the class conditional representation [bbcrevisit term use multiclass]. Simply set $H = H_1 \bigotimes I_{\Omega_1} + H_2 \bigotimes I_{\Omega_2}$.

$$\begin{aligned} K(\beta) = e^{\beta H} = e^{(\beta H_1 \bigotimes I_{\Omega_1} + H_2 \bigotimes I_{\Omega_2})} \Rightarrow \\ \frac{d}{d\beta}K(\beta) = H(K_1(\beta) \bigotimes K_2(\beta)) \end{aligned}$$

The kernels thus constructed can be used to drive a diffusion on a graph by letting H be the familiar graph Laplacian. Furthermore, the continuum limit of infinite data can be analyzed in within the framework of a discreet stochastic process much the way the convergence of finite element solutions of PDE's takes place.

0.3 Kernel Density Estimation

To define the empirical distribution function of a sample of size N - place mass $1/N$ at each member of the sample. This forms a nonparametric estimate of the

marginal density $P(X)$. This is a singular form of kernel smoothing for density estimation. If ψ belongs to some nice class of function, and $\int_{\infty}^{-\infty} \psi(x)dx = 1$, we can form a parametric estimator for the pdf of a process from a sample population of size N by calculating

$$p(x;\theta) = \frac{1}{N\theta}\sum_{i-1}^{N} \phi(\frac{x - X_n}{\theta}) \tag{0.3.1}$$

If ϕ happens to be a density then $p(x,\theta)$ is also a density. Letting $\theta \to 0$ for the right kernel, we get the empirical density of the sample population. The mean squared error of the estimator expressed as a bias term and a variance term is

$$Errp(x;\theta) = E[p(x;\theta) - p(x)]^2 = E[p(x;\theta) - p(x)]^2 = (E[p(x;\theta)] - p(x))^2 + Var[p(x;\theta)] \tag{0.3.2}$$

0.4 Distance Functions & The Affinity Matrix

There are two key ingredients to forming the affinity matrix; a distance function, and a convention for which pairs to consider. If all or too many pairs are compared, sparse methods will not be possible. We can include n nearest points, all points within ϵ of x, or some other criteria. For instance when working with spatial data, the diffusion can be done on a graph formed by a tessellation of the locations. This is exactly how numerical solutions of heat diffusions are done.

BBCREVISIT - Gram Matrix & Mercers Theorem - Make sure we connect to SVM section.

If the feature vector is a histogram, the L^2 distance is not meaningful, in this case use the χ^2

0.5 The Graph Cut

Spectral clustering is a relaxation to an NP-hard problem of finding an optimal way to cut a graph. Here we describe some various cut criteria. Let

$$X_1, \ldots, X_n \ \in \mathbb{R}^n$$
$$W_ij = s(X_i, X_j)d_i = \sum_{j} W_{ij}|A| = card(A)vol(A) = \sum_{i \in A} d_i$$

BBCREISIT - Find ref, fill in defs, and possibly demo on small graphs.

Definition 0.5.1 (Min Cut)**.**

$$mincut(A, B) = \sum_{i \in A\, j \in B} W_ij$$

Definition 0.5.2 (Balanced Cut / Ratio Cut).

$$mincut(A, B)\frac{1}{|A|}\frac{1}{|B|}$$

Definition 0.5.3 (n Cut).

$$mincut(A, B)\frac{1}{vol(A)}\frac{1}{vol(B)}$$

0.6 Graph Spectra

Graph spectral methods are some of the most successful heuristic approaches to partitioning algorithms in solving sparse linear systems, clustering and, ranking problems. Eigenvalues of the graph Laplacian are used to transform a combinatorial optimization problem to a continuous one, typically a SDP problem. Recent advances in SDP optimization techniques have opened new avenues of research in combinatorial optimization. For instance, isoperimetric properties of a graph are used to find efficient communication networks, and fast convergence of Markov Chains.

0.7 Matrix Factorization

Many forms of matrix factorization can be cast as an optimization problem that involves minimization of generalized Bregman divergences[43]. Factorization algorithms such as NNMF, Weighted SVD, Exponential Family PCA, , pLSI, Bregman co-clustering [6] can be cast in this framework. The approach uses an alternating projection algorithm for solving the optimization problem which allows for generalizations that include row, column, or relaxed cluster constraints. A brief description of the algorithm is given below. The description of a generalized Bregman divergence can be found in [21].

0.8 PCA and its generalization to the Exponential Family

PCA finds linear combinations of the variables that correspond to directions of maximal variance in the data. Typically this is performed via a singular value decomposition (SVD) of the data matrix $A \in R^{n,m}$, or via an eigenvalue decomposition if A is a covariance matrix in which case $A \in R^{n,n}$. Representing the data in the directions of maximum variance allows for a dimension reduction that preserves information. Principal component directions are uncorrelated which can be useful. PCA has the disadvantage that components are usually linear combinations of all variables. Weights in the linear combination data elements are non-zero. Sparse PCA is an attempt to find a low dimension representation of the data that explainers most of the variance.

Here we describe a generalization of Principal component analysis (PCA) to the Exponential Family of probability distributions. PCA is a popular dimensionality reduction technique that seeks to find a low-dimensional subspace passing close to a given set of points

$$\{x_i\} \subset \mathbb{R}^n$$

. The procedure is to solve the optimization problem that minimizes the sum of squared differences of the data points to the projections on a subspace spanned by the empirical variance after centering the data to have mean 0;

$$\sum_{i=i}^{n} \|x_i - \theta_i\|_{\ell^2}^2$$

. The choice of ℓ^2 norm here codifies the assumption of Gaussian data. An alternate interpretation of the algorithm is finding the parameters θ_i that maximizes the log likelihood of the data which corresponds to

$$\sum_{i=i}^{n} \|x_i - \theta_i\|_{\ell^2}^2$$

. The goal of PCA is to find the the true low dimensional distribution of the data given the assumption that data is corrupted by Gaussian noise. Bregman divergences

$$D_\phi(A, B) = \phi(A) - \phi(B) - \nabla\phi(B)(A - B)$$

offer a framework to extend PCA [and other spectral dimension reduction techniques] to the entire Exponential Family. Here ϕ is a striclty convex function. The roles of

Let θ_i be the natural parameter for dimension i, with Exponential distribution P_θ. Then the conditional expectation is given by

$$logP_\theta(x|\theta) = logP_0(x) + x\theta - G(\theta) \dot{:} G \ni \int P_\theta dx = 1$$

We can model multivariate data where the conditional distribution can vary along the feature space. The common feature of this PCA model and GLZ regression is the derivative of G which is familiar link function and the loss function which is appropriate for $P_\theta(x|\theta)$. The non linear relationship in the GLZ regression model data is captured by the link function $h = \frac{d}{d\theta}G(\theta)'$. This feature is also passed on to the generalized PCA. Instead of projecting on to a linear subspace, a Bregman divergence is used as the distortion measure. This gives a convex optimization problem to solve which can be shown to converge. In [5] a dual function to ϕ is defined by the relationship $\phi(g(\theta)) + G(\theta) = h(\theta)\theta$ which is used to write the log likelihood as a Bregman divergence

$$\log P(x|\theta) = -logP_0(x) - \phi(x) + D_\phi(x, h(\theta))$$

. Typically x is a vector but extending to matrices is straightforward.

Sparse PCA [CS Notes from http://ugcs.caltech.edu/ srbecker/

Classic PCA is sensitive to outliers. Convex methods can be use to address this by attempting to factor the data X as a sum of a low rank component L and a sparse component S by solving

$$\underset{min}{S, L} \|L\|_* + \lambda \|S\|_{\ell_1} : X = L + S$$

See http://cvxr.com/tfocs/demos/rpca/ for a demo of this in action with the popular cvx software package.

0.9 Manifold Learning

There are numerous machine learning techniques which accomplish some form of dimensionality reduction. Manifold learning uses principal curves and manifolds to encode a natural geometric framework for nonlinear dimensionality reduction. These methods construct low-dimensional data representation using a cost function that retains local properties. Contrasting methods such as MDS employ proximity data via a similarity or distance matrices. The important ISOMAP [24]algorithm extends MDS by capturing geodesic measurements of non-local pairs on the data manifold M via an multi-scale approximation. Non-local distances are approximated via a shortest path on a K nearest neighbor clustering of the data. Effectively a ball in data space is used to represent a cluster, and a graph is then constructed to encode the non-local information. The connectivity of the data points in the neighborhood graph are the nearest k Euclidean neighbors in the feature space. Dijkstra's algorithm for computing shortest paths with weights is used to construct the proximity matrix from the neighborhood graph. The top n eigenvectors encode the coordinates in the low dimensional Euclidean space. Choosing the correct number of neighbors is an essential component to an accurate representation. Other shortest path algorithms that may be employed to calculate the geodesic distances are listed below:

- Dijkstra's algorithm finds the single-pair, single-source, and single-destination shortest path.
- Johnson's algorithm finds all pairs shortest paths
- Bellman-Ford algorithm single source problem and allows negative edge weights.
- Floyd-Warshall algorithm solves all pairs shortest paths.
- A* search algorithm solves the single pair shortest path problem.

In [8] a sampling condition is given which bounds the quality of the manifold embedding based on the quality of the neighborhood graph.

0.10 Graph Laplacian

[10], [14], [17], [22], [10], [14], [29], [26]

Let G be a connected simple graph with vertex set $V = 1, 2, ..., n$, edge set E and let each edge be associated with a positive number, called the weight of the edge. The above graph is called a weighted graph. An unweighted graph is just a weighted graph with each of the edges bearing weight 1. The weight $w(i)$ of a vertex V_i is the sum of the weights of the edges incident with it. There are a number of ways in which the Laplacian matrix L is defined; the combinatorial Laplacian, the normalized Laplacian and the unsigned Laplacian. Spectra from graph matrix representations may be obtained from the adjacency matrix A and the various Laplacian discretizations. Spectra can also be derived from the heat kernel matrix and path length distribution matrix.

The matrix representation of the graph Laplacian has a significant effect on the spectrum. Attributes may be accounted for by by a complex number that encodes the edge attributes. The node attributes may be encoded in the diagonal elements. The complex graph Laplacian matrix is Hermitian, and hence it has real eigenvalues and complex eigenvectors. Graph feature vectors can be embedded in a pattern space by PCA, MDS, and LDA(linear discriminant analysis). Attribute graphs may be characterized by the application of symmetric polynomials to the real and complex components of the eigenvectors. [37] This gives rise to permutation invariants that can be used for pattern vectors. Partitioning a graph into three pieces, with two of them large and connected, and the third a small separator set can be accomplished using the second eigenvector [the Feidler Vector] of the graph Laplacian. In the case or sparse graphs, the first few eigenvectors can be efficiently computed using the Lanczos algorithm [see section below on ARPAC]. This graph partitioning algorithm can be extended to give a hierarchical subdivision of the graph.

0.11 Distance Metrics

A measure of similarity between data points is a vital component to clustering algorithms. The suitability of any given measure is dependent on the generative process providing the data.

0.12 Markov Chains

Let $X \in \mathcal{M}$, $P(x, y)$ the transition probability for an irreducible Markov chain. If P is reversible relative to π then we have that $Q(x, y) = \Pi(x)P(x, y) = \Pi(y)P(y, x) \forall x, y, \in X$. This implies that π is the stationary distribution. Bounding the rate of convergence to the stationary distribution is related to L_G and isoperimetric problems.

Detailed balance is a property of a Markov chain where $\pi(x)P(x, y) = \pi(y)P(y, x) \Rightarrow$ reversibility. This is stronger than the requirement of a sta-

tionary distribution. It is the property that implies for every closed cycle of states there is no flow of probability.

There are four aspects of Markov chain stability.

1. π-irreducible
2. small set
3. Harris recurrence
4. Geometric Ergodicity

For reversible Markov chains, the rate of convergence to Π for a finite state chain is determined by the second eigenvalue of P.

There are two types of bounds for the rates of convergence

1. graph theoretic Cheeger type - see Diaconis and Stook
2. Spectral

The rate of convergence to π for finite state Markov chains is determined by the second eigenvalue λ_2 of P.

BBCREVISIT CLEAN UP LOTS

Theorem 0.12.1 (Perron Frobenius - $P \in \mathbb{P}^{n\times n}$)**.** *There is a simple largest eigenvalue, and it's eigenvector is positive.*

The growth of P^k is determined by the Perron eigenvalue. Frobenious extended the theorem to include a class of non negative matrices. This is achieved through the concept of reducibility. P is irreducible if

$$\neg\exists\ \Lambda\ \ni\ \Lambda^\dagger L\Lambda = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}$$

where Λ is a projection $\Lambda^2 = \Lambda$ The irreducible matrices L have a simple largest eigenvalue λ_n, now called the Perron-Frobenius eigenvalue whose right eigenvector components are all positive. We can put this in Markov chain terminology; irreducibility is equivalent to the existence of a unique stationary distribution.

BBC Make the Markov Chain section nice.

Hammerlsy Clifford

The HammersleyClifford theorem gives necessary and sufficient conditions for when a positive probability distribution can be represented as a Markov Random Field. It states that a probability distribution that has a positive mass or density satisfies one of the Markov properties with respect to an undirected graph G if and only if it is a Gibbs random field. The density of the Gibbs random field can be factorized over the cliques (complete subgraphs) of G.

0.13 Spectral Graph Theory

Modern spectral graph theory increasingly takes insights from geometry. Discrete analogues of isoperimetry results and heat flow on manifolds are just a few examples being put to use in modern applications. The normalized graph Laplacian is used to aid in consistency between spectral geometry and stochastic processes. We consider connected graphs $G = (E, V)$ in this work, in which case we can define the normalized graph Laplacian as $\mathcal{L} = T^{\frac{1}{2}} L T^{\frac{-1}{2}} = I - T^{\frac{1}{2}} A T^{-\frac{1}{2}}$, where A is the adjacency matrix, L is defined by

$$L(u,v) = \begin{cases} d_v & : u = v \\ -1 & : u \sim v \\ 0 & : u \nsim v \end{cases} \tag{0.13.1}$$

and $T = diag\{d_1, \cdots, d_n\}$ where d_v is the degree of vertex v.

$\mathcal{L}$ is a difference operator :

$$\mathcal{L} = \frac{1}{\sqrt{d_u}} \sum^{v:u\sim v} (\frac{g(u)}{\sqrt{d_u}} - \frac{g(v)}{\sqrt{d_v}}) \tag{0.13.2}$$

$$Vol(G) = \sum_{v\in V}^{d_v} = Tr(T) \tag{0.13.3}$$

$$\sigma(\mathcal{L}) \in \mathbb{R}^+ \tag{0.13.4}$$

$$ker(\mathcal{L}) = span\{T^{\frac{1}{2}} \not\perp\} \tag{0.13.5}$$

0.14 Diffusion Maps

[11], [13], [15], [27], [28], [34].

Spectral clustering involves constructing a Markov chain over a graph is constructed over the graph of the data and using the sign of the first non-constant eigenvector for graph cuts and cluster localization. This approach can be generalized to higher-order eigenvectors yielding a multi-resolution view of the data. Using multiple eigenvectors allows one to embed and parameterize the data in a lower dimensional space. Examples of this procedure include LLE, Laplacian & Hessian Eigenmaps. The common theme among these approaches is that eigenvectors of a Markov process can encode coordinates of the data set on a low dimensional manifold in a Euclidian space. The advantage over conventional methods is that the representation is non-linear and they preserve local structure. Kernel eigenmap embeddings can be generalized into a diffusion framework where a discrete Laplacian acts on a low dimensional representation space. This allows for a true multi-scale parametrization. Iterating a Markov process involves computing power of the transition matrix to run a random walk of the graph forward in time. By construction a one parameter map defining the diffusion and specifying boundary conditions the full power of diffusions on

a smooth manifold may be brought to bear on parameterizing the geometry of the data. Different boundary conditions and diffusion operators give rise to a discrete approximations of familiar stochastic PDE's.

Let $(X, \mathcal{A}, \mu)$ be a measure space and $k : X \times Y \longrightarrow \mathbb{R}$ a kernel function.

$$\begin{aligned} d(x) &= \int_X k(x,y) d\mu(y) && (0.14.1) \\ P(x,y) &= \frac{k(x,y)}{d(x)} && (0.14.2) \\ (D_t(x,y))^2 &= \|P_t(x,\cdot) - P_t(y,\cdot)\|_{L^2(X, \frac{d\mu}{\pi})} && (0.14.3) \\ \pi(\mu) &= \frac{d(y)}{z \in Z^{d(z)}} && (0.14.4) \\ \pi(x)p(x,y) &= \pi(y)p(y,x) && (0.14.5) \end{aligned}$$

$D_t(x,y)$ is the functionally weighted L^2 distance between the 2 posteriors $\mu \to P_t(x,u)$ and $\mu \to P_t(y,u)$. This is related to isoperimetry. Think about what happens as the cardinality of paths connecting x and y is increased. D_t can be computed using the eigenvalues of P.

$$D_t(x,y) = \sqrt{\sum_{\lambda \geq 1} \lambda_l (\phi_l(x) - \phi_l(y))^2} \tag{0.14.6}$$

We can define an embedding in Euclidian space via

$$\Psi_t(x) = \{\lambda_1^t \phi_1(x), ... \lambda_{s(\delta,t)}^t \phi_{s(\delta,t)}(x)\} \tag{0.14.7}$$

0.15 Spectral Geometry

Spectral Geometry concerns itself with the relationships between a geometric structure and the spectra of a differential operator, typically the Laplacian. Inferring the geometry from the spectra is a type of inverse problem since two non isometric manifolds may share the same spectra. Going the other way, we encounter isoperimetric inequalities and spectral gap theorems. "Can One Hear the Shape of a Drum?" was the of an article by Mark Kac in the American Mathematical Monthly 1966. The frequencies at which a drum vibrate depends on its shape. The elliptic PDE $\nabla^2 A + kA = 0$ tells us the frequencies if we know the shape. These frequencies are the eigenvalues of the Laplacian in the region. Can the spectrum of the Laplacian tell us the shape if we know the frequencies? Hermann Weyl showed the eigenvalues of the Laplacian in the compact domain Ω are distributed according to $N(\lambda) \sim (2\pi)^{-d)} \omega_d \lambda^{\frac{d}{2}} vol(\Omega$

The Laplace Beltrami operator is the generalization of $\nabla \circ \nabla = \Delta$ to $\mathcal{M}$

$$\Delta f = tr(H(f))$$

In the exterior calculus we have $\Delta f = d^* d\, f$.

//BBCREVISIT - Fill this out and check The Laplacian of a Gaussian has well known applications in image processing. Given $f(x, y)$, we get a scale space representation when we convolve by

$$g(x, y, t) = \frac{e^{x^2+y^2}}{2\pi t}$$

$$L(x, y, t) = g(x, y, t) * f(x, y)$$

Applying Δ to $L(x, y, t)$ gives response to blobs of extent $\sqrt{t}$

There is a well known connection between diffusion processes and Schrodinger operators;

$$H = \nabla^2 + V(x)\Phi \in L^2(\mathbb{R}^n)$$
$$H\Phi = E\Phi$$
$$E = \sigma(H)$$

0.16 Concentration of Measure

[4], [7], [9], [20], [30], [33], [42], [45].

Familiar tools used when when dealing with additive functions of independent random variables are the CLT, LLN, and the inequalities of Markov, Chebychev, and Chernoff. When the differences are not independent we rely on the theory of martingales and use inequalities like Azuma's to provide concentration bounds.

Let (X, Σ, μ) be a measure space and let f be an real-valued measurable function defined on X. Then for any number $t > 0 \in (R$ we have

$$\mu(x \in X | f(x) \geq t) < \frac{1}{t} \int_X |f(x)| d\mu$$

If we let μ be a probability measure - $\mu(X) = 1$, then the above is equivalent to $P(|X| \geq a) \leq \frac{E(|X|)}{a}$ commonly known as Markov's inequality.

Another familiar concentration inequality is the Chebychev inequality has it's origins as a measure theoretic inequality;

$$\mu(x \in X : |f(x)| \geq t) \geq \frac{1}{t^2} int_X |f|^2 d\mu$$

When X has finite first moment μ and non-zero second moment σ, we have the more familiar

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2} \forall k > 0$$

The isoperimetric inequality concerns the relationship between the length l of a closed curve and the area a of the planar region that it encloses. Specifically, $4\pi a \leq l^2$. Equality holds in the case that the curve is a circle. The isoperimetric

problem is to determine a plane figure of the largest area whose boundary has a given length. F

Federer [19] is a good reference for a measure theoretic generalization to higher dimensions. We make a few remarks here that will be expanded on later.

Proposition 0.16.1 (Isoperimetric Inequality In $\mathbb{R}^n$)**.** *Let μ be Lebesgue measure in $\mathbb{R}$ and X in$\mathbb{R}$ $\ni$ $\mu(cl(X) < \infty$, then*

$$n\omega_n^{\frac{1}{n}}\mu(cl(X))^{\frac{n-1}{n}} \leq M^{n-1}(\partial X)$$

M is the Minkowski content, which is the Hausdorff measure of ∂X for rectifiable ∂X. The proof relies on the BrunnMinkowski theorem which states that $\mu(A+B)^{\frac{1}{n}} \geq \mu(A)^{\frac{1}{n}} + \mu(A)^{\frac{1}{n}}$ where set addition in $\mathbb{R}$ is in the sense of Minkowski. This addition behaves well with respect to the convex hull; $\forall A, B \in \mathbb{R}\ co(A+A) = co(A) + co(A)$

For smooth domains general Isoperimetry inequality is equivalent to a Sobolev inequality on $\mathbb{R}$.

Proposition 0.16.2 (Sobolev inequality)**.** *Let u be a continuously differentiable real-valued function on $\mathbb{R}$ with compact support. Then for $1p < n$ there is a constant C depending only on n and p such that*

$$||u||_{L^{\frac{pn}{n-p}}} \leq C||Du||_{L^p}$$

The Sobolev embedding theorem relies on the

Theorem 0.16.3 (Hardy Littlewood Sobolev fractional integration theorem)**.** *Let $0 < \alpha < n$ and $1 < p < q < \infty$ and let $I\alpha = (\Delta)\alpha/2$ be the Riesz potential*

$$I_\alpha f(x) = \frac{1}{C_\alpha}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy$$

Then, for $q = \frac{pn}{n-\alpha p}$ there exists a constant C depending only on p such that $||I_\alpha f||_q \leq C||f||_p$

The HardyLittlewoodSobolev lemma implies the Sobolev embedding by the relationship between the Riesz transforms and the Riesz potentials. The Riesz potential defines an inverse for a power of the Laplace operator on Euclidean space.

The Chernoff and Hoeffding bounds tell us that the average of n iid random variables $X_1, X_2, \ldots, Xn$ is tightly concentrated around its mean if X_i are bounded and n is sufficiently large. hat about $G(X_1, X_2, \ldots, X_n)$? The feature of the average which gives rise to tight concentration is that is is Lipschitz. The following concentration bound applies to any Lipschitz function of iid normal random variables. See Ledoux (2001, page 41, 2.35).

High dimensional space is mostly empty. This is more commonly called the *"curse of dimensionality"*. One way to get around the curse of dimensionality is to find interesting projections. Many common algorithms such as principal

components, multidimensional scaling, and factor analysis fall into this category. Huber [23] placed many of these in to a common framework called projection pursuit.

Logarithmic Sobolev inequalities have a close relationship with the concentration of measure phenomena. There are two major types of concentration; Gaussian and Exponential. [see Ledoux]

Let $(e^{-At})_{t\geq 0} = (T_t)_{t\geq 0}$ be a symmetric Markov semigroup on $L^2(X, d\mu)$ with generator A defined on a σ-finite measure space $(X, d\mu)$. $(T_t)_{t\geq 0}$ is ultracontractive if for any $t > 0$, there exists a finite positive number $a(t)$ such that, for all $f \in L^1$:

$$\|T_t f\|_\infty \leq a(t)\|f\|_1. \tag{0.16.1}$$

An equivalent formulation (by interpolation) of ultracontractivity is that for any $t > 0$, there exists a finite positive number $c(t)$ such that, $\forall f \in L^2$,

$$\|T_t f\|_\infty \leq c(t)\|f\|_2 \tag{0.16.2}$$

Also by duality, the inequality (0.16.2) is equivalent to

$$\|T_t f\|_2 \leq c(t)\|f\|_1 \tag{0.16.3}$$

It is known that, under the assumptions on the semigroup $(T_t)_{t\geq 0}$, (0.16.2) implies (0.16.1) with $a(t) \leq c^2(t/2)$ and (0.16.1) implies (0.16.2) with $c(t) \leq \sqrt{a(t)}$.

We say that the generator A satisfies LSIWP (logarithmic Sobolev inequality with parameter) if there exist a monotonically decreasing continuous function $\beta : (0, +\infty) \to (0, +\infty)$ such that

$$\int f^2 \log f \, d\mu \leq \epsilon Q(f) + \beta(\epsilon)\|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \tag{0.16.4}$$

for all $\epsilon > 0$ and $0 \leq f \in \mathrm{Quad}(A) \cap L^1 \cap L^\infty$ where $\mathrm{Quad}(A)$ is the domain of $\sqrt{A}$ in L^2 and $Q(f) = (\sqrt{A}f, \sqrt{A}f)$.

This inequality is modeled on the Gross inequality [].

In [**?**],[**?**], the authors show that LSIWP implies ultracontractivity property under an integrability condition on β. This condition can be enlarged and be stated as follows:

Theorem 0.16.4. *Let $\beta(\epsilon)$ be a monotonically decreasing continuous function of ϵ such that*

$$\int f^2 \log f \, d\mu \leq \epsilon Q(f) + \beta(\epsilon)\, \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \tag{0.16.5}$$

for all $\epsilon > 0$ and $0 \leq f \in Quad(A) \cap L^1 \cap L^\infty$. Suppose that for one $\eta > -1$,

$$M_\eta(t) = (\eta + 1)t^{-(\eta+1)}) \int_0^t s^\eta \beta\left(\frac{s}{\eta + 1}\right) ds \tag{0.16.6}$$

is finite for all $t > 0$. Then e^{-At} is ultracontractive and

$$\|e^{-At}\|_{\infty,2} \leq e^{M_\eta(t)} \tag{0.16.7}$$

for all $0 < t < \infty$.

0.17 The Condition Number of a Markov Chain

0.18 Generalized Chebyshev Bounds on Quadratic Sets via Semidefinite Programming

Boyd et al Vandenberghe et al. [48] provide a simplified development of an algorithm to compute the lower bound on the probability of a set which is defined by quadratic inequalities. That algorithm is discussed here.

$$\min(1 - \sum_{i=1}^{m} \lambda_i) \ni Tr(A_i z_i) + 2b_i^T z_i + c_i \lambda_i \geqslant 0 \quad \forall i = 1, ..., m \tag{0.18.1}$$

$$\sum_{i=1}^{m} \begin{bmatrix} z_i & z_i \\ z_i & \lambda_i \end{bmatrix} \succeq 0 \tag{0.18.2}$$

$$C = \{x \in \mathbb{R} : x^T A_i x + 2b_i^T + c_i < 0 : i = 1, ..., m\} \tag{0.18.3}$$

$$\min E[f_0(X)] \ni E[f_i(X)] = a_i : i = 1, ..., m \tag{0.18.4}$$

moment constraints

Let

$$\bar{x} \in \mathbb{R}^n S \subset S^n \ni S \succeq \bar{x}\bar{x}^T \tag{0.18.5}$$

and define

$$P(C, \bar{x}, S) = inf_{\mathcal{P}(\mathbb{R}^n)}\{P(X \in C) \mid E[X] = \bar{x} E[XX^T] = S\} \tag{0.18.6}$$

The optimization problem is to find $P \in \mathcal{P}(\mathbb{R}^n)$ - a probability density function which maximizes the probability of the convex set C and satisfies the moment constraints.

0.19 Bregman Divergences

NNMA is the approximation of a non-negative matrix A by a low rank matrix BC where $B \succ 0$ and $C \succ 0$. Bregman divergences are a robust distortion measure for this matrix factorization. Formally $D_\phi(A, BC) = \phi(A) - \phi(BC) - \nabla\phi(BC)(A - BC)$ measures the quality of the factorization relative relative to a convex penalty function.

Modeling of relational data can be abstracted out to the factorization in a low dimensional representation of a data matrix $(X_i j)$ where links [or relations] are represented as an nxm matrix X where $X_{i,j}$ indicates whether a relation exists between entities of type i, j. Let f be a link function and X be a factorization of X into a low rank approximation $X \approx UV^T : U \in R^{mxk}, v \in R^{mxk}$. The link function f can be interpreted as in GLM which gives extends exponential models to matrices. A simple example is choosing the identity link which and minimizing in the ℓ^2 norm gives rise to the SDV and the Gaussian model for the data $X_i j$. Similarly we can extend to Bernoulli, Poisson, Gamma, error distributions.

Many forms of matrix factorization can be cast as an optimization problem that involves minimization of generalized Bregman divergences [43]. Factorization algorithms such as NNMF, Weighted SVD, E xponential Family PCA, , pLSI, Bregman co-clustering [6] can be cast in this framework. The approach uses an alternating projection algorithm for solving the optimization problem which allows for generalizations that include row, column, or relaxed cluster constraints. A brief description of the algorithm is given below. The description of a generalized Bregman divergence can be found in [21].

Let $\phi S \in \mathbb{R}^n \to \mathbb{R}$, $D_\phi(x, y) = \phi(x) - \phi(y) - < x - y, \nabla\phi(y) >$ be the Bregman divergence. Let $\chi = X_i \in S \in dblr^d$ be a random variable and take the encoding $X_i \to S$ so the rate is zero and the code book is 1. The rate distortion is $E_\nu[D_\phi(\chi, s)] = min\mathcal{S}inS \sum_{i=1}^{n} \nu_i D_\phi(x_i, x)$ which we call the Bregman information for the random variable X.

Theorem 0.19.1. *$\mu = E_\nu[X]$ is the unique minimizer*

0.20 Sparse Representation

A Gaussian distribution is often an accurate density model for low dimensional data, but very rarely for high-dimensional data. High dimensional data is less likely to be Gaussian, because of the high degree of independence this demands. Recall the a Gaussian is a rotation of a distribution with completely independent coordinates. In a typical high dimensional application, one may be able to find a few features that are approximately independent, but generally as more features are added the dependencies between them will grow.

Diaconis and Freedman showed that for *most* high dimensional point clouds, *most* low dimensional orthogonal projections are a mixture of normal spherically symmetric distributions.

Lemma 0.20.1 (Poincare Lemma)**.** *If σ_n is uniform on $\sqrt{n}S_{n-1} \in \mathbb{R}^n$, $d < n$ and*

$$\Pi_{d,n}(x_1, \ldots, x_n) \to (x_1, \ldots, x_n)$$

is the canonical projection, then for fixed d, as $n \to \infty$, we have that $\Pi_{d,n}$ converges weakly towards a centered reduced Gaussian distribution on $\mathbb{R}^d$

Proof [See pp55 Some Aspects of Brownian Motion : Some Recent Martingale Problems]. Uee LLN. If $(X_1, X_2, \ldots, X_n)$ iid $N(0,1)$, then

$$\frac{1}{n}\rho_n^2 =: \frac{1}{n}\sum_{i=0}^{n} x_i^2 \to 1 \to \infty$$

If we define $\tilde{X}_{(n)} = (X_1, X_2, \ldots, X_n) = \frac{1}{\sqrt{n}}\rho_n\theta_n$ where $\theta_n \sim \sigma_n$ a uniform distribution on $\sqrt{n}S_{n-1}$. Then the lemma follows from the equation $\tilde{X}_{(n)} = \frac{1}{\sqrt{n}}\rho_n\Pi_{d,n}(\theta_n)$.

Sparse PCA [CS Notes from http://ugcs.caltech.edu/ srbecker/

Classic PCA is sensitive to outliers. Convex methods can be use to address this by attempting to factor the data X as a sum of a low rank component L and a sparse component S by solving

$$\underset{min}{S, L} \|L\|_* + \lambda \|S\|_{\ell_1} : X = L + S$$

See http://cvxr.com/tfocs/demos/rpca/ for a demo of this in action with the popular cvx software package.

0.21 Compressed Sensing

In [31] bounds are established on the relative error for the limiting probabilities for a perturbation of a finite Markov chain. This improves on the traditional eigenvector perturbation approach by exploiting the constraints of the problem. We collect some of these results in this section and see if they can be applied to analyse the stability of spectral clustering algorithms.

Let T be the transition matrix of a finite Markov chain $\mathcal{C}$. $A = I - T$, and $\mathcal{C}$ be a perturbation to $\mathcal{C}$ where $\tilde{T} = T - E$ is the transition matrix of $\mathcal{C}$. Let ω be the limiting probability, $\omega = \lim_{n\to\infty} T^n x$. Define $A = I - T$ and $A^\sharp$ the generalized inverse of A. Then $W = \lim_{n\infty} \frac{I+T+T^2+\ldots+T^{n-1}}{n} = I - AA^\sharp$ is the limiting matrix of $\mathcal{C}$. Every row of W is ω

We state a few relations without proof from Meyers paper

$$(A + E^\sharp) = A - A^\sharp EA^\sharp(I + EA^\sharp)^{-1} - W(I + EA^\sharp)^{-1}A^\sharp(I + EA^\sharp)^{-1} \quad (0.21.1)$$

This combined with the expression for $\tilde{W}$ yields

$$\tilde{W} = W(I + EA^\sharp)^{-1} = W - WEA^\sharp(I + EA^\sharp)^{-1} \quad (0.21.2)$$

The above gives a condition for the limiting matrix to be invariant under a perturbation; $W = \tilde{W} \iff range(E) \in range(A)$. It also allows us to write $\omega - \tilde{\omega} = \omega EA^\sharp(I + EA^\sharp)^{-1}$, so

$$\|\omega - \tilde{\omega}\| \leq \left\|EA^\sharp\right\| \left\|(I + EA^\sharp)^{-1}\right\| \quad (0.21.3)$$

and when $\left\|EA^{\sharp}\right\| \leq 1$ we can use the familiar Taylor expansion

$$\left\|(I + EA^{\sharp})^{-1}\right\| \leq \frac{1}{1 - \|EA^{\sharp}\|} \tag{0.21.4}$$

to obtain an expression for the relative error in ω for a given relative error in A

$$\frac{\|\omega - \tilde{\omega}\|}{\|\omega\|} = \frac{\frac{\|E\|}{\|A\|}\kappa(\mathcal{C})}{1 - \frac{\|E\|}{\|A\|}\kappa(\mathcal{C})} \tag{0.21.5}$$

0.22 Ultracontractivity

0.23 Generalized Uncertainty Principles

The sparse recovery problem can be stated as follows. Given an unknown signal $f\ in\mathbb{C}^n$, when can we recover f from a set of k linear measurements $\Phi f\ in\mathbb{C}^k\ k < n$? This problem is underdetermined and we are interested in the sparsest solution.

$$min\left\|f*\right\|_0 : \Phi f* = \Phi f$$

This problem is not convex, and in fact is generally NP-Hard [18] and [35]. Note that the norm here is not the usual ℓ_0 norm, here we abuse notation $\|g\|_0$ to mean $cardg_i! = 0$, the size of the support. This problem can be relaxed to a convex problem by using the ℓ_1 norm.

$$min\left\|f*\right\|_1 : \Phi f* = \Phi f$$

This problem is not differentiable at points where $f*_i = 0$ and will require a general convex solver. Usually the problem is solved in practice by recasting the minimization as a linear program

$$min < f*, f >: \Phi f* = f\ f* \succeq 0$$

bbcrevisit

Much work has been done to determine when these two problems have the same solution, i.e. when is exact recovery possible. One sufficient condition for exact recovery is the RIP condition of Candes and Tao (see below).

Definition 0.23.1 (RIP (Candes & Tao))**.** Let $A \in M_{m\times n}(\mathbb{R})$ and $p \in [1, n]$, then we say that A has the restricted isometry property if

$$\exists\delta_p\ \ni\ \forall m, pA_s \in A$$
$$(1 - \delta_p)\left\|y\right\|_{\ell^2}^2 \leq \left\|A_s y\right\|_{\ell^2}^2 \leq (1 + \delta_p)\left\|y\right\|_{\ell^2}^2 \leq$$

RIP is a property which classifies a matrix as being close to orthogonal. δ_p is referred to as the RIC, and is NP-Hard to compute. Compressed sensing decoders are guaranteed to recover the sparsest solution to $Y = Ax$ when A is close to an isometry. Bounds on the RIC are available for some classes of random matrix ensembles.

0.24 Random Matrix Theory & OP

RMT concerns itself with the eigenvalue statistics of large matrices with random entries. We define the eigenvalue counting measure of a matrix Hn as;

$$\mu_H(A) = \frac{|\lambda_i \in A|}{n} = N_{1_A,H}$$

More generally a eigenvalue statistic $N_{f,H} = \frac{trf(H)}{n}$ For many types of random matrices we have a CLT;

$$\frac{N_{f,H} - \int f(\lambda) dN(\lambda)}{\sigma_{f,n}} \rightarrow_d N(0,1)$$

$GUE(n)$ Start with Gaussian measure

$$\gamma^n(A) - \frac{1}{(\sqrt{2\pi})^n} \int_A e^{\frac{-1}{2}} \|X\|^2 d\lambda^n$$

Weiener space is the Hilbert space $L^{2,0}[0,1]$ of upon which a Gaussian measure can be defined. The inner product on Weiner space is

$$< \sigma_i, \sigma_j >= \int_0^1 < \dot{\sigma_i}, \dot{\sigma_j} > dt$$

where $\dot{\sigma_i} = \frac{d\sigma)i}{dt}$ The Wiener measure is a Gaussian measure.

Let H be Hermitian, $Z_{GUE(n)} = 2^{\frac{n}{2}} \pi^{\frac{n^2}{2}}$ We can write

$$\gamma^n(A) - \frac{1}{Z_{GUE(n)}} \int_A e^{\frac{-1}{2}} tr(H^2) d\mu$$

There are two domains of eigenvalue statistics, local and bulk. Locally we are concerned with level spacing ,$\Delta\lambda = \lambda_i - \lambda_{i-1}$, and edge statistics $P_{TW}(\lambda_1)$, $\lim_{n\to\infty} P_{TW}(\lambda_n)$. The Tracy Widom law P_{TW} gives edge statistics.

The empirical spectral measure of H is

$$\mu_H = \frac{|eigsH \in A|}{n}$$

The CDF of $H(n)$ is $N_n(\lambda)$ and as $n \to \infty$ is $N_n(\lambda) \to W(\lambda)$ where W is the Winger Semi-Circle Law.

Bulk statistics look at the determinantal point process $E(\lambda_0) = \sum_j \delta(n\rho(\lambda_0(\lambda_j - \lambda_0)$. The kernel of the process is the sine kernel $K(x,y) = \frac{sin(\pi(x-y))}{\pi(x-y)}$. $E(\lambda)$ captures the statistics of the eigenvalues in the vicinity of λ_0. Recall the joint densities of a determinantal point process with kernel K are given by $\rho_n(x_1, ..., x_n) = det(K(x_i, x_j)_{1 \le i,j \le j})$

0.25 The Hamburger Moment Problem - HMP

Given a sequence m_i the HMP seeks to find a measure that generates the moments.

$$\exists \mu : m_n = \int_{-\infty}^{+\infty} x^n d\mu$$

The answer is affirmative when the Hankel kernel $A \succ 0$, which is equivalent to $\sigma(A) \in \mathbb{R}^+$

$$A = \begin{pmatrix} m_0 & m_1 & \dots & \\ m_1 & m_2 & m_3 & \\ m_2 & m_3 & m_4 & \\ \vdots & & & \ddots \end{pmatrix}$$

Recall that A Hankel matrix is a square matrix with constant skew-diagonals; $A_{i,j} = Ai-1, j+1$ A Hankel matrix is an upside-down Toeplitz matrix.

The Hilbert matrix $Hi, j = \frac{1}{i+j-1}$ is a special case of a Hankel matrix. The Hilbert matrices are canonical examples of ill-conditioned matrices. They arise in the expression for the Grammian matrix of powers of x; $H_{ij} = \int 01 x^{i+j-2} dx$ which shows up in the least squares approximation by polynomials.

The solutions to the HMP are either unique or infinite in number and form a convex set in $\mathcal{H}$

Let

$$\Delta_n = \begin{pmatrix} m_0 & m_1 & \dots & m_n \\ m_1 & m_2 & \dots & m_{n-1} \\ & & \ddots & \\ m_n & m_{n+1} & & m_{2n} \end{pmatrix}$$

, then $A \succ 0 \Rightarrow det(\Delta_n) \geq 0 \forall n$ If $det(\Delta_n) = 0$ then $((H), <, >)$ is finite dimensional and T is self adjoint.

A gives us a sesquilinear form on $(H) = (l)^2$ via $< x, y >= \hbar x^T A y$.

Let T be a shift operator. The HMP is closely related to OP in $\mathbb{R}$. Gram Schmidt gives basis ϕ_i in which T has tridiagonal Jacobi.

The Caley transform $Q(T) = (I - T)(I + T)^{-1}$ shows the connection to the Nevanlina class of functions (sub-harmonic).

0.26 Random Matrix Ensembles

[1], [2], [3] , [16], [44], [46]

The classical ensembles of random matrix theory are GOE, GUE, GSE, Wishart, and MANOVA. These correspond to the weight functions of the equilibrium measure of the orthogonal polynomials Hermite, Laguerre,and Jacobi. The Jacobians of the well known matrix factorizations are used to compute the joint eigenvalue densities of these ensembles. The distribution of eigenvalues of the GOE ensemble follow the well know Winger Semi-circle distribution. The joint densities up to a constant factor are listed below:

- Hermite
- Laguerre
- Jacobi

We generated histograms in Matlab for samples from the GOE, GUE, GSE, Wishart, and MANOVA ensembles. The joint PDF of a generic Gaussian random matrix is given by,

$$P(M) = G_\beta(n,m) = \frac{1}{2\pi^{\frac{\beta n m}{2}}} \exp^{\frac{-1}{2}\|M\|_F}$$

where β encodes the dimension of the field. Note this leaves open the possibility to generalize to non integer β.

The table below describes how to generate from the common ensembles starting from a sample $A \in G_\beta(n,n)$

$$GOE\{M|M = \frac{A+A^T}{2}, A \in G_1(n,n)\}$$
$$GUE\{M|M = \frac{A+A^\dagger}{2}, A \in G_2(n,n)\}$$
$$GSE\{M|M = \frac{A+A^\ddagger}{2}, A \in G_4(n,n)\}$$

The CS decomposition is a matrix factorization equivalent to four SVD's which correspond to rotation problems $\begin{pmatrix} X \to Y & X^\perp \to Y \\ X \to Y^\perp & X^\perp \to Y^\perp \end{pmatrix}$ Which can be compactly written $[X|X^\perp]^T[Y|Y^\perp] = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} * \begin{pmatrix} C & S \\ -S & C \end{pmatrix} * \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$ Where U, S are unary.

The Tracy-Widom law of order one is the limiting distribution of the largest eigenvalue of a Wishart matrix with identity covariance when properly scaled. This has some application to weighted directional graphs. The largest eigenvalue of the adjacency matrix of a random d-regular directed graph follows the Tracy-Widom law. The kernels of integrable operators describe the asymptotic eigenvalue distribution of self-adjoint random matrices from the unitary ensembles. Consider the discreet operator $K(n,m) : l^2(N) \to l^2(M)$ where $K(n,m) = \frac{(<Ja(m),a(n)>}{m-n}$ the discrete Bessel kernel and kernels arising from the almost Mathieu equation. The celebrated paper of Tracy and Widom [46] investigated integral kernels of the form

$$K(x,y) = \frac{f(x)g(y) - f(y)g(x)}{x-y} : x \neq y f(x), g(x) \in L^2(0,\infty)$$

are solutions to the system of ODE's

$$\frac{d}{dx}\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} \alpha(x) & \beta(x) \\ -\gamma(x) & -\alpha(x) \end{pmatrix} * \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$$

Let $\phi_i(x)$ be an orthogonal basis in a Hilbert Space $\mathcal{H}$ where

$$\Gamma_\phi = \{\phi_(j+k-1)\}_{j,k=1}^{\infty}$$

is the induced Hankel Matrix.

Let $(L) : \mathcal{H} \to \mathcal{H}$ be compact. Then $(L) : f \mapsto \sum_{n=1}^{N} \omega_n < \phi_n, f > \psi_n$ where $\{\phi_i\}_{i=1}^{N}$

0.27 The Tracy Widom Law

The Tracy-Widom distribution is related to to determinantal stochastic processes. A process following this law is distributed as the largest point of a point process on the real line where the kernel K is the so-called Airy kernel. In addition to describing the edge spectrum of random matrices, it arises in several place in combinatorial for instance the longest increasing subsequences of random permutations is described by the Tracy Widom law. In addition to the eigenvalues of random matrices, this type of point process is used in models such as fluctuations in first and last passage percolation, and the asymmetric exclusion process.

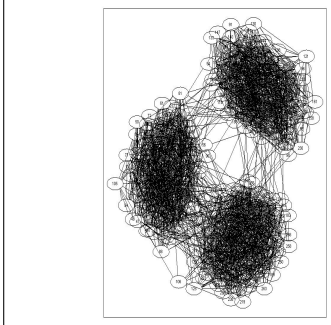
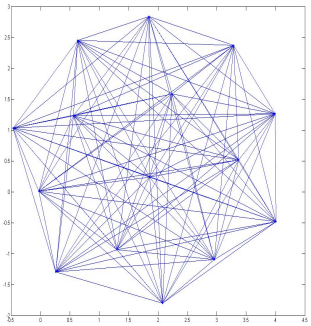
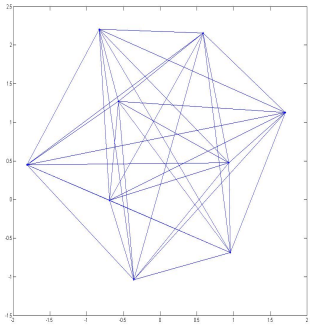
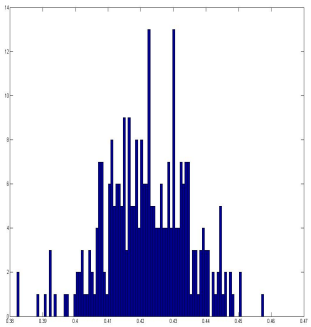
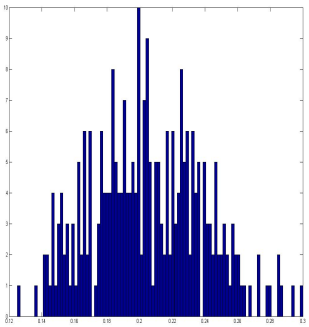
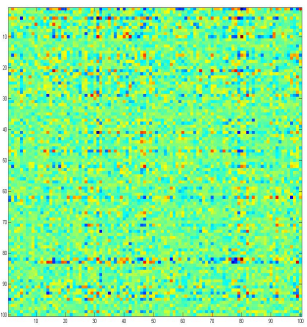
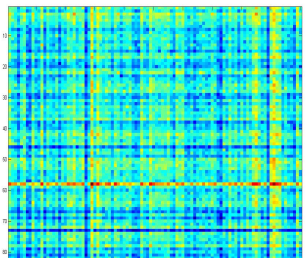
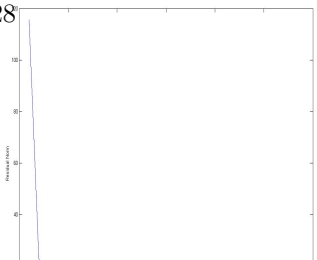
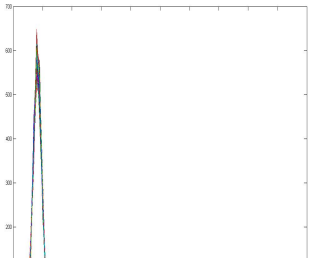
The path configuration of random viscous walkers is related to the Young tableaux. Statistical problems related to the Young tableaux include random growth, point processes, random permutation, and the random word problem. the asymptotic distribution of scaled variables from these models are described by the Tracy Widom distribution which is the limit distribution for the largest eigenvalue of $X \in GUE$.

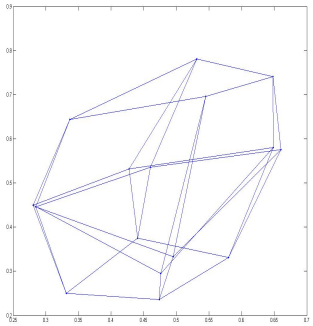
Chapter 1

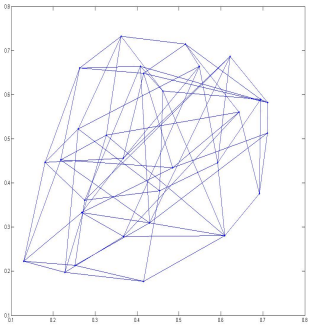
Figures & Images

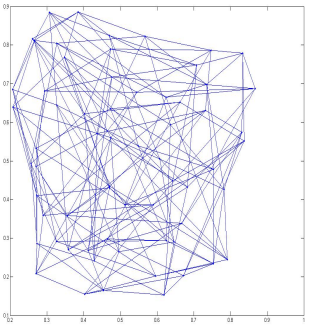
1.1 Modeling first & last steps: Plot the data, plot the results.

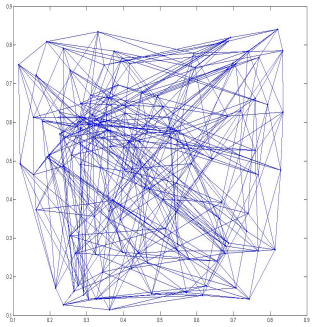
1.1.1 Figures and Numerical Results from Experiments in Networks, Small World Models, and Spectral Graph Theory

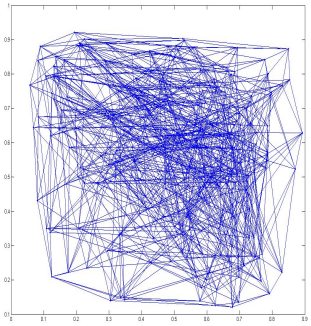










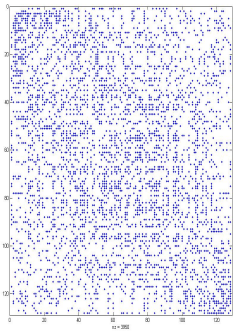


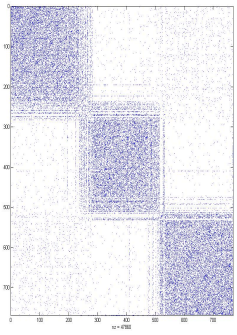


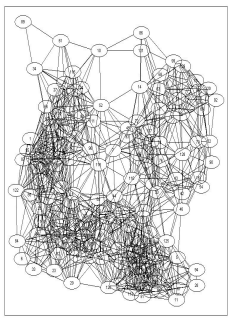


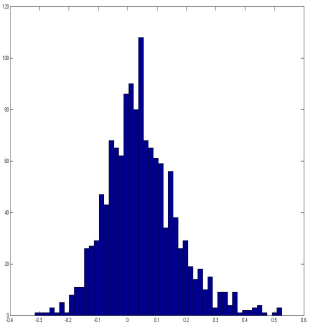


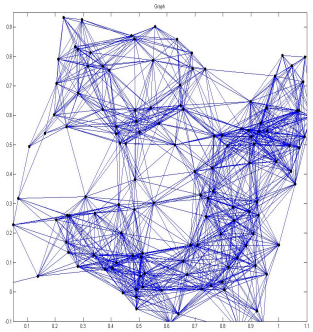


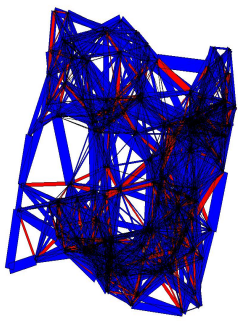


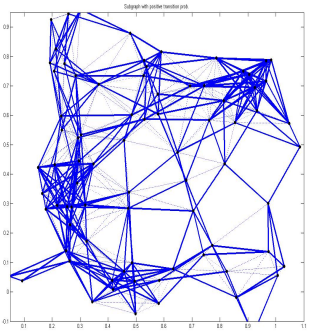


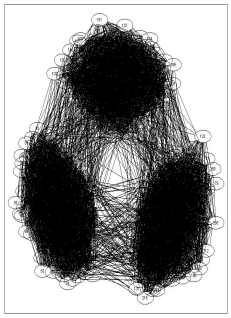


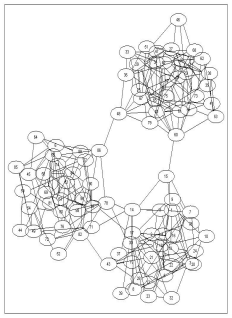


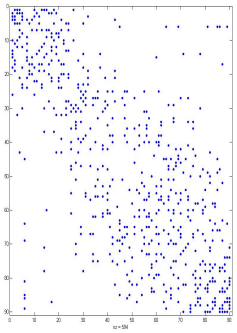


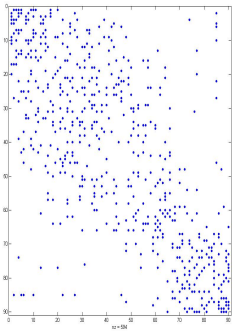


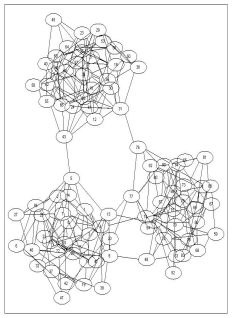


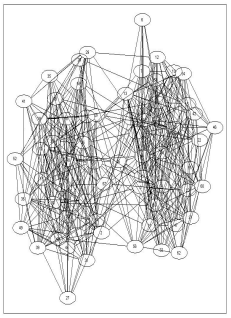


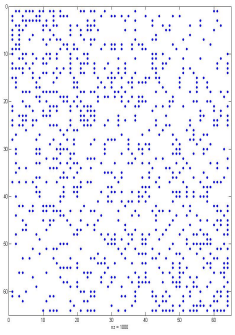


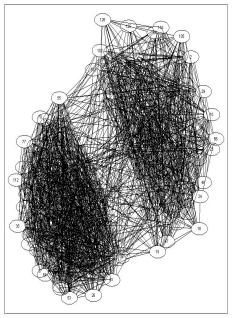


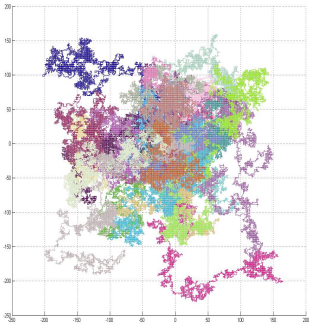


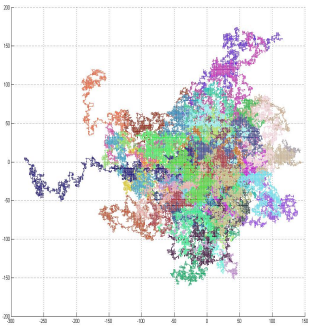


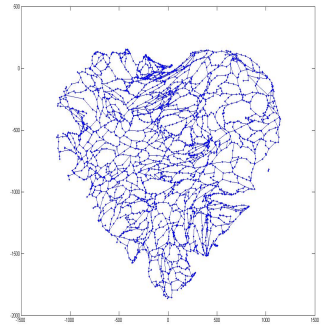


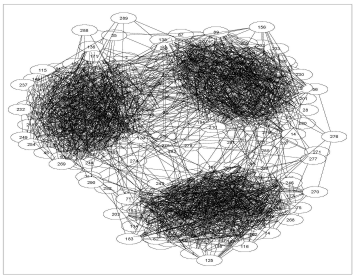


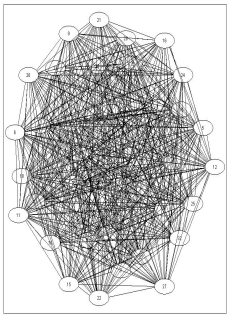


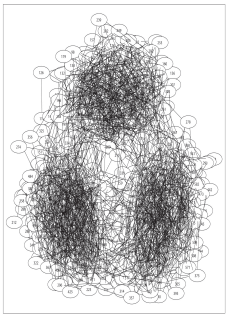


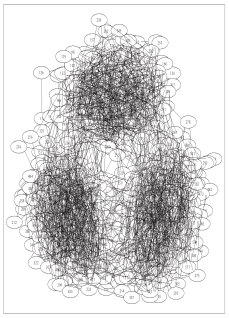


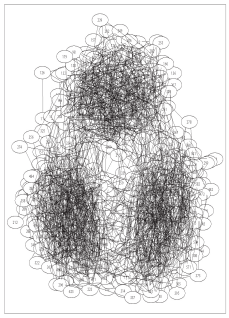


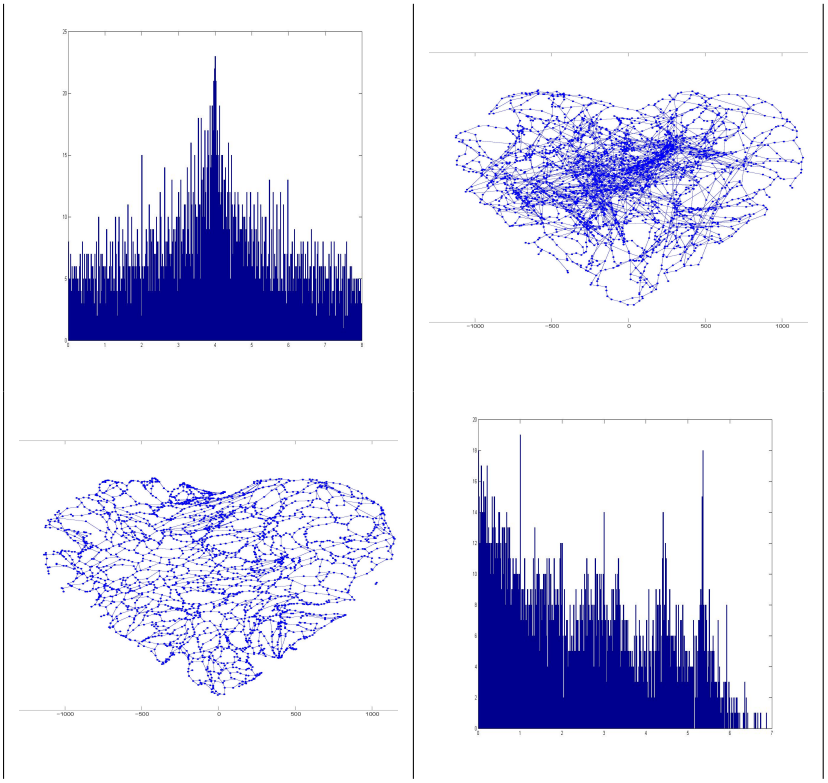




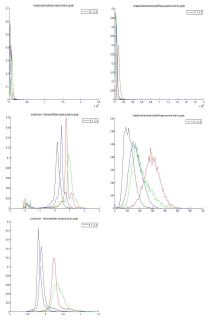


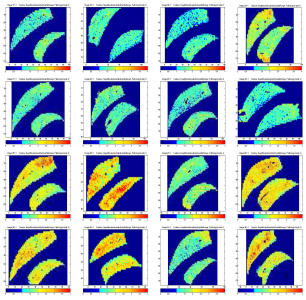


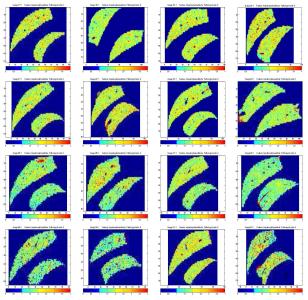


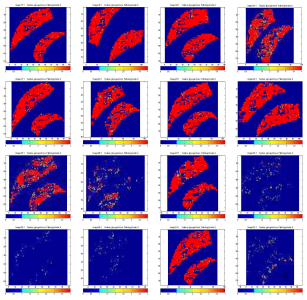


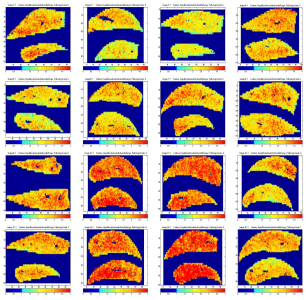
1.1.2 Cell Segmentation & Automated Pathology

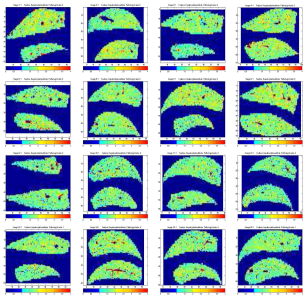


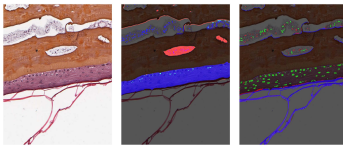












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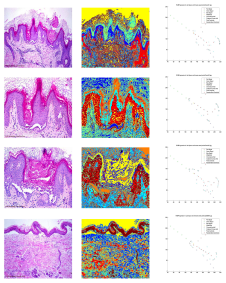

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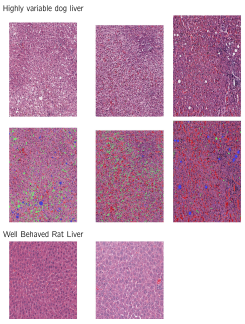

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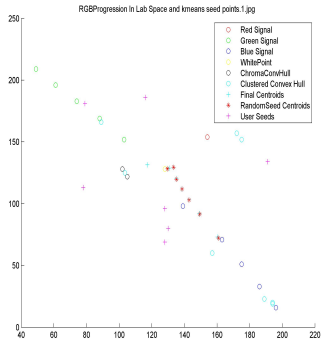
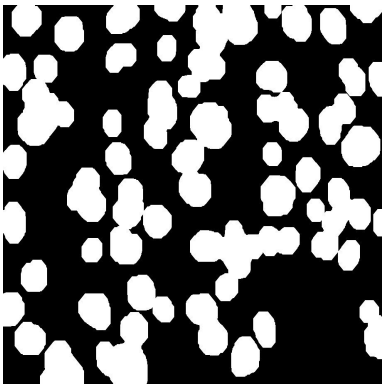
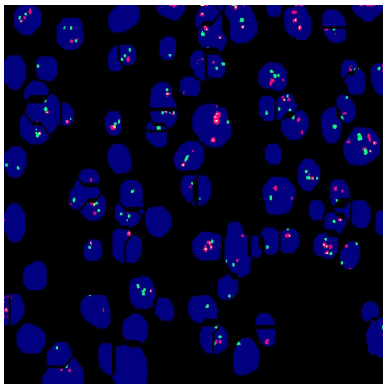
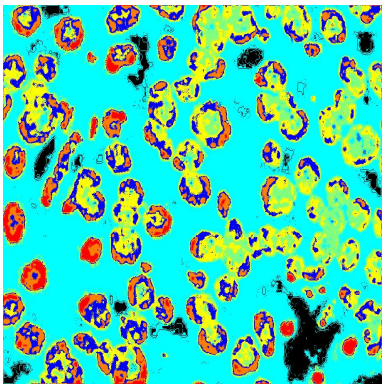
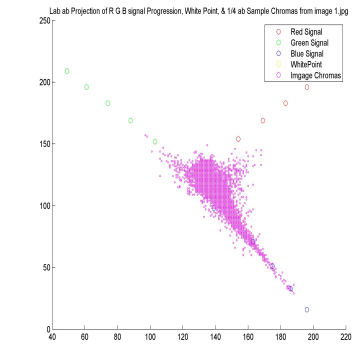
14

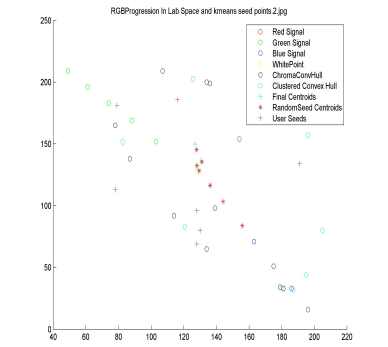

RGBProgression In Lab Space and kmeans seed points.1.jpg
Red Signal
Green Signal
Blue Signal
WhitePoint
ChromaConvHull
Clustered Convex Hull
Final Centroids
RandomSeed Centroids
User Seeds

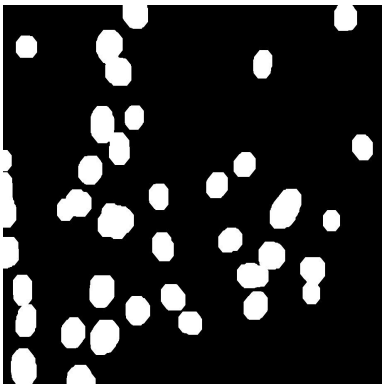


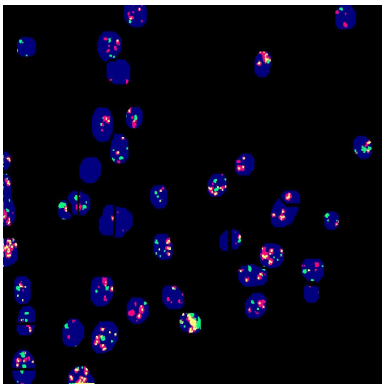


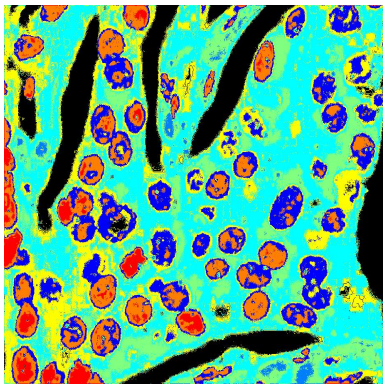


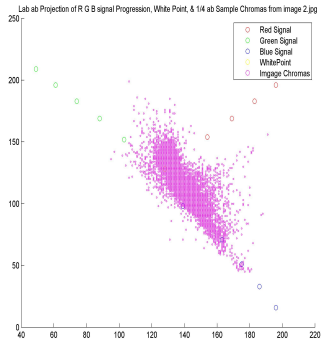

Lab ab Projection of R G B signal Progression, White Point, & 1/4 ab Sample Chromas from image 1.jpg
Red Signal
Green Signal
Blue Signal
WhitePoint
Imgage Chromas

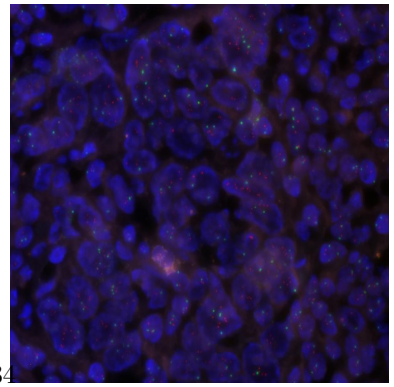

RGBProgression In Lab Space and kmeans seed points.2.jpg
Red Signal
Green Signal
Blue Signal
WhitePoint
ChromaConvHull
Clustered Convex Hull
Final Centroids
RandomSeed Centroids
User Seeds

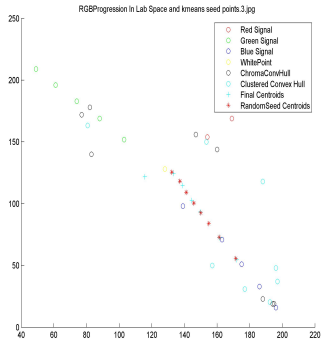


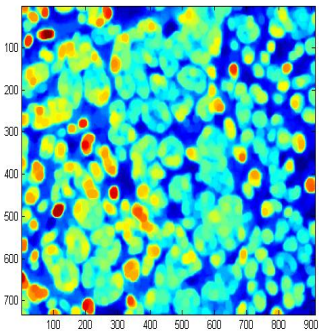


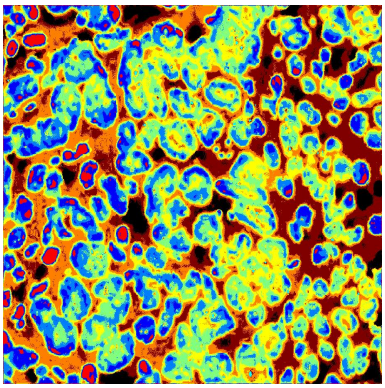


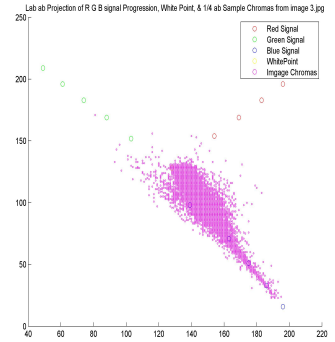

Lab ab Projection of R G B signal Progression, White Point, & 1/4 ab Sample Chromas from image 2.jpg
Red Signal
Green Signal
Blue Signal
WhitePoint
Imgage Chromas

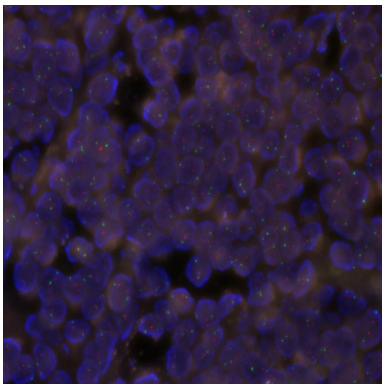


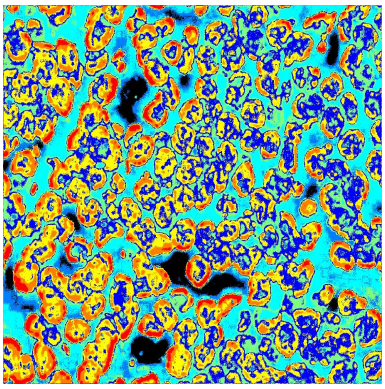

RGBProgression In Lab Space and kmeans seed points.3.jpg
Red Signal
Green Signal
Blue Signal
WhitePoint
ChromaConvHull
Clustered Convex Hull
Final Centroids
RandomSeed Centroids

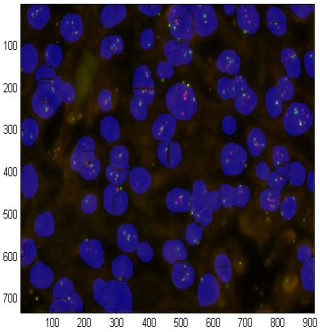


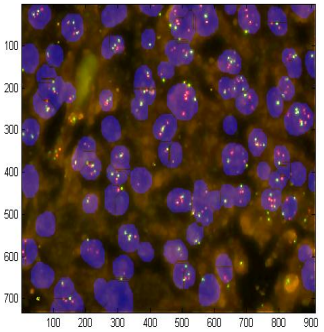


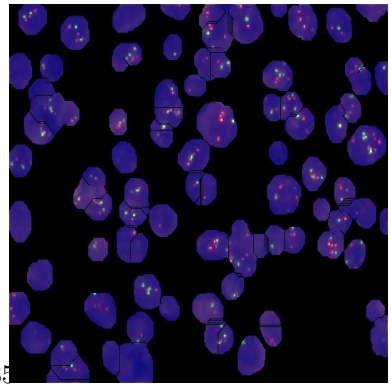

Lab ab Projection of R G B signal Progression, White Point, & 1/4 ab Sample Chromas from image 3.jpg
Red Signal
Green Signal
Blue Signal
WhitePoint
Image Chromas

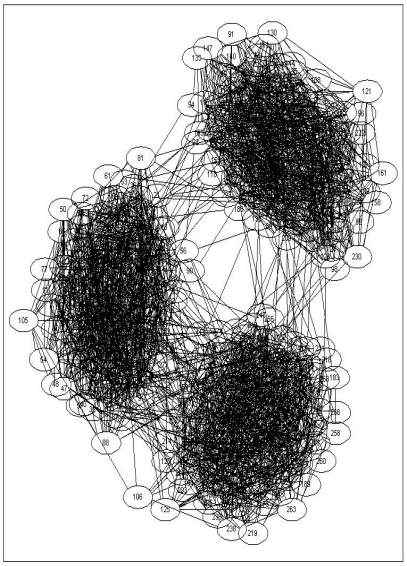


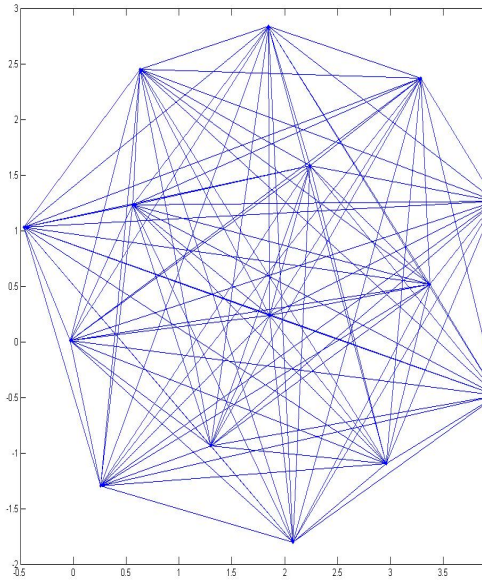


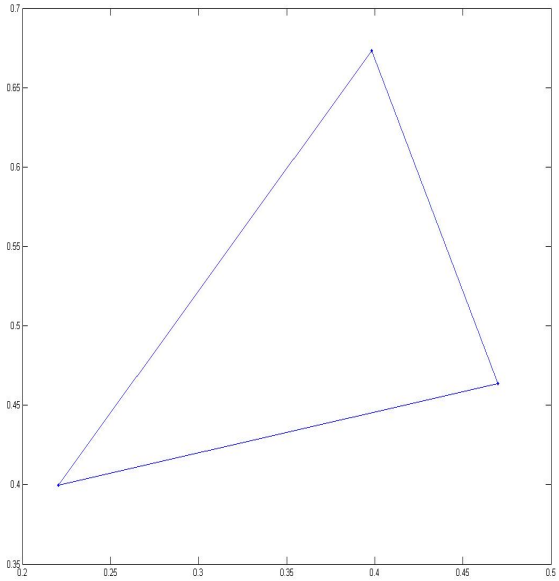


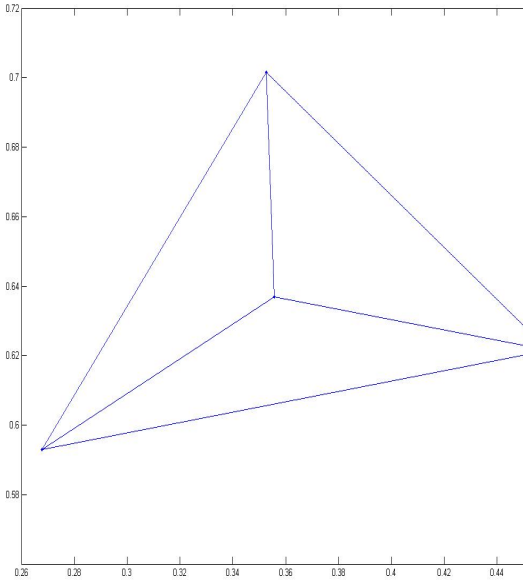


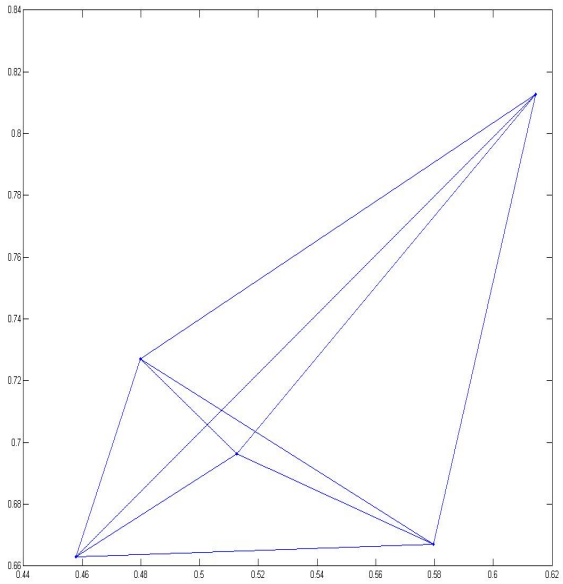








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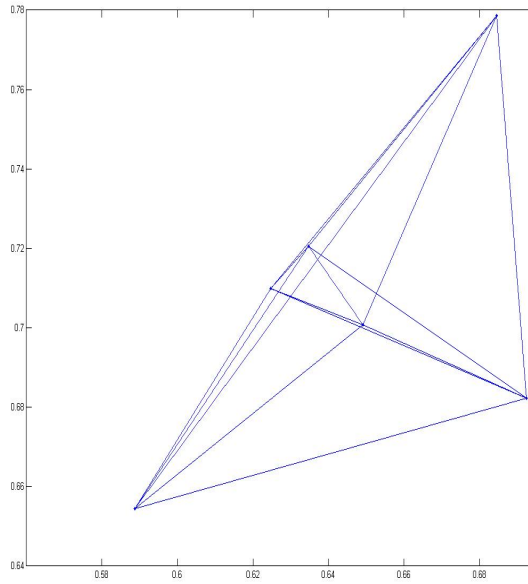
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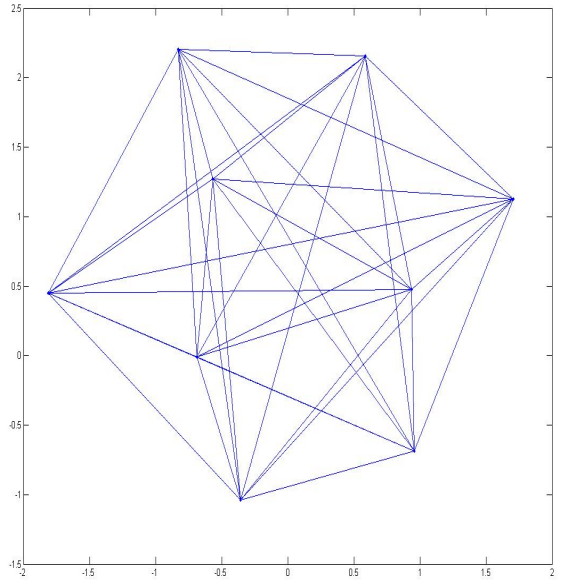
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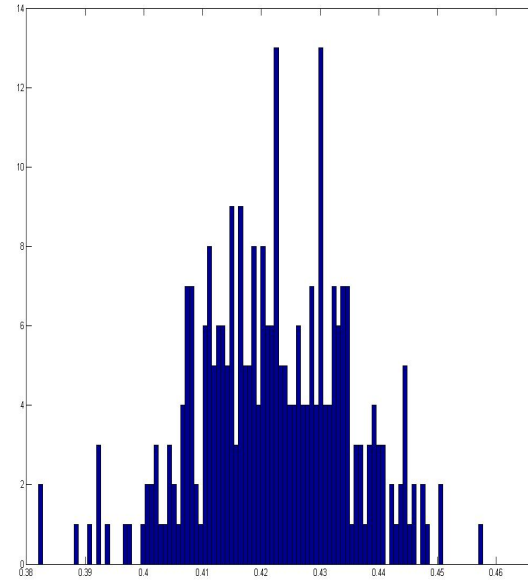
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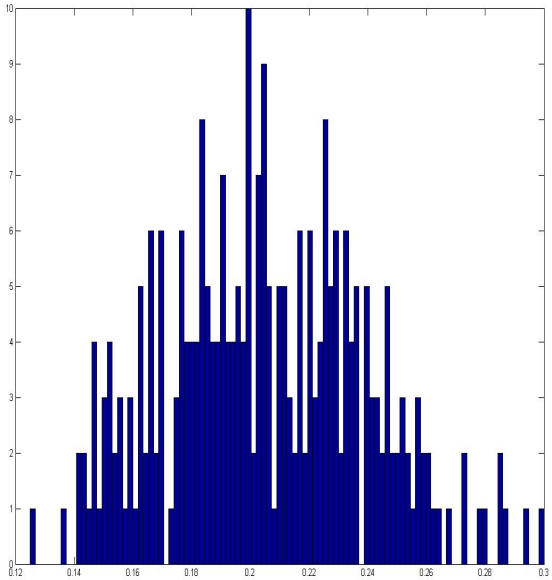

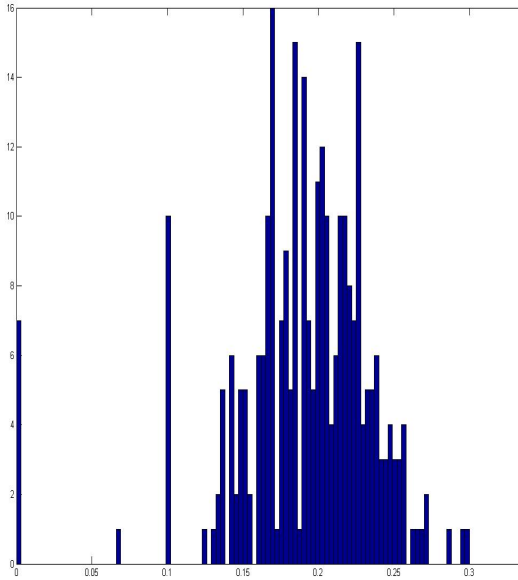

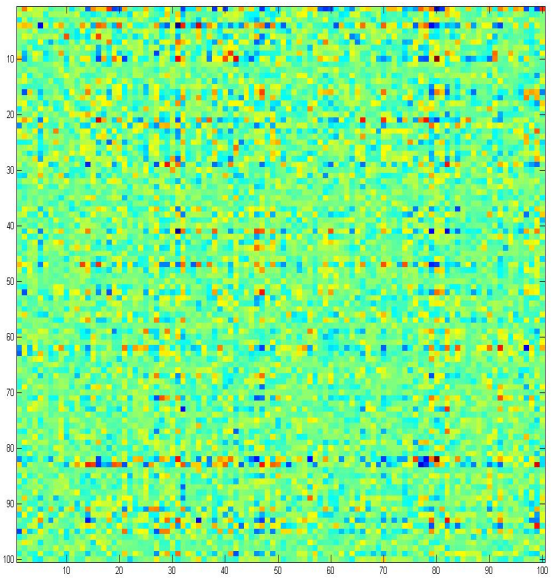

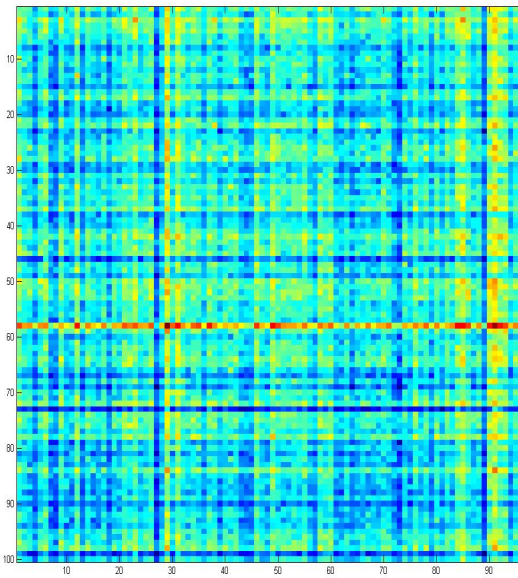

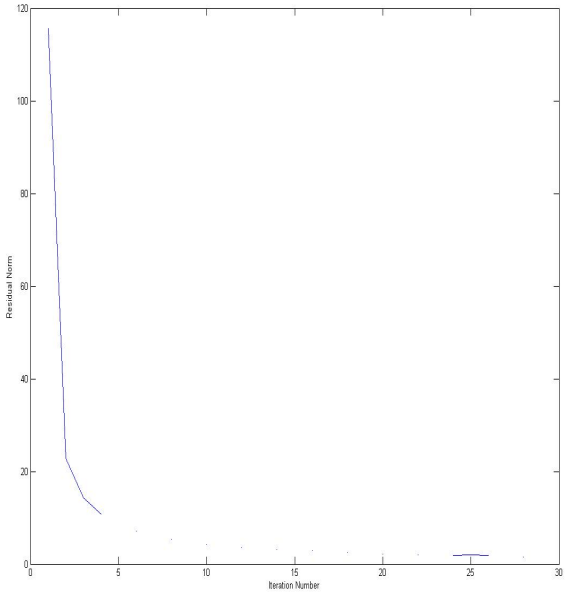
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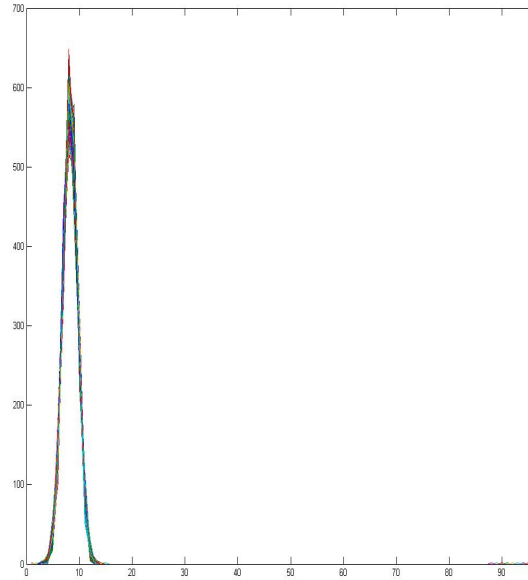

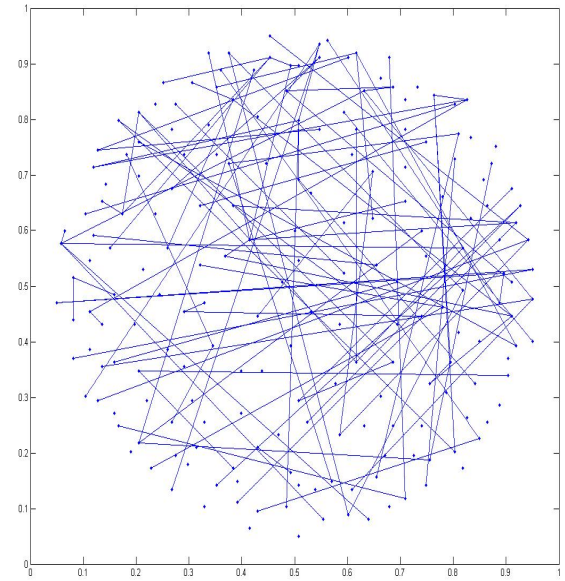

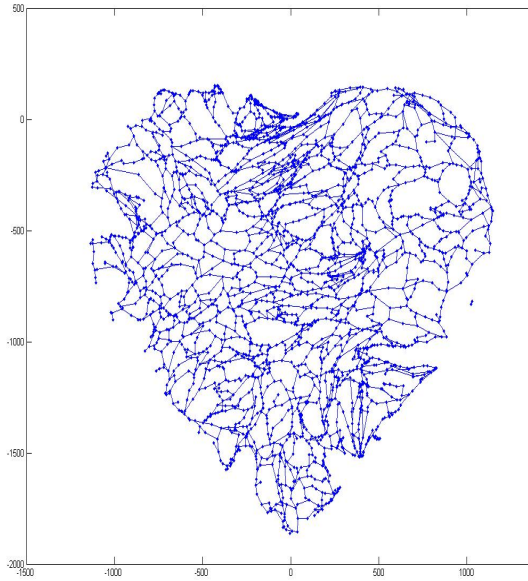

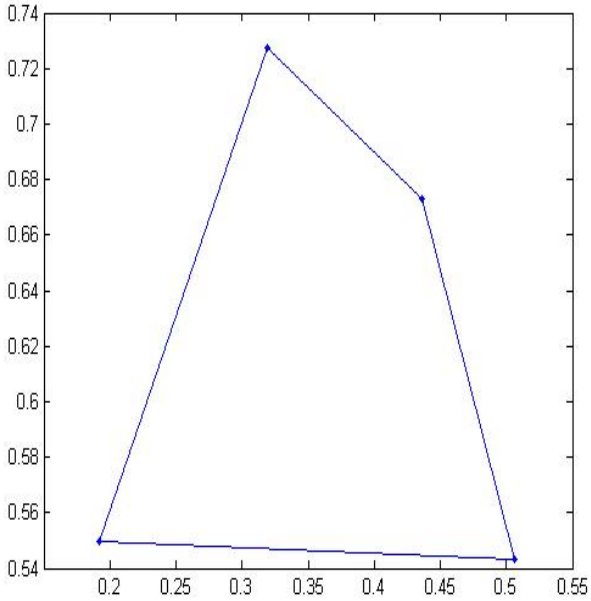
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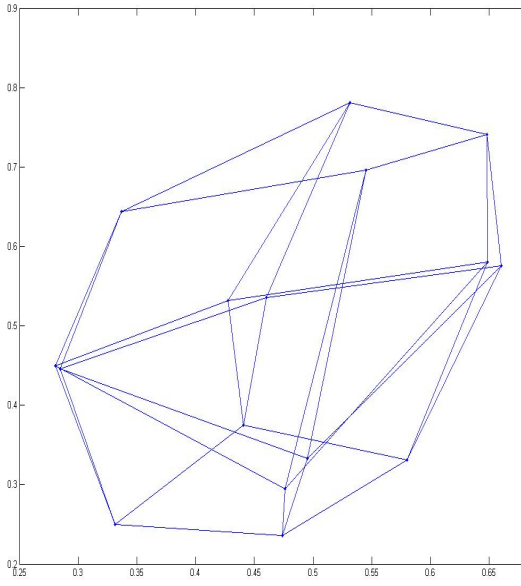

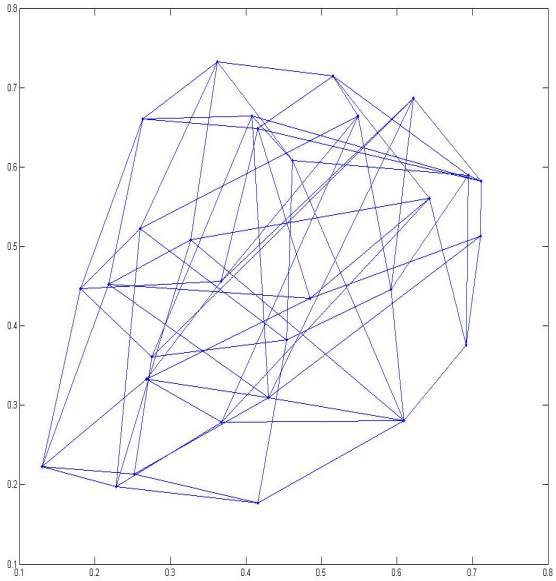

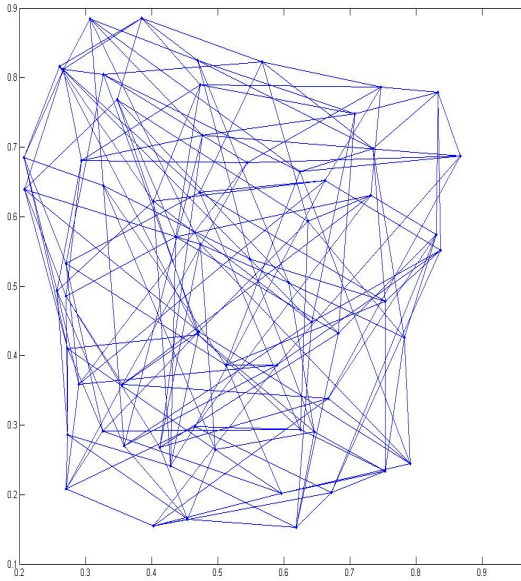

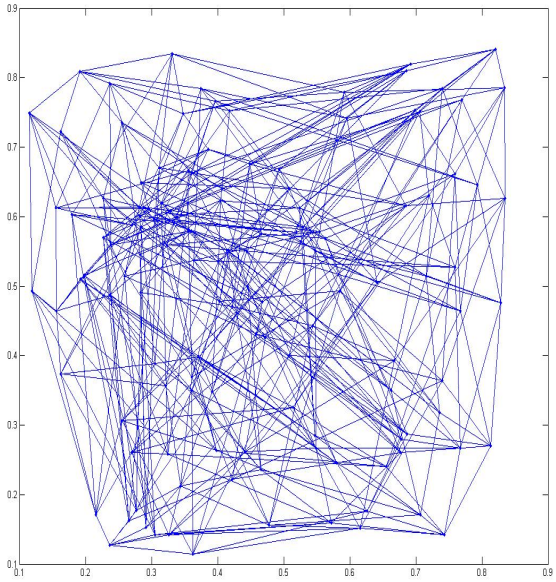

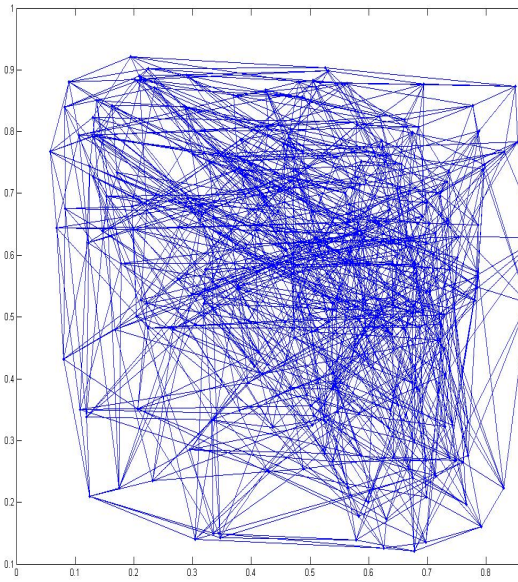

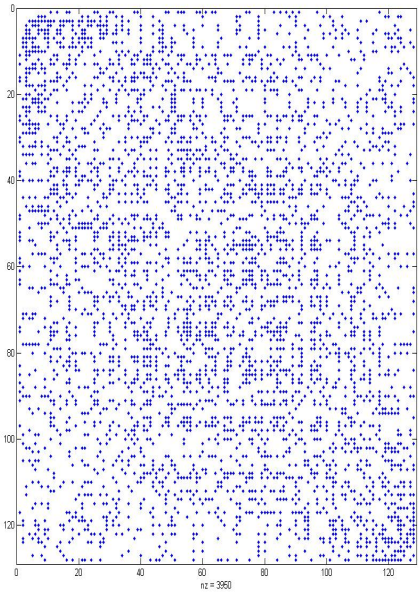
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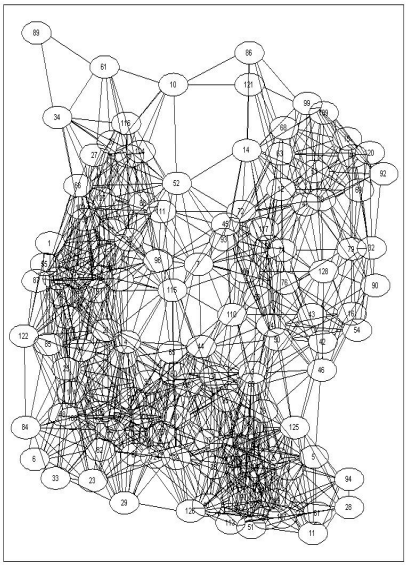

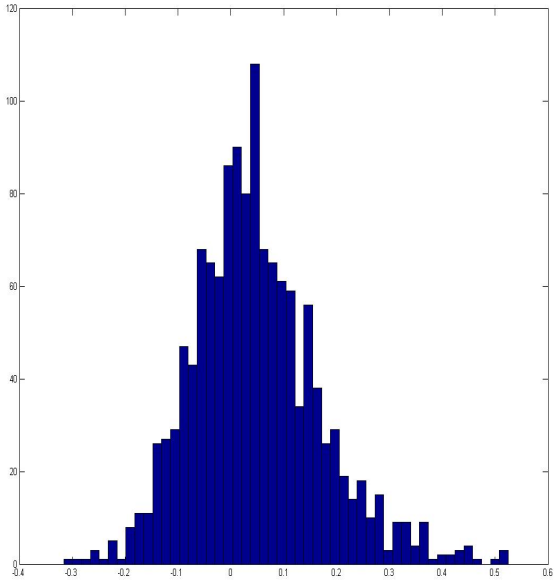

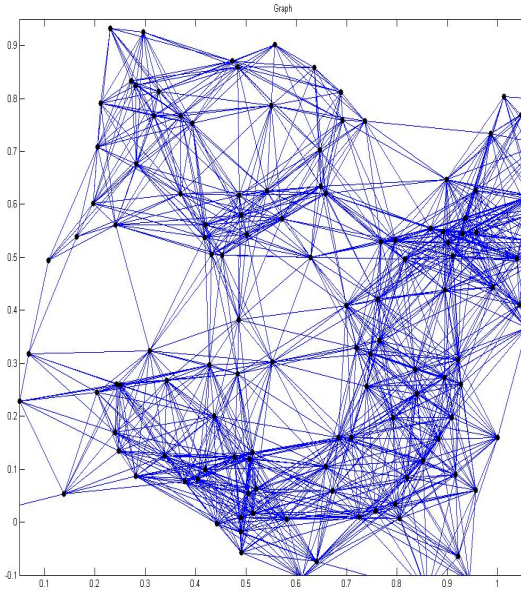
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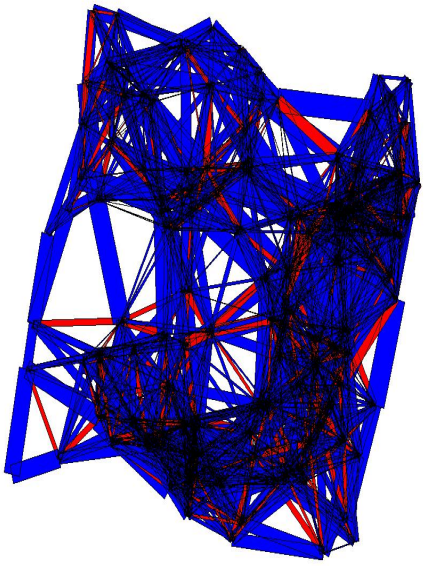

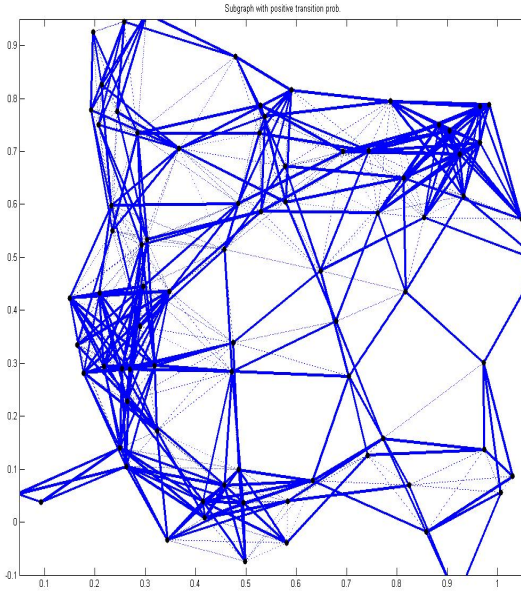
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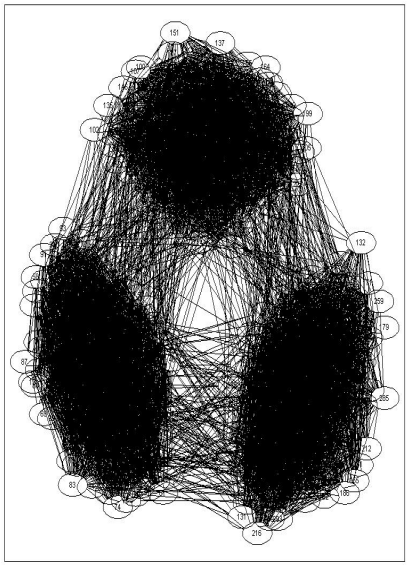

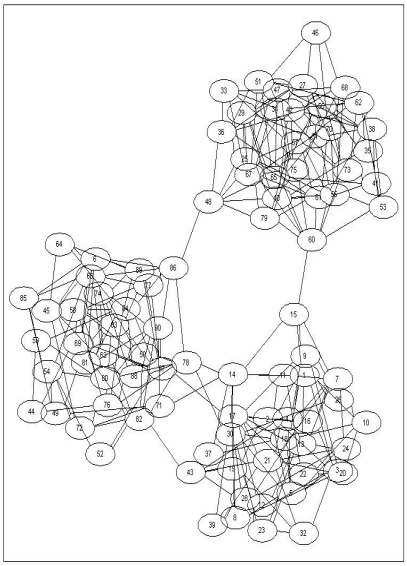

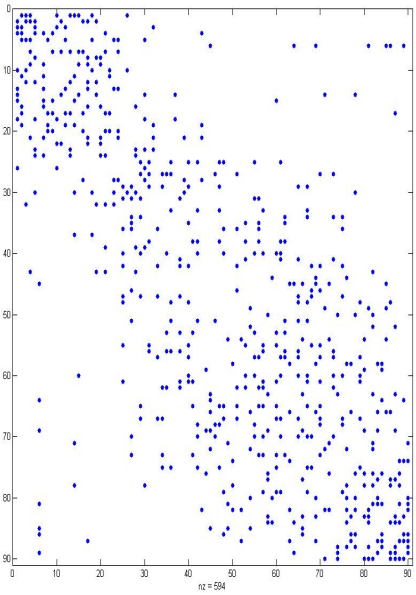
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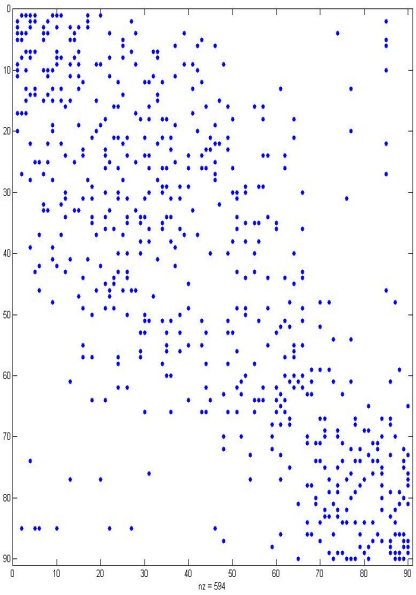
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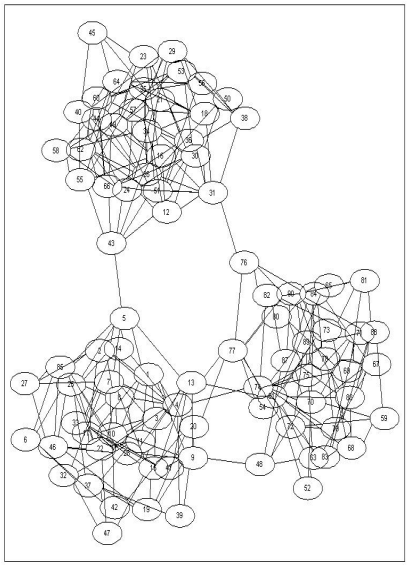

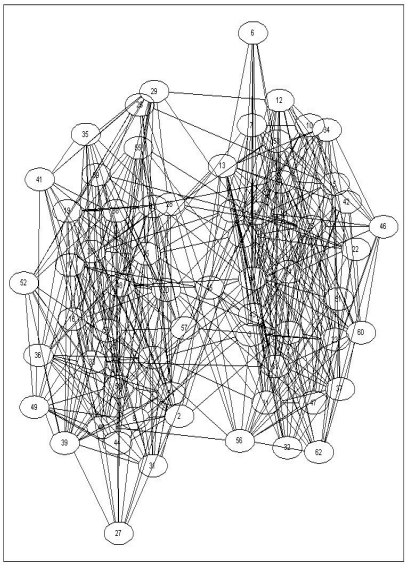

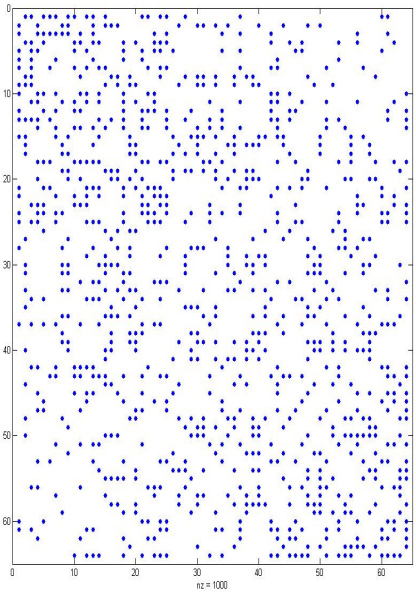
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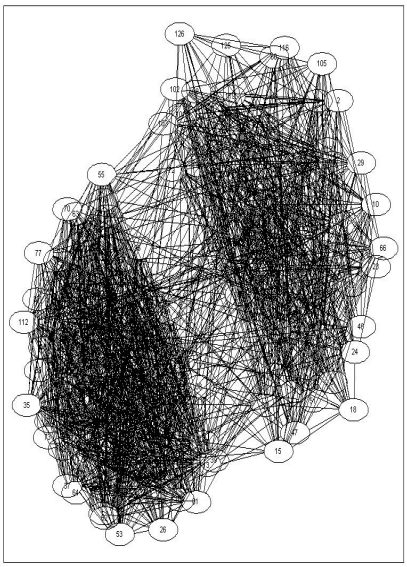

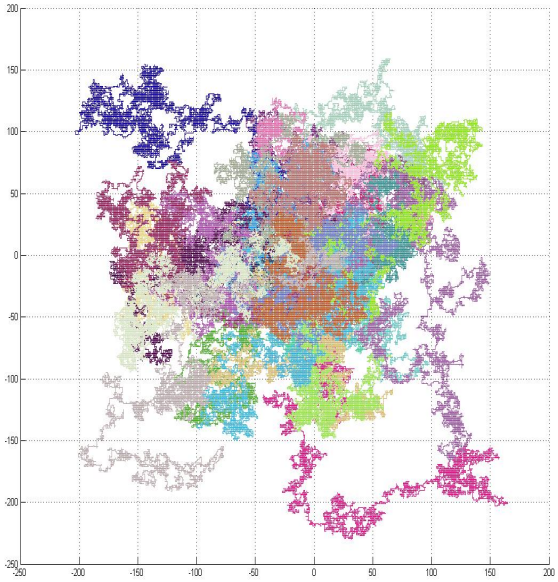
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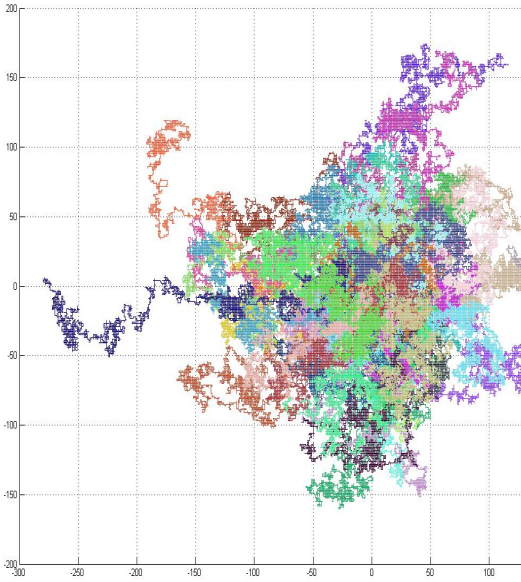
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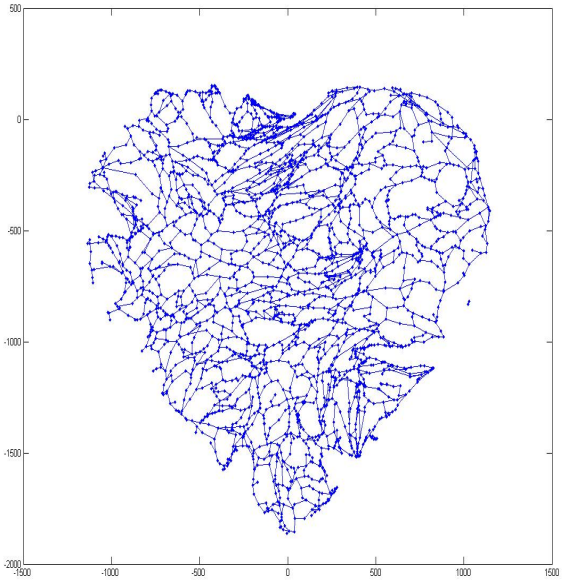
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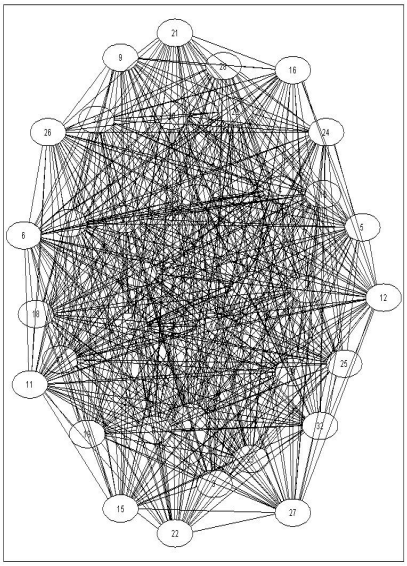
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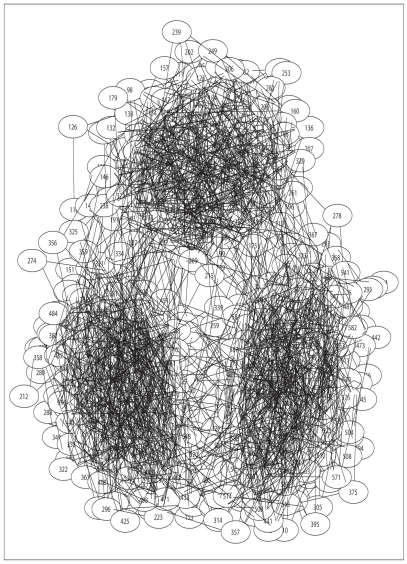

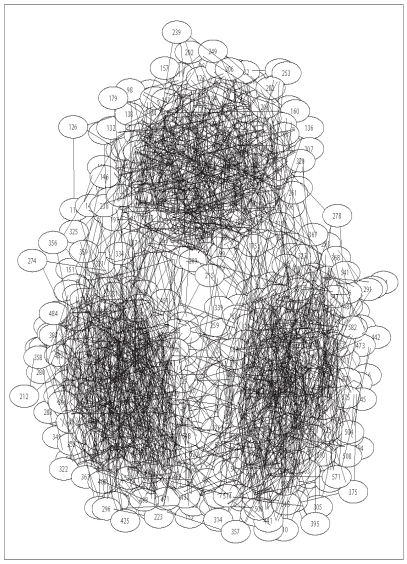

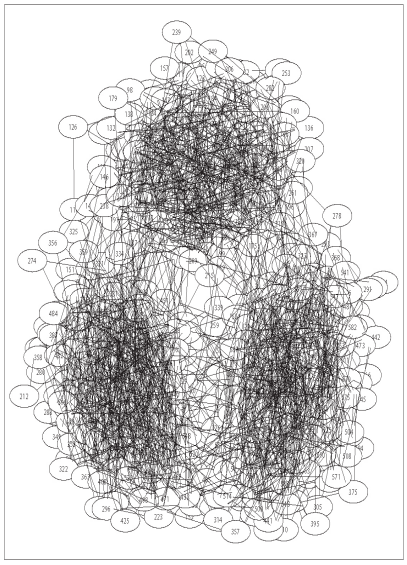

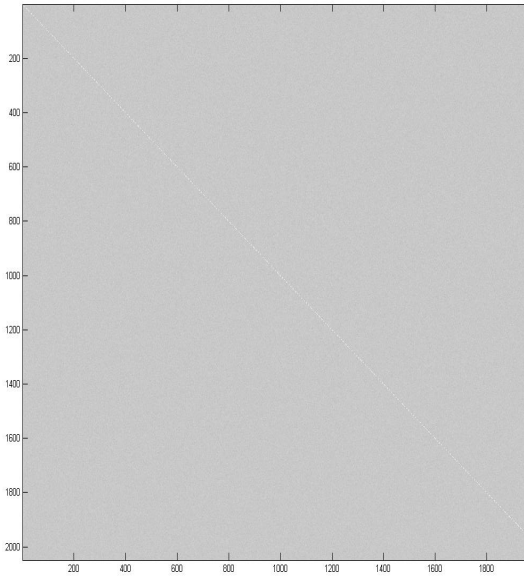

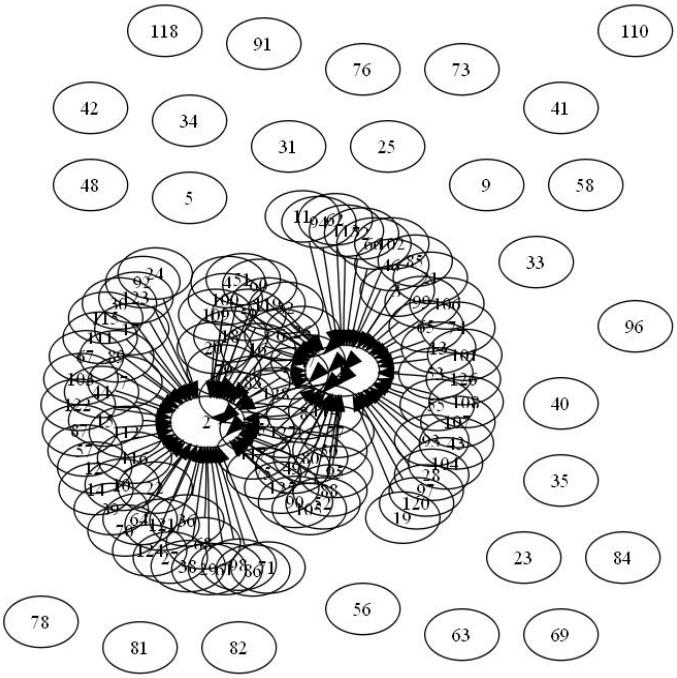
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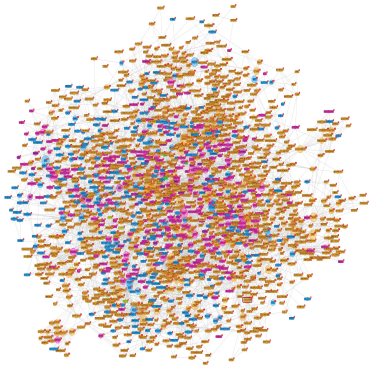
Chapter 2

Natural Language Processing



A word association graph using the based on the WordNet database.

2.1 Introduction

This document is primarily meant to server as a brief tutorial to Natural Language Processing (NLP) as it applies to the problems of classification and prediction of text. The statistical part comes from the use of corpora and probabilistic models in prediction and classification. Computational linguistics is an interdisciplinary field dealing with the statistical and/or rule-based modeling of natural language from a computational perspective. Computational linguistics is primarily concerned with developing algorithms and software for intelligently processing language data and is typically views as a subset of AI. For our purposes we refer to classification as the process of estimating a categorical variable and prediction as the process of estimating a continuous variable. For example , assigning a probability for the next word in a string of words is classification,

while assigning a vulgarity index to a string of words is regarded as prediction. A canonical open problem in Computational Linguistics is the automatic translation of one language into another, a task which requires understanding the morphological and syntactical structure of both languages. Other problems in this field include :

- Computational complexity of natural language
- Computational semantics
- Computer-aided corpus linguistics
- Development of parsers for natural languages
- Development of taggers to identify parts of speech
- Word Sense Disambiguation

Some specific problems this document is concerned with are:

- Word Completion. Given a sequence of letters, find the word that best completes the sequence.
- Word Prediction. Given a sequence of words, find the word that best completes the sequence.
- Document & Text Classification.

While we are primarily interested in a statistical approach to natural language processing, a basic to understanding of syntax and semantics is required to handle the many 'outlier' cases one will encounter. Modern English has many exceptional rules that must be dealt with. This document contains two appendices with definitions that reader will want to be familiar with. Two important tasks in NLP are the decoding of syntactic and semantic information. These are discussed broadly below.

2.1.1 Semantics

Linguistic semantics refers to the meanings that language elements express. It typically focuses on the relation between signifiers, such as words, phrases, signs and symbols, and what they stand for. The term is used in ordinary language to denote a problem of understanding that comes down to word selection or connotation. In written language, such things as paragraph structure and punctuation have semantic content. The basic unit of semantics is sense. One word may have many different meanings. Deriving the meaning from a context - called word sense disambiguation - is a key task in computational linguistics.

2.1.2 Lexicography

Lexicography is divided into two related disciplines, practical and theoretical. Practical lexicography is the art or craft of compiling, writing and editing dictionaries. Theoretical lexicography is the scholarly discipline of analyzing and describing the semantic, syntagmatic and paradigmatic relationships within the lexicon (vocabulary) of a language, developing theories of dictionary components and structures linking the data in dictionaries, the needs for information by users in specific types of situation, and how users may best access the data incorporated in printed and electronic dictionaries.

2.1.3 Corpus linguistics

Corpus linguistics is the study of language as expressed in samples (corpora) or "real world" text. This method represents a digestive approach to deriving a set of abstract rules by which a natural language is governed or else relates to another language. Originally done by hand, corpora are now largely derived by an automated process.

2.2 A Quantification the Semantic Information in WordNet

WordNet [4] is a lexical database for the English language. It groups English words into sets of synonyms called synsets, provides short, general definitions, and records the various semantic relations between these synonym sets. We are interested analyze the statistics of the WordNet semantic graph with the aim to inject sematic information into statistical models. A "semantic sweetenting" of sorts.

2.3 N-gram Models

N-grams are an important part of NLP tasks like tagging parts of speech, natural language generation [chatterbot engines], plagiarism identification, and character prediction in cell phone text input. To get an idea of how an n-gram model is built we will use example n-grams extracted from Anna Kerannina [AK]. The phrase *"for the first time"*, a 4-gram, appears 27 times the body of the novel:

- saw *for the first* time that inner life of an old, noble,
- and to make her an offer. And only then *for the first time* the
- In Moscow he had *for the first time* felt, after his luxurious and
- *for the first time* did Vronsky realize clearly the fact that
- seeing her *for the first time* with all her defects.
- him *for the first time.* She was conscious herself that her
- *For the first time* the question presented itself to him of the
- feeling. *For the first time* he pictured vividly to himself her

- Levin put on his big boots, and, *for the first time*, a cloth
- And *for the first time* the idea clearly presented itself that it
- he was going. He felt utterly wretched. *For the first time* in
- mowing this summer *for the first time.*
- of the men who led this life; but today *for the first time*,
- eagerness, recollecting her son's existence *for the first time*
- his nose, and *for the first time* in his life he felt on the point
- death. Death, the inevitable end of all, *for the first time*
- moment. And *for the first time*, for an instant, she felt for
- with him or separately, that *for the first time* he clearly
- Now *for the first time* he heard her words with pleasure, and did
- Vronsky *for the first time* experienced a feeling of anger against
- Vassenka Veslovsky, obviously *for the first time* in his life
- "And here we've met *for the first time* since we met at your
- Jew, or that *for the first time* in his life he was not following
- talk which he was hearing *for the first time.* The complexity of
- it." And now *for the first time* Anna turned that glaring light
- for a muslin garment, and going *for the first time* into the frost
- Then, *for the first time*, grasping that for every man, and

There are 14200 unique words in the text and 368588 words total. The frequencies of the unigrams are for 2636, the 16579, first 348, time 564 The unigram *for* appears 2636 times. The bigram *for the* appears 433 times. The trigram *for the first* appears 48 times.

We'll come back to these numbers in a moment. Frequencies of the n-grams are presented along with the text in the sequel.

First we need to discuss a subtlety in predicting text. The two strings *for the* and *for the* are very different when it comes to predicting what comes next. The bigrams that match the latter in our example text are *for the 433, for them 36, for their 21,for these 6,for themselves 6,for they 3, for theirs 1* Where there is only one possible bigram matching the former string. Intra word prediction requires unigram frequencies. The possible matches for continuing *the* are the 16576, they 998, there 965,them 841,their 689,then 410 We use these counts to define conditional probabilities of the form:

$$P(\text{"}the\text{"}|\text{"}the\text{"}) = 1$$
$$P(\text{"}the\text{"}|\text{"}the\text{"}) = 16579/(16579 + 998 + 841 + 689 + 401)$$
$$P(\text{"}they\text{"}|\text{"}the\text{"} = 998/(16579 + 998 + 841 + 689 + 401)$$

We refer to the conditional g in $P(w|g)$ as the context, $g = (w_1, ..., w_i, ..., w_n)$ Using the chain rule this is broken into a product of conditional probabilities,

$$P(w_1, ..., w_i, ..., w_n) = P(w_n|w_1..., w_i, ..., w_{n-1}) * P(w_1, ..., w_i, ..., w_{n-1})$$
$$= P(w_n|w_1..., w_i, ..., w_{n-1}) * P(w_{n-1}|w_1..., w_i, ..., w_{n-2}) * P(w_1, ..., w_i, ..., w_{n-2})$$
$$\dots$$
$$= P(w_1) * P(w_2|w_1) * P(w_3|w_1w_2)...P(w_n|w_2...w_{n-1})$$

The conditional probabilities are computed using n-grams;

- unigram $P(w_i|g) = P(w_i)$
- bigram $P(w_i|g) = P(w_i|w_{i-1})$
- trigram $P(w_i|g) = P(w_i|w_{i-1}w_{i-2})$

and so forth. Estimates are obtained from n-gram frequency counts in a training corpus. to compute the unigram conditional probabilities, $P(w) = \frac{card(w)}{N}$ Where N is the total number of tokens in the corpus. Bigram conditionals are computed likewise, except that we normalize by the total number of bigrams with the first word $P(w_i|w_{i-1}) = \frac{card((w_{i-1},w_i))}{\sum_{w_n \in B} card((w_{i-1},w_n))}$

We can calculate the conditionals with our test corpus. We use $*$ as a wild card.

$$P(for, the, first, time) = P(for)P(the|for)P(first|for, the)P(time|for, the, first)$$

$$P(for, the, first, time) = \frac{the = 16576}{words - 368588} \frac{for, the = 433}{for, * = 2663} \frac{for, the, first = 48}{for, the* = 433} \frac{for, the, first, time = 31}{for, the, first, * = 48}$$

$$P(for, the, first, time) = 0.0005235149$$

$P(forthefirsttime)$ as calculated above is bigger than the ratio of *for the first time* to all quad-grams $31/342132 = .0000906083$

2.4 Appendix : Syntax

Syntax is the rule set for constructing sentences in natural languages. The word is the smallest unit of syntactical structure and a related to each other through rules of grammar. Traditionally, grammar and syntax dictate that every sentence has a subject and predicate that form the nuclear part of a sentence The parts of a sentence decorating subject and predicate are termed extranuclear. There are eight traditional parts of speech (POS) in the English language:

- Verb : action or state *Buddha* ***climbed*** *the wall.*
- Noun : thing or person *jade*
- Adjective : describes a noun *Nice jade Buddha*
- Adverb : describes a verb, adjective or adverb *Buddha climbed slowly*
- Pronoun : replaces a noun
- Preposition : links a noun to another word
- Conjunction : joins clauses or sentences or words
- Interjection : short exclamation, sometimes inserted into a sentence

A more comprehensive POS list used in tagging, stemming, and inflecting text is presented below. This is from the well known PENN database. The Penn Treebank Project annotates naturally-occuring text for linguistic structure [3].

Coordinating conjunction	CC
Cardinal number	CD
Determiner	DT
Existential there	EX
Foreign word	FW
Preposition or subordinating conjunction	IN
Adjective	JJ
Adjective, comparative	JJR
Adjective, superlative	JJS
List item marker	LS
Modal	MD
Noun, singular or mass	NN
Noun, plural	NNS
Proper noun, singular	NNP
Proper noun, plural	NNPS
Predeterminer	PDT
Possessive ending	POS
Personal pronoun	PRP
Possessive pronoun	PRP$
Adverb	RB
Adverb, comparative	RBR
Adverb, superlative	RBS
Particle	RP
Symbol	SYM
to	TO
Interjection	UH
Verb, base form	VB
Verb, past tense	VBD
Verb, gerund or present participle	VBG
Verb, past participle	VBN
Verb, non-3rd person singular present	VBP
Verb, 3rd person singular present	VBZ
Wh-determiner	WDT
Wh-pronoun	WP
Possessive wh-pronoun	WP$
Wh-adverb	WRB

One will generally encounter rule-sets to transform between parts of speech in stemming and inflection engines.

2.4.1 Grammatical Syntactic Definitions

adjective

An adjective is a word whose main syntactic role is to modify a noun or pronoun, giving more information about the noun or pronoun's referent. The adjective order in English is:

- article or pronouns used as adjectives
- quality

- size
- age
- shape
- color
- proper adjective (often nationality or other place of origin)
- purpose or qualifier

One of the syntactic rules for English prescribes that adjectives describing size precede adjectives pertaining to age. Another syntactic rule for adjective is that adjectives describing color come after shape. Thus the phrase *"My poor fat old round *"* is a well formed phrase while *"My old square red *"* is not.

adjunct

Adjuncts are optionally omissible parts of a sentence that do not affect the remainder when removed. The sentence: *I won the bike race in the park while it was raining last Saturday.* Has adjuncts *in the park*, *while it was raining*, and *last Saturday.*

adverb

An adverb is a modifying part of speech describing verbs, other adverbs, adjectives, and phrases. They are used to describe how, where, when, how often, and why something happens. Adverbs of manner: *carefully, correctly, eagerly, easily, fast, loudly, patiently, quickly, quietly, and well.*

Adverbs of place: *abroad, anywhere, downstairs, here, home, in, nowhere, out, outside, somewhere, there, underground, upstairs.*

Adverbs of purpose : *so, so that, to, in order to, because, since, accidentally, intentionally, purposely.*

Adverbs of frequency: *always, every ,never ,often, rarely ,seldom, sometimes, usually.*

Adverbs of time : *after, already, during, finally, just, last, later, next, now, recently, soon*

Adverbs of manner are usually formed by adding *ly* to adjectives. Softly comes from soft, usually come from usual. The suffix *wise* and *ways* may be used to derive adverbs from nouns. Some adverbs are formed from nouns or adjectives by appending the prefix *a*. There are a number of other suffixes in English that derive adverbs from other word classes, and there are also many adverbs that are not morphologically indicated at all. Adverbs in English are inflected in terms of comparison like adjectives. The comparative and superlative forms of some single-syllable adverbs that do not end in *ly* are generated by adding *er* and *est* . Others, especially those ending *ly*, are periphrastically compared by the use of more or most – while some accept both forms, e.g. oftener and more often are both correct. Adverbs also take comparisons with as ... as, less, and least. Not all adverbs are comparable; for example in the sentence *He wore red yesterday* it does not make sense to speak of "*more yesterday*" or "*most yesterday*".

appositive

Apposition is a grammatical construction in which two elements, normally noun phrases, are placed side by side, with one element serving to define or modify the other. When this device is used, the two elements are said to be in apposition. For example, in the phrase *my friend Alice* the name *Alice* is in apposition to *my friend*. Apposition is a figure of speech of the scheme type, and often results when the verbs (particularly verbs of being) in supporting clauses are eliminated to produce shorter descriptive phrases. This makes them often function as hyperbatons, or figures of disorder, because they can disrupt the flow of a sentence. For example, in the phrase: *My dog, a Papillon with big ears,...*, it is necessary to pause before the parenthetical modification *Papillon with big ears*.

article

An article is a word that combines with a noun to indicate the type of reference being made by the noun. Articles specify the grammatical definiteness of the noun, in some languages extending to volume or numerical scope. English articles are *the, and, a, an*. The word *some* is used as a functional plural of *a, an*. Articles are considered a special category of adjectives. Generally common nouns are expressed as definite or indefinite and singular or plural. Every noun must be accompanied by the article, if any, corresponding to its definiteness, and the lack of an article (considered a zero article) itself specifies a certain definiteness. This is in contrast with optional adjectives and determiners. The compulsory nature of articles makes them the most frequently use words.

aspect

The grammatical aspect of a verb is a grammatical category that defines the temporal flow, or lack thereof, in a given action, event, or state (in a given situation). Commonly the distinction is in how the speaker views the situation, either as unitary and bounded *I ate* or as on-going and unbounded *I was eating*. The distinction here is not in the situation itself, but in the speaker's portrayal of it. Other common aspectual distinctions include whether the situation is repetitive or habitual *I used to eat* or has continuing relevance *I have eaten*. Any one language will have at most a subset of the attested aspectual distinctions made in the world's languages. Aspect can be a difficult concept to convey and understand intuitively. Because they both convey some sense of time, aspect is often confused with the closely-related concept of tense. While tense relates the time of a situation to some other time, commonly the time of speaking, aspect conveys other temporal information, such as duration, completion, or frequency, as it relates to the time of action. Thus tense refers to temporally when while aspect refers to temporally how. Aspect can be said to describe the texture of the time in which a situation occurs, such as a single point of time, a continuous range of time, a sequence of discrete points in time, etc, whereas tense indicates its location in time. The concept of aspect is best illustrated by example. Consider the following sentences: *I eat*, *I am eating*, *I have eaten*, and *I have been eating*. All are to some degree in the present tense, as they describe the present situation, yet each conveys different information or points of view as to how the action pertains to the present. As such, they differ in aspect.

auxiliary verb

An auxiliary is a verb functioning to give further semantic or syntactic information about the main or full verb following it. An auxiliary verb alters the basic form of the main verb to make it have one or more of the following functions: passive voice, progressive aspect, perfect aspect, modality, or dummy. Every clause has a finite verb which consists of a full verb non-auxiliary and optionally one or more auxiliary verbs. Examples of finite verbs include *write* (no auxiliary verb), *have written* (one auxiliary verb), and *have been written* (two auxiliary verbs). The primary auxiliary verbs in English are *to be* and *to have*; other major ones include *shall*, *will*, *may* and *can*.

case

The case of a noun or pronoun is a change in form that indicates its grammatical function in a phrase, clause, or sentence. For example, a noun may play the role of subject *I kicked the ball*, of direct object *John kicked me*, or of possessor *My ball*. More formally, case has been defined as a system of marking dependent nouns for the type of relationship they bear to their heads.

clause

A clause is a pair or group of words that consists of a subject and a predicate, although in some languages and some types of clauses the subject may not appear explicitly as a noun phrase. It may instead be marked on the verb (this is especially common in null subject languages). The most basic kind of sentence consists of a single clause. More complicated sentences may contain multiple clauses, including clauses contained within clauses. Clauses are often contrasted with phrases. Traditionally, a clause was said to have both a finite verb and its subject, whereas a phrase either contained a finite verb but not its subject (in which case it is a verb phrase) or did not contain a finite verb. Hence, in the sentence "I didn't know that the dog ran through the yard," "that the dog ran through the yard" is a clause, as is the sentence as a whole, while "the yard," "through the yard," "ran through the yard," and "the dog" are all phrases. However, modern linguists do not draw the same distinction, as they accept the idea of a non-finite clause, a clause that is organized around a non-finite verb.

closed class word

Closed class is a word class to which no new items can normally be added, and that usually contains a relatively small number of items. Typical closed classes found in many languages are adpositions (prepositions and postpositions), determiners, conjunctions, and pronouns. Contrastingly, an open class offers possibilities for expansion. Typical open classes such as nouns and verbs can and do get new words often, through the usual means such as compounding, derivation, coining, borrowing, etc. A closed class may get new items through these same processes, but the change takes much more time. The closed class is normally viewed as part of the core language and is not expected to change. Most readers can undoubtedly think of new nouns or verbs entering their lexicon, but it's very unlikely that they can recall any new prepositions or pronouns appearing in the same fashion

comparative

The comparative is the form of an adjective or adverb which denotes the degree or grade by which a person, thing, or other entity has a property or quality greater or less in extent than that of another, and is used in this context with a subordinating conjunction, such as than, as...as, etc. If three or more items are being compared, the corresponding superlative needs to be used instead.

complement

Complement is used with different meanings. The primary meaning is a word, phrase or clause which is necessary in a sentence to complete its meaning. We find complements which function as an argument (i.e. of equal status to subjects and objects) and complements which exist within arguments. Both complements and modifiers add to the meaning of a sentence. However, a complement is necessary to complete a sentence; a modifier is not. For example, "Put the bread on the table" needs "on the table" to make it complete. In most dialects of English, you cannot merely put something; you need to put it somewhere. In this context, the phrase "on the table" is a complement. By contrast, "The bread on the table is fresh." does not require "on the table" to be complete, so here, the phrase "on the table" is a modifier. A modifier, unlike a complement, is an optional element of a sentence

compound noun and adjective

A compound is a lexeme (less precisely, a word) that consists of more than one stem. Compounding or composition is the word formation that creates compound lexemes (the other word-formation process being derivation). Compounding or Word-compounding refers to the faculty and device of language to form new words by combining or putting together old words. In other words, compound, compounding or word-compounding occurs when a person attaches two or more words together to make them one word. The meanings of the words interrelate in such a way that a new meaning comes out which is very different from the meanings of the words in isolation.

conjugation

Conjugation is the creation of derived forms of a verb from its principal parts by inflection (regular alteration according to rules of grammar). Conjugation may be affected by person, number, gender, tense, aspect, mood, voice, or other grammatical categories. All the different forms of the same verb constitute a lexeme and the form of the verb that is conventionally used to represent the canonical form of the verb (one as seen in dictionary entries) is a lemma. Inflection of nouns and adjectives is known as declension. Conjugated forms of a verb are called finite forms. In many languages there are also one or more forms that remain unchanged with all or most of grammatical categories: the non-finite forms, such as the infinitive or the gerund. A table giving all the conjugated variants of a verb in a given language is called a conjugation table or a verb paradigm. Although conjugation tables are a useful tool for the beginner in a foreign language, they fail in irregular verbs. This limitation is particularly prominent in Latin-derived languages like French and Italian. The availability of high power computers has made possible to replace the conjugation tables with conjugation algorithms, that can handle without difficulty the conjugation and

the grammar analysis of any verb. However, these are much less useful in understanding the structure of the conjugation forms of a given language. A regular verb has a set of conventions for conjugation (paradigm) that derives all forms from a few specific forms or principal parts (maybe only one, such as the infinitive in English), in spelling or pronunciation. A verb that has conjugations deviating from this convention is said to be an irregular verb. Typically the principal parts are the root and/or several modifications of it (stems). Conjugation is also the traditional name of a group of verbs that share a similar conjugation pattern in a particular language (a verb class). This is the sense in which teachers say that Latin has four conjugations of verbs. This means that any regular Latin verb can be conjugated in any person, number, tense, mood, and voice by knowing which of the four conjugation groups it belongs to, and its principal parts

conjunction

A conjunction is a part of speech that connects two words, phrases or clauses together. This definition may overlap with that of other parts of speech, so what constitutes a "conjunction" should be defined for each language. In general, a conjunction is an invariable grammatical particle, and it may or may not stand between the items it conjoins.

The definition can also be extended to idiomatic phrases that behave as a unit with the same function as a single-word conjunction (as well as, provided that, etc.). 1 Coordinating conjunctions 2 Correlative conjunctions 3 Subordinating conjunctions

Coordinating conjunctions, also called coordinators, are conjunctions that join two or more items of equal syntactic importance, such as words, main clauses, or sentences. In English the mnemonic acronym FANBOYS can be used to remember the coordinators *for, and, nor, but, or, yet*, and *so*. These are not the only coordinating conjunctions; various others are used, including *and nor* (British), *but nor* (British), *or nor*(British), *neither* as in *They don't gamble; neither do they smoke*

for: presents a reason *He lost all his money, for he gambled too long and*: presents non-contrasting item(s) or idea(s) *They gamble, and they smoke. or*: presents an alternate item or idea *Every day they gamble or they smoke. nor*: presents a non-contrasting negative idea *They don't gamble, nor do they smoke. but*: presents a contrast or exception *They gamble, but they don't smoke. yet*: presents a contrast or exception *They gamble, yet they don't smoke. so*: presents a consequence *He gambled too long, so he lost all his money.* Correlative conjunctions are pairs of conjunctions that work together to coordinate two items. English examples include both and, [n]either [n]or, and not [only] but [also], whether... or.

Examples: *Either do your work or prepare for a trip to the office. Not only is he handsome but he is also brilliant. Neither the basketball team nor the football team is doing well. Both the cross country team and the swimming team are doing well.*

dangling modifier

A dangling modifier, a specific case of which is the dangling participle, is an error in sentence structure whereby a grammatical modifier is associated with a word other than the one intended, or with no particular word at all. For example, a writer may have meant to modify the subject, but word order makes the modifier seem to modify an object instead. Such ambiguities can lead to unintentional humor or difficulty in understanding a sentence. A typical example of a dangling modifier is illustrated in

the sentence *Turning the corner, a handsome school building appeared.* The modifying clause *Turning the corner* is clearly supposed to describe the behavior of the narrator (or other observer), but grammatically it appears to apply to nothing in particular, or to the school building. Similarly, in the sentence *At the age of eight, my family finally bought a dog*, the modifier *At the age of eight* "dangles" in mid-air, attaching to no named person or thing.

declension

Declension is the inflection of nouns, pronouns, adjectives, and articles to indicate number (at least singular vs. plural), case (nominative or subjective, genitive or possessive, etc.), and gender.

determiner

A determiner is a noun-modifier that expresses the reference of a noun or noun-phrase in the context, rather than attributes expressed by adjectives. This function is usually performed by articles, demonstratives, possessive determiners, or quantifiers. *The* girl is *a* student. I've lost *my* keys. *Some* folks get *all* the luck. *Which* book is that? I only had *two* drinks. I'll take *that* one. *Both* windows were open.

dual

Dual is a grammatical number that some languages use in addition to singular and plural. When a noun or pronoun appears in dual form, it is interpreted as referring to precisely two of the entities (objects or persons) identified by the noun or pronoun. Verbs can also have dual agreement forms in these languages.

expletive

The word expletive is currently used in three senses: syntactic expletives, expletive attributives, and "bad language". Expletive is a term for a meaningless word filling a syntactic vacancy (syntactic expletives). Sometimes an explicative refers to meaningless, filler, or bad language (expletive attributives), distinguishing this from meaningful use.

Function word

Function words (grammatical words or autosemantic words) are words that have little lexical meaning or have ambiguous meaning, but instead serve to express grammatical relationships with other words within a sentence, or specify the attitude or mood of the speaker. Words that are not function words are called content words (or open class words or lexical words): these include nouns, verbs, adjectives, and most adverbs, although some adverbs are function words (e.g., then and why). Dictionaries define the specific meanings of content words, but can only describe the general usages of function words. By contrast, grammars describe the use of function words in detail, but treat lexical words in general terms only.

Function words might be prepositions, pronouns, auxiliary verbs, conjunctions, grammatical articles or particles, all of which belong to the group of closed-class words. Interjections are sometimes considered function words but they belong to the group of open-class words. Function words might or might not be inflected or might have

affixes. Function words belong to the closed class of words in grammar in that it is very uncommon to have new function words created in the course of speech, whereas in the open class of words (that is, nouns, verbs, adjectives, or adverbs) new words may be added readily (such as slang words, technical terms, and adoptions and adaptations of foreign words). See neologism. Each function word either gives some grammatical information on other words in a sentence or clause, and cannot be isolated from other words, or it may indicate the speaker's mental model as to what is being said. Grammatical words, as a class, can have distinct phonological properties from content words. Grammatical words sometimes do not make full use of all the sounds in a language. The following is a list of the kind of words considered to be function words:

- articles : the and a.
- pronouns : inflected in English, as he - him, she - her, etc.
- adpositions : uninflected
- conjunctions : uninflected
- auxiliary verbs : forming part of the conjugation and are always inflected
- interjections : uninflected
- particles : convey the attitude of the speaker and are uninflected, as if, then, well, however, thus, etc.
- expletives : take the place of sentences, among other functions.
- pro-sentences : yes, okay, etc.

gender

Grammatical genders are classes of nouns reflected in the behavior of associated words; every noun must belong to one of the classes and there should be very few that belong to several classes at once. Modern English is normally described as lacking grammatical gender.

infinitive

An infinitive is the name for certain verb forms that exist in many languages. In the usual (traditional) description of English, the infinitive of a verb is its basic form with or without the particle to: therefore, do and to do, be and to be, and so on are infinitives. In most uses, infinitives are non-finite verbs. They function as other lexical categories - usually nouns - within the clauses that contain them, for example by serving as the subject of another verb. They do not represent any of the verb's arguments (as employer and employee do). They are not inflected to agree with any subject. They cannot serve as the only verb of a declarative sentence. They do not have tense, aspect, moods, and/or voice, or they are limited in the range of tenses, aspects, moods, and/or voices that they can use. They are used with auxiliary verbs.

modal particle

Modal particles are always uninflected words, and are a type of grammatical particle. Their function is that of reflecting the mood or attitude of the speaker or narrator, in that they are not reflexive but change the mood of the verb.

modifier

In grammar, a modifier (or qualifier) is an optional element in phrase structure or clause structure; the removal of the modifier typically doesn't affect the grammaticality of the construction. Modifiers can be a word, a phrase or an entire clause. Semantically, modifiers describe and provide more accurate definitional meaning for another element. In English, adverbs and adjectives prototypically function as modifiers, but they also have other functions. Moreover, other constituents can function as modifiers as the following examples show: //bbcrevisit this need reworking adverb in verb phrase: *Put it **quietly** in the box.* adverb in adverb phrase : *He put in down **very quietly**.* adverb in adjective phrase :*He was very **gentle**.* adverb in determiner phrase :***Even more** people were there.* adverb in prepositional phrase *It ran **right up the tree**..* adjective in noun phrase : *It was **a nice house**.* noun in noun phrase: *His desk was in **the faculty office**.* verb phrase in noun: ***The swiftly flowing waters** carried it away.* clause in noun phrase: *I saw the man whom we met yesterday].* clause in noun phrase)She's [the woman with the hat]. preposition phrase in noun phrase)It's not [that important]. determiner in adjective phrase) [A few more] workers are needed. determiner in determiner phrase)We've already [gone twelve miles]. noun phase in verb phrase)She's [two inches taller than I].noun phrase in verb adjective phrase)

A premodifier is a modifier placed before the head (the modified component). A postmodifier is a modifier placed after the head, for example: land mines (pre-modifier) mines in wartime (post-modifier)

mood

Grammatical mood (also mode) is one of a set of morphologically distinctive forms that are used to signal modality. It is distinct from grammatical tense or grammatical aspect, although these concepts are conflated to some degree in many languages, including English and most other modern Indo-European languages, insofar as the same word patterns are used to express more than one of these concepts at the same time (see Tense-aspect-mood).

Currently identified moods include conditional, imperative, indicative, injunctive, optative, potential, subjunctive, and more. Infinitive is a category apart from all these finite forms, and so are gerunds and participles.

noun

A noun is a member of a large, open lexical category whose members can occur as the main word in the subject of a clause, the object of a verb, or the object of a preposition. A proper noun or proper name is a noun representing unique entities (such as London, Jupiter, or Toyota), as distinguished from common nouns which describe a class of entities (such as city, planet, person or car). Count nouns are common nouns that can take a plural, can combine with numerals or quantifiers (e.g., one, two, several, every, most), and can take an indefinite article (a or an). Examples of count nouns are chair, nose, and occasion. Mass nouns (or non-count nouns) differ from count nouns in precisely that respect: they can't take plural or combine with number words or quantifiers. Examples from English include laughter, cutlery, helium, and furniture. For example, it is not possible to refer to a furniture or three furnitures. This is true even though the pieces of furniture comprising furniture could be counted. Thus the distinction between mass and count nouns should not be made in terms of what sorts of things the nouns refer to, but rather in terms of how the nouns present these

entities. Collective nouns are nouns that refer to groups consisting of more than one individual or entity, even when they are inflected for the singular. Examples include committee, herd, and school (of fish). These nouns have slightly different grammatical properties than other nouns. For example, the noun phrases that they head can serve as the subject of a collective predicate, even when they are inflected for the singular. Concrete nouns refer to physical entities that can, in principle at least, be observed by at least one of the senses (for instance, chair, apple, Janet or atom). Abstract nouns, on the other hand, refer to abstract objects; that is, ideas or concepts (such as justice or hatred). While this distinction is sometimes exclusive, some nouns have multiple senses, including both concrete and abstract ones; consider, for example, the noun art, which usually refers to a concept (e.g., Art is an important element of human culture) but which can refer to a specific artwork in certain contexts (e.g., I put my daughter's art up on the fridge).

Some abstract nouns developed etymologically by figurative extension from literal roots. These include drawback, fraction, holdout, and uptake. Similarly, some nouns have both abstract and concrete senses, with the latter having developed by figurative extension from the former. These include view, filter, structure, and key. In English, many abstract nouns are formed by adding noun-forming suffixes (-ness, -ity, -tion) to adjectives or verbs. Examples are happiness (from the adjective happy), circulation (from the verb circulate) and serenity (from the adjective serene).

Nouns and noun phrases can typically be replaced by pronouns, such as he, it, which, and those, in order to avoid repetition or explicit identification, or for other reasons. For example, in the sentence *Janet thought that he was weird*, the word *he* is a pronoun standing in place of the name of the person in question. The English word one can replace parts of noun phrases, and it sometimes stands in for a noun. *John's car is newer than the one that Bill has.* But one can also stand in for bigger subparts of a noun phrase. For example, in the following example, *one* can stand in for new car. *This new car is cheaper than that one.*

- number
- object
- open class word
- part of speech
- particle
- person
- phrase
- phrasal verb
- plural
- predicate (also verb phrase)
- predicative (adjectival or nominal)
- preposition
- personal pronoun
- pronoun
- Restrictiveness
- sentence (linguistics)

- singular
- subject
- superlative
- tense
- uninflected word
- verb
- voice
- wh-movement
- word order
-
-

2.4.2 Grammar

A formal grammar (sometimes simply called a grammar) is a set of rules of a specific kind, for forming strings in a formal language. The rules describe how to form strings from the language's alphabet that are valid according to the language's syntax. A grammar does not describe the meaning of the strings or what can be done with them in whatever context,only their form. Formal language theory, the discipline which studies formal grammars and languages, is a branch of applied mathematics. Its applications are found in theoretical computer science, theoretical linguistics, formal semantics, mathematical logic, and other areas. A formal grammar is a set of rules for rewriting strings, along with a "start symbol" from which rewriting must start. Therefore, a grammar is usually thought of as a language generator. However, it can also sometimes be used as the basis for a "recognizer"-a function in computing that determines whether a given string belongs to the language or is grammatically incorrect. To describe such recognizers, formal language theory uses separate formalisms, known as automata theory. One of the interesting results of automata theory is that it is not possible to design a recognizer for certain formal languages.

2.4.3 Parsing

Parsing is the process of recognizing an utterance (a string in natural languages) by breaking it down to a set of symbols and analyzing each one against the grammar of the language. Most languages have the meanings of their utterances structured according to their syntax-a practice known as compositional semantics. As a result, the first step to describing the meaning of an utterance in language is to break it down part by part and look at its analyzed form (known as its parse tree in computer science, and as its deep structure in generative grammar).

2.4.4 Generative Grammar

The hypothesis of generative grammar is that language is a structure of the human mind. The goal of generative grammar is to make a complete model of this inner language (known as i-language). This model could be used to describe all human language and to predict the grammaticality of any given utterance (that is, to predict whether the utterance would sound correct to native speakers of the language). This approach to language was pioneered by Noam Chomsky. Most generative theories (although not all of them) assume that syntax is based upon the constituent structure of sentences. Generative grammars are among the theories that focus primarily on the form of a sentence, rather than its communicative function.

Among the many generative theories of linguistics, the Chomskyan theories are:

- Transformational Grammar (TG) (Original theory of generative syntax laid out by Chomsky in Syntactic Structures in 1957)
- Government and binding theory (GB) (revised theory in the tradition of TG developed mainly by Chomsky in the 1970s and 1980's).
- Minimalist program (MP) (a reworking of the theory out of the GB framework published by Chomsky in 1995)

2.4.5 Collocation

Within the area of corpus linguistics, collocation defines a sequence of words or terms that co-occur more often than would be expected by chance. The term is often used in the same sense as linguistic government.

Collocation defines restrictions on how words can be used together, for example, which prepositions are used with ("governed by") particular verbs, or which verbs and nouns are typically used together. An example of this (from Michael Halliday) is the collocation strong tea. While the same meaning could be conveyed through the roughly equivalent powerful tea, the fact is that tea is thought of being strong rather than powerful. A similar observation holds for powerful computers, which is preferred over strong computers.

Collocations are examples of lexical units. Collocations should not be confused with idioms although both are similar in that there is a degree of meaning present in the collocation or idiom that is not entirely compositional. With idioms, the meaning is completely non-compositional whereas collocations are mostly compositional.

Collocation extraction is a task that extracts collocations automatically from a corpus, using computational linguistics.

2.4.6 Semantic prosody

Semantic prosody, also discourse prosody, describes the way in which certain seemingly neutral words can be perceived with positive or negative associations

through frequent occurrences with particular collocations.

An example given by John Sinclair is the combination, set in, which has a negative prosody: rot is a prime example for what is going to set in. Other well-known examples are cause, which is also used mostly in a negative context (accident, catastrophe, etc.), though one can also say that something "caused happiness".

In recent years, linguists have found many hidden associations affecting the neutrality of language, through the use of corpus linguistics and concordancing software. The software is used to arrange Key Words in Context from a corpus of several million words of naturally-occurring text. The collocates can then be arranged alphabetically according to first or second word to the right or to the left. Using such a method, Elena Tognini-Bonelli found that the word largely occurred more frequently with negative words or expressions, while broadly appeared more frequently with positive ones. Lexicographers have often failed to allow for semantic prosody when defining a word, although with the recent development and increasing use of computers, the field of corpus linguistics is now being combined with that of lexicography.

Prosody has also been used to analyze discourse structure. Discourse is not a mere concatenation of utterances; talk is organized in sections through relations between discourse segments, topicality, or other ways. Prosody has been found to correlate with these structures of discourse, notably via key (the pitch of a first prominent syllable in an utterance).

2.4.7 Root

The root is the primary lexical unit of a word, which carries the most significant aspects of semantic content and cannot be reduced into smaller constituents. Content words in nearly all languages contain, and may consist only of, root morphemes. However,sometimes the term "root" is also used to describe the word minus its inflectional endings, but with its lexical endings in place. For example, chatters has the inflectional root or lemma chatter, but the lexical root chat. Inflectional roots are often called stems, and a root in the stricter sense may be thought of as a monomorphemic stem. The traditional definition allows roots to be either free morphemes or bound morphemes. Root morphemes are essential for affixation and compounds.

The root of a word is a unit of meaning (morpheme) and, as such, it is an abstraction, though it can usually be represented in writing as a word would be. For example, it can be said that the root of the English verb form running is run, or the root of the Spanish superlative adjective amplisimo is ampl-, since those words are clearly derived from the root forms by simple suffixes that do not alter the roots in any way. In particular, English has very little inflection, and hence a tendency to have words that are identical to their roots. But more complicated inflection, as well as other processes, can obscure the root; for example, the root of mice is mouse (still a valid word), and the root of interrupt is, arguably, rupt, which is not a word in English and only appears in derivational forms (such as disrupt, corrupt, rupture, etc.). The root rupt is written as if it were a word,

but it's not.

2.4.8 Stem

A stem is a part of a word. The term is used with slightly different meanings. In one usage, a stem is a form to which affixes can be attached. Thus, in this usage, the English word friendships contains the stem friend, to which the derivational suffix -ship is attached to form a new stem friendship, to which the inflectional suffix -s is attached. In a variant of this usage, the root of the word (in the example, friend) is not counted as a stem. In a slightly different usage, which is adopted in the remainder of this article, a word has a single stem, namely the part of the word that is common to all its inflected variants. Thus, in this usage, all derivational affixes are part of the stem. For example, the stem of friendships is friendship, to which the inflectional suffix -s is attached. Stems may be roots, e.g. run, or they may be morphologically complex, as in compound words (cf. the compound nouns meat ball or bottle opener) or words with derivational morphemes (cf. the derived verbs black-en or standard-ize). Thus, the stem of the complex English noun photographer is photo -graph - er, but not photo. For another example, the root of the English verb form destabilized is stabil-, a form of stable that does not occur alone; the stem is de-stabil-ize, which includes the derivational affixes de- and -ize, but not the inflectional past tense suffix -(e)d. That is, a stem is that part of a word that inflectional affixes attach to.

2.4.9 Morpheme

A morpheme is the smallest component of word, or other linguistic unit, that has semantic meaning. The term is used as part of the branch of linguistics known as morpheme-based morphology. A morpheme is composed by phoneme(s) (the smallest linguistically distinctive units of sound) in spoken language, and by grapheme(s) (the smallest units of written language) in written language. The concept of word and morpheme are different, a morpheme may or may not stand alone. One or several morphemes compose a word. A morpheme is free if it can stand alone (ex: "one", "possible"), or bound if it is used exclusively alongside a free morpheme (ex: "im" in impossible). Its actual phonetic representation is the morph, with the different morphs ("in-", "im-") representing the same morpheme being grouped as its allomorphs.

2.4.10 Lexeme

A lexeme is an abstract unit of morphological analysis in linguistics, that roughly corresponds to a set of forms taken by a single word. For example, in the English language, run, runs, ran and running are forms of the same lexeme, conventionally written as RUN. A related concept is the lemma (or citation form), which is a particular form of a lexeme that is chosen by convention to represent a canonical form of a lexeme. Lemmas are used in dictionaries as the headwords, and other forms of a lexeme are often listed later in the entry if they

are not common conjugations of that word. A lexeme belongs to a particular syntactic category, has a certain meaning (semantic value), and in inflecting languages, has a corresponding inflectional paradigm; that is, a lexeme in many languages will have many different forms. For example, the lexeme RUN has a present third person singular form runs, a present non-third-person singular form run (which also functions as the past participle and non-finite form), a past form ran, and a present participle running. (It does not include runner, runners, runnable, etc.) The use of the forms of a lexeme is governed by rules of grammar; in the case of English verbs such as RUN, these include subject-verb agreement and compound tense rules, which determine which form of a verb can be used in a given sentence.

2.4.11 Word Structure : affix,prefix, suffix

An affix is a morpheme that is attached to a word stem to form a new word. Affixes may be derivational, like English *ness* and *pre*, or inflectional, like English plural *s* and past tense *ed*. They are bound morphemes by definition; prefixes and suffixes may be separable affixes. Affixation is, thus, the linguistic process speakers use to form new words (neologisms) by adding morphemes (affixes) at the beginning (prefixation), the middle (infixation) or the end (suffixation) of words.

A prefix is an affix which is placed before the stem of a word. Particularly in the study of Semitic languages, a prefix is called a preformative, because it alters the form of the words to which it is affixed.

- unhappy : un is a negative or antonymic prefix.
- prefix, preview : pre is a prefix, with the sense of before
- redo, review : re is a prefix meaning again

A suffix is an affix which is placed after the stem of a word. Examples are case endings, which indicate the grammatical case of nouns or adjectives, and verb endings, which form the conjugation of verbs. Suffixes can carry grammatical information (inflectional suffixes) or lexical information (derivational suffixes). An inflectional suffix is sometimes called a desinence.

- *Girls* where the suffix *s* marks the plural.
- *He makes* where suffix *s* marks the third person singular present tense.
- *It closed* where the suffix *ed* marks the past tense

Inflection changes grammatical properties of a word within its syntactic category. *The weather forecaster said it would clear today, but it hasn't cleared at all.* the suffix *ed* inflects the root-word *clear* to indicate past tense.

Inflectional English suffixes:

s third person singular present *ed* past tense *ing* progressive/continuous *en* past participle *s* plural *en* plural (irregular) *er* comparative *est* superlative *n't* negative

In the sentence *The weather forecaster said it would be clear today, but I can't see clearly at all* the suffix *ly* modifies the root-word clear from an adjective into an adverb.

Derivation can also form a semantically distinct word within the same syntactic category. *The weather forecaster said it would be a clear day today, but I think it's more like clearish!* The suffix *ish* modifies the root-word clear, changing its meaning to "clear, but not very clear".

English derivational suffixes : *ian ize/ise fy ly ful able/ible hood ness less ism ment ist al ish*

2.4.12 Lemma

In linguistics a lemma (plural lemmas or lemmata) is either of two things:

- Morphology, lexicography: the canonical form, dictionary form, or citation form of a set of words (headword); e.g., in English, run, runs, ran and running are forms of the same lexeme, with run as the lemma.

- Psycholinguistics: abstract conceptual form that has been mentally selected for utterance in the early stages of speech production, but before any sounds are attached to it. A lemma in morphology is the canonical form of a lexeme. Lexeme, in this context, refers to the set of all the forms that have the same meaning, and lemma refers to the particular form that is chosen by convention to represent the lexeme. In lexicography, this unit is usually also the citation form or headword by which it is indexed. Lemmas have special significance in highly inflected languages such as Czech. The process of determining the lemma for a given word is called lemmatisation.

The psycholinguistics interpretation refers to one of the more widely accepted psycholinguistic models of speech production, referring to an early stage in the mental preparation for an utterance. Here, lemma is the abstract form of a word that arises after the word has been selected mentally, but before any information has been accessed about the sounds in it (and thus before the word can be pronounced). It therefore contains information concerning only meaning and the relation of this word to others in the sentence.

2.4.13 Differences Between a Stem and a Lemma

A stem is the part of the word that never changes even when morphologically inflected, whilst a lemma is the base form of the verb. For example, from "produced", the lemma is "produce", but the stem is "produc-." This is because there are words such as production. In linguistic analysis, the stem is defined more generally as the analyzed base form from which all inflected forms can be formed. When phonology is taken into account, the definition of the unchangeable part of the word is not useful, as can be seen in the phonological forms of the words in the preceding example: "produced", "production". Some lexemes

have several stems but one lemma. For instance "to go" (the lemma) has the stems "go" and "wend". (The past tense is based on a different verb, "to wend". The "-t" suffix may be considered as equivalent to "-ed".)

2.4.14 Lexicon

The lexicon of a language is its vocabulary, including its words and expressions. More formally, it is a language's inventory of lexemes. The lexicon includes the lexemes used to actualize words. Lexemes are formed according to morpho-syntactic rules and express sememes. In this sense, a lexicon organizes the mental vocabulary in a speaker's mind: First, it organizes the vocabulary of a language according to certain principles (for instance, all verbs of motion may be linked in a lexical network) and second, it contains a generative device producing (new) simple and complex words according to certain lexical rules. For example, the suffix *able* can be added to transitive verbs only, so that we get *readable* but not *cryable*. Usually a lexicon is a container for words belonging to the same language.

2.4.15 Morphology

Morphology is the identification, analysis and description of the structure of morphemes and other units of meaning in a language like words, affixes, and parts of speech and intonation/stress, implied context (words in a lexicon are the subject matter of lexicology). Morphological typology represents a way of classifying languages according to the ways by which morphemes are used in a language from the analytic that use only isolated morphemes, through the agglutinative ("stuck-together") and fusional languages that use bound morphemes (affixes), up to the polysynthetic, which compress lots of separate morphemes into single words.

Words are generally accepted as being the smallest units of syntax,and are related to other words by rules of grammar. For example, English speakers recognize that the words dog and dogs are closely related - differentiated only by the plurality morpheme "-s," which is only found bound to nouns, and is never separate. Speakers of English (a fusional language) recognize these relations from their tacit knowledge of the rules of word formation in English. They infer intuitively that dog is to dogs as cat is to cats; similarly, dog is to dog catcher as dish is to dishwasher (in one sense). The rules understood by the speaker reflect specific patterns (or regularities) in the way words are formed from smaller units and how those smaller units interact in speech. In this way, morphology is the branch of linguistics that studies patterns of word formation within and across languages, and attempts to formulate rules that model the knowledge of the speakers of those languages. A morpheme is the smallest component of word, or other linguistic unit, that has semantic meaning. The term is used as part of the branch of linguistics known as morpheme-based morphology. A morpheme is composed by phoneme(s) (the smallest linguistically distinctive units of sound)

in spoken language, and by grapheme(s) (the smallest units of written language) in written language.

The concept of word and morpheme are different, a morpheme may or may not stand alone. One or several morphemes compose a word. A morpheme is free if it can stand alone (ex: "one", "possible"), or bound if it is used exclusively alongside a free morpheme (ex: "im" in impossible). Its actual phonetic representation is the morph, with the different morphs ("in-", "im-") representing the same morpheme being grouped as its allomorphs. The word "unbreakable" has three morphemes: "un-", a bound morpheme; "break", a free morpheme; and "-able", a free morpheme. "un-" is also a prefix, "-able" is a suffix. Both "un-" and "-able" are affixes. In morphology, a bound morpheme is a morpheme that cannot stand alone as an independent word. A free morpheme is one which can stand alone. Most English language affixes (prefixes and suffixes) are bound morphemes, e.g., -ment in "shipment", or pre- in "prefix". Many roots are free morphemes, e.g., ship- in "shipment", while others are bound. The morpheme ten- in "tenant" may seem free, since there is an English word "ten". However, its lexical meaning is derived from the Latin word tenere, "to hold", and this or related meaning is not among the meanings of the English word "ten", hence ten- is a bound morpheme in the word "tenant".

There are some distinguished types of bound morphemes. A unique morpheme is one with extremely limited distribution so that it occurs in only one word. A popular example is cran- in cranberry" (hence the term "cranberry morpheme"), although this example is something of a technicality given that it is an alteration or contraction of the free morpheme "crane". Unique morphemes are examples of the linguistic notion of fossilization: loss of productivity or usage of grammar units: words, phrases, parts of words. Besides fossilized root morphemes, there are also fossilized affixes (suffixes and prefixes).

2.4.16 Inflection

In grammar, inflection or inflexion is the modification of a word to express different grammatical categories such as tense, mood, voice, aspect, person, number, gender and case. Conjugation is the inflection of verbs; declension is the inflection of nouns, adjectives and pronouns.

Inflection can be overt or covert within the same language. An overt inflection expresses grammatical category with an explicitly stated suffix. In English, the word "lead" is not marked for either person or number, and is only marked for tense in opposition to "led" (i.e. is not specifically future tense). The whole clause, however, achieves all the grammatical categories by the inclusion of extra words. This is covert inflection (or periphrasis). The process typically distinguishes lexical items (such as lexemes) from functional ones (such as affixes, clitics, particles and morphemes in general) and has functional items acting as markers on lexical ones.

Lexical items that do not respond to overt inflection are invariant or uninflected; for example, "must" is an invariant item: it never takes a suffix or changes form to signify a different grammatical category. Its category can only

be determined by its context. Uninflected words do not need to be lemmatized in linguistic descriptions or in language computing. On the other hand, inflectional paradigms, or lists of inflected forms of typical words (such as sing, sang, sung, sings, singing, singer, singers, song, songs, songstress, songstresses in English) need to be analyzed according to criteria for uncovering the underlying lexical stem (here s*ng-); that is, the accompanying functional items (-i-, -a-, -u-, -s, -ing, -er, -o-, -stress, -es) and the functional categories of which they are markers need to be distinguished to adequately describe the language.

Constraining the cross-referencing of inflection in a sentence is known as concord or agreement. For example, in "the choir sings", "choir" and "sings" are constrained to the singular number; if one is singular, they both must be.

Languages that have some degree of overt inflection are inflected languages. The latter can be highly inflected, such as Latin (overtly), or weakly inflected, such as English (covertly), depending on the presence or absence of overt inflection. And, historically, English was traditionally described as a non-inflected Indo-European language.

2.4.17 Derivation

Derivation is the process by which new words, as with *happiness* and *unhappy* from *happy*, or *determination* from *determine*. A contrast is intended with the process of inflection, which uses another kind of affix in order to form variants of the same word, as with *determine/determine-s/determin-ing/determin-ed*. A derivational suffix usually applies to words of one syntactic category and changes them into words of another syntactic category. For example, the English derivational suffix *ly* changes adjectives into adverbs

2.4.18 Inflection Versus Derivation

Inflection is the process of adding inflectional morphemes (smallest units of meaning) to a word, which indicate grammatical information (for example, case, number, person, gender or word class, mood, tense, or aspect). Derivation is the process of adding derivational morphemes, which create a new word from existing words, sometimes by simply changing grammatical category (for example, changing a noun to a verb). Words generally are not listed in dictionaries (in which case they would be lexical items) on the basis of their inflectional morphemes. But they often are listed on the basis of their derivational morphemes. For instance, English dictionaries list readable and readability, words with derivational suffixes, along with their root read. However, no traditional English dictionary lists book as one entry and books as a separate entry nor do they list jump and jumped as two different entries.

2.4.19 Inflectional Morphology

Languages that add inflectional morphemes to words are sometimes called inflectional languages, which is a synonym for inflected languages. Morphemes

may be added in several different ways:

- Affixation, or simply adding morphemes onto the word without changing the root
- Reduplication, doubling all or part of a word to change its meaning
- Alternation, exchanging one sound for another in the root

Suprasegmental variations, such as of stress, pitch or tone, where no sounds are added or changed but the intonation and relative strength of each sound is altered regularly. For an example, see Initial-stress-derived noun.

Affixing includes prefixing (adding before the base), and suffixing (adding after the base), as well as the much less common infixing (inside) and circumfixing (a combination of prefix and suffix).Inflection is most typically realized by adding an inflectional morpheme (that is, affixation) to the base form (either the root or a stem).

2.4.20 Fossilization

In linguistic morphology, fossilization refers to two close notions. One is preserving of ancient linguistic features which have lost their grammatical functions in language. Another is loss of productivity of a grammatical paradigm (e.g., of an affix), which still remains in use in some words. Examples of fossilization include fossilized morphemes and fossil words. A fossil word is an obsolete word which remains in currency because it is contained within an idiom still in use. It can also occur for phrases, such as in point ('relevant'), which is retained in the larger phrases case in point and in point of fact, but is not otherwise used outside of a legal context.

English language examples: Ulterior, as in 'ulterior motives' Ilk, as in 'of that ilk' Fro, as in 'to and fro' Sleight, as in 'sleight of hand' Yore, as in 'days of yore' Coign, as in 'coign of vantage' Deserts, as in 'just deserts' Craw, as in 'sticks in one's craw' Fettle, as in 'in fine fettle' Kith, as in 'kith and kin' Spick, as in 'spick and span' Loggerheads as in 'at loggerheads' Offing, as in 'in the offing' Shrift, as in 'short shrift' Amok, as in 'run amok'

2.4.21 Stemming

Stemming is the process for reducing inflected (or sometimes derived) words to their stem, base or root form - generally a written word form. The stem need not be identical to the morphological root of the word; it is usually sufficient that related words map to the same stem, even if this stem is not in itself a valid root. The algorithm has been a long-standing problem in computer science; the first paper on the subject was published in 1968. The process of stemming, often called conflation, is useful in search engines for query expansion or indexing and other natural language processing problems. Stemming programs are commonly referred to as stemming algorithms or stemmers.

Brute Force Stemming

Brute force stemmers employ a lookup table which contains relations between root forms and inflected forms. To stem a word, the table is queried to find a matching inflection. If a matching inflection is found, the associated root form is returned. Brute force approaches are criticized for their general lack of elegance in that no algorithm is applied that would more quickly converge on a solution. In other words, there are more operations performed during the search than should be necessary. Brute force searches consume immense amounts of storage to host the list of relations (relative to the task). The algorithm is only accurate to the extent that the inflected form already exists in the table. Given the number of words in a given language, like English, it is unrealistic to expect that all word forms can be captured and manually recorded by human action alone. Manual training of the algorithm is overly time-intensive and the ratio between the effort and the increase in accuracy is marginal at best.

Production Stemming

Some programs attempt to automatically generate the table of root and inflected forms. A production algorithm attempts to infer the probable inflections for a given word. For example, if the word is "run", then the algorithm might automatically generate the forms "running" and "runs". In the traditional sense of the concept of stemming, this algorithm is its reverse process. Rather than try and remove suffixes, the goal of a production algorithm is to generate suffixes. Later, a brute force algorithm can simply query the automatically generated table of word relations to find the root form of a word. There are many types of heuristic, experimental techniques for identifying inflected forms of words. Some algorithms work phonetically, looking at the final syllables in a word. Some are rather brute force, using rules that seem a lot like normalization rules, by inspecting the last few characters. Others are similar to the process of lemmatisation, which takes advantage of the additional knowledge about the part of speech of the given word to limit what types of suffixes are considered when generating inflections for the word.

Suffix Stripping

Suffix stripping algorithms do not rely on a lookup table that consists of inflected forms and root form relations. Instead, a typically smaller list of "rules" are stored which provide a path for the algorithm, given an input word form, to find its root form. Some examples of the rules include: " if the word ends in 'ed', remove the 'ed' " if the word ends in 'ing', remove the 'ing' " if the word ends in 'ly', remove the 'ly' Suffix stripping approaches enjoy the benefit of being much simpler to maintain than brute force algorithms, assuming the maintainer is sufficiently knowledgeable in the challenges of linguistics and morphology and encoding suffix stripping rules. Suffix stripping algorithms are sometimes regarded as crude given the poor performance when dealing with exceptional

relations (like 'ran' and 'run'). The solutions produced by suffix stripping algorithms are limited to those lexical categories which have well known suffixes with few exceptions. This, however, is a problem, as not all parts of speech have such a well formulated set of rules. Lemmatisation attempts to improve upon this challenge.

Stochastic Stemming and Lemmatization Algorithms

Stochastic algorithms involve using probability to identify the root form of a word. Stochastic algorithms are trained (they "learn") on a table of root form to inflected form relations to develop a probabilistic model. This model is typically expressed in the form of complex linguistic rules, similar in nature to those in suffix stripping or lemmatisation. Stemming is performed by inputting an inflected form to the trained model and having the model produce the root form according to its internal ruleset, which again is similar to suffix stripping and lemmatisation, except that the decisions involved in applying the most appropriate rule, or whether or not to stem the word and just return the same word, or whether to apply two different rules sequentially, are applied on the grounds that the output word will have the highest probability of being correct (which is to say, the smallest probability of being incorrect, which is how it is typically measured). Some lemmatisation algorithms are stochastic in that, given a word which may belong to multiple parts of speech, a probability is assigned to each possible part. This may take into account the surrounding words, called the context, or not. Context-free grammars do not take into account any additional information. In either case, after assigning the probabilities to each possible part of speech, the most likely part of speech is chosen, and from there the appropriate normalization rules are applied to the input word to produce the normalized (root) form.

Stochastic N-Gram Stemming

To further explain, consider the well known technique of n-gram analysis. Here the term gram is a unit of measurement of text that may refer to a single character, a single syllable, or a single word. Which one applies is dependent upon which author or programmer is describing the algorithm (and their background, as a linguist is more likely to be referring to syllables, but a computer scientist characters and a software company words). The prefix n is, as usual in computer science jargon, representing 'any number of', or a 'variable number of'. There are frequently used subsets of n-grams, such as bigrams (a.k.a digrams, bi-grams, di-grams), representing a sequence of two grams (two characters or two words or two syllables in a row, consecutively in the text). Trigrams (three units) are also popular. In language modeling, one could write a software program that stores a large table of all word bigrams found in a large body of text (called a corpus). The algorithm would scan word by word, like a sliding window, across the text from the beginning, storing two word sequences as it moves along until the last word of the last document is reached. On the output,

one has what is called a bigrams table. Each row in the table may be called a tuple, storing the first word, and then the second word. Additional characteristics may be stored about each bigram, such as its frequency (the total count of the times the whole bigram was discovered) in whatever body of texts was used to generate the table. The table may also store the frequency of each individual word. For example, this sentence contains bigrams like "for example" and "this sentence". From the table an algorithm can garner the probability of the second word following the first word by assessing each word's frequency and the frequency of the bigrams and perhaps other criteria like the total number of bigrams or the total number of bigrams where the first word is the same. For example, one would be able to state conclusions like: the presence of the word post indicates a probability of the following word being office of 50%, because the sequence occurred this way 50% of the time in the text (where post occurred with and without office following it). In the bi-gram relationship, the first word can be understood as the context that describes the second word. It qualifies the second word, providing additional insight into its meaning. Humans use the technique of examining the context of words to determine the meaning of written words quite frequently. Words can have multiple meanings, called senses. The bigram-provided context provides a way to distinguish which sense of the word is used (well, most of the time, as literary puns and the like are the obvious exception). As mentioned elsewhere, the stemming algorithm can use this context as additional information to assist in determining the outcome of the algorithm (whether or not the current word is stemmed, which of one or more possible stems is appropriate, whether to use substitution or stripping, whether the word is a verb or noun or some other lexical category, etc). Because the algorithm used probabilities in determining the outcome, it is a stochastic algorithm. Entering into the realm of stochastic algorithms may greatly complicate the stemming algorithm, making it much harder to develop, distribute and maintain. There is considerable debate over its benefits. If a simple suffix stripper can get to about 80% accuracy from a few days or weeks of programming, then how much more valuable is it that a context-driven n-gram stemmer can achieve 90% accuracy given several months of work? There are also many problems with stochastic stemmers. What if the body of text used to derive the probabilities of the bigrams is not a representative sample of all documents ever written in a given language (and it is likely not)? What if the algorithm's bigram table 'becomes corrupted', by learning from 'bad text'? Reconfiguring a brute force algorithm that employs a lookup table is as simple as going in and removing a row from the table. But how does one reconfigure a probabilistic network (this is analogous to some of the problems behind neural networks)?

2.5 Appendix : Semantics

Words can be related by various attributes such as similarity, membership, or by lexical relations through morphological constructs. Morphological categorization into different grammatical categories such as tense, mood, voice, aspect,

person, number, gender and case is generally accomplished through the rules of inflection. We use WordNet as a source for semantic relationships, and discuss the concepts of semantics within the framework of the WordNet model.

2.5.1 Word Relationships in the WordNet database

WordNet [4] is a lexical database for the English language. It groups English words into sets of synonyms called synsets, provides short, general definitions, and records the various semantic relations between these synonym sets. Synonymy is a lexical relation between word forms. It is assigned a central role in WordNet. Semantic relations are indicated by pointers to other words. WordNet is organized by semantic relations. A semantic relation is a relation between meanings and since meanings can be represented by synsets. It is natural to think of semantic relations as pointers between synsets. It is characteristic of semantic relations that they are reciprocated: if there is a semantic relation R between meaning x, x0, . . . and meaning y, y0, . . ., then there is also a relation R between y, y0, . . . and x, x0, ...

Semantic Relations Found In WordNet:

Nouns

- hypernyms: Y is a hypernym of X if every X is a (kind of) Y (canine is a hypernym of dog, because every dog is a member of the larger category of canines)
- hyponyms: Y is a hyponym of X if every Y is a (kind of) X (dog is a hyponym of canine)
- coordinate terms: Y is a coordinate term of X if X and Y share a hypernym (wolf is a coordinate term of dog, and dog is a coordinate term of wolf)
- holonym: Y is a holonym of X if X is a part of Y (building is a holonym of window)
- meronym: Y is a meronym of X if Y is a part of X (window is a meronym of building)

Verbs

- hypernym: the verb Y is a hypernym of the verb X if the activity X is a (kind of) Y (to perceive is an hypernym of to listen)
- troponym: the verb Y is a troponym of the verb X if the activity Y is doing X in some manner (to lisp is a troponym of to talk)
- entailment: the verb Y is entailed by X if by doing X you must be doing Y (to sleep is entailed by to snore)
- coordinate terms: those verbs sharing a common hypernym (to lisp and to yell)

Adjectives

- elated nouns
- similar to
- participle of verb
- Adverbs
- root adjectives

Nouns are organized from a set of base classes in WordNet - so called 'Primes' act, action, activity natural object animal, fauna natural phenomenon artifact person, human being attribute, property plant, flora body, corpus possession cognition, knowledge process communication quantity, amount event, happening relation feeling, emotion shape food state, condition group, collection substance location, place time motive

Antonym

The antonym of a word x is sometimes not-x, but not always. For example, rich and poor are antonyms, but to say that someone is not rich does not imply that they mustbe poor; many people consider themselves neither rich nor poor. Antonymy, which seems to be a simple symmetric relation, is actually quite complex, yet speakers of English have little difficulty recognizing antonyms when they see them. Antonymy is a lexical relation between word forms, not a semantic relation between word meanings. For example, the meanings rise, ascend and fall, descend may be conceptual opposites, but they are not antonyms; [rise/fall] are antonyms and so are [ascend/descend], but most people hesitate and look thoughtful when asked if rise and descend, or ascend and fall, are antonyms. Such facts make apparent the need to distinguish between semantic relations between word forms and semantic relations between word meanings. Antonymy provides a central organizing principle for the adjectives and adverbs in WordNet, and the complications that arise from the fact that antonymy is a semantic relation between words are better discussed in that context.

Hyponym

hyponymy/hypernymy is a semantic relation between word meanings: e.g., maple is a hyponym of tree, and tree is a hyponym of plant. Much attention has been devoted to hyponymy/hypernymy (variously called subordination/superordination, subset/superset, or the *ISA* relation).

- member holynym
- substance holonym
- part holonym

Meronym

The part-whole (or HASA) relation, known to lexical semanticists as meronymy/holonymy. The meronymic relation is transitive (with qualifications) and asymmetrical, and can be used to construct a part hierarchy (with some reservations, since a meronym can have many holonyms).

- member meronym
- substance meronym
- part meonym

Entailment

In logic, entailment, or strict implication, is properly defined for propositions; a proposition P entails a proposition Q if and only if there is no conceivable state of affairs that could make P true and Q false. Entailment is a semantic relation because it involves reference to the states of affairs that P and Q represent. The term will be generalized here to refer to the relation between two verbs V1 and V2 that holds when the sentence Someone V1 logically entails the sentence Someone V2; this use of entailment can be called lexical entailment. Thus, for example, snore lexically entails sleep because the sentence He is snoring entails He is sleeping; the second sentence necessarily holds if the the first one does. Lexical entailment is a unilateral relation: if a verb V1 entails another verb V2, then it cannot be that case that V2 entails V1. The exception is that where two verbs can be said to be mutually entailing, they must also be synonyms, that is, they must have the same sense. For example, one might say both that The Germans beat the Argentinians entails The Germans defeated the Argentinians, and that The Germans defeated the Argentinians entails The Germans beat the Argentinians. However, we find such statements rather unnatural. Negation reverses the direction of entailment: not sleeping entails not snoring, but not snoring does not entail not sleeping. The converse of entailment is contradiction: If the sentence He is snoring entails He is sleeping, then He is snoring also contradicts the sentence He is not sleeping. The entailment relation between verbs resembles meronymy between nouns, but meronymy is better suited to nouns than to verbs. To begin with, in order for sentences based on the formula An x is a part of a y to be acceptable, both x and y must be nouns. It might seem that using the nominalizing gerundive form of the verbs would convert them into nouns, and as nouns the HASA relation should apply.

Other WordNet Relation Types

- cause
- also see
- derived from

- attribute
- relational adj
- similar to
- verb group
- participle
- attribute
- derivationally related
- domain topic
- member topic
- domain region
- member region
- domain usage
- member usage
- pertainym
- same
- equivalent
- subsuming
- instance
- antiequivalent
- antisubsuming
- antiinstance

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[2] http://xw2k.nist.gov/dads *NIST Dictionary of Algorithms and Data Structures*

[3] http://www.cis.upenn.edu/ treebank/ *The Penn Treebank Project*

[4] http://wordnet.princeton.edu/ *WordNet - a large lexical database of English*

[5] http://www.wikipedia.org/

Chapter 3

Image Processing

3.1 Definitions and Concepts

Color Science concerns itself with the characterization of perception of color stimuli, the synthesis of stimuli from perception, and the processing of color information. Characterization of perceptions from stimuli involves measurement and descriptive processes. Color reproduction and processing form the basis of most digital applications of color science with the aim to provide a stimulus giving rise to a target perception. Tristimulus colorimetry is based on three color matching functions which define the primary colors which are mixed to produce a range of stimuli. The perception of an isolated stimulus is matched to an additive mixture of three light sources called primaries R, G, B. The perception induced by a stimulus is characterized by three values X, Y, Z related to the luminance of the primaries. Linearly independent light sources are used to create a color measurement system where the tristimulus values encode the amount of each primary required to reproduce a color stimuli. To calculate a tristimulus value we integrate the spectral distribution of the light source $\phi(\lambda)$ multiplied by the color matching function

$$\begin{aligned} X &= \int R(\lambda)\phi(\lambda)d\lambda \\ Y &= \int G(\lambda)\phi(\lambda)d\lambda \\ Z &= \int B(\lambda)\phi(\lambda)d\lambda \end{aligned}$$

Tristimulus values relate the radiance from a stimulus to the color matching functions. For color reproduction a set of primaries are fixed and the tristimulus vector in that color space provides luminances of the primaries required to generate the stimuli. Color processing is any modification of the color content of a digital image. In order to make the processing task intuitive it has to be done in a representation correlated with the perceptual dimensions of color. X, Y, Z tristimulus values are not uniform when describing differences in colors. The CIE Lab color space is a non-linear transform of tristimulus values to a perceptually uniform color space. The transform from tristimulus values to Lab was designed to reproduce the response of the human eye. CIE Lab describes all the colors visible to the human visual system and serves as a device independent color model. Uniform changes in the Lab color space correspond to uniform changes in color perception.

Chromaticity is an specification of the quality of a color regardless of its luminance. In the Lab color space this is an easy projection on to the ab-plane. The white point of an illuminant is a neutral reference point in the ab-plane.

An absolute color space is one in which which the perceptual difference between colors relates to distances between colors as represented by points in the color space. Another

Figure 3.1: Spectra of CIE Daylight and Fluorescent Illuminants

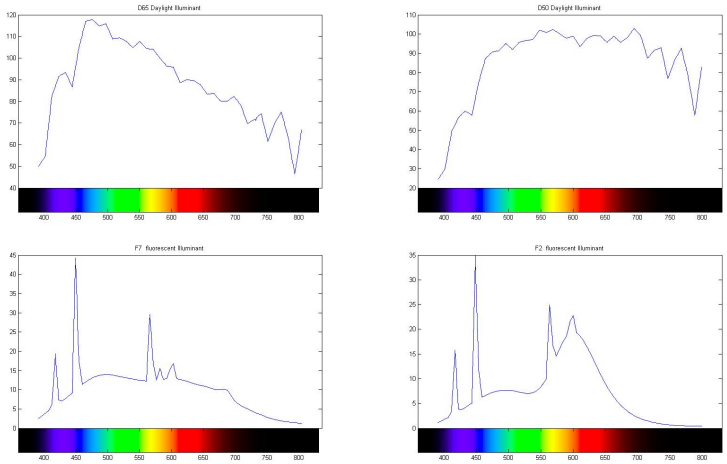
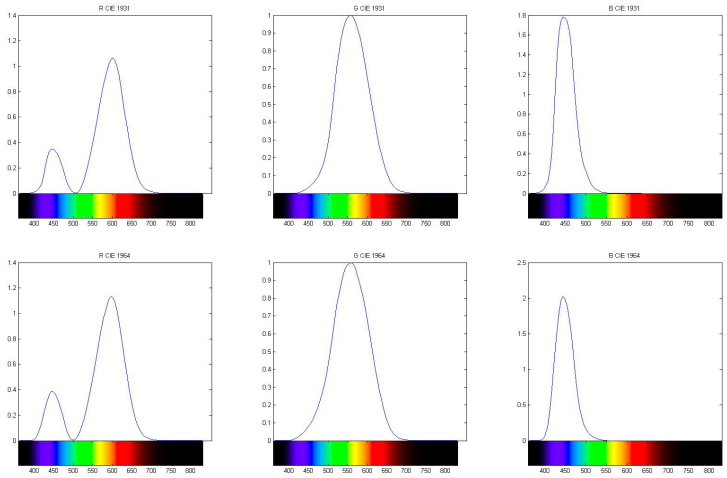


Figure 3.2: Color Matching Functions for CIE Standard Observer

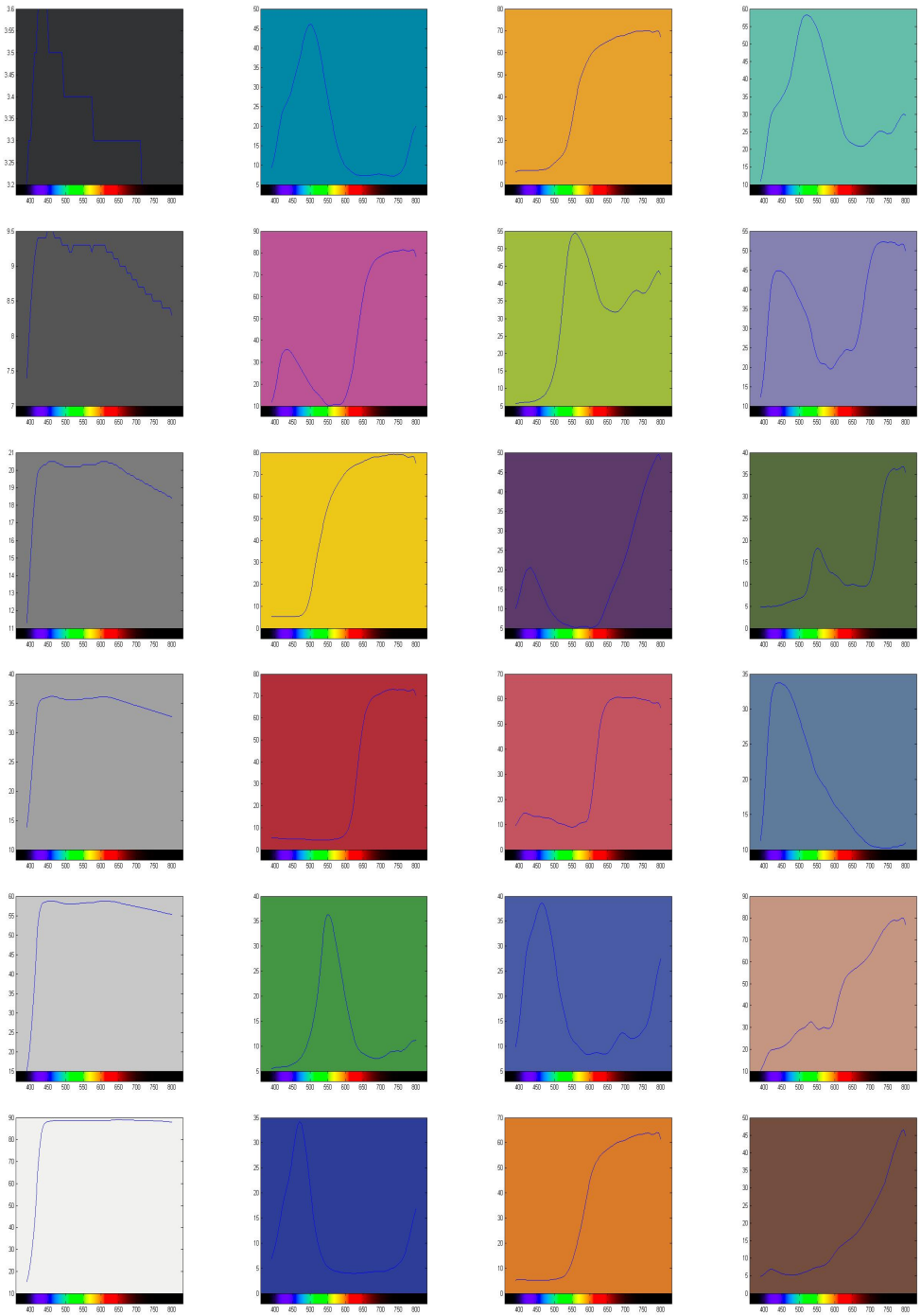


common definition of an absolute color space is one in which the colors characterized without having to specify an illuminant. CIEXYZ and sRGB are absolute color spaces. CIE Lab is absolute if one specifies the white point. There are no simple formulas for conversion between raw RGB values and Lab without the use of an ICC profile or specification of the spectral properties of the primaries and the illuminant. Converting between non-absolute color spaces or between absolute and non-absolute color spaces has no visual meaning.

An absolute color space can be reversibly converted into another absolute color space but gamut limitations may prevent all of the colors from making the round trip. Gamut mapping algorithms are typically tuned to preserve application sensitive colors or to minimize overall error.

3.1.1 Reflective and Transmissive Models of Color Perception

The spectral reflectance values obtained from a Macbeth Color chart.



3.1.2 Additive and Subtractive Color Models

3.1.3 Gamut Mapping

Definition definitions used by the CIE TC 8-03 on gamut mapping are given below.

Image: two-dimensional stimulus containing pictorial or graphical information whereby the original image is the image to which its reproductions are compared in terms of some characteristics.

Color reproduction medium: a medium for displaying or capturing color information, e.g. a CRT monitor, a digital camera or a scanner. Note that in the case of slide scanning, the color reproduction medium is not the scanner but the combination of scanner, stains and glass substrate.

Color gamut: a range of colors achievable on a given color reproduction medium under a given set of viewing conditions.

Color gamut boundary: a surface determined by color gamut extreme values.

Gamut boundary descriptor: an overall way of approximately describing a gamut boundary. Line gamut boundary: the points of intersections between a gamut boundary and a given line along which mapping is to be carried out.

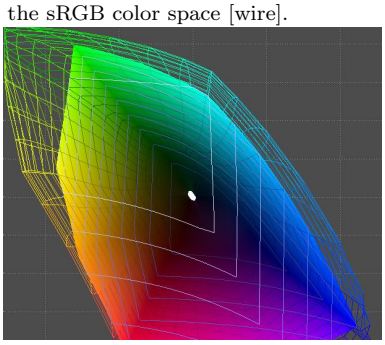
Color gamut mapping: a method for assigning colors from the reproduction medium to colors from the original medium or image.

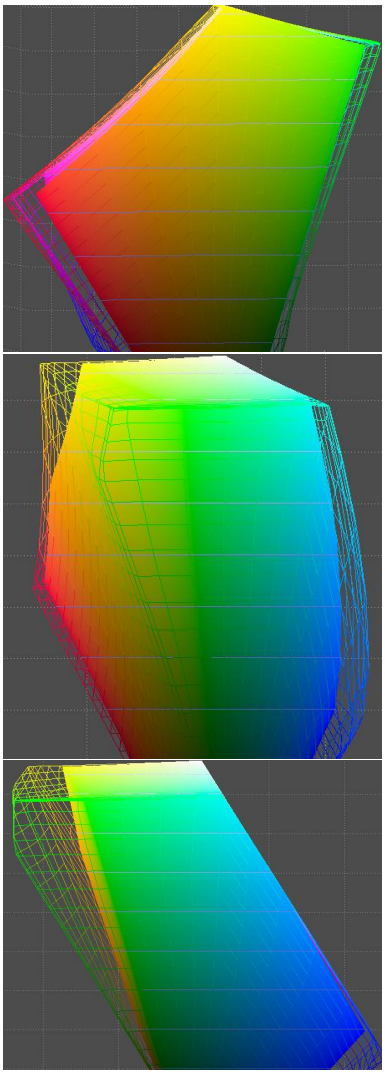
Color reproduction intent: the desired relationship between color information in original and reproduction media. As a number of solutions to cross-media reproduction intents can be pursued by gamut mapping. The most generic of these are accuracy and pleasantness but it is also possible to define others for specific application (e.g. to provide an accurate reproduction of corporate identity colors while giving pleasant results for others). Note that reproduction intents are also referred to as rendering intents.

Accurate reproduction intent: aims to maximize the degree of similarity between the original image and a reproduction of it as far as possible, given the constraints of the color reproduction media involved. Note that the characteristic of accurate reproduction is intrinsically relative (i.e. reproduction versus original).

Pleasant reproduction intent: aims to maximize the reproductions correspondence with preconceived ideas of given image should look according to an individual whereby this criterion encompasses contrast, lack of artifacts, sharpness, etc. Note that unlike accuracy, pleasantness is absolute at least as far as given observer understands it at a given moment.

The images below display various perspectives of the gamut of a scanner device [solid] and the sRGB color space [wire].





3.1.4 Rendering Intents

There are 4 rendering intents defined by the International Color Consortium (ICC), which are specifically defined for the purposes of cross media reproduction using color management systems. The intents are related to the gamut mapping techniques.

Perceptual intent: the full gamut is compressed to fill the gamut of the destination device. Gray balance is preserved, the relationship between the colors is not altered but colorimetric accuracy might not be preserved, as all the colors are moved. Saturation intent: it converts the saturated colors in the source gamut to the saturated colors in the destination gamut. This transformation is done at the expense of accuracy in hue and lightness.

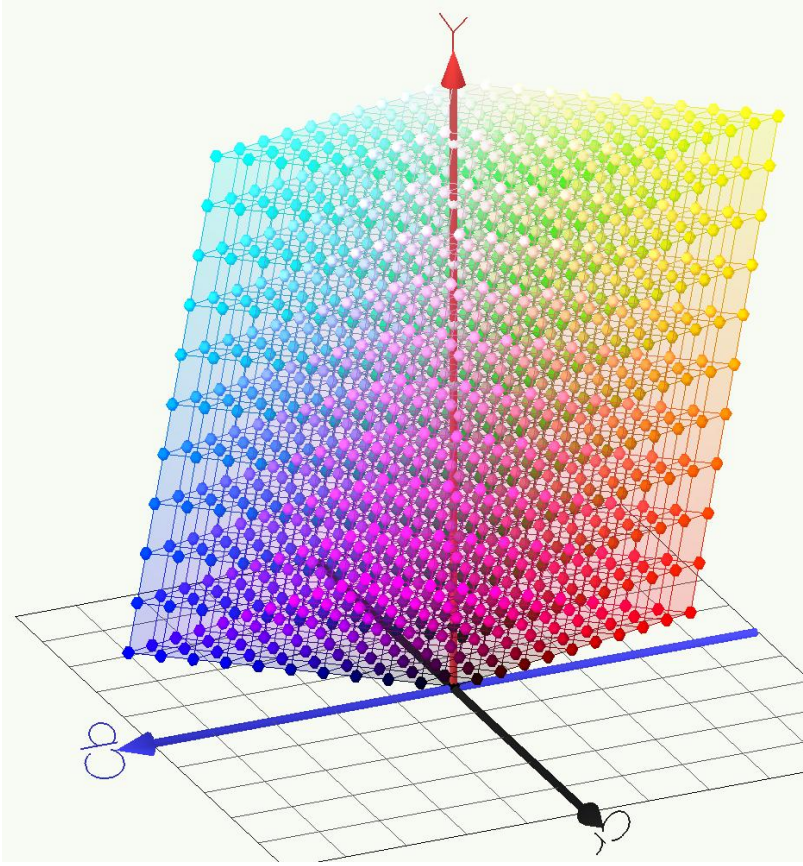
Relative colorimetric intent: only the colors that are outside the destination gamut are clipped to the destination gamut boundaries. It may result that two different colors of the source gamut are mapped into the same color in the destination profile. The relative colorimetric intent takes into account the fact that our eyes always adapt to the white of a medium.

The white on the output is the white of the medium and not the white of the source profile.

Absolute colorimetric intent: it is similar to the relative colorimetric intent except that the white point of the source and destination are the same. This mode is used for proofing, where we want to simulate the output of one printer to another device.

3.1.5 Affine Transforms of RAW RGB values

There are a number of non-absolute color transformations that are used in image processing for convenience. Compression rates in the encoding of JPEG and JPEG 2000 are increased when the RGB values are converted to YCbCr by an affine transform of the RGB values that places most of the color information in the second two components. These types of transforms are reversible except for floating point roundoff errors.



3.1.6 Non-Linear Transforms of RAW RGB values

Color transforms that combine a gamma correction with a rotation of color values are faster that a full gamut mapping transform. The goal of a gamma correction is to obtain an accurate tone scale reproduction. The associated rotation is usually performed to move luminance data to one channel so that a the non-linear operation takes place in 1 dimension.

3.2 Digital Color Management Q and A

1. Light sources are characterized by their spectral power distributions. The spectral power distribution $\rho(\lambda)$ is the fraction of the total power emitted from a source at wavelength λ.
2. Daylight is a mixture of direct sunlight, and light that is scattered and diffracted by the atmosphere (skylight). The power distribution of daylight changes according to weather, time of day, and atmospheric contamination.
3. For the purposes of color measurement, objects are characterized by their spectral reflectance $R(\lambda)$ which is the the fraction of incident light at wavelength λ that is reflected from a point on the object.
4. A green object will not always appear to be green. This can happen under the following circumstances;
   - if the spectral power distribution of the light source has low power in the same are of the spectrum where the spectral reflectance is high, then the object will appear to be different from green.
   - if the object is fluorescent and is exposed to a light source that excites the particular wavelength that causes a shift in the reflectance distribution away from green.
   - colorblind observer will not detect a green object to be green.
5. A color stimulus is a color of light to detected by some observational means. The color stimulus is characterized by conditions of the light and the objects being observed. The stimulus at wavelength λ is the product of the spectral power distribution of the light source and the spectral reflectance of the object; $S = \rho(\lambda) * R(\lambda)$
6. The human audio and visual systems are fundamentally different in terms of frequency domain processing and the spatial resolution capability. Spatial location of a stimulus is much better with the visual system. The audio system is capable of decoding and resolving any waveform with frequency components less than about 26kHz using a single detector. The human visual system uses three different receptors for light. These detectors have some overlap in their sensitivities - the overlap in the sensitivity gives some degree of freedom to the resolving process. It's underdetermined, so there are multiple inverse solutions to a particular measurement of a stimulus - $M(S)$ - to the human visual system. This property is called metamerism. An imaging system may be able to resolve stimuli that the human visual system can distinguish, this can complicate output if not handled. Metamerism simplifies imaging systems because it reduces the number of colors that must be faithfully reproduced. Some imaging systems get by with 256 colors - down from the 11 million that can be detected by the human visual system. Think of this as a lossy compression/decompression.
7. Objects may not appear the same under differing light sources because of metamerism. The stimuli might be the same for both under one light source even though the reflectance differs. Changing the light source changes $\rho(\lambda)$ which may elicit different stimulus responses $S$for the two objects.
8. Colorimetry is the process of encoding and measuring colors for retrieval and display in imaging systems.
9. True or false: The spectral responses of the CIE standard observer are identical to the cone responses of the average human observer. " Why, or why not?

   False. The CIE trichromatic color matching functions are a linear combination of any average cone response functions.
10. What information is needed to compute CIE tristimulus values?

    The spectral power distribution of the light source $S(\lambda)$, the spectral reflectance of the object,$R(\lambda)$, the CIE color matching functions; $\overline{x}(\lambda), \overline{y}(\lambda), \overline{z}(\lambda)$, and a normalizing

constant are integrated over λ in $[380, 780]$.

$$X = \int R(\lambda)S(\lambda)\overline{x}(\lambda)d\lambda$$

$$Y = \int R(\lambda)S(\lambda)\overline{y}(\lambda)d\lambda$$

$$Z = \int R(\lambda)S(\lambda)\overline{z}(\lambda)d\lambda$$

11. Would it be possible to build an output device, such as a video projector, using the primaries defined in the CIE XYZ system? " Why, or why not?

    No. The primaries in the CIE XYZ system were chosen to give the color matching function non-negative values. This requires the primaries to have negative power in some region of the spectrum, which can't really be realized in a light source.

12. What does a CIE x, y chromaticity diagram show? " What does it not show?

    If the X, Y, X tristimulus values are normalized , the plot of $X/(X + Y + Z), Y/(X + Y + Z)$ gives the qualities of a color stimulus with the luminance normalized out.

13. True or false: The appearance of a color can be described by a set of CIE X, Y, Z tristimulus values.

    False. The CIE coordinate system is useful for describing color differences for small differences in stimuli, but not for describing the appearance of a color.

14. True or false: The appearance of a color can be described by transforming its CIE X, Y, Z tristimulus values to CIELAB or CIELUV color space.

    False. The appearance of a color is dependent on the viewing conditions. CIELAB and CIELUV color spaces are good for measuring differences in color.

15. Describe an appropriate use for a Status A densitometer. Describe an inappropriate use for a Status A densitometer.

    An appropriate use of a Status A densiometer would be to measure or to monitor the optical density of the output of a three channel imaging system. An inappropriate use would be to do colrimetry calculations base on desiometer readings. The Status A densiometer response functions are tuned to the narrow bandwidth supported by dyes. Also using the Status A readings to compare output from different imaging systems is not advised. The Status A readings from different systems may agree, but that does not imply the systems generate images with the same appearance.

16. What are the three basic functions required in all imaging systems? Define each of these functions.

    Capture, signal processing, and image formation. Capture is the process of detecting light stimuli from a scene. Signal processing is modifies the captured image for output. Image formation takes the processed signal and uses it to control the color forming elements of the output device.

17. Why is color separation trichromatic in most color-imaging systems? Describe an application in which color separation other than trichromatic might be required.

    Most color imaging systems are trichromatic because the human visual system is. X-ray capture is monochromatic, and some landsat imaging imaging systems may have more than three bands of capture.

18. How can color separation be accomplished in an electronic camera?

    Color separation in an electronic camera may be achieved by capturing image data on a CCD mosaic of RGB sensors. Light can also be optically separated into three bands and then sent to three different CCD sensors - one for each band.

19. How is color separation accomplished in photographic films?

    Color separation in photographic film is done through three or more light sensitive layers and filters.

20. What is meant by exposure factor? How can exposure-factor values be calculated?

    The exposure is calculated by integrating the spectral responsivity with the light spectrum and the spectral reflectance or transmittance of the object being imaged.

21. What is the basic function of color signal processing? Describe some of the transformations typically included in color signal processing.

    The basic function of color signal processing is to make an image suitable for viewing. Typical functions are signal amplification, non-linear modification of the neutral signal, and color matrix calculations, sharpening and noise reduction, and possibly compression.

22. What are the two basic types of color image formation? Are other types of color image formation possible? If so, describe at least one.

    The two basic types of color image formation are additive color mixing and subtractive color mixing. Other types are possible.

23. In an additive system, what color is formed by: Red and blue? Blue and green? Red and green? Red plus green and blue?

    In an additive color forming system, red and blue make magenta, blue and green make cyan, red and green make yellow, and red plus blue plus green make white.

24. In a subtractive system, what color is formed by: Cyan and yellow? Yellow and magenta? Magenta and cyan? Cyan plus magenta and yellow?

    In a subtractive color forming system, cyan and yellow make green, yellow and magenta make red, magenta and cyan make blue, and cyan plus magenta plus yellow make black.

25. Define a complete color-imaging system. Is a digital camera a complete color-imaging system? Why, or why not? Is a photographic slide film a complete color-imaging system? Why, or why not?

    A complete color imaging system is one that can perform capture, signal processing, and image formation. A digital camera is such a system. Capture and signal processing take place in the camera, and image formation takes place on the LCD screen on the back. Photographic slide film is a complete color imaging systems well. Capture takes place in the camera, signal processing and image formation takes place in the lab.

26. Describe an application in which an instrument, such as a colorimeter, would be preferred for assessing color quality.

    A colorimeter would be used to compare an original scene to a reproduction produced by a color imaging system.

27. Describe an application in which a human observer would be preferred for assessing color quality.

    Taking a picture with photographic film, and then viewing the reproduction later.

28. What aspect of the human visual system corresponds to: image capture signal processing image formation

    Image capture takes place in the rods and cones of the eye, signal processing and image formation both take place in the brain.

29. What is psychological signal processing?

    Psychological signal processing is the component which accounts for color memory, and color preference.

30. What is psychophysical signal processing?

    Psychophysical signal processing is the component of image processing in the human brain which takes into account effects like chromatic adaptation, lateral brightness adaptation, and the relative nature of luminance detection.

31. . Define each of the following: general-brightness adaptation lateral-brightness adaptation chromatic adaptation

    Brightness adaptation is how the eye responds to varying levels of illumination. Lateral brighness adaptation is how the eye has different sensitivities in different parts of the eye. Chromatic adaptation is how the eye responds to an average chromatic stimulus of a scene.

32. For each adaptation effect listed in question 6, give an example of how the effect might be encountered in a practical imaging situation.

    Brightness adaptation would be encountered when viweing conditions change from low to high light illumination. Lateral brightness adaptation would be encountered when focusing on different objects - closer objects are captured with the fovea. Chromatic adaptation occurs in incandecent illumination induces an increase in short wavelenght sensitivity.

33. Describe the relationship of CIE colorimetry to human color vision. What aspects of color vision are predicted well by standard CIE colorimetry? What aspects are not predicted well? Why not?

    CIE colorimetry can be used to predict if two stimuli will visually match under identical viewing conditions. CIE colorimetry can not be used to emulate signal processing of color formation functions.

34. For the purposes of color characterization, what characteristics of a color monitor must be measured or otherwise determined?

    The CIE x,y chromaticity coordinates of the monitor red, green, and blue primaries must be determined experimentally, or be given by the manufacturer. The location of the chromaticity coordinates determines the gamut of colors the monitor can produce by adding varying intensities of the red, green, and blue primaries. The color matching functions of the CIE standard observer can be obtained from the x,y chromaticity coordinates of the primaries.

35. What does monitor white point mean?

    The monitor white point is the chromaticity coordinates of the stimulus obtained by mixing the red, green, and blue primaries at full intensity - $(R, G, B) = (255, 255, 255)$ for an 8 bit monitor.

36. What does monitor grayscale tracking mean? Why is such tracking important?

    Monitor greyscale tracking is the ability to emit light of constant chromaticity but varying luminance when $R = G = B$.

37. Monitor and hardcopy images are said to be highly metameric. What does this mean? Describe a practical consequence of a high degree of metamerism. Describe a method for dealing with that consequence.

    For a monitor to reproduce a color stimulus with a monitor we add light from three sources. Metamerism is when the spectral power distribution of the light source produced by the monitor is very different from the spectral power distribution of the stimulus - even though the CIE tristimulus values of the two stimuli are the same. The spectral power distribution of the three sources determine the amount of each light source needed to produce a particular stimulus. For hard copy the spectral reflection density of the dye set used for the print is what determines the amount of each dye to use for a given stimulus.

    Because of the differences among individual responsivities differ from the CIE standard observer, two individuals observing a color on the monitor may disagree whether the stimulus matches a reference one. Color matching experiments can be done for each observe to correct for this.

38. Why should (or should not) logarithmic units be used in characterizing the grayscales of monitors?

    Taking the log of monitor luminance gives a better measure of differences at the low end of the luminance scale. The CIE L-a-b luminance is defined as $L^* = 116(\frac{Y}{Y_n})^{\frac{1}{3}} - 16$. Conveniently, the ...

39. What grayscale characteristic must a video camera have for the entire system to have a grayscale that is one-to-one with that of the original scene?

    The greyscale characteristic of the camera needs to be the inverse of the greyscale characteristic of the monitor.

40. In addition to an appropriate grayscale characteristic, what further characteristic must a video camera have for the entire system to have colorimetric color reproduction that is one-to-one with that of the original scene?

    The chromaticities of the non-neutral scene colors must match the output device chromaticities.

41. Describe the appearance of such colorimetric color reproduction.

    When the chromaticities of the input match those of the output device, the result is something the viewer would report as being less saturated than the original scene. This is because using the CIE standard observer color matching functions as camera sensitivities leads to chromatic errors. Since the CIE Standard Observer color matching functions are a linear transform of any other color matching functions, we can remove these errors by transforming the camera color matching function from the ones representing the monitor primaries to another set that accurately reproduces the color stimuli.

42. What additional factors must be taken into account in order to produce pleasing reproductions of original scenes?

    In addition to accurate colorimetric reproduction, psychophysical effects of the human visual system must be taken into account in order to produce visually pleasing results.

43. How can video signal processing be used to account for these factors?

    The greyscale characteristic can be modified to account for viewing flare. Also, the reproduction device is of a lower overall luminance than the original scene - which leads to a reduction in the perceived saturation. Boosting the greyscale characteristic accounts for this effect as well.

44. Describe the principal components of an ideal video system. How do practical systems correspond to this ideal? Why will the color gamut of any real CRT be limited?

    An ideal video system would have camera spectral sensitivities with all positive color matching functions so that all color could be realized by the monitor. There should be a color matrix transformation that converts the camera color matching functions to ones that correspond to the monitors. The gamut of a monitor will be limited because the transform from the CIE standard observe to monitor color matching functions will give colors with negative amounts of monitor RGB primaries. The CIE standard observer color matching functions were all positive, and all colors could be reproduced because the primaries were imaginary. The monitor primaries are not imaginary - this is the reason why the color matching functions of the monitor must have negative values.

45. Describe three possible methods for encoding video signals.

    We could just do the colorimetric matching. We could do the color matching and then let Ed tweak the signal processing so the results are pleasing. Lastly we could reverse engineer any imaging system to produce pleasing results - get data on the output, and taylor the signal processing to produce the characteristics the viewers have reported as pleasing. This last suggestion works for any given imaging system, but we would not be able to design the signal processing of an input or output device in isolation this way.

46. What general characteristics are shared by virtually all reflection media?

Most reflection-print media consist of three or more subtractive dyes used to form an image on some type of reflective support material.

47. What happens when a ray of light strikes the surface of a simple reflection medium?

    Some incident light will be reflected on the surface of the media, the rest will be refracted through the colorant layer and scattered off the reflective support. Some light scattered off the support will be reflected off the surface coating back to the colorant.

48. A neutral object is photographed under a first illuminant. A reproduction is made and viewed under a second illuminant. For the reproduction of the object to appear neutral, should its chromaticity match that of the first illuminant or that of the second illuminant? Why?

    The chromaticity of the reproduction neutral should be the the chromaticity of the illuminant the reproduction is viewed in. The observer will be chromatically adapted to the second illuminant when viewing the reproduction. If the chromaticity of neutral objects are reproduced with the chromaticity of the first illuminant, the observer will perceive the difference.

49. What is meant by viewing illuminant sensitivity? Why it is important in imaging applications?

    Viewing illuminant sensitivity is a measure of how much image dye components must be modified to produce a metamer when reproducing neutrals to be viewed under different illuminants. This is important in imaging systems because the same reproduction could be viewed under different illuminants, and the observer may perceive a difference under the two illuminants if the viewing illuminant sensitivity is high.

50. Why would the grayscale characteristic of a reflection-print system not be one-to-one with the original scene?

    If the viewing conditions of the scene are different from the viewing conditions of the reproduction, the grayscale would need to be modified. The contrast is boosted for the midrange and rolled off at the ends of the density range. This is done to accommodate for viewing flare and differences in the absolute luminance of the scene and the viewing environment of the reproduction.

51. System A has a higher maximum density and a lower minimum density than System B. Which system will require a higher mid-scale gamma?

    The dynamic range of system A is greater than system B. The midscale gamma of system B will need to be higher to produce dark and light tones at acceptable density.

52. If the taking illuminant and viewing illuminant of a reflection-print system differ in chromaticity, what is required in order to make a meaningful evaluation of reproduced chromaticity values?

    The observes state of chromatic adaptation in the viewing environment must be taken into account.

53. Why might the reproduced colorimetry of a reflection-print system differ from that of an original scene?

    To accommodate for a difference viewing illuminant and scene illuminant, the chromaticities could linearly transformed to produce a visual match for the tristimulus

values of the original color stimulus.

54. Is there a single best color-reproduction position that all reflection-print systems should attempt to realize? Why, or why not?

    The color reproduction of reflection print systems is dependent on differences in scene and viewing conditions and the adaptive state of the observer. There is no common method to account for all the different possibilities for scene and reproduction viewing environment.

55. What properties make photographic transparencies popular for use as input to color-imaging systems?

    High dynamic range, and narrow spectral sensitivities make slide films popular inputs to high performance imaging systems. Photographic transparencies are popular in the graphics arts and advertising industries since the image can be viewed directly on the media.

56. How do the relative red, green, and blue speeds of a tungsten-balanced photographic transparency film compare to those of a daylight-balanced photographic transparency film? Why?

    Tungston light sources have more long wavelength power than D_{55}. To compensate for this, the spectral sensitivities of the tungsten balanced films have lower sensitivity in the long wavelengths and higher sensitivity in the short wavelengths. The bandwidths of the three channels are the same for tungsten and D_{55} balanced film.

57. What does it mean to say that one dye is purer than another?

    If the bandwidth - or area of the spectrum a dye is sensitive to - of one dye is narrower than another, we say that dye is more pure.

58. If the same magenta dye is coated on a reflective and a transmissive support, how will the spectral properties of the resulting coatings differ? Describe one consequence of this effect in practical imaging applications.

    On a reflective media, the light passes through each dye layer at least twice before reaching the viewer. This allows for more absorption of light. The effective density of the magenta dye on the reflective support will be higher than the density on the transmissive media. On the reflective support, there will be more absorption of light at the boundary of the band the dye is sensitive to. This leads to a change in the chromaticity of the reproduced colors since the magenta dye on the reflective support will absorb more blue and red light than the same dye on a transmissive media. More color processing is required for the reflective media to correct for this shift.

59. Why is viewing-illuminant sensitivity of lesser importance for transmission media than it is for reflection media?

    Transmissive media are designed for both scene and viewing illuminants. Slide films and photographic transparancies are designed to be viewed under a known illumination - tungston for slide film and $D55$. Since the viewing illuminant is known ahead of time, the dyes can be tuned in ways that reflection media can't. More pure dyes which give a larger gamut of reproduced colors can be used. Note this does make transmissive more illuminant sensitive than reflection media - we could expect more color shifts in viewing transparant media with the wrong light source.

60. Why are 35mm slide films balanced cyan-blue, according to instrument measurements?

    To compensate for the incomplete chromatic adaptaion to the viewing illiminant that takes place when viewing slide images in a darkened room. The viewer does not completely adapt to the tungsten light source, so a colroimetric neutral would appear to have extra red light. To compensate for this extra cyan dye is added to absorb the unwanted red light.

61. Name two other ways in which the grayscale characteristic of a photographic transparency film differs from that of reflection-print system. Why do these differences exist?

    The dynamic range (D_{max}/D_{min}) of transparency film is higher than reflection media and the gamma is higher. The dynamic range is higher because of the more pure dye-set used in transmissive media. The gamma is higher to account for lateral brighness adaptation in the viewing environment. The dark surround of the slide image causes a perceived drop in contrast which is accounted for by increasing the gamma.

62. Why are the spectral sensitivities of photographic transparency films significantly different from any set of color-matching functions?

    Since the chemistry of the transparancy system is relatively fixed, the non-linear signal processing is built into the spectral sensitivities of the dyes used.

63. Why are image overall density and color balance somewhat less critical for photographic transparency films compared to reflection-print systems? How does this affect the color encoding of these films?

    Density and color balance are less important for transparancy film because the image is driving the viewers state of adaption. This allows the brain to do signal processing that has to be built into reflection media.

64. Describe at least three other complications of color encoding of photographic transparency films.

    The relationship between colorimetry and color perception is sensitive to viweing illmuninant. When scanning transparancies, the illuminant of the scanner determines the colors reproduced. If the scanner illuminant is different from that designed into the transparancy, the image will need to be color corrected.

65. What properties make photographic negatives well suited for use as input to color-imaging systems?

    The large dynamic range and low gamma of photographic negatives give scanners the ability to capture images covering a wide range of exposures with lower sensitivity sensors.

66. What is the major complication in using photographic negatives as input to color-imaging systems?

    Color negatives are not designed for human viewing so any imaging application that requires previewing images prior to output will require additional processing to present the images in human viewable form. Until recently scanning of negatives was slow and costly so previewing negatives required the viewer to be able to interpret negative images, or a set of positive images printed from the negatives.

67. What is the fundamental difference between a photographic negative system and photographic transparency system?

    Maximum image dye is formed at minimum exposure for a negative and maximum exposure for a transparency.

68. What does printing density mean?

    Printing density is the negative log of the R,G,B exposures for the photographic media used to record a negative image.

69. Why is it important in the measurement of photographic negatives?

    Greyscale and color reproduction of photographic negatives are designed around the reflectances of the printing dyes used.

70. How can printing density values be determined, using an ISO Status M densitometer for measurement?

    Apply a color matrix to the signal.

71. What is printing density metamerism?

    Print density metamerism occurs when two negative signals have different spetral transmission responses but produce the same optical printing densities.

72. Why are the red, green, and blue printing-density gammas matched for photographic negative films? What would happen if the were not matched?

    R,G,B print densities are parallel so netural tonescales are accurately reproduced when the negatives are printed.

73. What is the optimum gamma for a photographic negative film?

    It depends on the application, indoor portrate film has a lower gamma than consumer film.

74. What is the most difficult problem encountered in color encoding images from photographic negatives?

    Print density metamerism.

75. List some reasons why matrix correction may be required in a color imaging system.

    Transform color signal to a new color space.

76. What is printing density cross talk? How can such cross talk be corrected in optical-printing systems? How can cross talk be corrected in digital-printing systems?

    When two dyes have overlapping spectral sensitivities.

77. Predict the effect that increasing right-way blue-onto-green color correction will have on these colors: neutrals, reds, greens, blues, cyans, magentas, yellows, and skin tones.

Vision

Chapter 4

GIS

A rank of ranks study was done on 16 poverty and 16 development features from AFDB/OCED 2007 data on 53 African Nations. Taking missing values into account - countries scores were calculated by ranking across all features. The resulting data set was imported to ESRI ArcMap software to produce the poverty and development heat maps. The obvious spatial correlation is demonstrated in Moran Scatterplots of the heat map data.

Ranking countries is a old and open problem. See [1] for a good background discussion. Poverty and development country ranks were calculated by ranking the unweighted sum of the feature ranks [2]. The tables below list the features used.

Development Features	
GDP based on PPP valuation	GDP per Capita
Annual real GDP Growth	FDI Inflow 2005 ($ million)
Share of Consumption Lowest 10%	Telcom Main Line (Per 100)2005
com Mobile Lines (Per 100) 2005	Water supply coverage Total 2004
Water supply coverage Urban 2004	Water supply coverage Rural 2004
Sanitation coverage Total 2004	Sanitation coverage Urban 2004
Sanitation coverage Rural 2004	Number of Products Accounting for $>$ 75% Exports
GNI per capita, Atlas method (current US$)	2006 Softening of Regime

Poverty Features	
Population growth % 2000-2005	PopulationGrowth Rate 2005-2010
Infant Mortality /1000	Mortality Under Age 5
Gini Coefficient	Share of Consumption Highest 10%
Estimated adult illiteracy rate (%)	AIDS Prevalence
Corruption Index (0-1)	Inflation 2005
Inflation 2006(e)	Inflation 2007(p)
Inflation 2008(p)	2005 Change in Political Troubles
2006 Change In Political Troubles	2006 Hardening of Regime

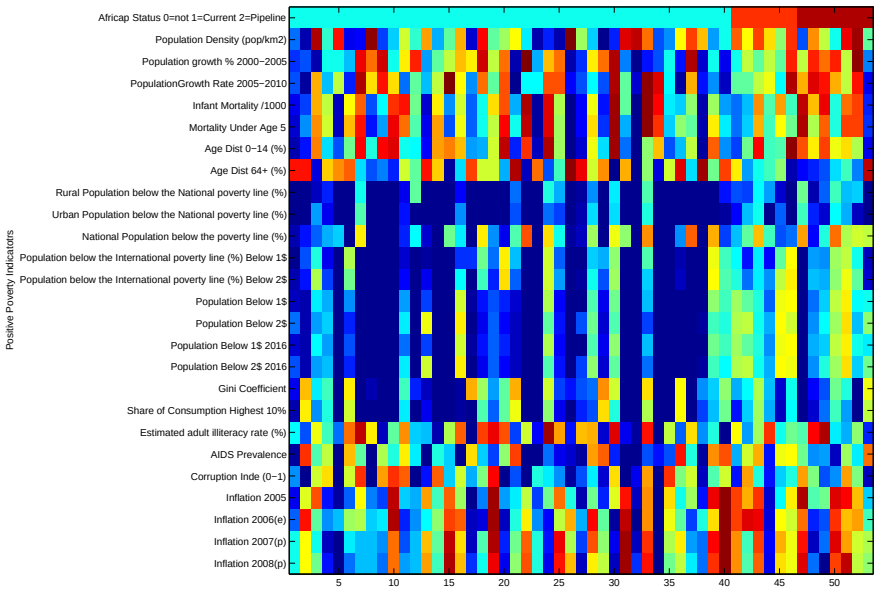
The two heat maps below show the rankings of 53 African nations in a larger data set.[3] Note missing values (dark blue) for many of the absolute poverty measures these were left out

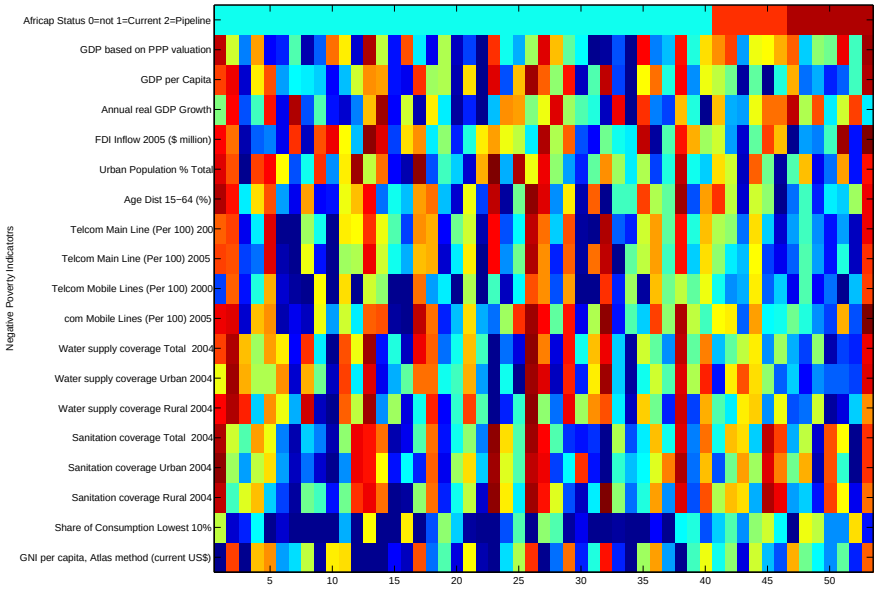
[1] Munda G., Nardo M. (2005), "Non-compensatory composite indicators for ranking countries: a defensible setting," EUR 21833 EN, European Commission.

[2] Tools for Composite Indicators Building. Nardo M., Saisana M., Saltelli A. and Tarantola S. (2005) European Commission, EUR 21682 EN, Institute for the Protection and Security of the Citizen, JRC Ispra, Italy.

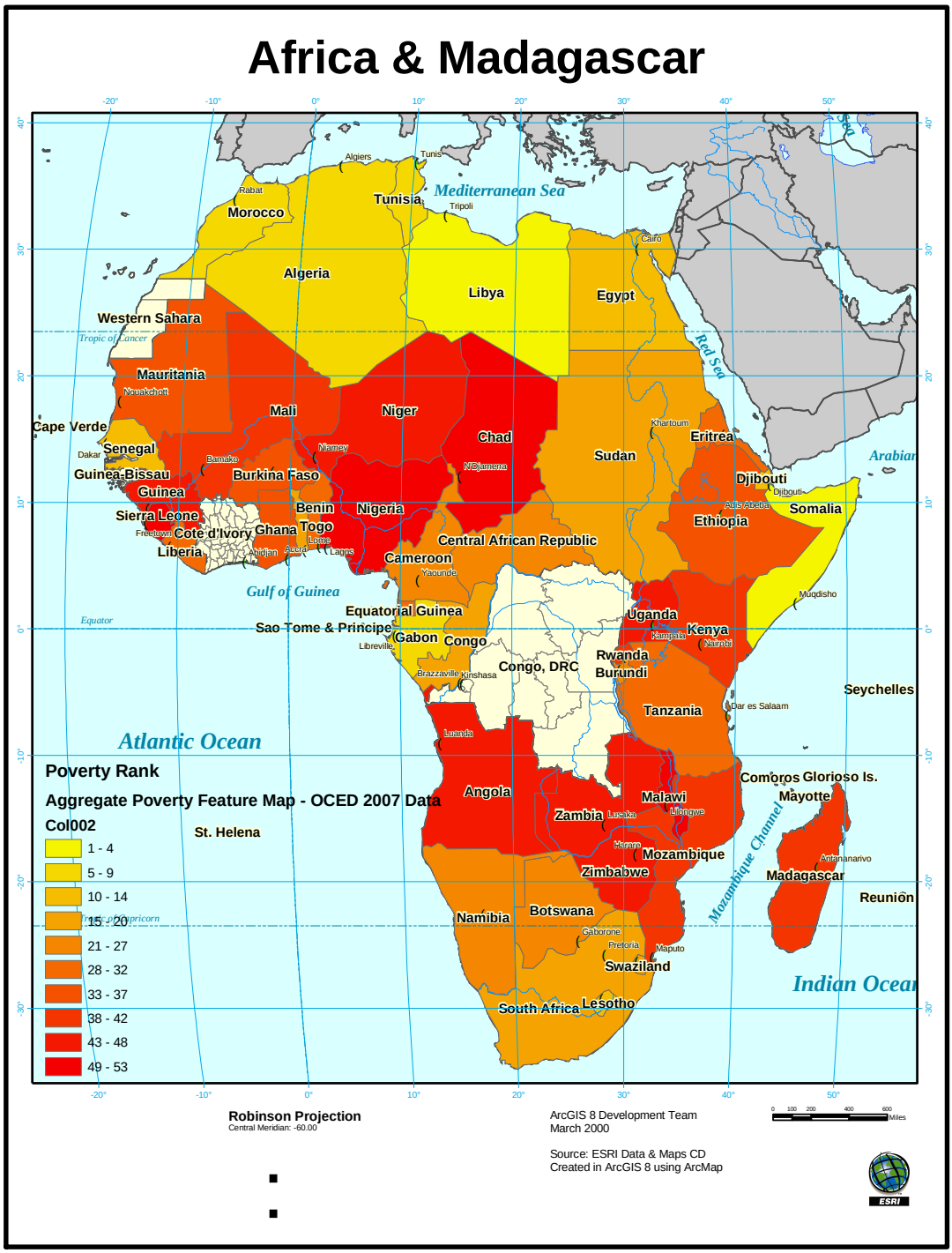
[3] AfDB/OECD 2007 Statistical Annex

of the rank study. Missing values were handled by exclusion instead of imputation. Missing values bias rank algorithms towards a higher false negative rate. Spatially interpolating feature values from neighboring countries works when the feature can be spatially modelled, say AIDS rate.

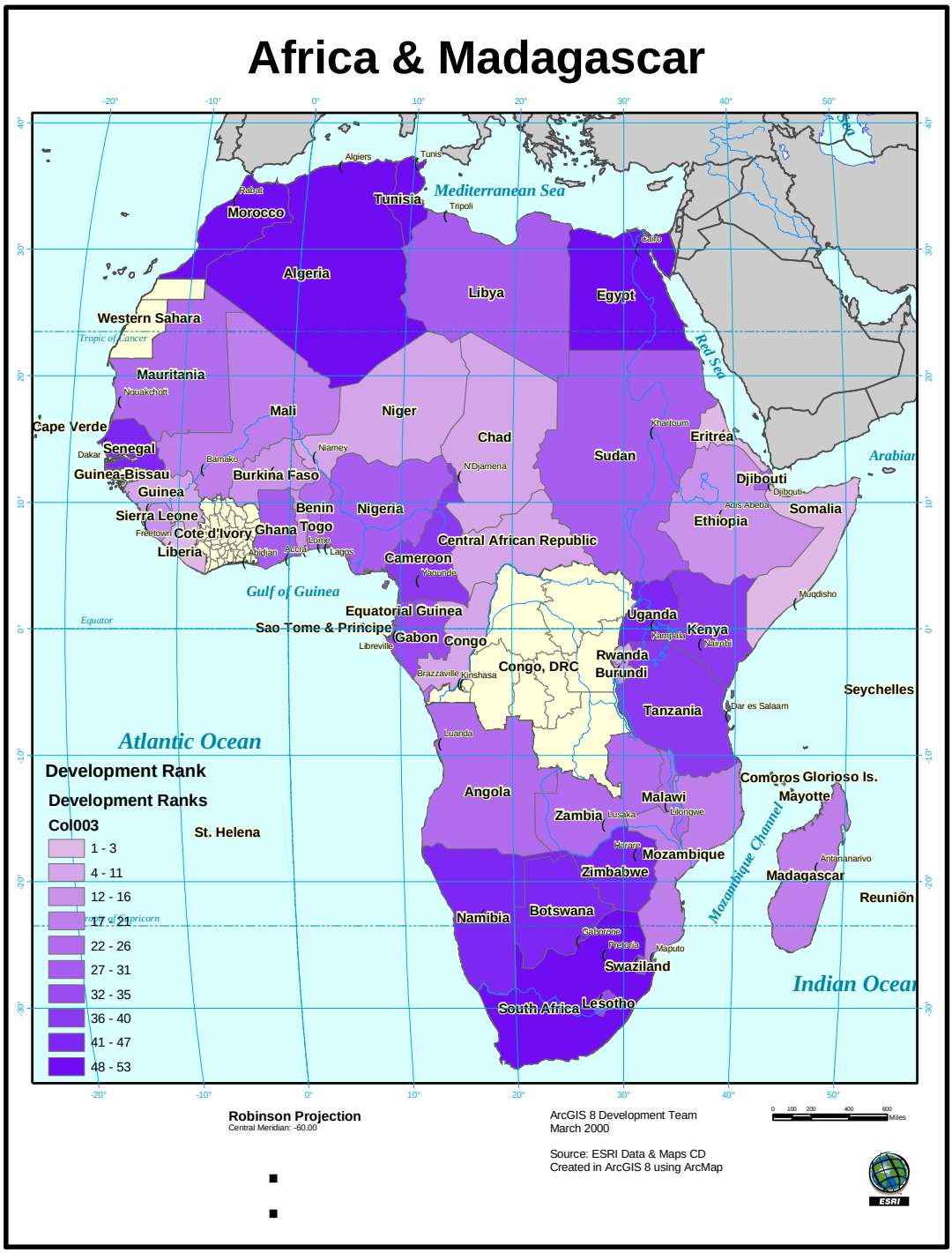



Africap Status 0=not 1=Current 2=Pipeline
GDP based on PPP valuation
GDP per Capita
Annual real GDP Growth
FDI Inflow 2005 ($ million)
Urban Population % Total
Age Dist 15−64 (%)
Telcom Main Line (Per 100) 200
Telcom Main Line (Per 100) 2005
Telcom Mobile Lines (Per 100) 2000
com Mobile Lines (Per 100) 2005
Water supply coverage Total 2004
Water supply coverage Urban 2004
Water supply coverage Rural 2004
Sanitation coverage Total 2004
Sanitation coverage Urban 2004
Sanitation coverage Rural 2004
Share of Consumption Lowest 10%
GNI per capita, Atlas method (current US$)
Negative Poverty Indicators
5
10
15
20
25
30
35
40
45
50

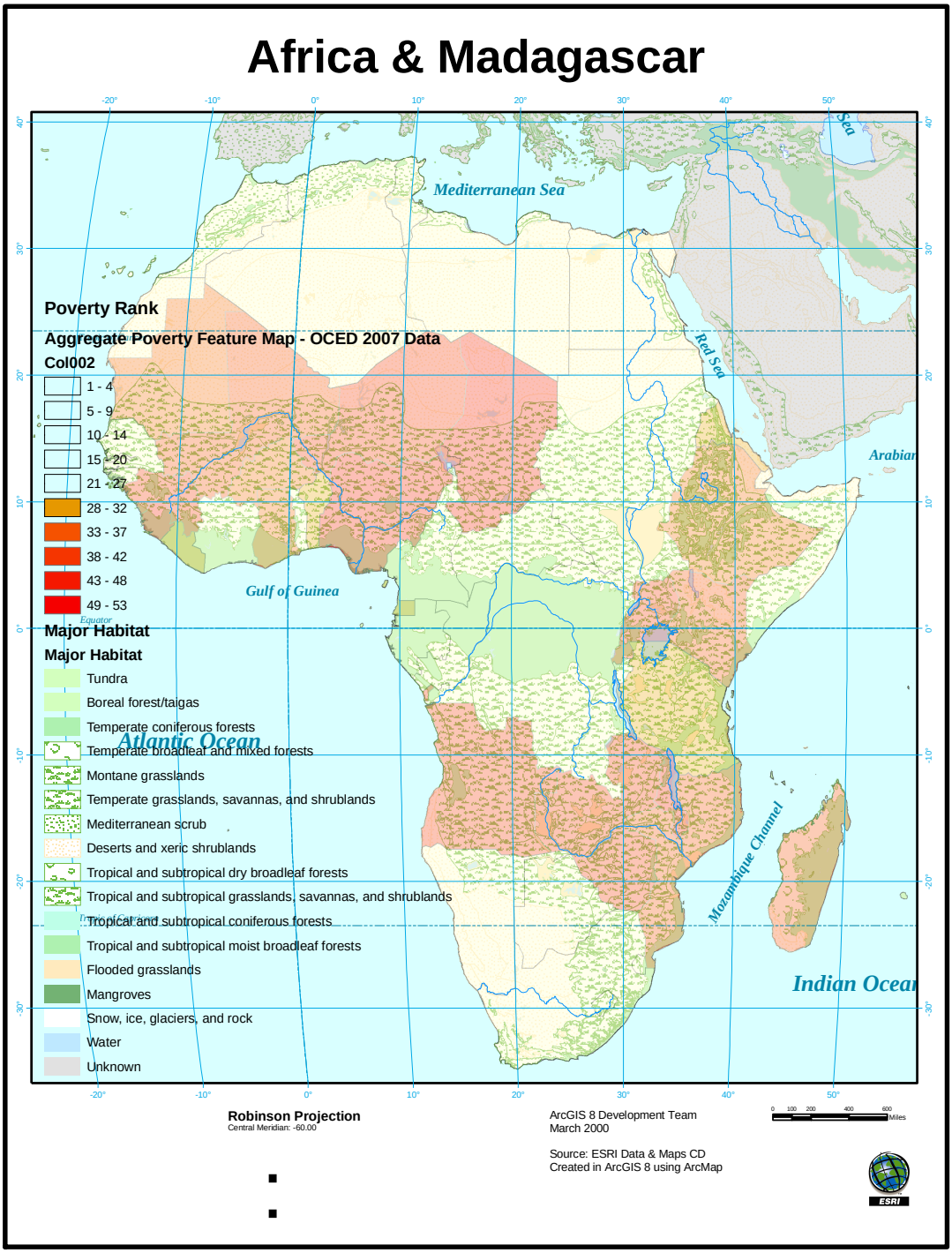
African poverty heat map.


Africa & Madagascar
Mediterranean Sea
Red Sea
Arabian
Gulf of Guinea
Atlantic Ocean
Indian Ocean
Mozambique Channel
Tropic of Cancer
Equator
Tropic of Capricorn
Algiers
Tunis
Tripoli
Cairo
Rabat
Morocco
Tunisia
Algeria
Libya
Egypt
Western Sahara
Mauritania
Nouakchott
Mali
Niger
Chad
Sudan
Khartoum
Eritrea
Cape Verde
Dakar
Senegal
Bamako
Niamey
N'Djamena
Guinea-Bissau
Guinea
Burkina Faso
Djibouti
Addis Ababa
Somalia
Ethiopia
Sierra Leone
Freetown
Cote d'Ivory
Ghana
Togo
Benin
Nigeria
Liberia
Abidjan
Accra
Lome
Lagos
Central African Republic
Cameroon
Yaounde
Equatorial Guinea
Sao Tome & Principe
Gabon
Libreville
Congo
Brazzaville
Congo, DRC
Kinshasa
Uganda
Kampala
Kenya
Nairobi
Rwanda
Burundi
Tanzania
Dar es Salaam
Mogadisho
Seychelles
Luanda
Angola
Zambia
Lusaka
Malawi
Lilongwe
Mozambique
Harare
Zimbabwe
Comoros
Glorioso Is.
Mayotte
Madagascar
Antananarivo
Reunion
Namibia
Botswana
Gaborone
Pretoria
Maputo
Swaziland
South Africa
Lesotho
St. Helena
Poverty Rank
Aggregate Poverty Feature Map - OCED 2007 Data
Col002
1 - 4
5 - 9
10 - 14
15 - 20
21 - 27
28 - 32
33 - 37
38 - 42
43 - 48
49 - 53
Robinson Projection
Central Meridian: -60.00
ArcGIS 8 Development Team
March 2000
Source: ESRI Data & Maps CD
Created in ArcGIS 8 using ArcMap
Miles
ESRI

African development heat map.


Africa & Madagascar
Mediterranean Sea
Red Sea
Arabian Sea
Gulf of Guinea
Atlantic Ocean
Indian Ocean
Mozambique Channel
Tropic of Cancer
Equator
Tropic of Capricorn
Algiers
Tunis
Rabat
Tripoli
Cairo
Morocco
Tunisia
Algeria
Libya
Egypt
Western Sahara
Mauritania
Nouakchott
Mali
Niger
Chad
Sudan
Eritrea
Khartoum
Cape Verde
Dakar
Senegal
Bamako
Niamey
N'Djamena
Guinea-Bissau
Burkina Faso
Guinea
Djibouti
Addis Abeba
Somalia
Ethiopia
Sierra Leone
Freetown
Benin
Nigeria
Cote d'Ivory
Ghana
Togo
Liberia
Abidjan
Accra
Lome
Lagos
Central African Republic
Cameroon
Yaounde
Muqdisho
Equatorial Guinea
Sao Tome & Principe
Gabon
Congo
Libreville
Uganda
Kampala
Kenya
Nairobi
Rwanda
Burundi
Congo, DRC
Brazzaville
Kinshasa
Seychelles
Tanzania
Dar es Salaam
Luanda
Comoros
Glorioso Is.
Mayotte
Angola
Malawi
Lilongwe
Zambia
Lusaka
Harare
Mozambique
Antananarivo
Madagascar
Reunion
Zimbabwe
Namibia
Botswana
Gaborone
Pretoria
Maputo
Swaziland
South Africa
Lesotho
St. Helena
Development Rank
Development Ranks
Col003
1 - 3
4 - 11
12 - 16
17 - 21
22 - 26
27 - 31
32 - 35
36 - 40
41 - 47
48 - 53
Robinson Projection
Central Meridian: -60.00
ArcGIS 8 Development Team
March 2000
Source: ESRI Data & Maps CD
Created in ArcGIS 8 using ArcMap
Miles
ESRI

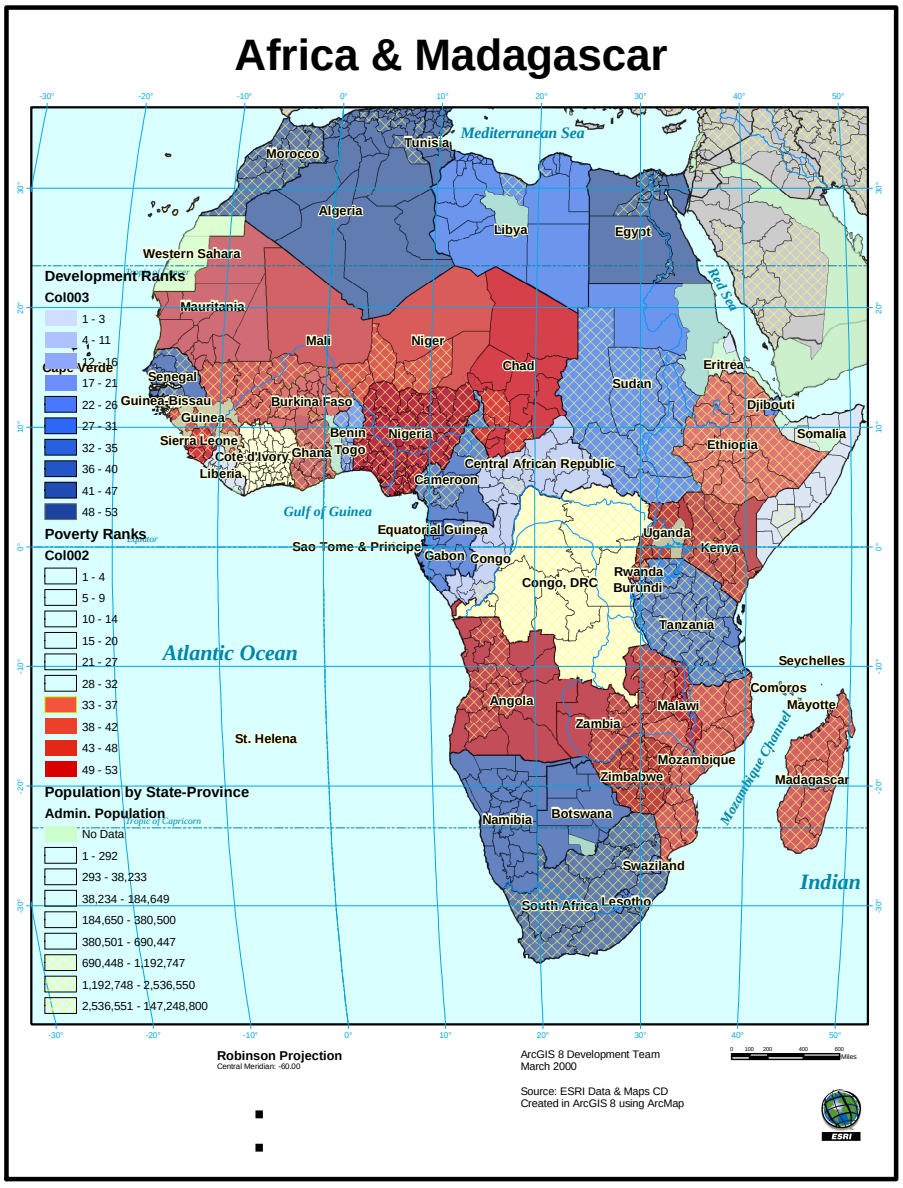
Adding Habitat to the poverty heat map.


Africa & Madagascar
Mediterranean Sea
Red Sea
Gulf of Guinea
Atlantic Ocean
Mozambique Channel
Indian Ocean
Arabian
Poverty Rank
Aggregate Poverty Feature Map - OCED 2007 Data
Col002
1 - 4
5 - 9
10 - 14
15 - 20
21 - 27
28 - 32
33 - 37
38 - 42
43 - 48
49 - 53
Major Habitat
Major Habitat
Tundra
Boreal forest/taigas
Temperate coniferous forests
Temperate broadleaf and mixed forests
Montane grasslands
Temperate grasslands, savannas, and shrublands
Mediterranean scrub
Deserts and xeric shrublands
Tropical and subtropical dry broadleaf forests
Tropical and subtropical grasslands, savannas, and shrublands
Tropical and subtropical coniferous forests
Tropical and subtropical moist broadleaf forests
Flooded grasslands
Mangroves
Snow, ice, glaciers, and rock
Water
Unknown
Robinson Projection
Central Meridian: -60.00
ArcGIS 8 Development Team
March 2000
Source: ESRI Data & Maps CD
Created in ArcGIS 8 using ArcMap
Miles
ESRI

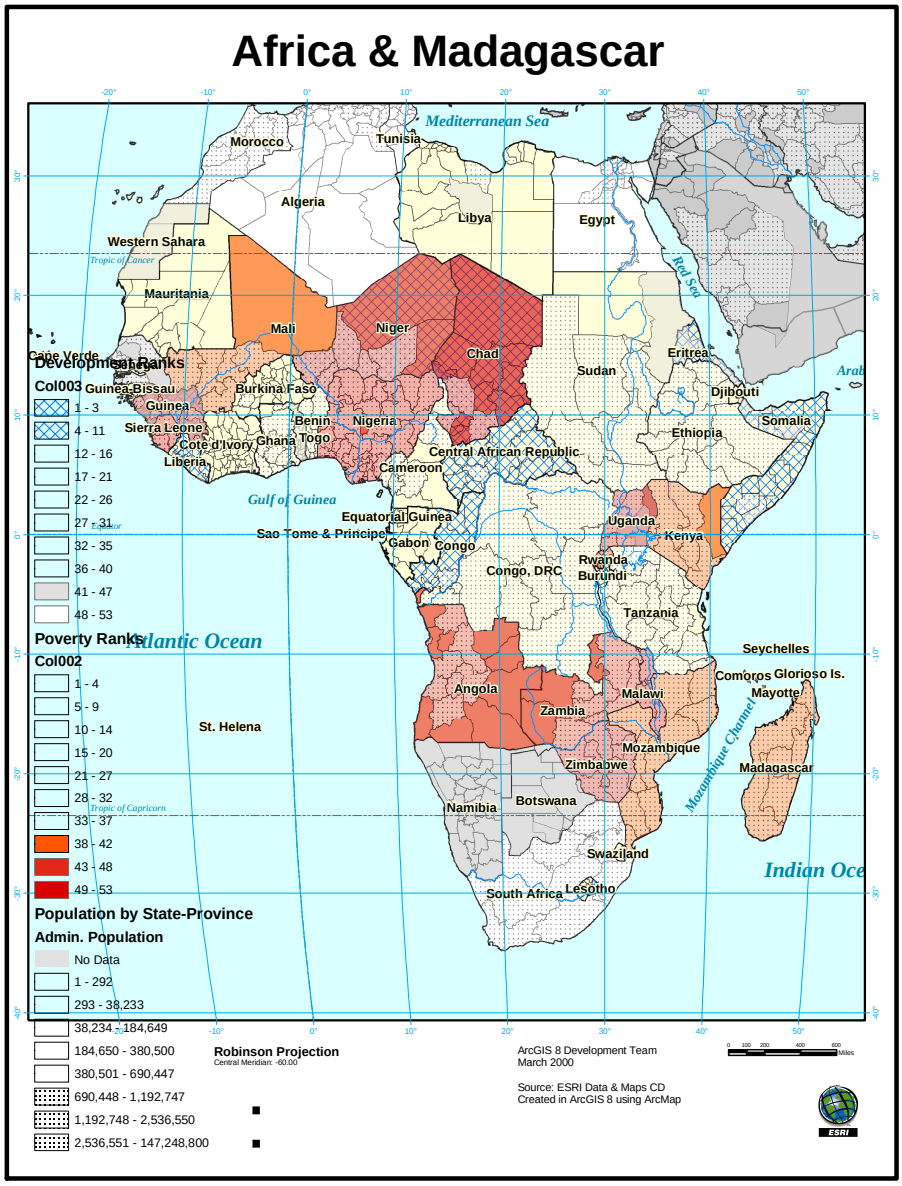
Overlap of high poverty rank and low development rank.


Africa & Madagascar
Mediterranean Sea
Red Sea
Arabian
Atlantic Ocean
Indian Ocean
Gulf of Guinea
Equator
Tropic of Cancer
Tropic of Capricorn
Mozambique Channel
St. Helena
Algiers
Tunis
Rabat
Tripoli
Cairo
Morocco
Tunisia
Algeria
Libya
Egypt
Western Sahara
Mauritania
Nouakchott
Mali
Niger
Cape Verde
Dakar
Senegal
Bamako
Niamey
Chad
N'Djamena
Khartoum
Eritrea
Sudan
Guinea-Bissau
Burkina Faso
Djibouti
Guinea
Benin
Nigeria
Sierra Leone
Freetown
Togo
Cote d'Ivory
Ghana
Liberia
Abidjan
Accra
Lome
Lagos
Central African Republic
Cameroon
Yaounde
Ethiopia
Addis Abeba
Somalia
Mogadishu
Equatorial Guinea
Sao Tome & Principe
Gabon
Libreville
Congo
Brazzaville
Kinshasa
Congo, DRC
Uganda
Kampala
Kenya
Nairobi
Rwanda
Burundi
Seychelles
Tanzania
Dar es Salaam
Luanda
Angola
Comoros
Glorioso Is.
Mayotte
Malawi
Lilongwe
Zambia
Lusaka
Harare
Mozambique
Zimbabwe
Antananarivo
Madagascar
Reunion
Namibia
Botswana
Gaborone
Pretoria
Maputo
Swaziland
South Africa
Lesotho
Development Ranks
Col003
1 - 3
4 - 11
12 - 16
17 - 21
22 - 26
27 - 31
32 - 35
36 - 40
41 - 47
48 - 53
Poverty Ranks
Col002
1 - 4
5 - 9
10 - 14
15 - 20
21 - 27
28 - 32
33 - 37
38 - 42
43 - 48
49 - 53
Robinson Projection
Central Meridian: -60.00
ArcGIS 8 Development Team
March 2000
Source: ESRI Data & Maps CD
Created in ArcGIS 8 using ArcMap
Miles
ESRI

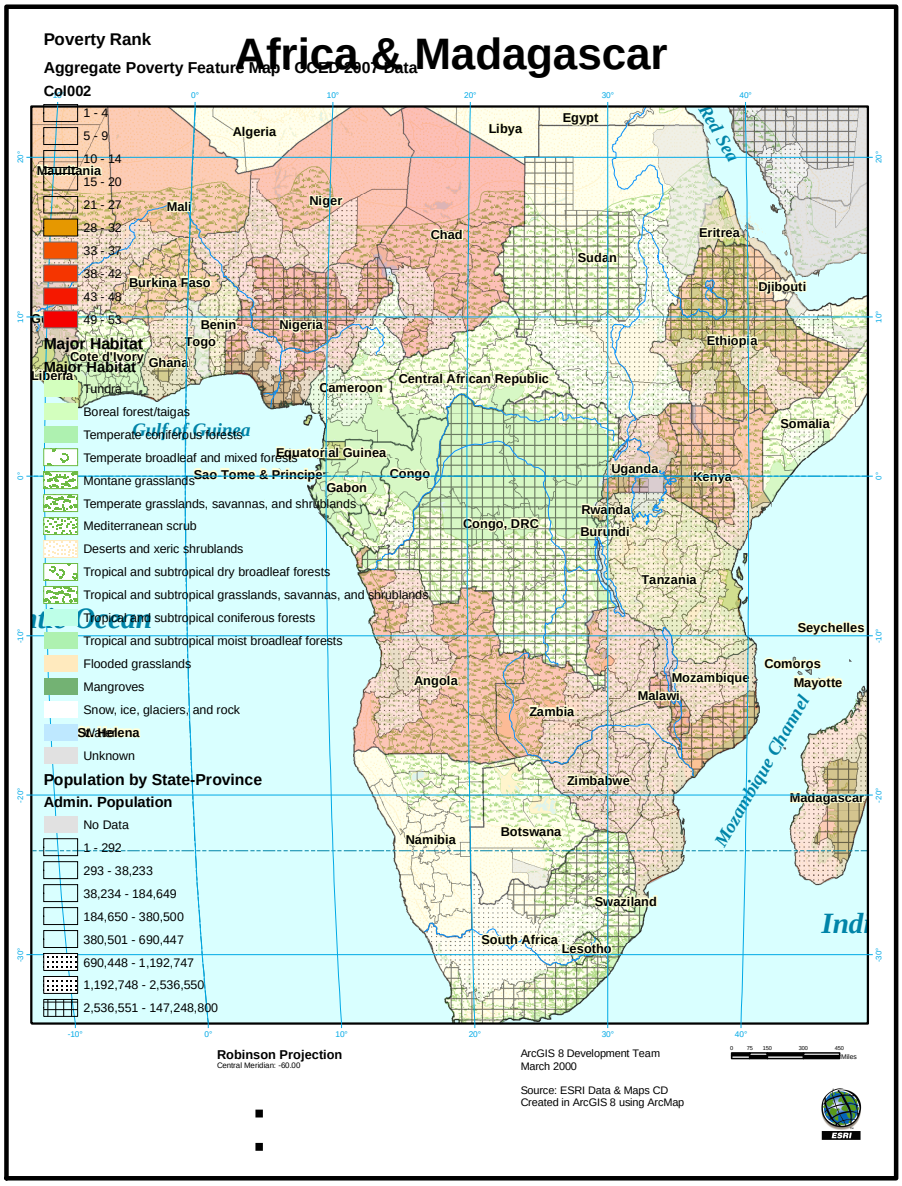
Poverty and Development ranks with State Population



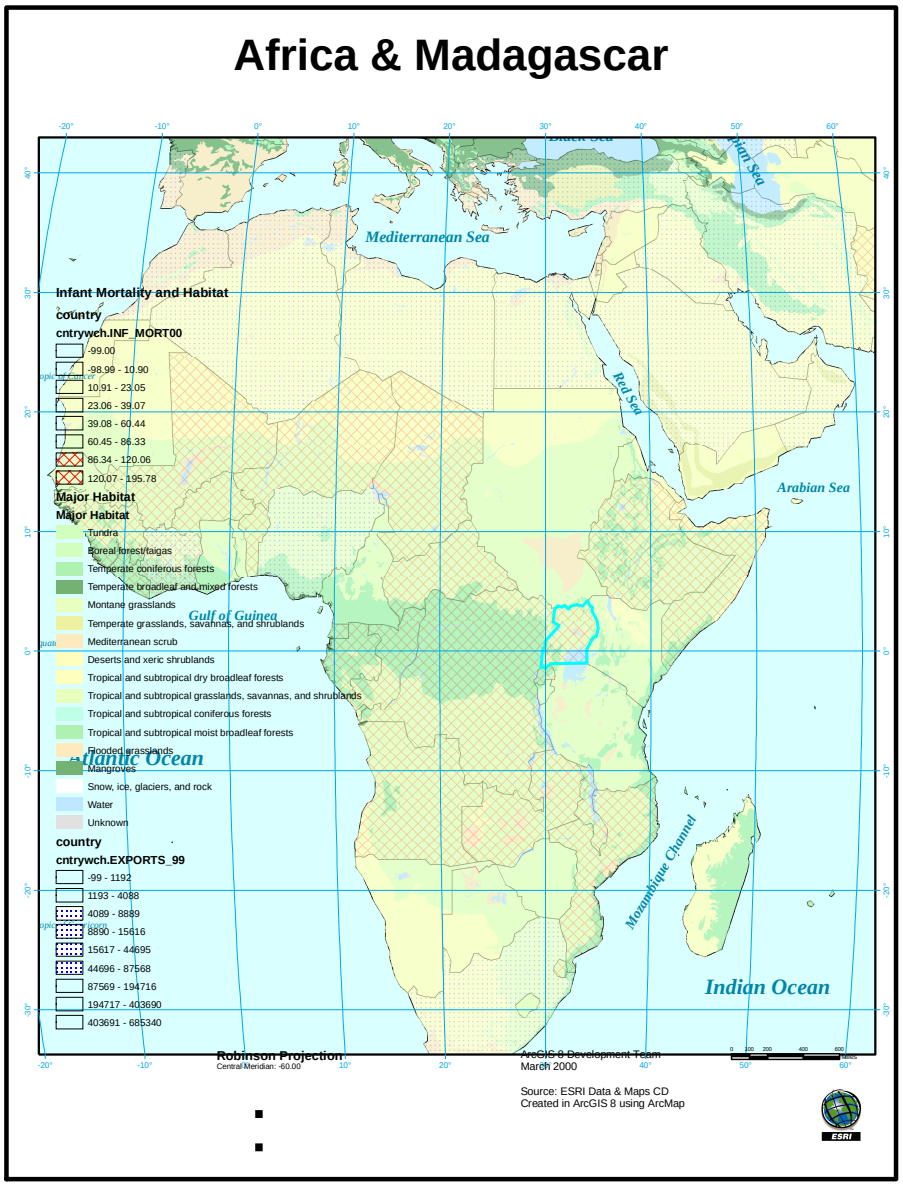
Another view of Poverty and Development ranks with State Population



Poverty, Habitat, and State Population (2000) - > 200% zoom recommended.



Older ESRI data showing similar spatial pattern using infant mortality and exports.



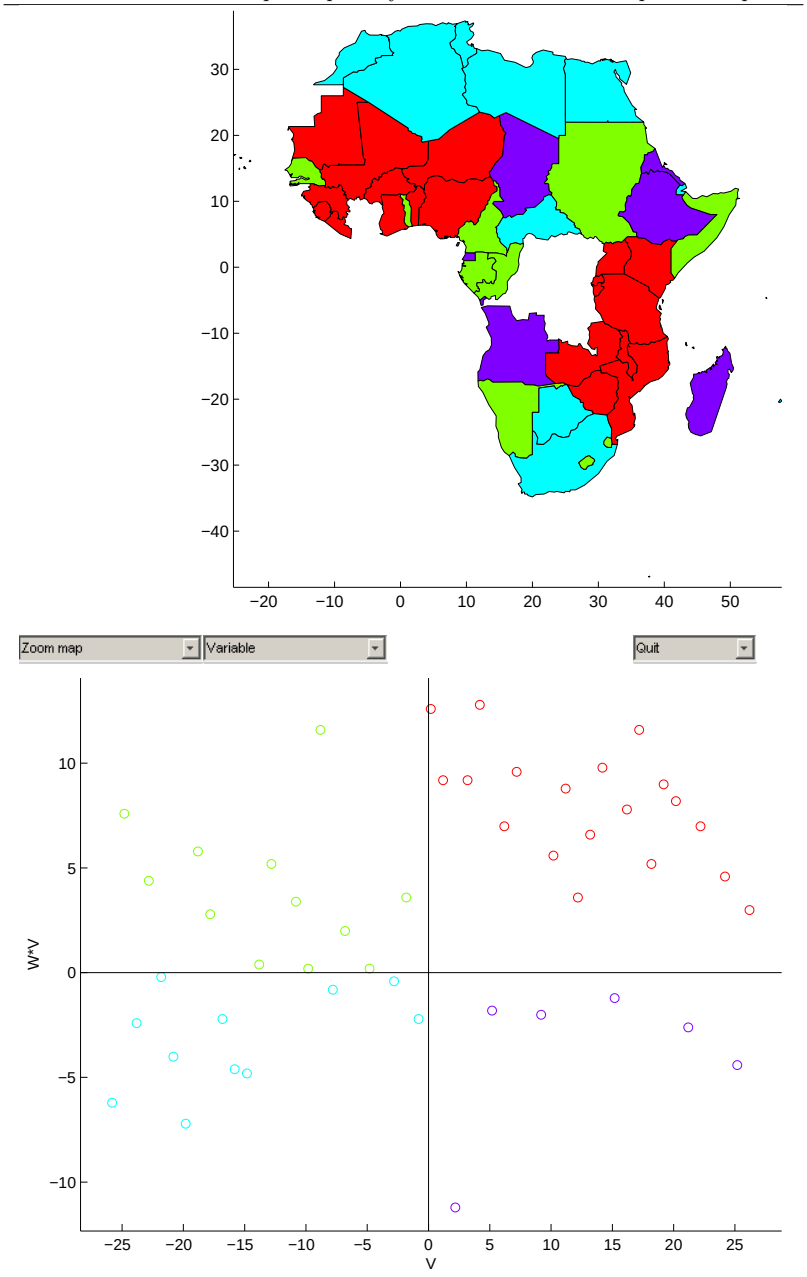
The following plots were generated in Matlab using Arc Mat ESRI reader and The Spatial Econometrics Toolbox [4]. The Moran Scatter plot provides a visual representation of spatial autocorrelation. The corresponding map shows the groupings. The plot relates the variable to the spatial z-score.

Linear Association

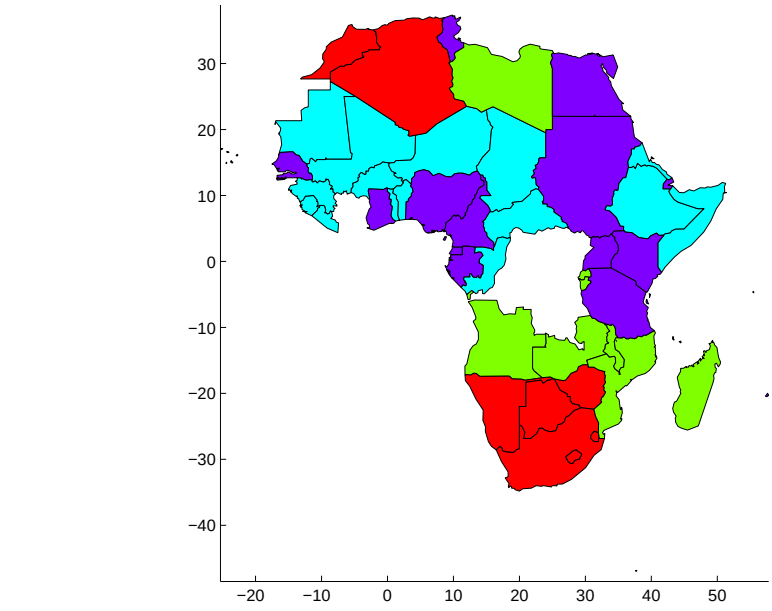
- + (Green)	+ + (Red)
- - (Cyan)	+ - (Blue)

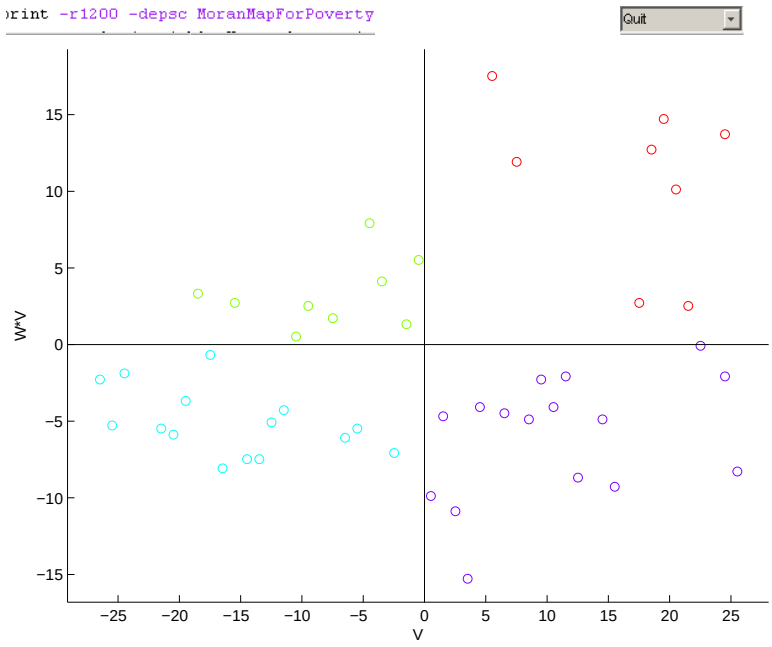
[4] Arc Mat, a Matlab toolbox for using ArcView Shape files for spatial econometrics and statistics James P. LeSage, R. Kelley Pace

Moran autocorrelation maps for poverty rank data. Red here is poor near poor.



Moran autocorrelation maps for development rank data. Red here is a development hot spot.





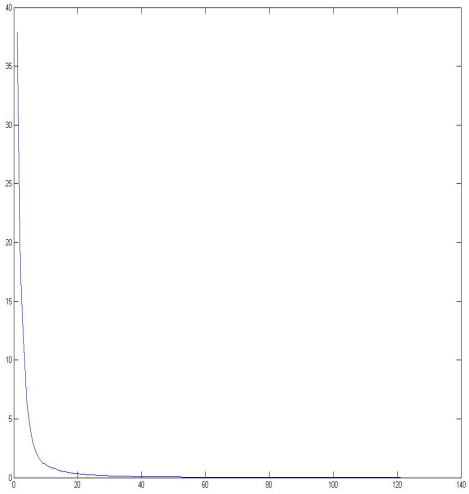
Chapter 5

Financial Engineering

5.1 Modeling High Dimensional Financial Instruments in Real Time. How to build a TV for RUT & SPX

In statistical thermodynamics, infinite systems exhibit scaling behavior that may break down in a finite setting. A power law correlation at a critical point is limited in a finite system while the mathematics of the infinite system give a singularity at the critical point. The long range dependence in real systems breaks down across regime change. Essentially, investors forget. Annealing and hysteresis models from solid state physics could be employed to explain bubble behavior and market memory. Of particular interest are models of random systems exhibiting long range interactions, critical phenomena, and frustration. Frustration is an important concept in social systems. Many competing interests with differing utility functions making decisions under uncertainty in a complex environment can lead to bubbles, crashes, and hysteresis. An example, consider a basket of equities $B = \{S_i\}$ weighted by $\omega = \{\omega_i\}$. Suppose options are traded on B and its components, and that we have a set of exogenous driving factors for the price evolution of the basket $\{\eta_j\}$.

The images below were obtained by fitting high frequency historical pricing data to a generalized multivariate elliptical model for returns. Models were fit for daily, five minute, and tic interval.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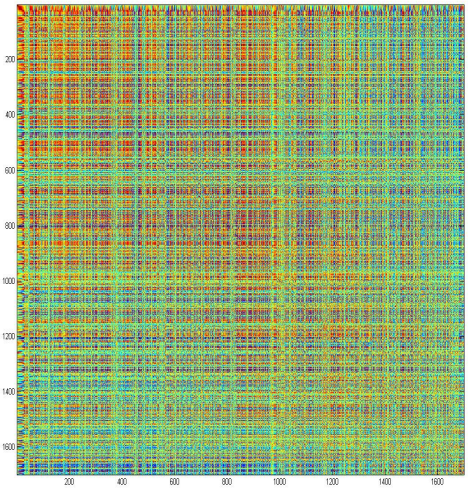
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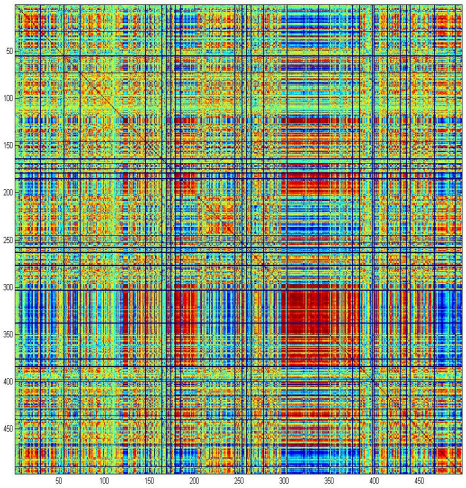
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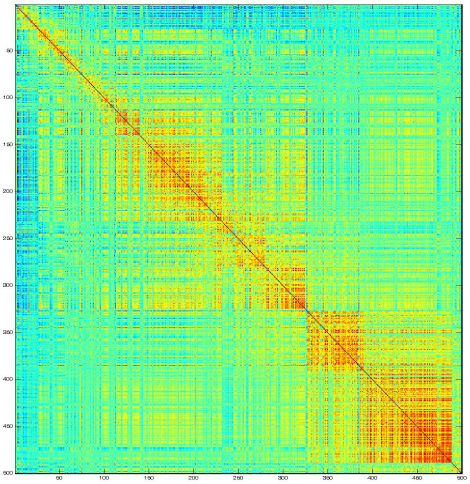
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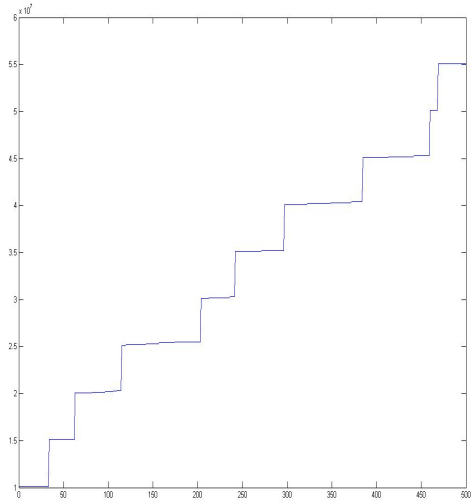
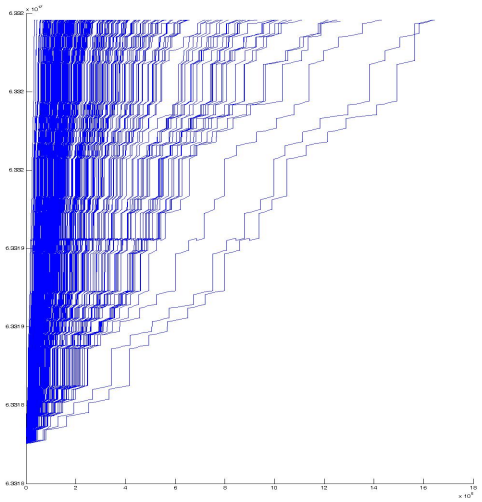




5.2 Time Series Analysis

X_t is a stochastic random process. If $E[X_t] = \mu$ and $COV[X_t, X_{t+k}] = \gamma_k$ are independent of t, then the process is second order stationary. Define The autoccorrelation function (ACF) $\rho_k = \frac{\gamma_k}{\gamma_0}$. The fourier transform of the autocovariance γ_k gives the distribution of variance over the frequency range $[0, \omega_{Nyquist}]$

5.3 Long Range Dependence, Rescaled Range, and the Hurst Exponent

Three important methods exist to model the long range dependence in $X(t)$; autocorrelation analysis, state space methods relying on embedding fractional differences (bbcsvc verify), and scaling laws. To form the rescaled range statistic, for a series of partial sums of scaled variances. $\widehat{\mu}(N,t_o) = \frac{1}{N}\sum_{t=t_o+1}^{t=t_o+N} X(t)\widehat{\sigma}^2(N,t_o) = \frac{1}{N}\sum_{t=t_o+1}^{t=t_o+N}(X(t)-\widehat{\mu}(N,t_o))^2$ now define the range for the scale τ by forming the partial sums of the deviations from the mean at time t_o resolution at scale N $Y(N,t_o,\tau) = \sum_{t=t_o+1}^{t=t_o+\tau} X(t) - \widehat{\mu}(N,t_o)\ \forall\ t \in [1,N]$ Compute the range from $max_\tau Y(N,t_o,\tau) - min_\tau Y(N,t_o,\tau)$ The RS statistic at t is obtained by rescaling the range of the process at a scale N by the variance at that scale and time;

$$RS(N) = \frac{\sum_{t=t_o+1}^{t=t_o+N} R(N,t_o)}{\sum_{t=t_o+1}^{t=t_o+N} \widehat{\sigma^2}(N,t_o)} \tag{5.3.1}$$

. Assuming a scaling law exists, $RS(N) \approx (aN)^H$ This is the same type of law defined for phase transitions in solid state statistical physics. Refer to the sections on simulated annealing. When $H = \frac{1}{2}$ we have standard Brownian motion, $H \in [0,\frac{1}{2})$ implies a mean reverting process, and $H \in (\frac{1}{2},1]$ means a long range dependence exists in the data.

Long Range Dependence (LRD) can be defined for non-Gaussian stable processes $X(t) = Y(t) + \varepsilon_t$ where ε is the noise. The process $X(t)$ is LRD if $\rho_{t,s}) = Corr(\varepsilon_t,\varepsilon_s) \sim |t-s|^{-H}(t-s) \to \infty$. Alternatively LRD can be defined if the characteristic function $\mathcal{F}(\varepsilon)(\omega) = \int_{-\infty}^{\infty} p(x)e^{-2\pi\omega x}dx$ has a pole at zero in (C) The characteristic function plays an important role in proving limit theorems and deriving estimators for LRD processes.

The approach outlined above has similarities with methods of solid state physics. In physics at the critical point for an infinite system, there exists discontinuities in the correlation function — long range correlations] and to get convergence in the calculations (bbcrevisit which)] a renormalizing transform is applied to bbcrevisit.

If $Y_i,\ldots,Y_n$ iid $N(\mu,\sigma)$ and $R = max_i Y_i - min_i Y_i$, $S^2 = \widehat{\sigma}^2$. S_i, Y_i are independent and we can forma new random variable called the Studentized Range, $Q_{n,\nu} = \frac{R}{S}$ It can be used in a multiple comparison settings. When $H = frac12$ above,

A probability space (Ω,F,P)

Refs: Dacorgna, Muller GARCH, HARCH IEMA

For FBM we have no Ito calculus. Care has to be taken when defining stochastic integrals. Many approaches exist, and no consensus yet exists on how to properly define such integral. The difficulties arise in the range $H \in [0,\frac{1}{2})$ where the sample path properties are more irregular than standard Brownian motion. By care, we mean deep results form measure theory and analysis need to be applied. Ref Embrechts Selfsimilar Processes.

5.4 Copula's

A copula is a multivariate cumulative distribution function on $[0,1]^n$ such that the marginals are uniform on $[0,1]$. Sklar's theorem provides

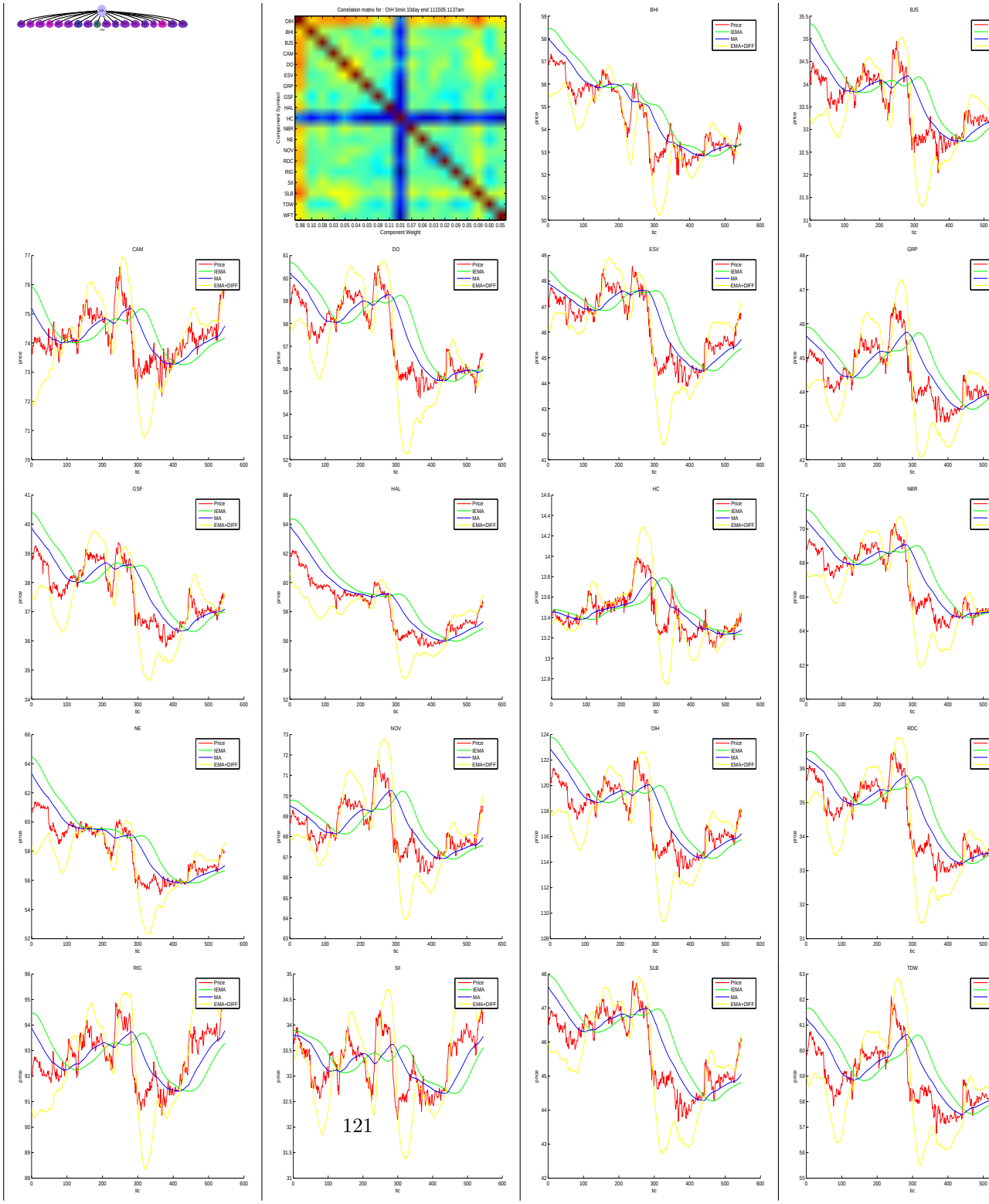
5.5 Risk

Risk is classified generally as credit, market, liquidity, and operational. Credit risk measures the potential loss due to the inability of a counterparty to meet obligations. Credit risk is classified as that due to credit exposure, the provability of counterparty default, and the

losses given counterparty default. Liquidity risk is caused by an unexpected large and stressful negative cash flow over a short period. Liquidity risk is assumed by individual firms and market participants. Firm or market participant in possession of illiquid assets may have to sell at a discount to meet cash flow of margin requirements. Market risk relates to the uncertainty of future earnings due to market conditions.

VaR is not coherent unless the underlying price process is normal. The expected shortfall ES is a more robust risk measure since it takes into account the information in the tail of the price process. VaR calculation requires marking to market, estimation of the distribution of return process, and finding a risk measure to calculate the actual risk. VaR in practice is typically calculated explicitly via assumptions such as that specified in the Basil accord, via Monte Carlo Simulation of future returns, or with historical market data. The models of [**?**] and [**?**] are combined to form the affine jump diffusion framework of Duffie, Pan, and Singleton (2000).

5.5.1 Iterated Exponential Filtering of high frequency TS data



Chapter 6

Copy Text From Literature

6.1 Big Data

We live in an era of "Big Data": science, engineering and technology are producing increasingly large data streams, with petabyte and exabyte scales becoming increasingly common. In scientific fields such data arise in part because tests of standard theories increasingly focus on extreme physical conditions (cf., particle physics) and in part because science has become increasingly exploratory (cf., astronomy and genomics). In commerce, massive data arise because so much of human activity is now online, and because business models aim to provide services that are increasingly personalized.

The Big Data phenomenon presents opportunities and perils. On the optimistic side of the coin, massive data may amplify the inferential power of algorithms that have been shown to be successful on modest-sized data sets. The challenge is to develop the theoretical principles needed to scale inference and learning algorithms to massive, even arbitrary, scale. On the pessimistic side of the coin, massive data may amplify the error rates that are part and parcel of any inferential algorithm. The challenge is to control such errors even in the face of the heterogeneity and uncontrolled sampling processes underlying many massive data sets. Another major issue is that Big Data problems often come with time constraints, where a high-quality answer that is obtained slowly can be less useful than a medium-quality answer that is obtained quickly. Overall we have a problem in which the classical resources of the theory of computatione.g., time, space and energytrade off in complex ways with the data resource.

Various aspects of this general problem are being faced in the theory of computation, statistics and related disciplineswhere topics such as dimension reduction, distributed optimization, Monte Carlo sampling, compressed sampling, low-rank matrix factorization, streaming and hardness of approximation are of clear relevancebut the general problem remains untackled. This program will bring together experts from these areas with the aim of laying the theoretical foundations of the emerging field of Big Data.

6.1.1 Parallel Optimization

Parallel and Distributed Algorithms for Inference and Optimization

Michael Mahoney (Stanford University; chair), Guy Blelloch (Carnegie Mellon University), John Gilbert (UC Santa Barbara), Chris R (Stanford University), Martin Wainwright (UC Berkeley).

Recent years have seen dramatic changes in the architectures underlying both large-scale and small-scale data analysis environments. For example, distributed data centers consisting of clusters of a large number of commodity machines, so-called cloud-computing platforms, and parallel multi-core architectures are all increasingly common. This, coupled with the

computations that are often of interest in large-scale analytics applications, presents fundamental challenges to the way we think about efficient and meaningful computation in the era of large-scale data. For example, when data are stored in a distributed manner, computation is often relatively-inexpensive, and communication, i.e., actually moving the data, is often the most precious computational resource. Alternatively, suboptimal solutions to optimization problems often lead to better behavior in downstream applications than optimal solutions. This workshop will address the state-of-the-art as well as novel future directions in parallel and distributed algorithms for large-scale data analysis applications. In addition to focusing on algorithmic questions, e.g., whether and how particular computations can be parallelized, the workshop will take a coordinated approach to exploring the many ties between large-scale learning and distributed optimization.

6.2 Nyström Method

Matrix Coherence and the Nystrom Method Ameet Talwalkar The Nystrom method is an efficient technique used to speed up large-scale learning applications by generating low-rank approximations. Crucial to the performance of this technique is the assumption that a matrix can be well approximated by working exclusively with a subset of its columns. In this work we relate this assumption to the concept of matrix coherence, connecting coherence to the performance of the Nystrom method. Making use of related work in the compressed sensing and the matrix completion literature, we derive novel coherence-based bounds for the Nystrom method in the low-rank setting. We then present empirical results that corroborate these theoretical bounds. Finally, we present more general empirical results for the full-rank setting that convincingly demonstrate the ability of matrix coherence to measure the degree to which information can be extracted from a subset of columns. Modern problems in computer vision, natural language processing, computational biology and other areas often involve datasets containing millions of training instances. However, several standard methods in machine learning, such as spectral clustering (Ng et al., 2001), manifold learning techniques (de Silva and Tenenbaum, 2003; Scholkopf et al., 1998), kernel ridge regression (Saunders et al., 1998) or other kernel-based algorithms do not scale to such orders of magnitude. In fact, even storage of the matrices associated with these datasets can be problematic since they are often not sparse and hence the number of entries is extremely large. As shown by Williams and Seeger (2000), the Nystrom method provides an attractive solution when working with large-scale datasets by operating on only a small part of the original matrix to generate a lowrank approximation. The Nystrom method has been shown to work well in practice for various applications ranging from manifold learning to image segmentation (Fowlkes et al., 2004; Platt, 2004; Talwalkar et al., 2008; Zhang et al., 2008). The effectiveness of the Nystrom method hinges on two key assumptions on the input matrix, G. First, we assume that a low-rank approximation to G can be effective for the task at hand. This assumption is often true empirically as evidenced by the widespread use of singular value decomposition (SVD) and principal component analysis (PCA) in practical applications. As expected, the Nystrom method is not appropriate in cases where this assumption does not hold, which explains its poor performance in the experimental results of Fergus et al. (2009). Previous work analyzing the performance of the Nystrom method incorporates this low-rank assumption into theoretical guarantees by comparing the Nystrom approximation to the best low-rank approximation, i.e., the approximation constructed from the top singular values and singular vectors of G (see Section 2 for further discussion) (Drineas and Mahoney, 2005; Kumar et al., 2009a). The second crucial assumption of the Nystrom method involves the sampling-based nature of the algorithm, namely that an accurate low-rank approximation can be generated exclusively from information extracted from a small subset of l n columns of G. This assumption is not generally true for all matrices. Michael Mahoney - videolectures.net Statistical Leverage Given an m x n matrix A and a rank parameter k, define the leverage of the i-th row of A to be the i-th diagonal element of the projection matrix onto the span of the top k left singular vectors of A. In this case, "high leverage" rows have a disproportionately large amount of the "mass" in the top singular vectors. Historically, this statistical concept (and generalizations of it) has found extensive applications in diagnostic regression analysis. Recently, this concept has also been

central in the development of improved randomized algorithms for several fundamental matrix problems that have broad applications in machine learning and data analysis. Two examples of the use of statistical leverage for improved worst-case analysis of matrix algorithms will be described. The first problem is the least squares approximation problem, in which there are n constraints and d variables. Classical algorithms, dating back to Gauss and Legendre, use O(nd2) time. We describe a randomized algorithm that uses only O(n d log d) time to compute a relative-error, i.e., 1+/-epsilon, approximation. The second problem is the problem of selecting a "good" set of exactly k columns from an m x n matrix, and the algorithm of Gu and Eisenstat provides the best previously existing result. We describe a two-stage algorithm that improves on their result. Recent applications of statistical leverage ideas in modern large-scale machine learning and data analysis will also be briefly described. This concept has proven to be particularly fruitful in large data applications where modeling decisions regarding what computations to perform are made for computational reasons, as opposed to having any realistic hope that the statistical assumptions implicit in those computations are satisfied by the data.

Chapter 7

Appendix Numerical Linear Algebra

$M_{m,n}(\mathbb{F})$ denotes the vector space of matrices over the field $\mathbb{F}$.

$$A \in M_{n,n}(\mathbb{F})\ b \in \mathbb{F}^n,\ \exists\ x \ni\ Ax = b\ \texttt{iff}\ det(A) = 0$$

$\mathbf{X} \in M_{mn}(\mathbb{R})$ is positive definite if $v^t X v > 0\ \forall\ v \in \mathbb{R}^n$. We can construct positive definite symmetric matrices by by forming $\mathbf{X}^t\mathbf{X}$ where $\mathbf{X}$ is an orthogonal (full rank) matrix.

Horner's rule is a method for evaluating a polynomial at a point in $O(n)$ time. Straightforward evaluation of a n degree polynomial is done in $O(n^2)$ time. Simply rewrite the function $f(x) = \sum_{i=0}^{n-1} a_i x^i$ as $f(x) = (\ldots(a_{n-1}x + a_{n-2})x + \ldots + a_1)x + a_0$

7.1 The min max characterization of eigenvalues

This is a variational characterization of the eigenvalues of compact operators on a Hilbert space. Let $H\ \ni\ H = H^\dagger$ The Rayleigh quotient is defined by $R(x) = \frac{<Hx,x>}{\|x\|^2}$ Let $\sigma(H) = \lambda_i$, then $\forall S_k \in \mathbb{R}^n$ we have

$$\underset{x \in S_k \|x\|=1}{max} < Hx, x > \geq \lambda_k$$

which implies

$$\underset{S_k}{inf} \underset{x \in S_k \|x\|=1}{max} < Hx, x > \geq \lambda_k$$

Equality above is achieved when $S_k = spn\mu_k$ where μ_k is the kth eigenvector of H.

The min max theorem is that

$$\lambda_1 \leq R(x) \leq \lambda_n$$

7.2 The Discrete Fourier Transform on $\ell^2(\mathbb{Z}_{N_1})$

Let $z \in \mathbb{Z}_{N_1}, z = (z(0), z(1), \ldots, z(N_1 - 1))$. We index from 0 instead of 1 for convenience of presenting the FFT. Define

$$\widehat{z(m)} = \sum_{k=0}^{N_1-1} z(k) e^{\frac{-2\pi\imath km}{N_1}}$$

The map$\hat{}: \ell^2(\mathbb{Z}_{N_1}) \to \ell^2(\mathbb{Z}_{N_1})$ is the Fourier Transform. The vectors

$$E_0, E_1, \ldots, E_{N_1-1} : E_m(n) = \frac{e^{\frac{-2\pi\imath mn}{N_1}}}{\sqrt{N_1}}$$

form an orthonormal basis for $\ell^2(\mathbb{Z}_{N_1})$. The vectors $\frac{E_0}{N_1}, \frac{E_1}{N_1}, \ldots, \frac{E_{N_1-1}}{N_1}$ form an orthogonal basis called the Fourier Basis.

Extend the indices over $\mathbb{Z}_{N_1}$ to $\mathbb{Z}$ by considering $\mathbb{Z}_{N_1}$ to be the algebraic group $\mathbb{Z} mod N_1$. Then we can define the translation operator

$$(R_l z)(n) = z(n-l).$$

We can also define the convolution operator with this extended notion of $\mathbb{Z}_{N_1}$;

$$z * w = \sum_{k=0}^{N_1-1} z(m-n)W(n)$$

The Fourier Multiplier Operator $T_{(m)}$ where $m \in \ell_2\mathbb{Z}_{N_1}$ is given by

$$T_{(m)} = (m\hat{z})\check{}$$

Fourier Inversion Formula:

$$z(m) = \frac{1}{N_1} \sum_{k=0}^{N_1-1} z\hat{(k)} e^{\frac{2\pi\imath km}{N_1}}$$

Parsevall's Relation:

$$< z, w >= \frac{1}{N_1} < \hat{z}, \hat{w} >$$

Plancherel's Formula: Parsevall's relation with $w = z$.
Representation in the Fourier Basis:

$$z = \sum_{k=0}^{N_1-1} z\hat{(k)} F_k$$

The effect of the translation operator is to rotate the phase of the Fourier Transform:

$$(R_l z)\hat{(k)} = e^{\frac{2\pi\imath kl}{N_1}} \widehat{z(k)}$$

The effect of conjugation is to reflect the Fourier Transform:

$$(\overline{z})\hat{(k)} = \overline{\hat{z}(-k)}$$

The Convolution Operator is equivalent to a Fourier Multiplier Operator:

$$b * z = (m\hat{z})\check{}: m = \hat{b}$$

7.3 Multiresolution analysis

Basis functions of a linear subspace $V_j \subset L^2(\Omega)$ are defined by a scaling function ϕ via the following procedure;

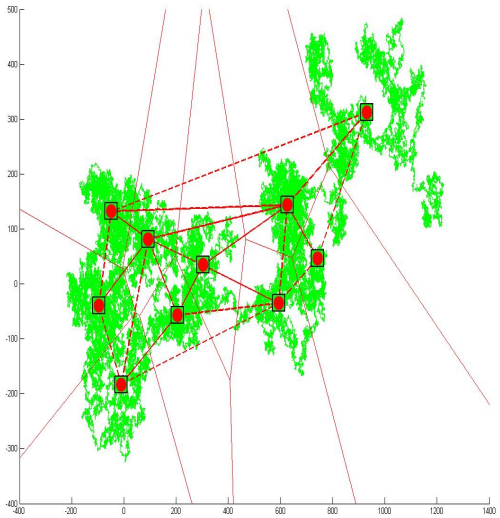
$$\phi_{ij}(x) = \phi(2^{-j}x - i)$$
$$V_j = span\{\phi_{ij}\}$$
$$W_{j+1} = V_j \; V_{j+1}^{\perp}$$
$$\ldots V_{j+1} \subset V_j \subset \ldots \subset V_0 \subset \ldots V_{-j} \subset \ldots$$
$$V_j = V_{j+1} \oplus W_{j+1} x \in W_{j+1} \Rightarrow \; \exists \, a_l \;\; x = \sum_l \{a_l\}\phi_{jl}$$

A basis for W_{ij} is constructed from a mother wavelet ψ.

7.4 Voroni Tesselations

A centroidal Voronoi tessellation is a Voronoi tessellation where the generating points are the centroids of the corresponding regions. Applications Voronoi tessellations can be found in image compression, clustering, quadrature, and finite difference methods. distribution of resources. The dual of the Voroni tessellation in $\mathbb{R}^2$ is the Delaunay triangulation.

The example below is a simulated example of resource allocation in $\mathbb{R}^2$ A partition of a Random walk in $\mathbb{R}^2$ obtained by calculating the Voroni tessellation and associated Delaunay triangulation on the k-means centroids.



7.5 The Matrix Exponential

The matrix exponential of a matrix $\mathbf{A}$ is defined as

$$\begin{aligned} e^{\mathbf{A}} &= \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}. \end{aligned}$$

The Pade approximation to $e^{\mathbf{A}}$ is

$$e^{\mathbf{A}} \approx R(\mathbf{A}),$$

with

$$R_{pq}(\mathbf{A}) = (D_{pq}(\mathbf{A}))^{-1} N_{pq}(\mathbf{A})$$

where

$$D_{pq}(\mathbf{A}) = \sum_{j=1}^{p} \frac{(p+q-j)!p!}{(p+q)!j!(p-j)!} \mathbf{A}^j$$

and

$$N_{pq}(\mathbf{A}) = \sum_{j=1}^{q} \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!}\,\mathbf{A}^j.$$

See [32] for a detailed accounting of this and other matters regarding the calculation of the matrix exponential.

Chapter 8

Software

Computers are getting more powerful over time but size of the problems we're solving scales with the increased performance. Tools for acquiring and storing data are improving at an even faster pace than processors. It turns out the communication is the real bottleneck to scaling many algorithms. The capacity of fast memory close to the computing resource (cache) grows very slowly in time.

8.1 BLAS

8.2 Atlas

8.3 MKL

8.4 fftw

8.5 Graphviz - Graph Visualization Software

Graphviz (http://www.graphviz.org) is open source graph visualization software. Graph visualization is a way of representing structural information as diagrams of abstract graphs and networks. It has important applications in networking, bioinformatics, software engineering, database and web design, machine learning, and in visual interfaces for other technical domains.

Graphviz is used to generate collaboration, inheritance, and call diagrams the KL documentation. There is an API that is used in the KL framework to facilitate graph visualization.

8.6 ARPACK

ARPACK++ is an object-oriented version of the Fortran ARPACK package. ARPACK is designed to compute a few eigenvalues and eigenvectors of large scale sparse matrices and pencils via the Arnoldi process for finding eigenvalues called. These methods utilize Krylov Subspace Projections for iterative solution that avoids matrix multiplication. ARPACK implements the implicit restarted Arnoldi method which reduces the storage requirements of the traditional Lanczos iteration for Hermitian matrices and Arnoldi iteration for general matrices. The key to the Krylov method is to calculate the linear subspace of $\mathbb{R}^{(n,n)}$ induced by span of the first m powers of the image of b under a linear operator A, $\kappa_m(A,b)|A \in \mathbb{R}^{(n,n)} b \; in \mathbb{R}^n =$

$\{b, Ab(A)^2b, \ldots (A)^m b\}$. This avoids direct matrix matrix operations when finding the first few eigenvector, eigenvalue pairs in a large system of linear equations.

8.7 ATLAS

Automatically Tuned Linear Algebra software.

8.8 METIS

METIS is a software library for finite element analysis and graph partitions. It also can be used to reduce the fill order of sparse matrices.

8.9 SDPA

SDPA is a software library for solving SDPs using on the Mehrotra-type predictor-corrector infeasible primal-dual interior-point method. It is implemented C++ language and utilizes the machine dependent BLAS such as Intel MKL, ATLAS. LAPACK routines are used for matrix computations. Efficient methods to compute the search directions exploiting the sparsity of the data matrices are implemented. Sparse or dense Cholesky factorization for the Schur complemetn matrix is automatically selected. The calculation of the Schur complement matrix is implemented in reentrant code. A sparse version of SDPA is available that uses METIS and SPOOLES libraries for finding a proper sparse structure of the problem.

8.10 SPOOLS

SPOOLES is a library for solving sparse real and complex linear systems of equations. SPOOLES can factor and solve square linear systems of equations with symmetric structure, and it can compute multiple minimum degree, generalized nested dissection and multisection orderings of matrices with symmetric structure. SPOOLES utilizes a variety of Krylov iterative methods. The preconditioner is a drop tolerance factorization.

8.11 SuperLU

SuperLU (http://crd-legacy.lbl.gov/ xiaoye/SuperLU/) is a general purpose library for the direct solution of large, sparse, nonsymmetric systems of linear equations on high performance machines. The library is written in C and is callable from either C or Fortran. The library routines will perform an LU decomposition with partial pivoting and triangular system solves through forward and back substitution. The LU factorization routines can handle non-square matrices but the triangular solves are performed only for square matrices. The matrix columns may be preordered (before factorization) either through library or user supplied routines. This pre-ordering for sparsity is completely separate from the factorization. Working precision iterative refinement subroutines are provided for improved backward stability. Routines are also provided to equilibrate the system, estimate the condition number, calculate the relative backward error, and estimate error bounds for the refined solutions.

8.12 SuiteSparse

Tim Davis' (http://www.cise.ufl.edu/ davis/welcome.html) collection of sparse matrix software. Tim is also the curator of The University of Florida Sparse Matrix Collection (http://www.cise.ufl.edu/research/sparse/matri
a must see for anyone interested in sparse matrices and visualization.

AMD: symmetric approximate minimum degree BTF: permutation to block triangular form CAMD: symmetric approximate minimum degree CCOLAMD: constrained column approximate minimum degree COLAMD: column approximate minimum degree CHOLMOD: sparse supernodal Cholesky factorization and update/downdate CSparse: a concise sparse matrix package CXSparse: an extended version of CSparse KLU: sparseLU factorization, for circuit simulation LDL: a simple LDL^T factorization UMFPACK: sparse multifrontal LU factorization RBio: MATLAB toolbox for reading/writing sparse matrices UFconfig: common configuration for all but CSparse SuiteSparseQR: multifrontal sparse QR

8.12.1 AMD

AMD is a set of routines for pre-ordering a sparse matrix prior to numerical factorization. It uses an approximate minimum degree ordering algorithm to find a permutation matrix P so that the Cholesky factorization $PAP^\dagger = LL^\dagger$ has fewer (often much fewer) nonzero entries than the Cholesky factorization of A. The algorithm is typically much faster than other ordering methods and minimum degree ordering algorithms that compute an exact degree . Some methods, such as approximate deficiency [Rothberg and Eisenstat 1998] and graph-partitioning based methods [Hendrickson and Rothberg 1999; Karypis and Kumar 1998; Pellegrini et al. 2000; Schulze 2001] can produce better orderings, depending on the matrix. The algorithm starts with an undirected graph representation of a symmetric sparse matrix . Node i in the graph corresponds to row and column i of the matrix, and there is an edge (i, j) in the graph if a_{ij} is nonzero. The degree of a node is initialized to the number of off diagonal non-zeros in row i, which is the size of the set of nodes adjacent to i in the graph.

8.12.2 UMFPACK

UMFPACK is a set of routines for solving systems of linear equations, $Ax = b$, when A is sparse and unsymmetric. It is based on the Unsymmetric-pattern MultiFrontal method. UMFPACK factorizes PAQ, $PRAQ$ and $PR^{-1}AQ$, into the product LU, where L and U are lower and upper triangular, respectively, P and Q are permutation matrices, and R is a diagonal matrix of row scaling factors (or $R = I$ if row-scaling is not used). Both P and Q are chosen to reduce fill-in (new nonzeros in L and U that are not present in A). The permutation P has the dual role of reducing fill-in and maintaining numerical accuracy (via relaxed partial pivoting and row interchanges). The sparse matrix A can be square or rectangular, singular or non-singular, and real or complex (or any combination). Only square matrices A can be used to solve $Ax = b$ or related systems. Rectangular matrices can only be factorize

Chapter 9

Appendix Statistics

9.1 Testing for normality and other distributions

Powerful inference methods can be employed when data is generated by a Gaussian process. This section describes techniques for testing the normality of a sample and comparing two samples.

Kolmogorov-Smirnov test uses the fact that the empirical cumulative distribution function is normal in the limit. It is a non-parametric and distribution free test. Given the empirical distribution

$$F_n(x) = \frac{1}{n}\sum_{n}^{i=1}\begin{cases} 1 : x_i \leq x \\ 0 : x_i > x \end{cases}$$

, and a test CDF

$$F(x)$$

the K-S test statistics are $D_n^+ = max(F_n(x) - F(x))$ and $D_n^- = min(F_n(x) - F(x))$ The generality of this test comes at a loss in precision near the tails of a distribution. The K-S statistics are more sensitive near points close to the median, and are only valid for continuous distributions.

The Kuipers test uses the statistic $D_n^+ + D_n^-$ and is useful for detecting changes in time series since the statistic is invariant in ???? transformation of the dependent variable F_n.

The Anderson-Darling test is based on the K-S test and uses the specific distribution to specify the ????critical values??? of the test.

The chi-squared is based on the sample histogram and allows comparison against a discrete distribution, but has the potential drawback of being sensitive to how the histogram is binned and requires more samples to be valid.

The Shapiro-Wilk test uses the expected values of the order statistics of $F(x)$ to calculate the test statistic. It is sensitive to data that are very close together, and numerical implementations may suffer from a loss of accuracy for large sample sizes.

K-S [Chakravarti, Laha, and Roy, (1967). Handbook of Methods of Applied Statistics, Volume I, John Wiley and Sons, pp. 392-394].

Shapiro-Wilk [Shapiro, S. S. and Wilk, M. B. (1965). "An analysis of variance test for normality (complete samples)", Biometrika, 52, 3 and 4, pages 591-611.]

DAgostino-Pearson

9.2 Regression Methods

Standard least squares regression consists in fitting a line through the data points (training points in learning theory) that minimizes the sum of square residuals. The underlying assumption is that the data and the response can be modeled by a linear relationship. In the

event that the model accurately captures the functional dependence of the response generated by the data, and under the assumptions that the data is corrupted by Gaussian noise, precise statistical inferences can be made on the model parameters.

Modifications to this standard model include nonlinear mapping of the input data, local fitting, biased estimators, subset selection, coefficient shrinking, weighted least squares, and basis expansion transformations.

9.3 Generalized Linear Models

Suppose we have n observations of k dimensional data denoted $\{x_i\}_{i=1}^{k}$ and for each observation we have a response y_i. We wish to fit the observations to the responses. Generalized Linear Regression is a modeling technique that allows for non normal distributions and models non-linear relationships in the training data. M-estimators are used to fit a generalized linear model Ref Huber (1964).

A linear model $Y = \Lambda(X) = X\beta + \epsilon$ fits a linear relationship between the dependent variables Y_i and the predictor variables X_i

$$Y_i = \Lambda(X_i) = b_o + b \circ X_i. \tag{9.3.1}$$

A generalized linear model $Y = g(\Lambda(X)) + \epsilon$ fits the data to $Y = g(X \circ W)$. Fitting the model consists of minimizing the objective function $\sum_{i=1}^{n} g(e_i) = \sum_{i=1}^{n} g(y_i - x_i\beta)$, where e_i are the residuals $y_i - x_i\beta$. We see that for ordinary least squares $g(e_i) = e_i^2$, and the usual matrix equations fall out by differentiating with respect to β. Carrying this out for general g

$$\sum_{i=1}^{n} \frac{\partial g(y_i - x_i\beta)}{\partial \beta} = 0 \tag{9.3.2}$$

gives the system of $k+1$ equations to solve for estimating the coefficients b_i. If we set$\alpha(x) = \frac{g'(x)}{x}$ and calculate the derivative above, we have to solve

$$\sum_{i=1}^{n} \omega(e_i)(y_i - x_i\beta)x_i = 0 \tag{9.3.3}$$

Which gives rise to a weighted least squares where the weights depend on the residuals - which depend on the coefficients - which depend on the weights. This suggests an iterative algorithm;

$$\beta^{\tau} = (X^t W^{(\tau-1)} X)^{-1} X^t W^{\tau-1} y \tag{9.3.4}$$

where $W_{ij}^{(\tau-1)} = \alpha(e_i^{(\tau-1}))$.

Several parameterizations are popular for the exponential family. The most general form of the distribution

$$p(x,\theta) = f(x,\theta)e^{g(x,\theta)} \in C^2(\mathbb{R} \otimes \mathbb{R}) \otimes C^2(\mathbb{R} \otimes \mathbb{R})$$

. The estimators derived below assume that f and g are separable,

$$p(x,\theta) = f(x)h(\theta)e^{\alpha(x)\beta(\theta)} \in C^2(\mathbb{R}) \otimes C^2(\mathbb{R}) \otimes C^2(\mathbb{R}) \otimes C^2(\mathbb{R})$$

.

From

$$\int_{x=-\infty}^{x=+\infty} p(x,\theta)dx \; = 1$$

we get

$$\frac{d}{d\theta}\, p(x,\theta) = 0 = \frac{d^2}{d^2\theta}\, p(x,\theta)d$$

Since the parametrization we have chosen for the exponential family allows, in the sequel we drop the notation for dependent variable and denote the derivative with a prime.

$$\frac{d}{d\theta}\, p(x,\theta) = \frac{d}{d\theta}\, fhe^{\alpha\beta} = h'fe^{\alpha\beta} + fh\alpha\beta' e^{\alpha\beta} = \big(\frac{h'}{h} + \alpha\beta'\big)p(x,\theta)$$

which gives

$$\int \frac{d}{d\theta} p(x,\theta)dx = \int \big(\frac{h'}{h} + \alpha\beta'\big)p(x,\theta)dx = \frac{h'}{h}\int p(x,\theta)dx + \beta' \int \alpha(x)p(x,\theta)dx = \frac{h'}{h} + \beta' E[\alpha(x)]$$

so that

$$E[\alpha(x)] = -\frac{h'}{h\beta'}$$

. Continuing along this vein,

$$0 = \int \frac{d^2}{d^2\theta}\, p(x,\theta)dx = \int \frac{d}{d\theta}\, \big(\frac{h'}{h} + \alpha\beta'\big)p(x,\theta)dx =$$

$$\int \big(\frac{h''}{h} - \frac{(h')^2}{h^2} + \alpha\beta''\big)p(x,\theta) + \big(\frac{h'}{h} + \alpha\beta'\big)\frac{d}{d\theta}\, p(x,\theta)\, dx =$$

$$\int \big(\frac{h''}{h} - \frac{(h')^2}{h^2} + \alpha\beta''\big)p(x,\theta) + \big(\frac{h'}{h} + \alpha\beta'\big)^2 p(x,\theta)\, dx =$$

$$\int \big(\frac{h''}{h} - \frac{(h')^2}{h^2} + \alpha\beta''\big)p(x,\theta) + \big(\frac{h'}{h} + \alpha\beta'\big)^2 p(x,\theta)\, dx =$$

$$\int \big(\frac{h''}{h} - \frac{(h')^2}{h^2} + \alpha\beta''\big)p(x,\theta) + (\alpha\beta' - E[\alpha(x)]\beta')^2 p(x,\theta)\, dx$$

Keeping in mind that

$$Var[ax] = E[(ax - E(ax)^2] = a^2 E[(x - E[x])^2] = a^2 Var[x]$$

we get the variance via

$$\big(\frac{h''}{h} - \frac{(h')^2}{h^2} + E[\alpha(x)]\beta''\big) + Var[\alpha(x)\beta'(\theta)] = \big(\frac{h''}{h} - \frac{(h')^2}{h^2} + E[\alpha(x)]\beta''\big) + (\beta')^2 Var[\alpha(x)] = 0$$

The score $U(x)$ is given by

$$U(x) = \frac{\partial}{\partial\theta}\, L(\theta,x) = \frac{\partial}{\partial\theta}\, \log\, p(x,\theta) = \frac{\partial}{\partial\theta}\, \big(\log h(\theta) + \log f(x) + \alpha(x)\beta(\theta)\big) = \frac{h'}{h} + \alpha\beta'$$

so

$$E[U(x)] = \beta' E[\alpha(x)] + \frac{h'}{h} = 0$$

. The Fisher Information $\mathcal{F}$ is defined

$$\mathcal{F} = Var[U(x)] = Var[\alpha\beta' + \frac{h'}{h}] = Var[\alpha\beta']$$

So from above we have

$$Var[U(x)] = Var[\alpha\beta'] = \big(-\frac{h''}{h} + \frac{(h')^2}{h^2} - E[\alpha(x)]\beta''\big)$$

. Now differentiating,

$$\frac{d}{d\theta}\, U(\theta,x) = \frac{h''}{h} - \frac{(h')^2}{h^2} + \alpha\beta''$$

$$E[U'(\theta,x)] = \frac{h''}{h} - \frac{(h')^2}{h^2} + E[\alpha]\beta'' = \frac{h''}{h} - \frac{(h')^2}{h^2} - \frac{\beta''h'}{\beta'} = -Var[U(x)]$$

. Note that if we write the parametrization of the separable exponential family as

$$p(x,\theta) = e^{\alpha(x)\beta(\theta) + \log(f(x)) + \log(h(\theta))}$$

then,

$$\frac{d^2}{d^2\theta}\, \log(h(\theta)) = \frac{d}{d\theta}\, \frac{h'}{h} = \frac{h''}{h} - \frac{(h')^2}{h^2}$$

.

A general form of the exponential distribution

$$\rho(x;\theta) = exp(\frac{x\theta - \xi(\theta)}{\sigma})\nu(x) \tag{9.3.5}$$

has a log likelihood for a random sample $\{X_i\}_{i=1...N}$ given functionally by

$$\mathcal{L}(\theta) = \sum_{i=1}^{N} [X_i\theta - \xi(\theta) + log(\nu(X_i))] \tag{9.3.6}$$

The scale parameter σ and θ are orthogonal parameters in that E [ ] The Generalized Linear model can

ρ' is referred to as a link function in the statistical literature. If $\rho'(x) = x(1)$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ are iid $N(\mu, \sigma)$ we have multiple linear regression. In classification problems or binomial models the logit $\rho'(x) = log(x/(1-x))$ link function is used. The logit is extended to the k category case by

$$\rho'(x_i|x_j j \neq i) = log(\frac{x_i}{1 - \sum\limits_{j \neq i} x_j}) \tag{9.3.7}$$

. The posterior probability densities $p_i(?)$ bbcrevisit (or p_i the probability of observing class i) of k classes are modeled by linear functions of the input variables x_i.

9.4 Fitting the GLM

Iteratively re-weighted least squares (IRLS) is used to for fitting generalized linear models and in finding M-estimators. The objective function

$$J(\beta^{i+1}) = argmin \sum w_i(\beta)|y_i - f_i(\beta)| \tag{9.4.1}$$

is solved iteratively using a Gauss-Newton or Levenberg-Marquardt (LM) algorithm. LM is an iterative technique that finds a local minimum of a function that is expressed as the sum of squares of nonlinear functions. It is a combination of steepest descent and the Gauss-Newton method. When the current solution is far from the minimum the next iterate is in the direction of steepest descent. When the current solution is close to the minimum the next iterate is a Gauss-Newton step.

Linear least-squares estimates can behave badly when the error is not normal. Outliers can be removed, or accounted for by employing a robust regression that is less sensitive to outliers than least squares. M-Estimators were introduced by Huber as a generalization to maximum likelihood estimation. Instead of trying to minimize the log likelihood

$$L(\theta) = \sum -log(p(x_i, \theta) \tag{9.4.2}$$

Huber proposed minimizing

$$M(\theta) = \sum \rho(x_i, \theta) \tag{9.4.3}$$

where ρ reduces the effect of outliers. Common loss function are the Huber, and Tukey Bisquare. For $\rho(x) = x^2$ we have the familiar least squares loss.

M estimators arise from the desire to apply Maximum Likelihood Estimators to noisy normal data, and to model more general distributions. They provide a regression that is robust against outliers in the training set, and allow for modeling of non-Gaussian processes. When ρ above is a probability distribution, we are preforming a maximum likelihood estimation.

The Huber function which is a hybrid L^2 L^1 norm

$$\rho_\eta(e_i) = \begin{cases} \frac{e_i^2}{2} & |e_i| \leq \eta \\ \eta|e_i| - \frac{\eta^2}{2} & |e_i| > \eta \end{cases} \tag{9.4.4}$$

The Tukey Bisquare estimator is given by

$$g_\eta(e_i) = \begin{cases} \frac{\eta^2}{6}(1 - [1 - \frac{e_i}{\eta}^2]^3) & |e_i| \leq \eta \\ \frac{\eta^2}{6} & |e_i| > \eta \end{cases} \tag{9.4.5}$$

Numerical procedures for doing this calculation are the Newton-Raphson method [see the section on root finding below], and Fisher-Scoring method [replace $\frac{\partial^2 \mathcal{L}(\theta)}{\partial\theta\partial\theta^t}$ with $E[\frac{\partial^2 \mathcal{L}(\theta)}{\partial\theta\partial\theta^t}]$.

For high dimensional data, many models may be fit in an attempt to find the simplest one that can explain the data.

In the language of statistical learning theory, the choice of a norm ρ is tantamount to choosing a loss function. Restricting the admissible functions to the one parameter family of exponential probability distributions defines the capacity via a functional form of the law of large numbers. [**?**]

9.5 Feature Subset Selection (FSS)

The goal of feature selection techniques to to improve the model building process by eliminating features that do not have discriminative power. Algorithms for feature selection either rank features or create subsets of increasing optimality. FSS should be contrasted with feature extraction techniques such as PCA, LLE, or Laplacian eigenmaps. The goal of feature extraction is to transform data from a high dimensional space to a low dimensional one while preserving the relevant information.

The statistical approach to feature selection most commonly used is stepwise regression. Common optimality criteria are FS schemes the Kolmogorov-Smirnov Test ,the t-test, the f-test, the Wilks Lambda Test and Wilcoxon Rank Sum Test.

Feature subset selection (FSS) is the process of determining which measurements will be used for classification. It's important to distinguish this process from a data dimension reduction process such as PCA which requires all the original measurements to compute the projection. The better FSS algorithms are recursive

Construct a pxM basis matrix H^T and transform feature vector $x' = H^T x$.

Generalize to L^2 with smoothing splines

Smoothing spline $RSS(f,\lambda) = \sum_{i=1}^{N}(y_i - f(x_i))^2 + \lambda \int f''(t)^2 dt$. where $f \in C^2(\mathbb{R})$ This is minimized in L^2 the first term measuring closeness of fit, and the second term penalizes curvature. $\lambda \to 0$ gives any function interpolating the data points $x_{ii\in 1,...N}$ an $\lambda \to \infty$ constrains f to be linear.

9.6 Longitudinal Data Analysis

Longitudinal data analysis is the observation of multiple subjects over repeated intervals. Binary repeated responses are typically modelled with a marginal or random effects model, which will be made precise below. Marginal Models are a generalization of the GLM presented above for correlated data. Here, the correlation is inter subject across time. Statistical analysis of longitudinal data must take into account that serial observations of a subject are likely to be correlated, time may be an explanatory variable, and that missing response data my induce a bias in the results.

Let X_{ij} be time varying or fixed covariates for the binary response Y_{ij} of subject $i \in 1,...n$ at time intervals $j \in t_1,...t_m$. By convention $X_{ij} \in \mathbb{R}x\mathbb{R}^{\prime}$ where the first dimension is the intercept. The marginal model is; $logit(E(Y_{ij}|X_{ij})) = X_{ij}^{\dagger}\beta$ and enforces the assumption that the relationship between the covariates and the response is the same for all subjects. Recall that for a binary response, $E(Y_{ij}|X_{ij}) = P(Y_{ij} = 1|X_{ij})$. The random effects model takes into account that the relationship between the covariates and response varies between subjects; $logit(P(Y_{ij} = 1|X_{ij})) = X_{ij}^{\dagger}\beta_i$ If it is know that only a subset of the covariates are involved in the inter-subject variability, we can set $\beta_i = \beta + \beta_i$ and write $logit(P(Y_{ij} = 1|X_{ij}, \beta_i)) = X_{ij}^{\dagger}\beta + OX_{ij}\beta_i$ Where the kernel of $O : \mathbb{R}^n \to \mathbb{R}^{n^{\prime}}$ is the span of the covariates that do not change between subjects. If $\lambda_i =_d N(0,\sigma)$ then the difference in the parameter vectors β in the two models differ according to σ.

The GEE method of fitting the marginal model is described in: [**? ? ?**]

The Survival Analysis is a form of longitudinal analysis that takes into consideration the amount of time an observation is made on a subject.

GLM's can be used to fit discrete longitudinal hazard models derived from survival analysis, see [**?**]. [**?** **?**] generalized that approach to account for an unobserved subject heterogeneity.

[**?** **?**] applied the hazard model of [**?** **?**] to the takeover hazard of large firms. A negative relationship between dual class ownership and value is empirically known, and that relationship can be explained by the lower takeover probability of the dual class firms. Dual class entities had a higher risk for takeover, but the hazard is lower since these firms use the dual class structure to change the capital structure in a way that allows the controlling shareholders to remain in control by reducing firm value.

The proportional hazards model can be discretized, but it is important to identify whether the process is truly a discrete process. In that case the link function should be the logit as the Marginal Model above specifies, rather than the log-log function of the discretized proportional model. The difference is the modelling of a probability transition in the former case versus a rate for the latter case.

Variable selection techniques for longitudinal data are relatively limited and most seem to rely on Wald type tests. Wald tests to include a variable are based on already computed maximum likelihood values. The Rao score test is used to include a covariate in the model building process. The Wald test calculates

$$z^2 = \frac{\widehat{\beta}}{stderr} =_d \chi^2$$

The likelihood ratio statistic for comparing two models $L_0 \in L_1$

$$-2\frac{L_0}{L_1} =_d \chi^2$$

is useful for backward stepwise variable subset selection. The degrees of freedom of the of the statistic is equal to the difference in dimension of the two models.

9.7 Discretization & Sheppard's Correction

W. Sheppard (1898) Derived an approximate relationship between the moments of a continuous distribution and it's discrete approximation. This provides a transformation to statistical estimators that correct for the binning of continuous data. As the scale at which datum are collected is increased, the variance of an estimate can become biased.

It is important to assess bias caused by grouping and to correct it if necessary. The bias of the approximate maximum likelihood estimator where observations are approximated by interval midpoints $O(w^2)$, where w is the bin width. A Sheppards correction can be used to reduce the bias to order $O(w^3)$,

Signal processing engineers often have to deal with such a quantization effect when designing finite precision systems, image processing being a particularly relevant example. The engineering community typically models the quantization noise $Q = [X] - X$, where $[X]$ is the quantized realization of X. One might be tempted to apply a Sheppard's correction to the moments of the quantized data, thinking that $Var(X) < Var([X])$ but it is possible to construct examples where Q and $[X]$ are independent, or where $Cov(X, Q)$ is such that $Var(X) > Var([X])$.

Shepard's correction is limited in that is doesn't apply to the first moment, and the frequencies of the first and last bins need to be low.

Expand $p(x;\theta)$ in a Taylor series and substitute in the Maximum Likelihood equations. [**?** **?**]

Suppose we have n realizations of iid RV's $X_1, \ldots, X_n$ and the data is collected on a discrete grid on the range of X $Ran(X) = \{[y_i - d_i/2, y_i + d_i/2]\}_{i=1}^{i=m}$ where the intervals are centered on the location where a measurement. The realized values $y_1, \ldots, y_m$ have probabilities $p_i = \int_{y_i - d_i/2}^{y_i + d_i/2} p(x;\theta)\ dx$ Expanding $p(x;\theta)$ in a Taylor series about y, $p(x;\theta) = \sum_{i=0}^{\infty} \frac{p^{(i)}(y)}{i!}(x-y)^i$.

9.8 Multidimensional Scaling

Multidimensional scaling (MDS) is an alternative to factor analysis. The aim of MDS and factor analysis is to detect meaningful underlying dimensions that explain similarities or dissimilarities data points. In factor analysis, the similarities between points are expressed via the correlation matrix. With MDS any kind of similarity or dissimilarity matrix may be used.

Given n observations $x_i{}_{i=1}^{n} \in \mathbb{R}^k$ and n^2 distances d_{ij} between them, MDS looks for n points $\xi_i{}_{i=1}^{n}$ in $dblr^l : l < k$ that preserve the distance relations. When a metric $\rho()$ exists for the similarity measure, gradient descent is used to minimize the MDS functional $S(\xi_1, \ldots, \xi_l) = \left(\sum_{i \neq j} d_{ij} - ||\xi_i - \xi_j||_\rho \right)^{\frac{1}{2}}$.

9.9 Principal Components

For a data set $\mathbf{X} \in M_{(N,m)}(\mathbb{R}) = x_1, x_2, \ldots x_N | x_i \in \mathbb{R}^m$, the first k principal components provided the best k dimensional linear approximation to that data set. Formally, we model the data via $f(\theta) = \mu + \mathbf{V}_k \theta | \mu \in \mathbb{R}^m, V_k \in O_{m,k}(\mathbb{R}), \theta \in \mathbb{R}^k$ so $f(\theta)$ is an affine hyperplane in $\mathbb{R}^m$

9.10 Evaluating classifier performance

Multi-class problems can be treated simultaneously or broken in to a sequence of two class problems. Cross validation is used both for classifier parameter tuning and for feature subset selection. Student-t and ANOVA can be used to evaluate the performance of classifiers against one another. The Student-t test compares two classifiers, while the ANOVA test can compare multiple classifiers against one another. Confusion matrices and ROC graphs are commonly employed visualization tools for assessing classifier performance. The rows of a confusion matrix add to the total population for each class, and the columns represent the predicted class. An ROC curve plots the TP rate against the FP rate. Often a curve in ROC space is drawn using classifier parameters for tuning purposes.

TN	FP
FN	TP

Table 9.1: Two class confusion matrix where the proportions are specified

Common performance metrics for the two class problem are sensitivity (TP), specificity (TN), precision (the proportion of predicted cases within a class that were correct), and accuracy (the overall proportion of correct predictions). These metric can be extended to more than two classes by defining $A = tr(C)/||C||_{L^\infty}$ where C is the confusion matrix. TP, FN, FP, TN are proportions defined for the two class problem.

9.11 Covariance Matrix Estimation

For numerical stability in regression algorithms, the covariance matrix needs to be positive definite. An well conditioned estimator for the covariance matrix of a process can be obtained by mixing the sample covariance with the identity matrix. This is a linear shrinkage estimator based on a modified Frobenius norm for $A \in M_{mn}$

$$||\mathbf{A}||_{\mathcal{F}} = \sqrt{\frac{tr(AA^t)}{n}} \tag{9.11.1}$$

Without loss of generality, set $\mu = 0$ and let $\widehat{\Sigma} = \alpha \mathbb{I} + \beta \mathbf{S}$ where $\mathbf{S} = \frac{\mathbf{X}^T \mathbf{X}}{n}$ is the sample covariance. We seek to minimize $E(||\widehat{\Sigma} - \Sigma||^2)$, but since we don't know the true population covariance matrix, we have to form an approximation.

Chapter 10

Appendix Probability

Let μ be a non-negative countably additive set function over a sigma algebra Ω of sets from a sample space S. In probability theory, Ω is the set of possible events E. Sets of measure zero denote impossible outcomes. An important feature of measures ν and μ that agree on sets of measure zero is the ability to define a derivative $\frac{d\nu}{d\mu}$, the Radon-Nikodym derivative. When $\nu(E) = 0 \; \forall \; E \in \Omega \mid \mu(e) = 0$ Alternatively, given a measure μ and a nonnegative measurable function f, a new measure can be defined by $\nu(E) = \int E \in \Omega f d\mu$. A random variable is a real valued function on a sample space into a metric space, $X : S \to \mathbb{R}^1$. Associated with a random variable is it's probability density function $f_X(x) = P(\{s \in S | X(s) = x\})$ operating on an algebra of sets generated by the sample space S. By definition, $f_X(x)$ is the sum of probabilities of the events in S that get mapped to $x \in \mathbb{R}$ by X. Let $\mathcal{B}(S)$ be the Borel sets on S, then X has density f if $P(X \in A) = \int_{A \in \mathcal{B}(S)} f(x)dx$ and distribution function $F(x) = P(X < x) = \int_{-\infty}^{x} f(y)dy$ so $F'(x) = f(x) \;\; a.e..$ $E(X) = \int xf(x)dx$ is the expectation of X. The characteristic function $\phi(x) = E(e^{itX})$ determines the distribution and is used in the proof of the CLT theorem, testing for symmetry, and conditional independence. We denote samples with lower case in this section. $x_1 \ldots x_n$ is a random sample of size n In $\mathbb{R}^n$ the random variate X has distribution function $F(x_i, \ldots, x_n) = P(X_1 < x_1, \ldots, X_n < x_n)$ and density $f(x_1, \ldots, x_n)$.

For parametric distributions, one is interested in the question of what value of a parameter best describes the data at hand. This obviously requires the assumption that the data derives from a family of distributions parameterized by one or more variables θ_k. If $x_1 \ldots x_n$ is a random sample from X with a distribution given by $p(x; \theta_1 \ldots \theta_k)$, we can think of the joint pdf of of the sample $L(x_i, \ldots, x_n) = \prod_{i=1}^{n} p(x_i; \theta_1 \ldots \theta_k)$ as being explained by the parameters. L is the likelihood of the data given the parameters. The maximum likelihood estimate is obtained by solving the set of equations;

$$\frac{\partial L(\theta_1 \ldots \theta_k)}{\partial \theta_1} = 0$$

$$\vdots$$

$$\frac{\partial L(\theta_1 \ldots \theta_k)}{\partial \theta_k} = 0$$

The statistical moments of a random variable X are defined $\mu_n = E[X^n] = \int_\Omega X^n P(X) dX$.

The characteristic function is the fourier transform of $P(X)$

$$\Phi(\omega) = \digamma\mathcal{F}[P(X)](\omega) = \int_{\infty}^{\infty} e^{i\omega X} P(X) dX.$$

Taking the logarithm and expanding in a MacLauren series, we can relate the statistical moments to the coefficients. Statistical moments are central or raw.

Common sample descriptive statistics relating to location, scale, tail size, and peakedness;

$$mean = \hat{\mu_1} = \frac{1}{n} \sum x_i$$

$$variance = sdd^2 = \hat{\mu_2} = \frac{1}{n-1} \sum (x_i - \hat{\mu_1}$$

$$skewness = \frac{\hat{\mu_3}}{\hat{\mu_2}^{3/2}}$$

$$kurtosis = \frac{\hat{\mu_4}}{\hat{\mu_2}^2}$$

The linear association between X_i and X_j is measured by the covariance

$$Cov_{ij} = \sigma_{ij} = \frac{1}{n-1} \sum_k (x_k i - \hat{\mu_1}_i)(x_{kj} - \hat{\mu_1}_j).$$

For a measure without dependence on units the scaled covariance is the correlation

$$C_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}}$$

The sampling distribution of an estimator is the probability distribution of the estimator under repeated sampling. The standard error of a measurement is essentially the standard deviation of the process by which the measurement was generated. When the underlying probability distribution of the generating process is known the standard error can be used to calculate confidence intervals. Otherwise Chebyshev's inequality can be used. The standard error of a sample from a population is the standard deviation of the sampling distribution and may be estimated as $\frac{\sigma}{\sqrt{n}}$

10.1 Univariate Probability Distributions

This section covers the properties of common univariate probability distributions.

10.1.1 Uniform, $U(\alpha, \beta)$

$X =_d U(\alpha, \beta)$ if $p(x; \alpha, \beta) = \frac{\chi_{[\alpha,\beta]}}{\beta - \alpha}$.

10.1.2 Exponential Class of Distributions

The exponential class of distributions are characterized by the functional for of the pdf;

$$p(x; \theta) = exp(\alpha(x)\beta(\theta) + \gamma(\theta) + \delta(x))$$

. This class of distributions form the basis of Generalized Linear Model Theory that is discussed below.

There is an alternate parametrization of the exponential family that explicitly includes a dispersion parameter ϕ. This is useful for count data where $E[X] = E[X^2] = \theta$ In general if $E[X^2] > E[X]$ we say the process or data is over dispersed. The parameter ϕ is usually fixed in practice. If we write

$$p(x; \theta, \phi) = exp(\frac{x\theta - \beta(\theta)}{\alpha(\phi)} + \gamma(x, \phi))$$

, the dispersion parameter ϕ for some common distributions;

$p(x;\theta,\phi)$	ϕ
$N(\mu,\sigma)$	σ^2
$IG(\mu,\sigma)$	σ^2
$Gamma(\theta,\phi)$	$\frac{1}{\phi}$
$Poisson(\theta)$	1
$Binomial(\theta)$	1
$NegativeBinomial(\theta,r)$	r

Normal/Gaussian, $N(\mu,\sigma)$

$X =_d N(\mu,\sigma)$ if

$$p(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(\frac{(x-\mu)^2}{2\sigma^2}$$

. This can be re-written in exponential form

$$p(x;\theta) = exp(\frac{x\theta}{2\sigma^2} - \frac{\theta^2}{2\sigma^2} - \frac{1}{2}log(2\pi\sigma^2) - \frac{x^2}{2\sigma^2})$$

Binomial

Setting

$$\alpha(x) = x \ \ ,\beta(\theta) = log(\frac{\theta}{1-\theta}) \ \ ,\gamma(\theta) = nlog(1-\theta) \ \ ,\delta(x) = log(\left(\begin{array}{c} n \\ y \end{array} \right))$$

gives us the binomial distribution $p(x;\theta) = \left(\begin{array}{c} n \\ x \end{array} \right)\theta^x(1-\theta)^{(n-x)}$

Negative Binomial

$$p(x;\theta,r) = \left(\begin{array}{c} x+r-1 \\ r-1 \end{array} \right)\theta^r(1-\theta)^x$$

Poisson

X is Poisson distributed if $p(x;\theta) = \frac{\theta^x e^{-\theta}}{x!}$ Setting $\beta(\theta) = log(\theta) \ \ ,\gamma(\theta) = -\theta \ \ ,\delta(x) = -log(x!)$ we get the exponential form of the Poisson distribution,

$$p(x;\theta) = exp(xlog(\theta) - \theta - log(x!))$$

The expected value and the variance of a Poisson distributed random variable is equal to θ. The higher moments of the Poisson distribution are the Touchard polynomials in θ. There is a combinatorial interpretation. When $E[X] = 1$ for a Poisson random variate then the i-th moment of X is equal to the number of partitions of a set of size n $\frac{1}{e}\sum_{n=0}^{\infty}\frac{n^i}{n!}$ via Dobinski. The normal distribution with mean θ and variance θ is a good approximation to the Poisson distribution for large θ.

Pareto

X is Pareto distributed if

$$p(x;\theta) = \theta x^{-\theta}.$$

Gamma

X is Gamma distributed if

$$p(x;\theta,\phi) = \frac{x^{\phi^{-1}}\theta^\phi e^{-x\theta}}{\Gamma(\phi)}.$$

Weibull

X follows the Weibull distribution if

$$p(x;\theta,\lambda) = \frac{\lambda x^{\lambda-1}}{\theta^{\lambda}} e^{(\frac{x}{\theta})^{\lambda}}.$$

Inverse Gaussian/ Wald Distribution

X follows the Inverse Gaussian distribution if

$$p(x;\theta) = \sqrt{\frac{\theta}{2\pi x^3 \sigma}} exp(-\frac{\lambda(x-\theta)^2}{2x\theta^2\sigma})$$

10.1.3 Generalized Extreme Value Distribution $GEV(\theta,\phi,\xi)$

This class of distributions includes the three limiting extreme value distributions of [**?**] and [**?**]. $X =_d GEV(\theta,\phi,\xi)$ if

$$p(x;\theta,\phi,\xi) = exp(-max\Big(\big(1+\xi\frac{x-\theta}{\phi}\big)^{-\frac{1}{\xi}},\ \ 0\Big)$$

. Where

10.1.4 Multinomial

Let

$$\Omega = \mathcal{B}(\prod_{i=0}^{i=\infty} \mathbb{Z}(K))$$

be the Borel Algebra generated by $\prod 1,2,\ldots,K$. This is the sample space of all realizations of experiments with K categorical outcomes. Equip $\mathbb{Z}(K) = i \in 1,\ldots,K$ with a measure $P(i) = \theta_i$. Let X_i be n iid copies of $X =_d p(i;\pi_1,\ldots,\pi_K)$. Now map $\mathbf{X} = (X_1,\ldots,X_n) \in \Omega \to \mathbf{Y} \in \mathbb{N}^K$ Then Y_i are counts of the number of elements of category i in the experiment with n observations. The multinomial distribution is given by

$$p(\mathbf{Y};n) = \frac{n!}{y_1!\ldots y_K!}(\theta_1)^{y_1}\ldots(\theta_K)^{y_K}$$

. This in not a member of the exponential family, but we can show that the multinomial distribution is the joint distribution of $Y_i =_d Poission(\theta_i')_{i=1,\ldots K}$ random variables conditional to their sum.

$$p(\mathbf{Y};\theta_1',\ldots,\theta_K') = \prod_{i=1}^{K} \frac{(\theta_i')^{y_i}\ e^{-\theta_i'}}{y_i!}$$

, set $n = Y_1 + \ldots + Y_K$. Writing $p(\mathbf{Y}|n) = p(\mathbf{Y};\theta_1',\ldots,\theta_K')/p(n)$ and noting that $n =_d Poisson(\sum_{i=1}^{K}\theta_i')$, we recover the multinomial distribution by simplifying and setting $\theta_i = \frac{\theta_i'}{\sum_{i=1}^{K}\theta_i'}$

10.1.5 $\chi^2(n)$

If X_i iid $N(0,1)$ and $Y_i = X_i^2$ then $Y = \sum_{i=1}^{n} Y_i =_d \chi^2(n)$. $E[Y] = n$ and $Var(Y) = E[(Y-\mu_Y)^2] = E[(Y-E[Y])^2] = 2n$. More generally, if $Y_i = X_i + \mu_i$ then

$$Y = \sum_{i=1}^{n}(Y_i)^2 = \sum_{i=1}^{n} X_i^2 + 2\sum_{i=1}^{n} X_i\mu_i + \sum_{i=1}^{n}\mu_i^2 =_d \chi^2(n,\lambda)$$

. Where $\lambda = \sum_{i=1}^{n}\mu_i$ is non-centrality parameter.

See the section on multivariate probability distributions for further information, but it is worth noting that if $X =_d N(\mu, \sigma)$ is multivariate and the variance covariance matrix $s\sigma$ is non-singular, then $(y-\mu)^T\sigma^{-1}(y-\mu) =_d \chi^2(n)$ and setting $\lambda = \mu^T\sigma^{-1}\mu$ we have $y^T\sigma^{-1}y =_d \chi^s(n, \lambda)$

10.1.6 Student-t $t(\nu)$

$X =_d \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\Gamma(\nu/2)}(1+x^2)^{-\frac{\nu+1}{2}}$

10.1.7 Generalized Inverse Gaussian $GIG(\lambda, \alpha, \beta)$

The GIG distributions are characterized by

$$p(x; \lambda, \theta, \sigma) = \big(\frac{\theta}{\sigma}\big)^{\frac{\lambda}{2}} x^{\lambda-1} \frac{1}{2K_\lambda(\sqrt{\theta\sigma})} exp(-\frac{1}{2}(\theta x^{-1} + \sigma x)$$

.

Note, $GIG(-1/2, \theta, \sigma) = IG(\theta, \sigma)$

The GIG family members arise as first passage time distributions of ordinary Brownian diffusions to a constant boundary.

10.1.8 Normalized Inverse Gaussian $NIG(\mu, \alpha, \beta, \delta)$

$X =_d NIG(\mu, \alpha, \beta, \delta)$ if

$$p(x; \mu, \beta, \alpha, \delta) = \frac{\delta\alpha}{\pi} exp\big(\delta\sqrt{\alpha^2-\beta^2} + \beta(x-\mu)\big)\frac{K_1(\alpha\ s_\delta(x-\mu))}{s_\delta(x-\mu)}$$

where $x \in \mathbb{R}$ $\mu \in \mathbb{R}$ $\delta > 0$ $0 \le |\beta| \le \alpha$ and $s_\delta(x) = \sqrt{\delta^2+x^2}$ and $K_1(x) = \frac{x}{4}\int_0^\infty exp -$ $\big(y + \frac{x^2}{4y}\big)y^{-2}\ dy$ is the modified Bessel function of the third kind. This family of distributions is infinitely divisible, $\exists$ X_t, a Levy process, $X_{t+\Delta t} - X_t =_d X_{\Delta t} =_d NIG(\mu, \alpha, \beta, \delta)$. X_t is a pure jump process, and

$$p(t; \alpha, \beta, \delta) = \left(\frac{\delta\alpha}{\pi|t|}\right)e^{\beta t}K_1(\alpha|t|)$$

. See Eberlein and Keller (1995) and Barndorff-Nielsen (1998)

10.1.9 Generalized Hyperbolic $GH(\lambda, \alpha, \beta, \delta, \mu)$

The parameters $\lambda, \alpha, \beta, \delta, \mu$ have the respective interpretation of tail heaviness, kurtosis, skewness, and scale, and location. The distribution includes the important classes GIG, NIG, IG, and can be characterized as a Normal variance-mean mixture NVMM parameterized by a GIG distribution. Formally set $U =_d GIG()$, then $X =_d NVMM()$ if $P(X|U=u) =_d N(\mu+\beta, u\Delta)$. This gives a stochastic representation $X = \mu + \beta Z + \sqrt{Z}Y$ where $Y =_d N(0,1)$ and $Z =_d GIG()$.

10.2 Limit Theorems

Limit Theorems use the notion of a basin of attraction for pdf's in some functional space $\mathcal{H}$. In $L^2(\Omega)$, we have $N(\mu, \sigma) \subset L^2(\Omega, \nu)$ is the basin of attraction for all pdf's satisfying the conditions of the CLT.

The CLT says that the series $\frac{\sum_{i=1}^n X_i}{n}$ converges in in probability to the mean of x_i. Cramer's theorem gives a bound on the probability of large deviation away from the mean in the series $\frac{\sum_{i=1}^n X_i}{n}$. The probability decays exponentially with a rate given by the Legendre transform of the cumulant generating function for X_i

Theorem 10.2.1 (Cramer's Theorem). *Let $X_1, X_2, \ldots$ be iid $E(X_i) = 0$, $E(X_i^2) = \sigma^2$, and $F_n(x) = P(\frac{1}{\sigma n^{\frac{1}{2}}} \sum_{i=1}^{n} X_i < x)$, then if $x > 1$ and $x = O(\sqrt{n})$ as $n \to \infty$ we have*

$$\frac{1 - F_n(x)}{1 - \Phi(x)} = exp(\frac{x^3}{\sqrt{n}}\lambda(\frac{x}{\sqrt{n}}))[1 + O(\frac{x}{\sqrt{n}})]$$

$\lambda(x) = \sum_{i=0}^{\infty} c_i x_i$ where the c_i depend on the moments of X_i. $\Phi(x)$ is the distribution function of $N(0,1)$.

Theorem 10.2.2 (Law of Large Numbers).

The CLT tells us that the pdf of the scaled mean of a sample approaches the normal distribution and the BerryEsseen theorem specifies the rate at which that happens. The CLT requires X_i to be iid, and with finite second moment and the Berry-Esseen theorem additionally requires a finite third moment.

Theorem 10.2.3 (BerryEsseen). *Let X_i be iid, $E(X_i^2) = \sigma$, $E(X_i^3) = \rho$, $Y_n = \frac{X_1 + \ldots + X_n}{n}$, and $F_n = \int \frac{Y_n \sqrt{n}}{\sigma}$ and Φ the CDF of $N(0,1)$, then*

$$|F_n(x) - \Phi(x)| \leq \frac{C\rho}{\sigma^3 \sqrt{n}} \tag{10.2.1}$$

10.3 Multivariate Probability Distributions

Let S_n be the unit sphere in $\mathbb{R}^n$ A random variate X is uniformly distributed on S_n when X is radially symmetric and $||X||_{L^2} = 1 a.s.$ The pdf of a radially symmetric random variable is necessarily of the form $f(x_1, ..., x_n) = g(||x||)$ for some $g \in [0, \infty) \ni \int_0^{\infty} n V_n r^{n-1} g(r) dr = 1$ where $V_n = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ is the volume of S_n. If X is radially symmetric , then $\frac{X}{|||X||}$ is uniformly distributed on S_n. If X is uniformly distributed on S_n then $(X_1^2, \ldots, X_n^2) =_{dist} (\frac{Y_1}{\kappa}, \ldots, \frac{Y_n}{\kappa})$ where Y_i iid $\Gamma(\frac{1}{2})$ with sum κ. If $N_1, \ldots, N_n$ iid normal, then $(N_1, \ldots, N_n)$ is radially symmetric with density $g(r) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{\frac{-r^2}{2}}$ This leads us to an algorithm for generating pseudo random variants on uniformly distributed on S_n ;

- Generate n iid $N(0,1)$
- Compute $\kappa = (\sum_i = 1^n N_i^2)^{\frac{1}{2}}$
- Return $(\frac{N_1}{\kappa}, \ldots, \frac{N_n}{\kappa})$

[**?**]

With a little linear algebra, the above can be generalized to a generator for $N(\mu, \Sigma) \in \mathbb{R}^n$. Consider $f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} x^T \dot{x}} x \in \mathbb{R}_n$, f has density of n iid $N(0,1)$ rv if

10.3.1 Multivariate Normal $N(\mu, \mathbf{\Sigma})$

$N(\mu, \mathbf{\Sigma})$ is arguably the most important and tractable multivariate probability distribution. The Gaussian distribution is separable via rotation. Precisely the rotation induced by PCA.

10.3.2 Wishart Distribution

The Wishart distribution $W(n)$ is the multivariate generalization of the $\chi^2(n)$ distribution. If $X_{(i)} \sim N(\mu, \mathbf{\Sigma})$, then $XX^t = S \sim W(N)$.

10.3.3 Elliptic $E(\mu, \mathbf{\Sigma})$

Elliptical distributions $E(\mu, \mathbf{\Sigma})$ extend the multivariate normal $N(\mu, \mathbf{\Sigma})$. They can be characterized as affine maps of spherical distributions. The density functions are defined by $p(x) = cg((x-\mu)'\Sigma^{-1}(x-\mu))$ Where $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $\Sigma \succ 0$ is positive definite. Many of the properties of the multivariate normal distribution are shared by the elliptical distributions. Linear combinations, marginal distributions and conditional distributions of elliptical random variables can largely be determined by linear algebra using knowledge of covariance matrix, mean and generator.

10.4 Statistical Dependence

Linear correlation is a natural dependence measure for multivariate normally and elliptically distributed random variables. Other dependence concepts include rank correlation, comonotonicity, and Brownian covariance.

10.5 Distance measures for probability distribution functions.

A number of distance measures for probability distance functions exist. Kullbak Lieber divergence:

$$J_D = \int_{\mathrm{x}} [p(\mathrm{x} \mid \omega_1) - p(\mathrm{x} \mid \omega_2)] \log \frac{p(x \mid \omega_1)}{p(x \mid \omega_2)} \mathrm{dx}$$

which simplifies to:

$$J_D = \tfrac{1}{2} (\mu_2 - \mu_1)^T \left(\Sigma_1^{-1} + \Sigma_2^{-1} \right) (\mu_2 - \mu_1) + \tfrac{1}{2} \mathrm{tr} \left\{ \Sigma_1^{-1} \Sigma_2 + \Sigma_2^{-1} \Sigma_1 - 2I \right\}$$

when

$$X_1 =_d N(\mu_1, \Sigma_1) \ , \ X_2 =_d N(\mu_2, \Sigma_2).$$

The Bhattacharyya distance :

$$J_B = -\log \int [p(\xi \mid \omega_1) p(\xi \mid \omega_2)]^{1/2} \mathrm{d}\xi$$

which simplifies to:

$$J_B = \tfrac{1}{8} (\mu_2 - \mu_1)^T \left(\frac{\Sigma_1 + \Sigma_2}{2} \right)^{-1} (\mu_2 - \mu_1) + \tfrac{1}{2} \log \frac{\left| \frac{1}{2}(\Sigma_1 + \Sigma_2) \right|}{(|\Sigma_1| \, |\Sigma_2|)^{1/2}}$$

when

$$X_1 =_d N(\mu_1, \Sigma_1) \ , \ X_2 =_d N(\mu_2, \Sigma_2).$$

The Matusita distance:

$$J_T = \left\{ \int \left[\sqrt{p(\xi \mid \omega_1)} - \sqrt{p(\xi \mid \omega_2)} \right]^2 \mathrm{d}\xi \right\}^{1/2}$$

which simplifies to:

$$J_T = \{ 2 [1 - \exp(-J_B)] \}^{1/2}$$

where J_B is the Bhattacharyya distance, when

$$X_1 =_d N(\mu_1, \Sigma_1) \ , \ X_2 =_d N(\mu_2, \Sigma_2).$$

The Patrick-Fisher distance:

$$J_P = \left\{ \int [p(\xi \mid \omega_1) P_1 - p(\xi \mid \omega_2) P_2]^2 \mathrm{d}\xi \right\}^{1/2}$$

which simplifies to:

$$J_P = \begin{array}{c}(2\pi)^d\,|2\Sigma_1|)^{-1/2} + ((2\pi)^d\,|2\Sigma_2|)^{-1/2} - \\ 2((2\pi)^d\,|\Sigma_1+\Sigma_2|)^{-1/2}\exp\left\{-\frac{1}{2}(\mu_2-\mu_1)^T(\Sigma_1+\Sigma_2)^{-1}(\mu_2-\mu_1)\right\}\end{array}$$

when

$$X_1 =_d N(\mu_1,\Sigma_1)\ \ ,\ \ X_2 =_d N(\mu_2,\Sigma_2).$$

Reference: pp257, et sqq., Devijver, P.A. & Kittler, J (1982) "Pattern Recognition: A Statistical Approach", Prentice Hall International, Englewood Cliffs, NJ.

10.6 $S_\alpha(\sigma,\beta,\mu)$ Stable Random Variates

A Levy process is a stochastic process with a drift, a diffusion, and a jump component. The LvyKhinchine representation of a Levy process X_t with parameters (a,σ^2,W) is given by

The Levy-Ito decompositon of a Levy process X_t is a decomposition of X_t into singular, absolutely continuous, and discrete processes

$$\begin{aligned} X_{ac} : X_{ac} &\ll X \\ X_s : X_s &\perp X \\ X_d : card(suppX_d) &= \aleph_0 \end{aligned}$$

via Lebesgue's decomposition theorem.

10.6.1 4 definitions of stable

- If
$$\exists\, C,D\ \forall\ A,Bs.t.AX_1+BX_2 =_d CX+D\ \forall\ X_1,X_2$$
independent copies of X, then $X\in S_\alpha(\sigma,\beta,\mu)$. Furthermore $\exists\,\alpha\in(0,2]s.t.C$ satisfies $C^\alpha = A^\alpha + B^\alpha$, for any stable RV and $\forall\ A,B$
- Stable RV's satisfy a general CLT. If
$$\forall\ n\geq 2\ \exists\ C_n>0D_n\in\mathbb{R}s.s.X_i+X_@\ldots X_n =_D C_nX+D_n$$
where X_i iid, then $X\in S_\alpha(\sigma,\beta,\mu)$
- If $\exists$ $iid\ RV\ Y_i$ and
$$d_n,a_n\in BBCREVISIT^n\mathbb{R}^n s.t\frac{\sum\limits^{n}_{1}Y_i}{d_n}+a_n =_d X$$
then $X\in S_\alpha(\sigma,\beta,\mu)$.
- If $\exists\,\alpha\in(0,2],\ \sigma\geq 0\ \beta\in[-1,1],\ \mu\in\mathbb{R}$ such that
$$E[e^{i\theta X}] = \int_\Omega e^{i\theta X}dX = exp\big(-\sigma^\alpha|\theta|^\alpha(1-i\beta\ sgn(\theta)\ tan\big(\frac{\pi\alpha}{2}\big)+i\mu\theta)\big)$$
when $\alpha\neq 1$ and when $\alpha=1$ we have
$$E[e^{i\theta X}] = exp(-\sigma|\theta|(1+i\frac{2}{\pi}\beta\ sgn(\theta)\ ln|\theta|+i\mu\theta))$$

10.6.2 Variance Gamma Process

$X_{VG}(t;\sigma,\nu,\theta)\theta\gamma(t;\nu)+\sigma W_{Y(t;\nu)}$ where $\gamma(t;\nu)$is a Γ process.

$$P_{\gamma(t;\nu)}(x) = \frac{x^{\frac{t}{\nu-1}}e^{-\frac{x}{\nu}}}{\nu^{t/\nu}\Gamma(t/\nu)}$$

$$\Phi_{VG}(\omega) = E(e^{i\omega X_{VG}}) = \frac{1}{(1-i\omega\nu\theta+\sigma^2\nu\mu^2/2)^{t/\nu}}$$

We can show that the Variance Gamma process is the difference of two independent Gamma processes $X_{VG} = \gamma_p - \gamma_n$ to obtain a new pdf

$$P_{\gamma(t;\nu)} = \begin{cases} \frac{1}{\nu|x|} e^{-|x|/\eta_p} & x < 0 \\ \frac{1}{\nu|x|} e^{-|x|/\eta_n} & x > 0 \end{cases}$$

.

10.7 Maximum Entropy

Entropy in the context of information theory is expressed in units of bits, the amount of uncertainty in a yes or no question. Formally, for a sequence $\{X_i\} \ni p_i$ is a priori/posteriori probability of observing X_i we define $H = -\sum_i p_i log_2(p_i)$. We can define the entropy of a probability distribution by $H = \int_{infty}^{\infty} p(x)log(p(x))dx$. We see the uniform distribution maximizes the entropy; if $p_i = \alpha \; \forall \; i$ then $\frac{\partial H}{\partial} = logp + 1/p = 0$

In this section we will use the term $EPDF_X$ to mean the empirical probability density function. There are a variety of univariate tests to help determine which parametric distribution your data belongs to. These fall under the category of Goodness of Fit testing. For a parametric family the null hypothesis $H_o : X =_d p(x|\theta)$ is tested against the alternative that X does not belong to the family $p(x|\theta)$ There are also family of test to determine whether two $EPDF$'s come from the same distribution.

Below we present a table of test and results from the KL libraries via a CDH port from the original Fortran Statlib libray. [1]

Omnibus Moments Test for Normality
Geary's Test of Normality
Calculate Extreme Normal Deviates
D'Agostino's D-Statistic Test of Normality
Kuiper V-Statistic Modified to Test Normality
Watson U^2-Statistic Modified to Test Normality
Durbin's Exact Test (Normal Distribution
Anderson-Darling Statistic Modified to Test Normality
Cramer-Von Mises W^2-Statistic to Test Normality
Kolmogorov-Smirnov D-Statistic to Test Normality
Kolmogorov-Smirnov D-Statistic (Lilliefors Critical Values)
Chi-Square Test of Normality (Equal Probability Classes)
Shapiro-Wilk W Test of Normality for Small Samples
Shapiro-Francia W' Test of Normality for Large Samples
Shapiro-Wilk W Test of Exponentiality
Cramer-Von Mises W^2-Statistic to Test Exponentiality
Kolmogorov-Smirnov D-Statistic to Test Exponentiality
Kuiper V-Statistic Modified to Test Exponentiality
Watson U^2-Statistic Modified to Test Exponentiality
Anderson-Darling Statistic Modified to Test Exponentiality
Chi-Square Test for Exponentiality(with E.P.C.)
Kotz Separate-Families Test for Lognormality vs. Normality

We only discuss the Anderson Darling and Kolmogorov-Smirnov tests.

The Anderson-Darling test determines whether a sample comes from a specified distribution. The sample data can is transformed to a uniform distribution and then a uniformity test

[1] I don't know who did the port. The following is a comment from the GRASS GIS tutorial on goodness of fit: *"The cdhc library was inspired by Johnson's STATLIB collection of FORTRAN routines for testing distribution assumptions. Some functions in cdhc are loosely based on Johnson's work (they have been completely rewritten, reducing memory requirements and number of computations and fixing a few bugs). Others are based on algorithms found in Applied Statistics, Technometrics, and other related journals."*

is then done on the transformed data. The test statistic is compared against pre-computed values for the assumed probability distribution.

The Kolmogorov-Smirnov is non-parametric a form of minimum distance estimation. It can be used to test a sample against a reference or to compare two samples against each other. In the one sided case the KS statistic calculates the distance between the $EPDF$ of a sample and a reference. In the two sided case the distance between the $EPDS$'f of the two samples are calculated. The KS test is robust to location and shape, making it

Omnibus tests evaluate whether the explained variance in a set of data is significantly greater than the unexplained variance. For example is the F-test in ANOVA. Omnibus tests of normality based on the likelihood ratio outperform the Anderson-Darling test statistic.

10.8 Simulation and modeling with the kl Software Framework

Class, interaction and collaboration diagrams are presented below for a modeling framework implemented by the author. The framework is implemented in C++. The simulation of various univariate and multivariate random number generators along with the distribution tests from the CDHC library are included as well.

Features of this framework include:

- utilizing optimized BLAS libraries
- the up to date methods for univariate random number generation
- wrappers for Intel performance primitives and GSL
- multiple memory management facilities

We will use the term $EPDF_X$ to mean the empirical probability density function. There are a variety of univariate tests to help determine which parametric distribution your data belongs to. These fall under the category of Goodness of Fit testing. For a parametric family the null hypothesis $H_o : X =_d p(x|\theta)$ is tested against the alternative that X does not belong to the family $p(x|\theta)$ There are also family of test to determine whether two $EPDF$'s come from the same distribution.

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