Generalized Hydrodynamics, Hamiltonian Systems, and the Toda Lattice

ChatGPT

December 15, 2025

Contents

1 Introduction

The **Toda lattice** is a famous integrable system in mathematical physics. It describes a one-dimensional chain of particles with nonlinear, exponential interactions between nearest neighbors. Its integrability and rich structure make it a central model in soliton theory, spectral analysis, and Hamiltonian mechanics.

In this note, we derive the **Lax pair** formulation of the Toda lattice, exploring how it was originally discovered. Along the way, we address key theoretical insights, dispel the notion that it was a product of trial and error, and show how the reformulation reveals the hidden linearity in a nonlinear system.

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2 1. The Classical Toda Lattice

2.1 1.1 Hamiltonian Formulation

The Toda lattice with $ N $ particles has the Hamiltonian:

$$H = \sum_{i=1}^{N} \left(\frac{1}{2} p_i^2 + a e^{-(q_{i+1} - q_i)} \right)$$

where:

- $ q_i $: position of the $ i $-th particle,
- $ p_i $: momentum of the $ i $-th particle,
- $ a $: interaction strength (often normalized to 1).

2.2 1.2 Equations of Motion

From Hamilton's equations:

$$\dot{q}_i = p_i$$
$$\dot{p}_i = a\left(e^{-(q_i - q_{i-1})} - e^{-(q_{i+1} - q_i)}\right)$$

These are **second-order nonlinear ODEs**. While not linear, the structure hints at something special due to the form of the exponential interactions.

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3 2. From Nonlinear to Linear: The Need for a New Formulation

3.1 2.1 Motivation

The Toda lattice exhibits **soliton-like behavior** and **regular wave patterns**, suggesting it may be **integrable**. In the late 1960s and early 1970s, Peter Lax introduced the idea of expressing nonlinear PDEs and ODEs as **isospectral matrix flows**:

$$\frac{dL}{dt} = [B, L]$$

This **Lax equation** implies that the eigenvalues of $ L $ are constants of motion — a signal of integrability. The question became: can we cast the Toda lattice in this form?

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4 3. Flaschka's Change of Variables

The breakthrough came in **1974**, when **H. Flaschka** introduced a clever reparameterization to simplify the Toda equations and reveal their underlying linear algebraic structure.

Define new variables:

$$a_i := \frac{1}{2}e^{-\frac{1}{2}(q_{i+1} - q_i)}, \quad b_i := -\frac{1}{2}p_i$$

This change turns the second-order nonlinear equations into a **first-order, bilinear system**:

$$\dot{a}_i = a_i(b_{i+1} - b_i) \tag{1}$$
$$\dot{b}_i = 2(a_i^2 - a_{i-1}^2) \tag{2}$$

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5 4. Why This Form is Significant

> *"These are now first-order, bilinear, and resemble commutator forms — perfect setup for matrix equations."*

5.0.1 First-order

Equations (1) and (2) are **first-order in time**, which makes them compatible with **matrix evolution equations** of the form $\dot{L} = [B, L]$.

5.0.2 Bilinear

Equation (1) is **bilinear**:

$$\dot{a}_i = a_i(b_{i+1} - b_i)$$

Equation (2) is quadratic:

$$\dot{b}_i = 2(a_i^2 - a_{i-1}^2)$$

These forms suggest a hidden **matrix product** structure.

5.0.3 Commutator-Like

The differences $ b_{i+1} - b_i $ and $ a_i^2 - a_{i-1}^2 $ are **typical of commutators** in tridiagonal matrices — a sign that the Toda lattice might be representable by a **Lax pair**.

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6 5. Constructing the Lax Pair

6.1 5.1 The Lax Matrix $ L $

Define a symmetric, tridiagonal matrix:

$$L = \begin{bmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & 0 & a_{N-1} & b_N \end{bmatrix}$$

6.2 5.2 The Skew-Symmetric Matrix $ B $

Define:

$$B = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \ddots & \vdots \\ 0 & -a_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & 0 & -a_{N-1} & 0 \end{bmatrix}$$

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7 6. Verifying the Lax Equation

Compute:

$$\dot{L} = [B, L] = BL - LB$$

7.1 6.1 Diagonal Entries

From matrix multiplication:

$$(\dot{L})_{ii} = 2(a_i^2 - a_{i-1}^2) = \dot{b}_i$$

7.2 6.2 Off-Diagonal Entries

For $i \neq j$, particularly $i, i+1$:

$$(\dot{L})_{i,i+1} = a_i(b_{i+1} - b_i) = \dot{a}_i$$

This confirms:

$$\dot{L} = [B, L] \Longleftrightarrow \left\{ \dot{a}_i = a_i(b_{i+1} - b_i) \ \dot{b}_i = 2(a_i^2 - a_{i-1}^2) \right.$$

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8 7. Insights Behind the Derivation

You asked:

> Was it trial and error or was there theory driving the solution?

8.0.1 Theoretical Motivation

- Lax's 1968 theory of isospectral flows
- Spectral theory of Jacobi matrices
- Hamiltonian structure and conservation laws
- The algebraic pattern in Flaschka's equations

8.0.2 Creative Guesses

- Exact substitution to define $ a_i $
- Form of the skew-symmetric matrix $ B $
- Choosing a matrix framework aligned with nearest-neighbor interactions

So: **theory-driven**, but **with insightful experimentation**.

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9 8. Consequences of the Lax Pair

- The Toda lattice is **completely integrable**
- The eigenvalues of $ L $ are **conserved**
- The solution can be obtained via the **inverse scattering transform**
- The model generalizes to:
  - **Quantum Toda systems**
  - **Lie algebraic Toda chains**
  - **Toda field theories**

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10 9. Conclusion

The discovery of the Lax pair for the Toda lattice reveals a profound insight: many nonlinear systems harbor linear, algebraic structures when recast properly. Flaschka's transformation and the resulting Lax equation show how spectral methods and matrix theory can unlock hidden symmetries and conservation laws in nonlinear dynamics.

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11 Appendix: Computing the Commutator $ [B, L] $

Let's verify that $ = [B, L] $ gives the Flaschka equations.

Let $ L $ and $ B $ be as defined above.

11.1 Diagonal Entries $ i = j $

$$(\dot{L})_{ii} = \sum_k B_{ik}L_{ki} - L_{ik}B_{ki}$$

Only terms with $ k = i-1, i+1 $ contribute. Explicitly:

$$(\dot{L})_{ii} = B_{i,i+1}L_{i+1,i} + B_{i,i-1}L_{i-1,i} - L_{i,i+1}B_{i+1,i} - L_{i,i-1}B_{i-1,i}$$

Plug in values:

$$\begin{aligned}
&= a_i a_i - a_{i-1}a_{i-1} + a_i a_i - a_{i-1}a_{i-1} \\
&= 2(a_i^2 - a_{i-1}^2) \\
&= \dot{b}_i
\end{aligned}$$

11.2 Off-Diagonal Entries $ i, i+1 $

$$(\dot{L})_{i,i+1} = \sum_k B_{ik}L_{k,i+1} - L_{ik}B_{k,i+1}$$

Only nonzero terms involve $ k = i $ and $ k = i+1 $:

$$= B_{i,i}L_{i,i+1} + B_{i,i+1}L_{i+1,i+1} - L_{i,i}B_{i,i+1} - L_{i,i+1}B_{i+1,i+1}$$

Simplifies to:

$$\begin{aligned} &= a_i b_{i+1} - b_i a_i = a_i(b_{i+1} - b_i) \\ &= \dot{a}_i \end{aligned}$$

The commutator indeed reproduces the Toda lattice dynamics in Flaschka variables.

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12 Paradigmatic Integrable Systems

A *paradigmatic integrable system* is a model that is simultaneously (i) analytically tractable, (ii) structurally representative of integrability, and (iii) rich enough to exhibit nontrivial nonlinear phenomena (e.g., solitons, factorized scattering, infinite hierarchies of conservation laws). Such systems function as benchmarks for perturbation theory, numerics, semiclassics, and hydrodynamic limits, and they clarify the geometric meaning of integrability through symplectic structure, action–angle variables, and (in many cases) Lax representations.

12.1 Key concepts: Liouville integrability and exact solvability

Liouville integrability (finite-dimensional classical systems). Consider a Hamiltonian system on a $2n$-dimensional phase space with canonical coordinates $(q_i, p_i)_{i=1}^n$ and Poisson bracket

$$\{f, g\} \;=\; \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right). \tag{1}$$

The system is *Liouville integrable* if there exist n functionally independent first integrals

$$I_1, \ldots, I_n \quad \text{with} \quad \{I_i, I_j\} = 0 \quad \forall\, i, j, \tag{2}$$

including the Hamiltonian itself (often $I_1 = H$). Independence means $dI_1 \wedge \cdots \wedge dI_n \neq 0$ on an open dense set.

A central consequence is the *Liouville–Arnold theorem*: on regular, compact common level sets

$$\mathcal{M}_{\mathbf{c}} \;=\; \{(q,p) \,:\, I_k(q,p) = c_k,\; k = 1, \ldots, n\}, \tag{3}$$

the invariant manifolds are n-tori $\mathbb{T}^n$, and there exist *action–angle* variables (J_i, θ_i) with

$$\dot{J}_i = 0, \qquad \dot{\theta}_i = \omega_i(\mathbf{J}), \qquad \theta_i(t) = \theta_i(0) + \omega_i(\mathbf{J})t, \tag{4}$$

so the motion is quasi-periodic. One may compute actions as integrals over fundamental cycles,

$$J_i \;=\; \frac{1}{2\pi} \oint_{\gamma_i} \mathbf{p} \cdot d\mathbf{q}. \tag{5}$$

Exact solvability and Lax pairs. Many paradigmatic models admit a *Lax representation*

$$\frac{d}{dt}L \;=\; [B, L], \tag{6}$$

where L and B are matrices/operators depending on the dynamical variables. Equation (6) implies the isospectrality of L:

$$\frac{d}{dt}\,\mathrm{Tr}(L^k) = 0 \qquad (k = 1, 2, \dots), \tag{7}$$

yielding a hierarchy of conserved quantities. In infinite-dimensional settings (integrable PDEs), Lax pairs underpin the inverse scattering transform (IST), which reduces nonlinear evolution to linear evolution of scattering data.

Non-chaotic dynamics. Integrability severely constrains phase-space transport: trajectories remain on invariant tori (finite-dimensional) or on infinite-dimensional analogs determined by conserved charges (PDE/quantum chains). This contrasts with chaotic systems, where invariant tori are typically destroyed and sensitivity to initial conditions dominates.

12.2 Classical paradigms

12.2.1 Toda lattice

The (nonperiodic) Toda lattice describes N particles with nearest-neighbor exponential interactions. With coordinates (q_i, p_i), a standard Hamiltonian is

$$H_{\mathrm{Toda}} \;=\; \sum_{i=1}^{N} \frac{p_i^2}{2} \;+\; \sum_{i=1}^{N-1} e^{-(q_{i+1}-q_i)}. \tag{8}$$

Hamilton's equations give

$$\dot{q}_i = p_i, \tag{9}$$

$$\dot{p}_i = e^{-(q_i - q_{i-1})} - e^{-(q_{i+1}-q_i)}, \tag{10}$$

(with boundary conventions). Introducing Flaschka variables

$$a_i = \tfrac{1}{2} e^{-\frac{1}{2}(q_{i+1}-q_i)}, \qquad b_i = -\tfrac{1}{2} p_i, \tag{11}$$

the equations become polynomial and admit a Lax form $\dot{L} = [B, L]$ with a tridiagonal Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & 0 & a_{N-1} & b_N \end{pmatrix}. \tag{12}$$

Consequently, the spectral invariants $\mathrm{Tr}(L^k)$ provide N conserved quantities in involution, establishing Liouville integrability.

Example: elastic soliton-like scattering. In the large-N/continuum intuition, localized compressions propagate and interact with near-elastic scattering, a hallmark of integrability. In the fully integrable picture, the "nonlinear normal modes" are encoded in the scattering data of L and evolve linearly, explaining the persistence of coherent structures.

12.2.2 KdV equation

The Korteweg–de Vries (KdV) equation for a scalar field $u(x,t)$,

$$u_t + 6uu_x + u_{xxx} = 0, \tag{13}$$

is an infinite-dimensional integrable Hamiltonian system. One Hamiltonian structure is

$$u_t = \partial_x \frac{\delta H}{\delta u}, \qquad H[u] = \int_{\mathbb{R}} \left(u^3 - \frac{1}{2}u_x^2 \right) dx, \tag{14}$$

with Poisson operator ∂_x. KdV admits a Lax pair (in one common form)

$$L = -\partial_x^2 + u(x,t), \qquad B = -4\partial_x^3 + 3(u\partial_x + \partial_x u), \tag{15}$$

so that $\dot{L} = [B, L]$ is equivalent to (13). The isospectral evolution of the Schrödinger operator L yields an infinite sequence of commuting integrals (mass, momentum, energy, *etc.*).

Example: one-soliton solution. KdV has traveling-wave solitons

$$u(x,t) = 2\kappa^2 \operatorname{sech}^2\!\big(\kappa(x - 4\kappa^2 t - x_0)\big), \tag{16}$$

moving at speed $4\kappa^2$ with amplitude $2\kappa^2$. Multi-soliton solutions exhibit phase shifts after interaction but preserve shapes and speeds, reflecting factorized scattering in the IST framework.

12.3 Quantum paradigms

12.3.1 Lieb–Liniger (1D Bose gas)

The Lieb–Liniger model describes N bosons on a line (or ring) with contact interactions:

$$H_{\mathrm{LL}} \;=\; -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \;+\; 2c \sum_{1\le i<j\le N} \delta(x_i - x_j), \qquad c \ge 0. \tag{17}$$

On a ring of length L, coordinate Bethe ansatz wavefunctions in each ordering sector are superpositions of plane waves with quasi-momenta $\{k_j\}$ satisfying the Bethe equations

$$e^{ik_jL} = \prod_{\ell\ne j} \frac{k_j - k_\ell + ic}{k_j - k_\ell - ic}, \qquad j = 1, \dots, N, \tag{18}$$

and energy $E = \sum_{j=1}^{N} k_j^2$. Integrability manifests in an extensive set of commuting conserved charges and in purely elastic, factorized scattering.

Example: Tonks–Girardeau limit. As $c \to \infty$, bosons become "impenetrable" and many observables map to free fermions (up to symmetrization), offering a controlled setting for comparing interacting and effectively free behavior.

12.3.2 Heisenberg XXZ spin chain

The spin-$\frac{1}{2}$ XXZ chain on L sites is

$$H_{\mathrm{XXZ}} \;=\; J\sum_{j=1}^{L}\left(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\Delta\,\sigma_j^z\sigma_{j+1}^z\right), \tag{19}$$

(with periodic boundary conditions). It is solvable by Bethe ansatz; the anisotropy Δ controls regimes from gapless criticality to gapped phases. Integrability is formalized via the Yang–Baxter equation and transfer matrices, which generate an infinite family of commuting operators (conserved charges) including H_{XXZ}.

Example: magnon excitations. In the ferromagnetic reference state, spin flips propagate as magnons whose scattering is elastic and factorized. The Bethe roots encode both the spectrum and correlation structures, enabling asymptotics of spin correlators and entanglement in certain regimes.

12.3.3 Hubbard chain

The 1D Hubbard model for electrons with on-site interaction is

$$H_{\mathrm{Hub}} \;=\; -t\sum_{j,\sigma}\left(c_{j,\sigma}^\dagger c_{j+1,\sigma}+c_{j+1,\sigma}^\dagger c_{j,\sigma}\right)\;+\;U\sum_j n_{j,\uparrow}n_{j,\downarrow}, \tag{20}$$

where $n_{j,\sigma}=c_{j,\sigma}^\dagger c_{j,\sigma}$. In one dimension, the model is integrable (Lieb–Wu solution) via a nested Bethe ansatz, producing coupled Bethe equations for charge and spin rapidities and revealing spin–charge separation in the low-energy physics.

Example: spin–charge separation. At low energies, excitations decompose into independent spin and charge modes with distinct velocities, a canonical phenomenon in 1D correlated electron systems and a natural output of integrability-based analysis.

12.4 Why paradigmatic integrable systems matter

Benchmarks for approximation and numerics. Exactly solvable models provide ground truth for testing time-evolution algorithms, tensor-network truncations, semiclassical methods, and kinetic/hydrodynamic closures.

Emergent phenomena: solitons, hydrodynamics, entanglement. Integrable PDEs (KdV) explain soliton formation and stability. Quantum integrable chains (XXZ, Hubbard) exhibit constrained thermalization and hydrodynamics governed by extensive conserved charges, motivating generalized ensembles and transport theory.

Deep structure: symmetry, geometry, and algebra. Classical examples (Toda) link integrability to isospectral flows and symplectic geometry; quantum examples link it to Yang–Baxter integrability, quantum groups, and commuting transfer matrices.

Experimental relevance. The Lieb–Liniger model, in particular, is closely connected to quasi-1D ultracold atomic gases, making integrability quantitatively testable and providing a controlled environment to explore near-integrable perturbations and their relaxation dynamics.

13 Hamiltonian Systems and the Toda Lattice

13.1 Overview of Hamiltonian Mechanics

Hamiltonian mechanics is a reformulation of classical mechanics based on energy functions and symplectic geometry. A system is described by:

- A set of **generalized coordinates** q_i and **conjugate momenta** p_i
- A **Hamiltonian function** $H(q, p)$, typically representing total energy
- **Hamilton's equations of motion**:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{21}$$

These equations preserve the symplectic structure of phase space and are foundational in both classical and quantum dynamics.

13.2 The Toda Lattice as a Hamiltonian System

The **Toda lattice** is a paradigmatic example of a Hamiltonian system, describing a one-dimensional chain of particles with nonlinear, exponential nearest-neighbor interactions.

13.2.1 Hamiltonian Form

The Hamiltonian of the classical Toda lattice is

$$H = \sum_n \left(\frac{p_n^2}{2} + e^{-(q_{n+1} - q_n)} \right). \tag{22}$$

Here:

- q_n: position of the n-th particle
- p_n: momentum conjugate to q_n

13.2.2 Equations of Motion

From Hamilton's equations, the time evolution is

$$\dot{q}_n = p_n, \qquad \dot{p}_n = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)}. \tag{23}$$

These define a chain of interacting particles governed by nearest-neighbor forces.

13.3 Integrability and Soliton Solutions

The Toda lattice is **completely integrable**, meaning it has as many conserved quantities as degrees of freedom. This allows for:

- Soliton solutions: stable, localized wave packets
- Lax pair formulation: a matrix representation $\dot{L} = [P, L]$ preserving the spectrum of L
- Change of variables (Flaschka variables) simplifying analysis:

$$a_n = \frac{1}{2} e^{-(q_{n+1} - q_n)/2}, \qquad b_n = -\frac{1}{2} p_n. \tag{24}$$

This leads to a hierarchy of commuting flows and integrals of motion.

14 Generalized Hydrodynamics and the Toda Lattice

14.1 What is Generalized Hydrodynamics (GHD)?

Generalized Hydrodynamics (GHD) is a modern theoretical framework developed to describe the large-scale (Euler-scale) dynamics of **integrable systems**, which possess infinitely many conservation laws.

GHD generalizes traditional hydrodynamics by incorporating a **continuum of conserved quantities**, such as quasi-particle modes. It describes these through a **root density function** $\rho(x, \theta, t)$, where θ parametrizes rapidity (momentum-like variables from the integrable structure).

The central equation of GHD is

$$\partial_t \rho(x, \theta, t) + \partial_x \Big(v^{\text{eff}}(\theta, \rho)\, \rho(x, \theta, t) \Big) = 0, \tag{25}$$

where v^{eff} is the effective velocity of excitations, depending on ρ itself due to interactions.

14.2 Hamiltonian Structure of GHD

GHD can be formulated as a **Hamiltonian field theory**:

- Define functionals $\mathcal{F}[\rho]$ over the root density.
- Introduce a **Poisson bracket** structure:

$$\{\mathcal{F}, \mathcal{G}\} = \int dx \int d\theta\, \rho(x, \theta, t) \left(\frac{\delta \mathcal{F}}{\delta \rho}\, \partial_x \frac{\delta \mathcal{G}}{\delta \rho} - \frac{\delta \mathcal{G}}{\delta \rho}\, \partial_x \frac{\delta \mathcal{F}}{\delta \rho} \right). \tag{26}$$

- The Hamiltonian is typically the **total energy functional**:

$$\mathcal{H}[\rho] = \int dx \int d\theta\, \epsilon(\theta)\, \rho(x, \theta, t). \tag{27}$$

- Hamilton's equations then yield the GHD evolution equation.

This structure applies to both classical and quantum integrable systems, and can be extended to include external forces, interactions, and inhomogeneities.

14.3 Application of GHD to the Toda Lattice

The Toda lattice is a classical integrable system to which GHD has been successfully applied. Key results include:

14.3.1 Gas and Chain Pictures

Two complementary descriptions are used:

- **Gas picture**: particles moving in continuous space
- **Chain picture**: particles fixed on a lattice with interactions

Both can be described using GHD by defining appropriate spectral densities and effective velocities.

14.3.2 Thermodynamic Bethe Ansatz (TBA)

Using the classical TBA, one can derive:

- The **generalized Gibbs ensemble (GGE)** for equilibrium
- The **effective velocity** $v^{\text{eff}}(\theta)$
- Hydrodynamic equations in GHD form

14.3.3 Correlation Dynamics and Transport

Linearized GHD has been used to:

- Compute **space-time correlation functions**
- Analyze **ballistic transport** and dynamical structure factors
- Derive exact results for spreading of perturbations in the Toda lattice

These results are consistent with and extend known results from soliton theory and numerical studies.

14.4 Summary Table

Feature	**Description**
Hamiltonian System	Toda lattice fits the standard framework with exponential interactions.
Integrability	Infinite conservation laws; Lax pair; solitons.
GHD Formalism	Hydrodynamics of root densities $\rho(x, \theta, t)$.
Hamiltonian Structure in GHD	Energy functional + Poisson brackets define evolution.
Application to Toda Lattice	Both gas and chain views; TBA-derived velocities; exact transport results.

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