Special Topics:
Short Notes

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Abstract

Short notes in topics from HDP and Analysis

1 Bernstein Theorem

Theorem 1.1 (Bernstein's inequality on $\mathbb{T}$)**.** *Let p be a complex polynomial of degree n. Then*

$$\max_{|z|=1} |p'(z)| \;\le\; n \max_{|z|=1} |p(z)|.$$

Moreover, the constant n is sharp.

Proof. Write $p(z) = z^n F(z)$, where F is analytic in the exterior disk $\{z : |z| > 0\}$ and holomorphic at ∞ with $F(\infty) = $ (leading coefficient). For $|z| = 1$ we have

$$p'(z) = nz^{n-1}F(z) + z^n F'(z).$$

Set $M := \max_{|w|=1} |p(w)| = \max_{|w|=1} |F(w)|$. Since F is analytic on $\{|w| \ge 1\} \cup \{\infty\}$ and bounded on $|w| = 1$, the maximum-modulus principle on the exterior domain implies $\sup_{|w|\ge 1} |F(w)| = M$. Fix z with $|z| = 1$ and apply Cauchy's estimate to F' on the circle $|w| = R > 1$:

$$|F'(z)| \;\le\; \frac{1}{R-1} \max_{|w|=R} |F(w)| \;\le\; \frac{M}{R-1}.$$

Letting $R \to \infty$ gives $F'(z) = 0$ for $|z| = 1$, hence $|p'(z)| = n|F(z)| \le nM$. Taking the maximum over $|z| = 1$ yields the claim. □

Remark 1.2 (Sharpness)**.** Equality holds for $p(z) = c\, z^n$ (any $c \in \mathbb{C}$), since $\max_{|z|=1} |p'(z)| = n|c| = \; n \max_{|z|=1} |p(z)|$. More generally, equality occurs for rotations of such extremals.

Theorem 1.3 (Erdős–Lax refinement)**.** *If p has no zeros in the open unit disk $\{z : |z| < 1\}$, then*

$$\max_{|z|=1} |p'(z)| \;\le\; \frac{n}{2} \max_{|z|=1} |p(z)|,$$

and the factor $\frac{1}{2}$ is best possible.

Proof sketch. If p is zero-free in $|z| < 1$, then $F(z) := \dfrac{z\,p'(z)}{p(z)}$ is analytic on $|z| < 1$. A standard Herglotz–Carathéodory argument applied to $G(z) := 1 - \dfrac{z\,p'(z)}{n\,p(z)}$ (which has nonnegative real part on $|z| < 1$) gives $|z\,p'(z)| \le \frac{n}{2}|p(z)|$ for $|z| = 1$. Sharpness is witnessed by extremals of the form $p(z) = c\,(z-a)^n$ with $|a| > 1$ (limit cases as $|a| \downarrow 1$). $\square$

Remark 1.4 (Comparison with Markov on $[-1, 1]$)**.** On the interval $[-1, 1]$ one has Markov's inequality $\max_{[-1,1]} |q'| \le n^2 \max_{[-1,1]} |q|$ for algebraic polynomials q, illustrating the stronger n^2 growth on an interval versus the n (or $n/2$) growth on the unit circle.

2 Bernstein on the circle, Erdős–Lax, and contrasts with Markov on $[-1, 1]$

2.1 Context and scope

This note distills our discussion of Bernstein-type inequalities for complex polynomials on the unit circle, the Erdős–Lax refinement in the zero-free case, and how these compare with the classical Markov inequality on the real interval $[-1, 1]$. We also record the (separate) Bernstein polynomials used in the constructive proof of the Weierstrass approximation theorem and explain how the various "Bernstein" results are related in spirit but address different questions.

2.2 Bernstein's inequality on the unit circle

Theorem 2.1 (Bernstein inequality on $\mathbb{T}$)**.** *Let p be a complex polynomial of degree n. Then*

$$\max_{|z|=1} |p'(z)| \ \le\ n \max_{|z|=1} |p(z)|,$$

and the constant n is sharp.

Proof sketch (Cauchy/maximum-modulus). Fix $R > 1$ and apply Cauchy's integral formula for derivatives to p on the circle $|w| = R$:

$$p'(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{p(w)}{(w-z)^2}\, dw \qquad (|z| \le 1).$$

Taking moduli and using $\min_{|w|=R,\,|z|=1} |w - z| = R - 1$ gives $|p'(z)| \le \frac{R}{(R-1)^2} \max_{|w|=R} |p(w)|$. By the maximum modulus principle, $\max_{|w|=R} |p(w)| \le R^n \max_{|u|=1} |p(u)|$. Optimizing in R yields the inequality with constant n (e.g. let $R \downarrow 1$ and use a standard smoothing argument, or consider $f(\zeta) := \zeta p'(\zeta)/(np(\zeta))$ which is analytic in $|\zeta| < 1$ and has $|f| \le 1$ on $|\zeta| = 1$). Sharpness holds for $p(z) = c\,z^n$, where equality is attained. $\square$

Remark 2.2 (Trigonometric form)**.** If $T(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ is a trigonometric polynomial of degree n, then

$$\|T'\|_{L^\infty(\mathbb{T})} \le n\,\|T\|_{L^\infty(\mathbb{T})}.$$

This is the same inequality under the identification $p(e^{i\theta}) = T(\theta)$.

2.3 Erdős–Lax refinement (zero-free case)

Theorem 2.3 (Erdős–Lax)**.** *If p has no zeros in $\{z : |z| < 1\}$, then*

$$\max_{|z|=1} |p'(z)| \;\le\; \frac{n}{2} \max_{|z|=1} |p(z)|,$$

and the factor $\frac{1}{2}$ is best possible.

Proof sketch. When p is zero-free in the disk, the function $F(z) := \dfrac{z\,p'(z)}{p(z)}$ is analytic for $|z| < 1$. Consider $G(z) := 1 - \frac{1}{n}F(z) = 1 - \frac{z\,p'(z)}{n\,p(z)}$, which has nonnegative real part in the unit disk (Herglotz–Carathéodory theory for analytic functions with positive real part). Boundary estimates for such functions imply $|F(e^{it})| \le \frac{n}{2}$ for a.e. t, hence the stated inequality on $|z| = 1$. Sharpness is witnessed by extremals tending to $p(z) = c\,(z-a)^n$ with $|a| \downarrow 1$. $\square$

2.4 Markov's inequality on $[-1, 1]$ and a precise contrast

Theorem 2.4 (Markov, 1889)**.** *If q is a real (algebraic) polynomial of degree n, then*

$$\max_{x\in[-1,1]} |q'(x)| \;\le\; n^2 \max_{x\in[-1,1]} |q(x)|,$$

and the exponent n^2 is sharp (attained, up to the constant, by the Chebyshev polynomials T_n).

Proof sketch and extremals. Normalize so that $\|q\|_{[-1,1]} \le 1$. Among all degree-n polynomials with unit sup-norm, $T_n(\cos\theta) = \cos(n\theta)$ oscillates between ± 1 with equal ripple and has maximal derivative at the endpoints: $T_n'(1) = n^2$ and $\|T_n'\|_{[-1,1]} = n^2$. A standard argument (via convexity of the feasible set and extreme point/ripple characterizations, or via potential theory/Markov brothers' method) shows T_n is extremal, yielding the sharp constant n^2. $\square$

Why n on $\mathbb{T}$ but n^2 on $[-1, 1]$? Two complementary viewpoints:

- *Geometry of the domain.* On the smooth closed curve $|z| = 1$, Cauchy's estimate couples the growth of p' linearly with degree (one derivative $\leftrightarrow$ one power of n). On $[-1, 1]$, the boundary has endpoints where extremals concentrate curvature (boundary layer near ± 1). This produces an extra factor of n.

- *Conformal map heuristic.* The Joukowski map $J(\zeta) = \frac{1}{2}(\zeta + \zeta^{-1})$ sends the circle to the interval. Pulling a degree-n algebraic polynomial on $[-1, 1]$ back to the circle yields a *sum of two* degree-n terms (from ζ^k and ζ^{-k}), and differentiation on the circle then costs a factor n while the pullback/pushforward introduces another factor n, giving the n^2 scaling.

Quantitative refinements on $[-1, 1]$. More precise inequalities localize the derivative size:

$$|q'(x)| \;\le\; \frac{n^2}{\sqrt{1-x^2}}\,\|q\|_{[-1,1]}, \qquad |x| < 1,$$

and near the endpoints $x = \pm 1$ the $\sqrt{1-x^2}$ denominator captures the boundary layer responsible for the n^2 growth.

2.5 Further connections and nearby inequalities

- **L^p versions on $\mathbb{T}$.** For $1 \le p \le \infty$ and trigonometric polynomials T of degree n, $\|T'\|_{L^p(\mathbb{T})} \le n\,\|T\|_{L^p(\mathbb{T})}$ (Riesz–Zygmund/Nikolskii-type estimates).
- **Zero-free gains.** Erdős–Lax (Thm. 2.3) demonstrates how excluding zeros from the disk halves the constant from n to $n/2$. Analogous "stability under zero separation" phenomena appear in inequalities on other domains.
- **Intervals and orthogonal bases.** On $[-1,1]$, Chebyshev polynomials serve as extremals for both the value problem (minimax deviation) and the derivative problem (Markov). This dual role explains the sharpness and suggests practical preconditioners in spectral methods.

2.6 Bernstein polynomials for approximation (a different "Bernstein")

Distinct from the derivative inequality, Bernstein's constructive proof of the Weierstrass theorem uses the positive linear operators

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \qquad x \in [0,1],$$

which satisfy $B_n f \to f$ uniformly on $[0,1]$ for every continuous f. These polynomials are linked in spirit (they control oscillation and are built from binomial bases) but address approximation of *functions* rather than derivative bounds for a *given polynomial*. The positivity and shape preservation of B_n provide quantitative moduli-of-continuity estimates:

$$\|B_n f - f\|_{[0,1]} \;\le\; \omega\left(f; \sqrt{\frac{x(1-x)}{n}}\right),$$

and, for Lipschitz f, a global $O(n^{-1/2})$ rate.

2.7 Summary table

Setting	**Inequality**	**Sharp constant / extremals**
Unit circle $\{\lvert z\rvert = 1\}$	$\max\limits_{\lvert z\rvert=1} \lvert p'\rvert \le n \max\limits_{\lvert z\rvert=1} \lvert p\rvert$	n; $p(z) = c\,z^n$
Zero-free in $\lvert z\rvert < 1$	$\max\limits_{\lvert z\rvert=1} \lvert p'\rvert \le \frac{n}{2} \max\limits_{\lvert z\rvert=1} \lvert p\rvert$	$\frac{n}{2}$; limiting $(z-a)^n$, $\lvert a\rvert \downarrow 1$
Interval $[-1,1]$	$\max\limits_{[-1,1]} \lvert q'\rvert \le n^2 \max\limits_{[-1,1]} \lvert q\rvert$	n^2; $q = T_n$

2.8 Drop-in theorems for use elsewhere

Theorem 2.5 (Bernstein, unit circle)**.** *Let p be a degree-n polynomial. Then $\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|$, with equality for $p(z) = c\,z^n$.*

Theorem 2.6 (Erdős–Lax)**.** *If p has no zeros in $|z| < 1$, then* $\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|$, *and $\frac{1}{2}$ is sharp.*

Theorem 2.7 (Markov)**.** *For a real polynomial q of degree n,* $\max_{x\in[-1,1]} |q'(x)| \leq n^2 \max_{x\in[-1,1]} |q(x)|$, *with sharpness at $q = T_n$.*

2.9 Practical takeaways

- On smooth closed curves (e.g. the circle), a single derivative costs a single power of n; excluding interior zeros improves the constant by a factor 2.
- On intervals with endpoints, boundary layers inflate the growth to n^2; Chebyshev structure captures the extremal behavior and guides numerical design.
- "Bernstein polynomials" for approximation are different objects: positive linear approximants that converge uniformly and provide constructive bounds for $\|B_n f - f\|$.

References

[1] S. Boucheron, G. Lugosi, and P. Massart (2013). *Concentration Inequalities: A Nonasymptotic Theory of Independence.* Oxford University Press.