1 Realization of the Cone of Positive Polynomials via SDP

The study of non-negative polynomials is central to real algebraic geometry and polynomial optimization. While the condition of global non-negativity is computationally intractable in general, it can be effectively approximated—and in many cases exactly realized—by the cone of sums of squares, which admits a tractable characterization using Semidefinite Programming (SDP).

1.1 The Cones $P_{n,2d}$ and $\Sigma_{n,2d}$

Let $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ denote the ring of polynomials in n variables with real coefficients. We focus on polynomials of even degree $2d$, as odd-degree polynomials cannot be globally non-negative (they must go to $-\infty$ in some direction).

We define two primary convex cones of interest:

- **The Non-negative Cone ($P_{n,2d}$):** The set of polynomials of degree at most $2d$ that are non-negative everywhere on $\mathbb{R}^n$:

$$P_{n,2d} = \{p(x) \in \mathbb{R}[x]_{2d} \mid p(x) \geq 0 \quad \forall x \in \mathbb{R}^n\}$$

- **The Sum of Squares (SOS) Cone ($\Sigma_{n,2d}$):** The set of polynomials that can be explicitly written as a sum of squares of other polynomials:

$$\Sigma_{n,2d} = \left\{ p(x) \in \mathbb{R}[x]_{2d} \mid p(x) = \sum_{i=1}^{k} h_i(x)^2, \quad h_i(x) \in \mathbb{R}[x]_d \right\}$$

It is evident that $\Sigma_{n,2d} \subseteq P_{n,2d}$. Hilbert's seminal work in 1888 established that this inclusion is strict for most cases (specifically, whenever $n \geq 3$ and $2d \geq 4$, with the exception of $n = 3, 2d = 4$).

1.2 The Gram Matrix Representation

The fundamental computational realization of $\Sigma_{n,2d}$ relies on the **Gram matrix** representation. This representation translates the algebraic structure of SOS polynomials into the linear algebraic structure of positive semidefinite matrices.

Let $\mathbf{v}_d(x)$ be the vector of all monomials in variables $x_1, \ldots, x_n$ up to degree d. The dimension of this vector is $N = \binom{n+d}{d}$. For example, if $n = 2$ and $d = 2$, a standard basis choices is:

$$\mathbf{v}_2(x, y) = [1, x, y, x^2, xy, y^2]^T$$

Any polynomial $p(x)$ of degree $2d$ can be written as a quadratic form in these monomials:

$$p(x) = \mathbf{v}_d(x)^T Q \mathbf{v}_d(x) = \sum_{i=1}^{N} \sum_{j=1}^{N} Q_{ij} v_i(x) v_j(x) \tag{1}$$

where $Q \in \mathbb{R}^{N \times N}$ is a symmetric matrix, referred to as the Gram matrix of $p(x)$.

Crucially, this representation is not unique. Different matrices Q can yield the same polynomial $p(x)$ because different products of monomials can result in the same term (e.g., $x_1 \cdot x_1 = x_1^2$, but $1 \cdot x_1^2$ is also x_1^2). The set of Gram matrices that represent a specific $p(x)$ forms an affine subspace of the space of symmetric matrices.

1.3 Semidefinite Programming Formulation

The connection between SOS polynomials and SDP is established by the following core theorem, which provides a completely tractable characterization of $\Sigma_{n,2d}$.

Theorem 1.1 (SOS-SDP Equivalence). *A polynomial $p(x) \in \mathbb{R}[x]_{2d}$ is a sum of squares if and only if there exists a symmetric positive semidefinite matrix $Q \succeq 0$ such that $p(x) = \mathbf{v}_d(x)^T Q \mathbf{v}_d(x)$.*

This theorem allows us to pose the membership problem $p(x) \in \Sigma_{n,2d}$ as a semidefinite feasibility problem:

$$\begin{aligned} \text{Find } & Q \in \mathbb{S}^N \\ \text{subject to } & Q \succeq 0 \\ & \mathbf{v}_d(x)^T Q \mathbf{v}_d(x) = p(x) \quad \text{(coefficient matching)} \end{aligned}$$

The equality constraint $\mathbf{v}_d(x)^T Q \mathbf{v}_d(x) = p(x)$ is equivalent to a set of linear equality constraints on the entries Q_{ij}. Specifically, if $p(x) = \sum_\alpha c_\alpha x^\alpha$, we require:

$$\sum_{i,j: v_i(x) v_j(x) = x^\alpha} Q_{ij} = c_\alpha \quad \forall \alpha$$

This is precisely the standard form of an SDP feasibility problem.

1.4 Explicit SOS Construction via Matrix Factorization

If a Gram matrix $Q \succeq 0$ is found, we can explicitly construct the polynomials $h_i(x)$ that comprise the sum of squares. This relies on the fact that any positive semidefinite matrix admits a factorization of the form $Q = L^T L$ (or similar decompositions like Cholesky or spectral).

Let $Q \succeq 0$ have rank r. We can perform a spectral decomposition:

$$Q = \sum_{i=1}^{r} \lambda_i \mathbf{u}_i \mathbf{u}_i^T$$

where $\lambda_i > 0$ are the non-zero eigenvalues and $\mathbf{u}_i \in \mathbb{R}^N$ are the corresponding eigenvectors.

Substituting this back into the Gram representation:

$$\begin{aligned} p(x) &= \mathbf{v}_d(x)^T \left(\sum_{i=1}^{r} \lambda_i \mathbf{u}_i \mathbf{u}_i^T \right) \mathbf{v}_d(x) \\ &= \sum_{i=1}^{r} \lambda_i \left(\mathbf{v}_d(x)^T \mathbf{u}_i \right) \left(\mathbf{u}_i^T \mathbf{v}_d(x) \right) \\ &= \sum_{i=1}^{r} \left(\sqrt{\lambda_i} \mathbf{u}_i^T \mathbf{v}_d(x) \right)^2 \end{aligned}$$

By defining the polynomials $h_i(x) = \sqrt{\lambda_i} \mathbf{u}_i^T \mathbf{v}_d(x)$, we arrive at the explicit SOS decomposition:

$$p(x) = \sum_{i=1}^{r} h_i(x)^2$$

The vector $\mathbf{u}_i$ serves precisely as the coefficient vector for the polynomial $h_i(x)$ in the basis $\mathbf{v}_d(x)$.

Remark 1.1. *While spectral decomposition is theoretically clean, in practice, a Cholesky decomposition $Q = L^T L$ (if $Q \succ 0$) or an LDL^T decomposition is often used numerically. If $Q = L^T L$, with rows of L denoted by ℓ_i, then $h_i(x) = \ell_i \cdot \mathbf{v}_d(x)$.*

1.5 Example

Consider the univariate polynomial $p(x) = 2x^4 + 2x^2 + 5$. Here $n = 1, 2d = 4$, so we use the basis $\mathbf{v}_2(x) = [x^2, x, 1]^T$. We seek $Q \in \mathbb{S}^3$ such that:

$$2x^4 + 2x^2 + 5 = \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}$$

Expanding the right side yields $q_{11}x^4 + 2q_{12}x^3 + (2q_{13} + q_{22})x^2 + 2q_{23}x + q_{33}$. Matching coefficients provides the linear constraints:

$$\begin{aligned} q_{11} &= 2 \\ 2q_{12} &= 0 \implies q_{12} = 0 \\ 2q_{13} + q_{22} &= 2 \\ 2q_{23} &= 0 \implies q_{23} = 0 \\ q_{33} &= 5 \end{aligned}$$

A feasible solution is $q_{13} = 0, q_{22} = 2$, giving the diagonal matrix $Q = \operatorname{diag}(2, 2, 5)$. Since all diagonal entries are positive, $Q \succeq 0$. The decomposition is immediate:

$$p(x) = (\sqrt{2}x^2)^2 + (\sqrt{2}x)^2 + (\sqrt{5})^2$$

demonstrating $p(x) \in \Sigma_{1,4}$.