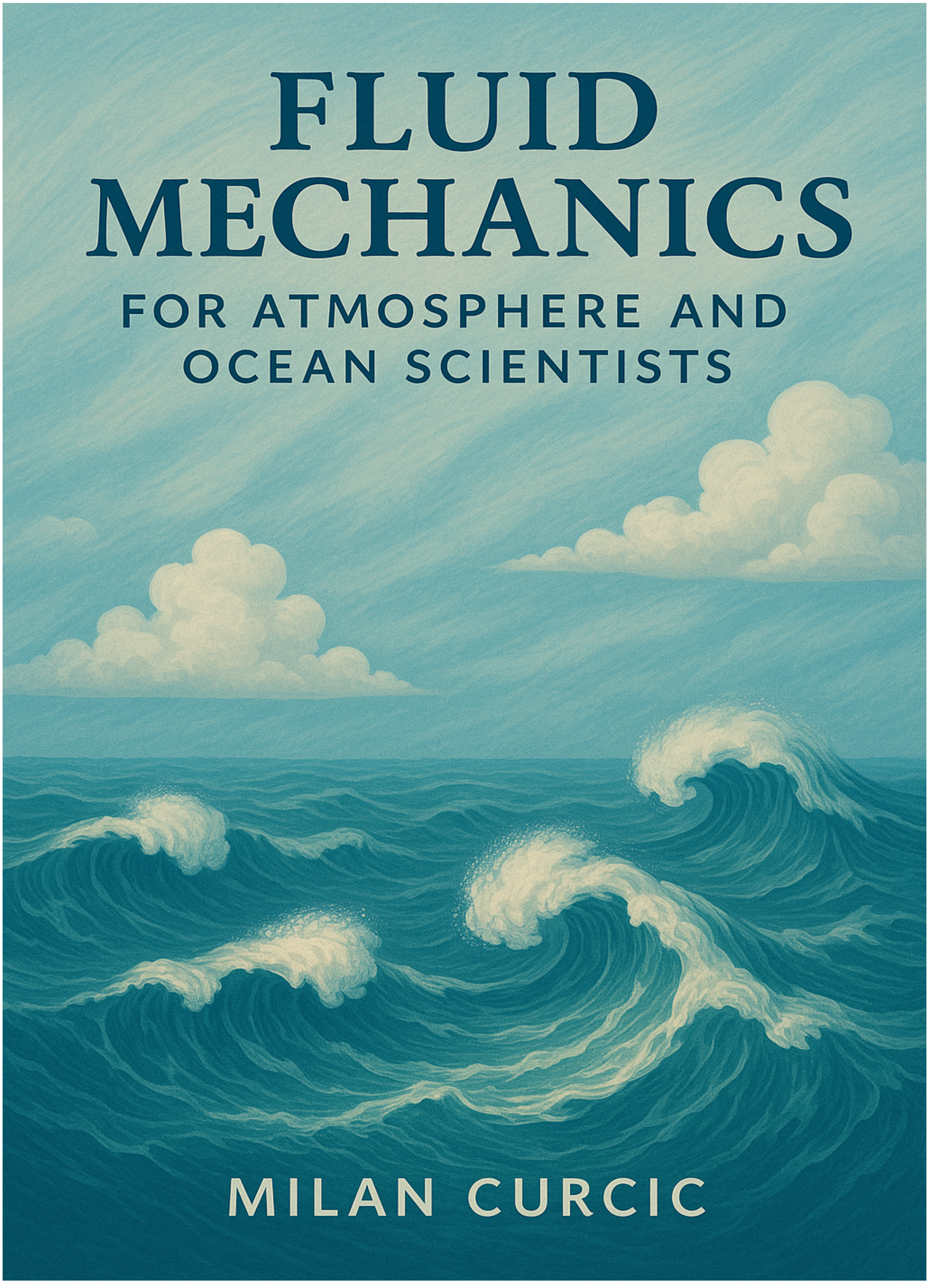
# FLUID MECHANICS

## FOR ATMOSPHERE AND OCEAN SCIENTISTS



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# Fluid Mechanics for Atmosphere and Ocean Scientists

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## Contents

*For Chris Piasecki, who understood the word "ocean"*

# Preface

This book grew out of lecture notes for a graduate-level fluid mechanics course that I teach at the University of Miami's Rosenstiel School of Marine, Atmospheric, and Earth Science. The course provides advanced undergraduate students or beginning graduate students with a solid foundation in fluid mechanics concepts that are essential for understanding ocean physics. While there are many excellent fluid mechanics textbooks available, most are written for either for engineering students with a focus on industrial applications, or for students of meteorology and oceanography that focus on geophysical flows. Many of our students at Rosenstiel go on to pursue research topics related to turbulence, boundary layers, and ocean surface waves, and these topics are well covered by different textbooks in these respective subdisciplines. This book bridges that gap by presenting a concise yet comprehensive treatment of classical, geophysical, turbulent, and wavy fluid mechanics processes that are relevant to atmospheric and ocean physics research.

The book progresses from fundamental concepts to increasingly complex topics. We begin with a review of vector calculus before introducing core fluid mechanics principles such as conservation of mass and momentum. The effects of Earth's rotation and density stratification, which are crucial for atmospheric and ocean dynamics, come next. We then explore simplified yet powerful models like the shallow water equations which allow an analytical examination of most common solutions. Later chapters cover turbulence, boundary layers, and surface gravity waves - phenomena that are ubiquitous in the ocean and that are becoming increasingly important in coupled weather-ocean prediction and climate projections.

I aim to balance mathematical rigor with physical intuition throughout the text. Detailed derivations are provided, but equal emphasis is placed on understanding the underlying physics. Examples and figures help illustrate key concepts. This textbook is a work in progress and continues to evolve. I welcome feedback from students and colleagues on how to best improve it. I thanks my students Susan Harrison, Jack Lee, Mia Vallee, and Jessie Yang for their contributions so far. Special thanks to Prof. Mike Brown who previously taught this course and who gave me valuable advice on preparing for it. My hope is that this textbook will serve as a useful resource for students beginning their journey into atmospheric and/or ocean physics research.

# 1 Introduction

## 1.1 What will you learn in this course

The course aims to provide students with a solid understanding of key fluid mechanics concepts that are used in ocean physics research. This course will first refresh you on the vector calculus needed to understand fluid mechanics, and introduce the Eulerian and Lagrangian views of the flow. We then proceed to derive the conservation equations for mass and momentum. We will then consider the effects of rotation and stratification, and look at some steady solutions of the rotating Navier-Stokes equations. Then, we will simplify the flow by looking at it as a thin layer of incompressible rotating fluid that flows over a variable bottom topography and a free surface. From there, we will study turbulence and boundary layers, and complete the course with the linear theory of surface gravity waves. By the end of the course, you will be proficient in applying fluid mechanical concepts and mathematical tools to solve many ocean physics research problems. It will also prepare you for more specialized courses on turbulence, waves, and atmospheric and oceanic circulation.

## 1.2 Reference textbooks

No single textbook out there covers all the topics that we need for this course. However, parts of this course are covered in detail by various textbooks. These lecture notes are based on the following textbooks:

1. *Fluid Mechanics*, $7^{th}$ Ed., by Kundu, Cohen, Dowling, and Capecelatro (Elsevier), for the classical;

2. *Atmospheric and Oceanic Fluid Dynamics* (AOFD) by Geoffrey Vallis (Cambridge University Press), for the geophysical;

3. *Turbulent Flows* by Stephen Pope (Cambridge University Press), for the turbulent;

4. *Water Wave Mechanics for Engineers and Scientists* by Dean and Dalrymple (World Scientific), for the wavy.

While the notes contain the distilled and required information for you to succeed in this course, please refer to these textbooks for more detailed explanations and examples. Over time, the lecture notes will evolve toward a unified, coherent, and self-contained book.

# 2 Review of vector calculus

In this section we will review the necessary concepts from vector calculus that we will use in this course. These include: scalars, vectors and tensors; gradient, divergence, and curl; line, surface, and volume integrals; and the Gauss and Stokes theorems.

## 2.1 Scalars, vectors, and tensors

In this book we will use three types of quantities to describe fluid properties: *scalars*, *vectors*, and *tensors*.

*Scalars* are completely described by their magnitude. Examples of scalars are temperature, pressure, or density. A value of 290 K, for example, completely describes the temperature of a fluid at some point in space and time. The fundamental scalar fields in fluid mechanics are the pressure, density, and in some derivations, the velocity potential. In the atmosphere, density is often represented with the air temperature and humidity scalars through the ideal gas law. In the ocean, density if typically represented with the water temperature and salinity scalars through the equation of state. The fundamental scalars for us are then, in this approximate order, pressure, density, temperature, water salinity, and air humidity. In equations, we will write scalars using italics, *e.g.* $T$, $p$, or $\rho$.

*Vectors* have a magnitude and a direction. Examples of vectors are velocity, acceleration, or force. In 3-dimensional Cartesian space with coordinates $(x, y, z)$, for example, vector $\mathbf{u}(x, y, z)$ can be described by its components

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{2.1}$$

where $u_x$, $u_y$, and $u_z$ (each a scalar) are the components of $\mathbf{u}$ in the $x$, $y$, and $z$ directions, respectively. This is the conventional notation, however, we will often write vectors inline as $\mathbf{u} = (u_x, u_y, u_z)$. The fundamental vector field of fluid mechanics is the velocity. Many other vector fields are derived from velocity, such as vorticity, acceleration, and force. In equations, we will write vectors using boldface, *e.g.* $\mathbf{u}$, $\mathbf{a}$, or $\mathbf{F}$.

The magnitude, or norm, of a vector $\mathbf{u}$ is written as $||\mathbf{u}||$ and calculated as

$$||\mathbf{u}|| = \sqrt{u_x^2 + u_y^2 + u_z^2} \tag{2.2}$$

Here we're working in 3-dimensional Cartesian space, but vectors can be defined in any number of dimensions, and the above definitions generalize exactly how you'd expect them to. The most ubiquitous vector field in fluid mechanics is the velocity. In atmospheres and oceans, we will often refer to the velocity as wind and current, respectively. Wind speed is thus the magnitude (norm) of the wind vector, and likewise for the current speed.

*Tensors* have magnitude, direction, and orientation. They are vectors that act on each respective surface orthogonal to the direction of the tensor. Arguably the most important tensor in fluid mechanics is the stress tensor. In 3-dimensional space, for example, a stress tensor can be described as:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \tag{2.3}$$

In this notation and index ordering, i.e. $\tau_{ij}$, the first index ($i$) refers to the direction of the stress component, and the second index ($j$) refers to the direction of the normal to the surface. In other words, each row of the tensor contains the three components of a vector, and each column contains the three surface normals that the stress component is acting on. For example, $\tau_{xy}$ is the stress in the x-direction and is acting on the surface whose normal is in the y-direction (and which lies in the x-z plane).

One special type of tensor is the *identity tensor* $\mathbf{I}$, which is a tensor that maps a vector onto itself. In Cartesian coordinates, it is given by:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.4}$$

It may be useful to think of scalars as $0^{th}$-order tensors, vectors as $1^{st}$-order tensors, and tensors as $2^{nd}$-order tensors.

## 2.2 Unit vectors

Unit vectors are vectors with magnitude of 1. A popular notation for unit vectors in Cartesian coordinates is $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, which point in the $x$, $y$, and $z$ directions, respectively. So, a vector $\mathbf{u}$ can be written as

$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k} \tag{2.5}$$

Notice that you can get the unit vector by dividing any vector by its magnitude, i.e. $\mathbf{u}/||u||$.

## 2.3 Vector operations

Two vectors can be added, subtracted, or multiplied. Although vector addition and subtraction are straightforward (simply add or subtract each of their respective scalar components), vector multiplication is more interesting. There are many ways to multiply two vectors, but the two most important ones for us are the *dot product* and the *cross product*.

### 2.3.1 Dot product

The dot product of two 3-dimensional Cartesian vectors $\mathbf{a}$ and $\mathbf{b}$ is an element-wise sum of their components (and thus, a scalar!):

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3 \tag{2.6}$$

More generally, the dot product of two n-dimensional vectors **a** and **b** is

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \tag{2.7}$$

The dot product is commutative, meaning that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

The magnitude of a dot product of two vectors is equal to the product of their magnitudes and the cosine of the angle $\theta$ between them:

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos\theta \tag{2.8}$$

To visualize this relationship, take one vector and project it onto the other. This projection is the magnitude of the vector times the cosine of the angle between them. Now, one vector and the projection of the other onto the first vector are pointing in the same direction, so their dot product is the product of their magnitudes. It can be useful to think of a dot product as collapsing the two vectors into a single scalar that contains contributions from each of their components.

The following listing shows how to manually compute the dot product of two vectors in Python using the built-in arithmetic operators:

```python
import numpy as np

# initialize two vectors; specific values are arbitrary.
a = np.array([1, 2, 3])
b = np.array([4, 5, 6])

c = 0  # initialize the result variable
for i in range(a.size):  # loop over indices of the vector
    c += a[i] * b[i]  # multiply elements and add to the result
```

Notice that this an exact implementation of the right-hand side of Eq. (2.7). The NumPy library, however, allows element-wise multiplication of vectors, which is both more computationally efficient and more concise:

```python
c = np.sum(a * b)  # multiply element-wise and sum up the components
```

Notice that this is an exact implementation of the middle part of Eq. (2.7). Even though the dot product is simple to implement, as we did above, NumPy provides a function that is even more concise, and likely the most efficient way to compute the dot product:

```python
c = np.dot(a, b)
```

Although it's important to understand how to implement the fundamental vector operations by hand, and do it yourself at least once, in practice it's best to use established libraries such as NumPy, as they are well tested and optimized for computational efficiency.

### 2.3.2 Cross product

The cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as:

$$\mathbf{a} \times \mathbf{b} = det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \tag{2.9}$$

where $det(\mathbf{M})$ means the *determinant of matrix* $\mathbf{M}$.

Using the so-called *rule of Sarrus*, the cross product can be calculated as:

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} \tag{2.10}$$

or:

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \tag{2.11}$$

The result of a cross product is a vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Its orientation in space is determined by the right-hand rule: if you point your right thumb in the direction of $\mathbf{a}$ and your index finger in the direction of $\mathbf{b}$, then your middle finger will point in the direction of $\mathbf{a} \times \mathbf{b}$.

The magnitude of the cross product is equal to the product of the magnitudes of the two vectors times the sine of the angle between them:

$$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||||\mathbf{b}|| \sin\theta \tag{2.12}$$

So, the magnitude of the cross product is largest when the two vectors are orthogonal. Unlike the dot product, the cross product is anticommutative, meaning that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

In fluid mechanics, a cross product will often come up when we are interested in the rotation of a vector field. For example, vorticity is the curl of the velocity field.

## 2.4 Matrix multiplication

Occasionally, we will need to multiply a vector by a matrix, or, a matrix by a matrix. As a vector is a special case of a matrix in which either the number of rows or columns is 1, the same rules of matrix multiplication will apply when we multiply a vector by a matrix or a matrix by a matrix. These operations are not commutative, meaning that the order of multiplication matters.

Take two matrices $\mathbf{A}$ and $\mathbf{B}$ such that

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{2.13}$$

and:

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{2.14}$$

The result of their multiplication is a matrix **C** given by:

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix} \tag{2.15}$$

That is, the entry $c_{ij}$ of the product is obtained by multiplying term-by-term the entries of the $i$-th row of **A** and the $j$-th column of **B**, and summing these products. In other words, $c_{ij}$ is the dot product of the $i$-th row of **A** and the $j$-th column of **B**. Although the matrices are not required to be square, the number of columns of **A** must be equal to the number of rows of **B**.

## 2.5 Total and partial derivatives

We will denote total and partial derivative operators (for example, in time $t$) as $\frac{d}{dt}$ and $\frac{\partial}{\partial t}$. Scalars, vectors, and tensors alike can be differentiated with respect to any variable. A derivative of a vector is simply a vector of derivatives of its components:

$$\frac{d\mathbf{u}}{dt} = \left(\frac{du_x}{dt}, \frac{du_y}{dt}, \frac{du_z}{dt}\right) = \frac{du_x}{dt}\mathbf{i} + \frac{du_y}{dt}\mathbf{j} + \frac{du_z}{dt}\mathbf{k} = \begin{bmatrix} \frac{du_x}{dt} \\ \frac{du_y}{dt} \\ \frac{du_z}{dt} \end{bmatrix} \tag{2.16}$$

$$\frac{\partial\mathbf{u}}{\partial t} = \left(\frac{\partial u_x}{\partial t}, \frac{\partial u_y}{\partial t}, \frac{\partial u_z}{\partial t}\right) = \frac{\partial u_x}{\partial t}\mathbf{i} + \frac{\partial u_y}{\partial t}\mathbf{j} + \frac{\partial u_z}{\partial t}\mathbf{k} = \begin{bmatrix} \frac{\partial u_x}{\partial t} \\ \frac{\partial u_y}{\partial t} \\ \frac{\partial u_z}{\partial t} \end{bmatrix} \tag{2.17}$$

and likewise for tensors.

## 2.6 Gradient, divergence, and curl

Now, we introduce another operator that builds on top of previous concepts to describe how scalar and vector fields vary in space. This operator is called *del* and is denoted by the symbol $\nabla$ (pronounced "nabla"):

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{2.18}$$

Written as above, $\nabla$ cannot stand on its own but must be applied as an operator to a field. A good way to think about $\nabla$ is as of a *differential operator*, which itself is a 3-dimensional vector that can operate on scalars or vectors. Specifically:

- $\nabla p$ is as vector that is a gradient of a scalar field $p$; it quantifies how $p$ changes in space.
- $\nabla \cdot \mathbf{u}$ is a scalar that is the divergence of a vector field $\mathbf{u}$; it quantifies how $\mathbf{u}$ flows out of a point.
- $\nabla \times \mathbf{u}$ is a vector that is the curl of a vector field $\mathbf{u}$; it quantifies how $\mathbf{u}$ rotates around a point.

Although, strictly speaking, one is a symbol and the other is an operator, $\nabla$ ("nabla") and "del" are often used interchangeably when reading equations out loud.

### 2.6.1 Gradient

The gradient of a scalar field $T$ is a vector field that points in the direction of the greatest rate of increase of $T$. It is denoted by $\nabla T$ and is defined as

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} \tag{2.19}$$

Gradient of a scalar field is a vector that points in the direction of the steepest increase of that field, and its magnitude is the rate of that increase. Imagine hiking up a hill; the gradient of the terrain is a vector that is pointing toward the steepest incline, and its magnitude is the steepness of that incline.

### 2.6.2 Divergence

The divergence of a vector field $\mathbf{u}$ is a scalar field that describes the rate at which the vector field flows out of a point. It is denoted by $\nabla \cdot \mathbf{u}$ and is defined as

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{2.20}$$

Divergence of a vector field is a scalar that describes how much the vector field is expanding or contracting at a point. Negative divergence is called convergence.

### 2.6.3 Curl

The curl of a vector field $\mathbf{u}$ is a vector field that describes the rotation of the vector field. It is denoted by $\nabla \times \mathbf{u}$ and is defined as

$$\nabla \times \mathbf{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k} \tag{2.21}$$

Curl of a vector field is another vector that is orthogonal to the original vector field and quantifies how much the vector field is rotating around a point. When curl is zero, the vector field is said to be *irrotational*.

### 2.6.4 Laplacian

The Laplacian is a second-order differential operator that can be applied to both scalar and vector fields. It measures the rate at which field varies in space and is defined as:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.22}$$

Applied to a scalar field $T$, it is:

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \tag{2.23}$$

Applied to a vector $\mathbf{u} = (u, v, w)$, it is applied to each component:

$$\nabla^2 \mathbf{u} = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{bmatrix} \tag{2.24}$$

The Laplacian of a scalar is thus a scalar and the Laplacian of a vector is a vector. In some literature you will see the Laplacian written as $\Delta$, but here we will use $\nabla^2$ to avoid confusion with the $\Delta$ that we use to denote a finite increment.

### 2.6.5 Useful vector identities

Curl of a gradient of a scalar field is always zero:

$$\nabla \times (\nabla T) = 0 \tag{2.25}$$

Further, divergence of a curl of a vector field is always zero:

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0 \tag{2.26}$$

Finally, curl of a curl of a vector field is:

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} \tag{2.27}$$

Some of these identities will come handy when we derive the conservation of vorticity laws.

## 2.7 Gauss and Stokes theorems

The most useful in our work will be variants of the *Gauss and Stokes theorems*. The Gauss theorem relates a volume integral of a divergence of a vector field to a surface integral of that vector field. The Stokes theorem relates a surface integral of the curl of a vector field to a line integral of that vector field. Here, they are stated for reference, and we'll explore their meaning and application in more detail as we use them to derive the fundamental equations for fluid flows.

### 2.7.1 Gauss theorem

The Gauss theorem states that the volume integral of the divergence of a vector field $\mathbf{u}$ over a volume $V$ is equal to the surface integral of $\mathbf{u}$ over the surface $A$ that encloses $V$:

$$\int_V \nabla \cdot \mathbf{u} dV = \oint_A \mathbf{u} \cdot d\mathbf{A} \tag{2.28}$$

In other words, the rate of change of the fluid mass within a volume is equal to the flow normal through the surface that encloses that volume. This form of Gauss's theorem is also known as the *divergence theorem*. It will come in handy when we derive the conservation of mass (continuity) equation.

### 2.7.2 Stokes theorem

The Stokes theorem states that the surface integral of the curl of a vector field $\mathbf{u}$ over a surface $A$ is equal to the line integral of $\mathbf{u}$ over the boundary of $A$:

$$\int_A (\nabla \times \mathbf{u}) \cdot d\mathbf{A} = \oint_{\partial A} \mathbf{u} \cdot d\mathbf{l} \tag{2.29}$$

In other words, the rotation rate of the fluid over a surface area is equal to the flow velocity integrated around the boundary of that surface.

## Summary

In this chapter, we reviewed:

- Scalars, vectors, and tensors;
- Vector algebra: dot product ($\mathbf{a} \cdot \mathbf{b}$) and cross product ($\mathbf{a} \times \mathbf{b}$);
- Derivatives: total ($\frac{d}{dt}$) and partial ($\frac{\partial}{\partial t}$);
- Gradient, divergence ($\nabla \cdot \mathbf{u}$), and curl ($\nabla \times \mathbf{u}$);
- Gauss theorem that relates volume and surface integrals: $\int_V \nabla \cdot \mathbf{u} dV = \oint_A \mathbf{u} \cdot d\mathbf{A}$;
- Stokes theorem that relates surface and line integrals: $\int_A (\nabla \times \mathbf{u}) \cdot d\mathbf{A} = \oint_{\partial A} \mathbf{u} \cdot d\mathbf{l}$.

These concepts will serve as the basic building blocks for everything that follows in the remainder of this course.

## Exercises

1. Pick your favorite programming language (or ask for a recommendation for one). Write a program that defines a scalar, a vector, and a tensor, and assign numerical values to them. Print the values to the screen. Is there a difference in how you define them in your program?

2. What is the dot product of two orthogonal vectors? How about the dot product of a vector with itself? Please write out the solution step by step.

3. Write a program that calculates the cross product of two vectors. Please implement your solution using the basic arithmetic operations such as addition and multiplication. Then, see if your programming language or one of its software libraries provides a function to do this. Can you verify your implementation by comparing its output to that of the library function?

4. How would you calculate a derivative of a quantity (scalar, for example) in a computer program, e.g. $\frac{\partial a}{\partial x}$? Consider that you can approximate a derivative as a difference between two values of the quantity at two points in space. In other words, assume $\partial a \approx \Delta a = a(x_2) - a(x_1)$, and similar for $x$.

5. Write a computer program that calculates the gradient of a scalar field, and the divergence and curl of a vector field.

6. Draw example vector fields that are: (a) non-divergent and irrotational, (b) divergent and irrotational, (c) non-divergent and rotational, and (d) divergent and rotational.

# 3 Fluid kinematics

Fluid kinematics describe the fluid motion without considering the forces that cause that motion. We will explore two main views of the flow: the *Lagrangian* view, which follows individual fluid particles, and the *Eulerian* view, which observes the flow at fixed points in space. Although the Eulerian (fixed-point) view is more commonly used in the theory and simulation of fluid flows, the Lagrangian (particle-following) view will be essential when deriving some of the fundamental equations, as well as for understanding where certain features of the flow come from. Both approaches are often used together in numerical simulations. Flows are typically simulated in the Eulerian framework on a fixed grid, and for many applications the flow is analyzed *a posteriori* and/or visualized in the Lagrangian framework. For example, picture a high-speed flow simulation around an aircraft that is modeled on a fixed grid, and particle-following trajectories drawn to visualize the turbulent wake behind the vessel. Another example is the Lagrangian evolution of an oil spill in the ocean or a volcanic plume in the atmosphere, derived from Eulerian simulation output.

We will also introduce some useful concepts to describe the flow, namely the *velocity potential* and the *stream function*. These two scalar quantities are complementary to the vector field of velocity and together provide a complete description of the flow.

## 3.1 Lagrangian and Eulerian derivatives of a fluid property

We will start by first drawing a distinction between the Lagrangian and Eulerian derivatives. Consider a 3-dimensional quantity $\varphi$ that varies in space and time such that $\varphi = \varphi(x, y, z, t)$. This can be a scalar, a vector, or a tensor, however, to keep things simple, suppose $\varphi$ is a scalar field. Let's find its rate of change. Since it depends on $x$, $y$, $z$, and $t$, the rate of change of $\varphi$ along each of these dimensions must be taken into account. So, the total change of $\varphi$ (let's call it $\delta\varphi$, where $\delta$ is a small but finite increment) over spatial and temporal increments $\delta x$, $\delta y$, $\delta z$, and $\delta t$, is the sum of changes along each of these dimensions:

$$\delta\varphi = \frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z + \frac{\partial\varphi}{\partial t}\delta t \tag{3.1}$$

Divide by $\delta t$ to obtain:

$$\frac{\delta\varphi}{\delta t} = \frac{\partial\varphi}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial\varphi}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial\varphi}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial\varphi}{\partial t} \tag{3.2}$$

Recall the definition of $\nabla$ (Eq. 2.18) and let the finite increment $\delta t$ approach $dt$ (and likewise for $\delta x$, $\delta y$, and $\delta z$), to obtain:

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt} + \frac{\partial\varphi}{\partial t} \tag{3.3}$$

The above is equivalent to applying the chain rule to $\varphi$ with respect to time and assuming that the spatial dimension variables are functions of time ($\varphi = \varphi(x(t), y(t), z(t), t)$). Recognize that by stating the dependence of position on time, we are implicitly stating that we are following a fluid particle. Then, recognize that the velocity in each direction is the rate of change of the position in that direction:

$$\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial x} + v\frac{\partial \varphi}{\partial y} + w\frac{\partial \varphi}{\partial z} \tag{3.4}$$

which states that the total change of $\varphi$ is due to the local (at fixed point in space) change over time, and due to spatial variations of $\varphi$ as the fluid particle moves through them. Finally, recall the definition of $\nabla$ (Eq. 2.18) to obtain:

$$\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi \tag{3.5}$$

The term $\frac{d\varphi}{dt}$ is called the *total derivative* of $\varphi$. It is also called a *Lagrangian derivative*, or *material derivative*, since it follows the motion of a fluid particle. The term $\frac{\partial \varphi}{\partial t}$ is called the *Eulerian derivative*, or *partial derivative* of $\varphi$ with respect to time. The term $\mathbf{u} \cdot \nabla \varphi$ describes how $\varphi$ changes due to its spatial variation and the flow of the fluid.

Although the term $\mathbf{u} \cdot \nabla \varphi$ is the dot product of $\mathbf{u}$ and $\nabla \varphi$, the Lagrangian derivative in Eq. 3.5 can be expressed as an operator:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \tag{3.6}$$

The parentheses on the right-hand side indicate that that term acts as an operator on a field. Like we stated for the operator $\nabla$ in the previous chapter, the total derivative operator $\frac{d}{dt}$ cannot stand on its own, but is instead applied to a field.

## 3.2 Lagrangian derivative of a volume

Consider a fluid parcel with a constant mass but whose volume may change over time and is $\int_V dV = V$. The total rate of change of that volume as it moves with the fluid is equal to the surface integral of the velocity field $\mathbf{u}$ through the surface $S$ that is bounding the volume $V$:

$$\frac{d}{dt}\int_V dV = \int_S \mathbf{u} \cdot d\mathbf{S} \tag{3.7}$$

Recall now the divergence theorem (Eq. 2.28) to obtain:

$$\frac{d}{dt}\int_V dV = \int_V \nabla \cdot \mathbf{u} dV \tag{3.8}$$

Now, for a volume parcel so small that $\int_V dV = \Delta V \to 0$, the velocity divergence can be considered to be constant over the volume, and the integral can be replaced by the volume itself:

$$\frac{d\Delta V}{dt} = \Delta V \nabla \cdot \mathbf{u} \tag{3.9}$$

We can derive a similar expression for the rate of change of a fluid property per unit volume $q$, such that $q\Delta V$ is the amount of that quantity in a fluid parcel with the volume $\Delta V$.

$$\frac{d}{dt}(q\Delta V) = \Delta V\frac{dq}{dt} + q\frac{d\Delta V}{dt} \tag{3.10}$$

Recall the material derivative of $\Delta V$ from Eq. 3.9 to obtain:

$$\frac{d}{dt}(q\Delta V) = \Delta V\frac{dq}{dt} + q\Delta V\nabla\cdot\mathbf{u} \tag{3.11}$$

$$\frac{d}{dt}(q\Delta V) = \Delta V\left(\frac{dq}{dt} + q\nabla\cdot\mathbf{u}\right) \tag{3.12}$$

This was for a fluid property that is defined per unit volume. Let's now do the same for some property $\varphi$ that is defined per unit mass, such that $\varphi\rho\Delta V$ is the amount of that quantity in the fluid parcel with the volume $\Delta V$ and density $\rho$ (and mass $\rho\Delta V$).

$$\frac{d}{dt}(\varphi\rho\Delta V) = \rho\Delta V\frac{d\varphi}{dt} + \varphi\frac{d(\rho\Delta V)}{dt} \tag{3.13}$$

However recall that our fluid parcel has constant mass, so $\frac{d(\rho\Delta V)}{dt} = 0$. Our total derivative becomes:

$$\frac{d}{dt}(\varphi\rho\Delta V) = \rho\Delta V\frac{d\varphi}{dt} \tag{3.14}$$

The Lagrangian derivative of a volume will come in handy when we derive the continuity equation in the next chapter.

## 3.3 Velocity potential

Velocity potential is defined as a scalar field $\phi$ such that the velocity field $\mathbf{u}$ is the gradient of $\phi$:

$$\mathbf{u} = \nabla\phi = \begin{bmatrix}\frac{\partial\phi}{\partial x}\\ \frac{\partial\phi}{\partial y}\\ \frac{\partial\phi}{\partial z}\end{bmatrix} \tag{3.15}$$

The concept of the velocity potential is useful in fluid mechanics because it is often easier to work with a scalar field than a vector field. We will revisit it later in Chapter 10 when we derive the equations of surface gravity waves.

## Summary

In this chapter, we covered:

- Lagrangian (material) and Eulerian (field) derivatives; the former follows a fluid parcel of constant mass as it moves through the flow field, while the latter is the rate of change at a fixed point (or volume) in space;
- The Lagrangian derivative of volume, as well as of a fluid property per unit volume and per unit mass.

We'll use these concepts in the next chapter where we derive the equations of continuity and motion.

# 4 Conservation of mass and momentum

In this chapter we will derive the fundamental equations for fluid flows: continuity, momentum, and energy. We start with the conservation of mass, which is the easiest to derive, but also arguably the most fundamental.

## 4.1 Conservation of mass

Recall from the previous chapter that we can take at least two perspectives on the fluid flow: the Lagrangian perspective, which follows a fluid parcel as it moves through space, and the Eulerian perspective, which observes the flow at fixed points in space. We can thus derive the conservation of mass, or commonly known as the *continuity*, from both perspectives. Let's start with the Eulerian perspective, as it may seem more intuitive to derive from first principles.

### 4.1.1 Eulerian derivation

Consider a fixed rectangular volume $\Delta V = \Delta x \Delta y \Delta z$ in three-dimensional space. The mass of the fluid in this volume is $\rho \Delta V$, where $\rho$ is the density of the fluid. The fluid enters the volume through the surfaces of the box, and the rate at which the mass enters the volume through a surface is given by the product of the density, the velocity component normal to the surface, and the area of the surface. Let's call this velocity $\mathbf{u}$ with components $u$, $v$, and $w$ in the $x$, $y$, and $z$ directions, respectively.

For simplicity, let's first consider only the $x$-component of the velocity. This scenario is illustrated in Fig. 4.1. The fluid mass flow rate into the volume through the left face is $\rho u \Delta y \Delta z$, and the mass flow rate out of the volume through the right face is $\rho(u + \frac{\partial u}{\partial x}\Delta x)\Delta y \Delta z$. The net mass increase in the control volume must be governed by the net mass inflow excess relative to the net mass outflow:

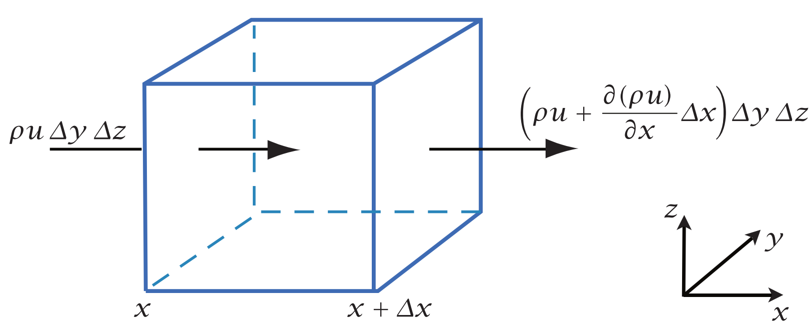


Fig. 1.1

Figure 4.1: Mass conservation in an rectangular Eulerian control volume. The mass convergence, $\partial(\rho u)/\partial x$ (plus contributions in the $y$ and $z$ directions), must be balanced by a density decrease. This is Fig. 1.1 in AOFD (Vallis, 2017).

$$\int_V \frac{\partial \rho}{\partial t} dV = \rho u \Delta y \Delta z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\Delta x\right)\Delta y \Delta z \tag{4.1}$$

$$\int_V \frac{\partial \rho}{\partial t} dV = -\frac{\partial(\rho u)}{\partial x}\Delta x \Delta y \Delta z \tag{4.2}$$

Now, if we allow the flow field to have components in the $y$ and $z$ directions as well, the equation becomes:

$$\int_V \frac{\partial \rho}{\partial t} dV = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\Delta V \tag{4.3}$$

Let $\Delta V \to 0$ to such that any field within $\Delta V$ is uniform to obtain:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{4.4}$$

This is the continuity equation in the Eulerian reference frame.

We're not constrained to a rectangular, fixed volume, however. We can derive this equation for an arbitrary control volume using the divergence theorem. The total rate of change of that volume as it moves with the fluid is equal to the surface integral of the velocity field $\mathbf{u}$ through the surface $S$ that is bounding the volume $V$ (Fig. 4.2). Mathematically, we can express this as:

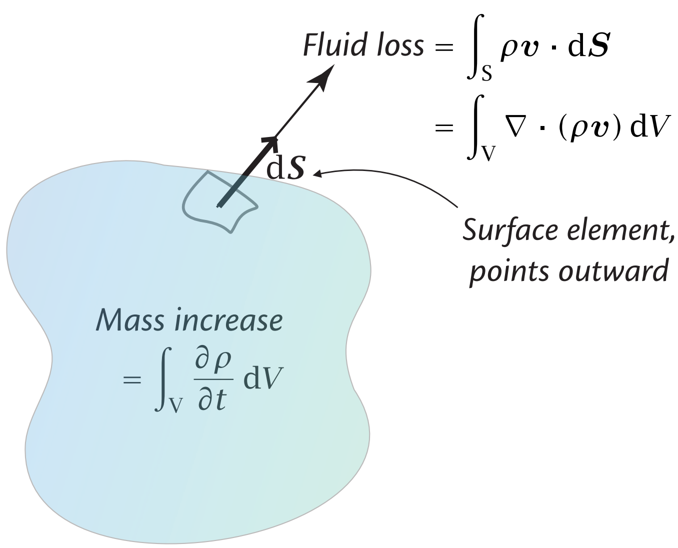


Fig. 1.2

Figure 4.2: Mass conservation in an arbitrary Eulerian control volume $V$ bounded by a surface $S$. The mass increase, $\int_V (\partial\rho/\partial t) dV$ is equal to the mass flowing into the volume, $-\int_S (\rho \mathbf{v}) \cdot d\mathbf{S} = -\int_V \nabla \cdot (\rho \mathbf{v}) dV$. This is Fig. 1.2 in AOFD (Vallis, 2017).

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_S \rho \mathbf{u} \cdot d\mathbf{S} \tag{4.5}$$

Now, recall the divergence theorem (Eq. 2.28) to obtain:

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla \cdot (\rho \mathbf{u}) dV \tag{4.6}$$

Let $\Delta V \to 0$ to integrate and drop $\Delta V$ on both sides to obtain Eq. 4.4, which is the Eulerian form of the continuity equation.

### 4.1.2 Lagrangian derivation

In the Lagrangian frame, we follow a fluid parcel as it moves through space. Its mass $\rho \Delta V$ is constant by definition, but its density or volume may change. Since the mass of the parcel is constant, its Lagrangian derivative is zero:

$$\frac{d}{dt}(\rho \Delta V) = 0 \tag{4.7}$$

Since the mass doesn't change, any change in the density of the parcel must be balanced by a change in its volume:

$$\Delta V \frac{d\rho}{dt} + \rho \frac{d\Delta V}{dt} = 0 \tag{4.8}$$

Recall that we've already derived the Lagrangian derivative of a volume of the fluid parcel (Eq. 3.9), which is the second term here. The equation becomes:

$$\Delta V \frac{d\rho}{dt} + \Delta V \rho \nabla \cdot \mathbf{u} = 0 \tag{4.9}$$

Finally, drop $\Delta V$ on both sides to obtain the Lagrangian form of the continuity equation:

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0 \tag{4.10}$$

Equations 4.4 and 4.10 are two fundamental expressions of the conservation of mass for a fluid. In one form or another, this equation is a critical component of all weather, ocean, and climate prediction models.

### 4.1.3 Continuity of an incompressible fluid

Liquids are nearly incompressible, and for them $\frac{d\rho}{dt} = 0$ is a good approximation. For an incompressible fluid, the continuity equation simplifies to:

$$\nabla \cdot \mathbf{u} = 0 \tag{4.11}$$

Although as simple as it gets, Eq. 4.11 is extremely important in fluid dynamics.

## 4.2 Conservation of momentum

Like the conservation of mass, the conservation of momentum is a fundamental concept in fluid mechanics. It allows us to predict how the fluid should accelerate due to its state (i.e. velocity and density) and due to the forces acting on it. Together, the continuity and momentum conservation equations form the core of most fluid prediction models, such as weather, ocean, and climate prediction models. We will derive the momentum equation in the remainder of this section. We'll start from the most basic form first and then incrementally introduce some common forces, such as the pressure gradient force, gravity, and viscosity.

### 4.2.1 The first step

To derive the momentum conservation equation, we will start from the second Newton's law, which states that the time rate of change of the momentum of a fluid particle is equal to the net force acting on it. For a fluid parcel of volume $\Delta V = \int_V dV$ whose momentum per unit mass is $\rho\mathbf{u}$, the momentum conservation equation is:

$$\frac{d}{dt}\int_V \rho\mathbf{u}dV = \int_V \mathbf{F}dV \tag{4.12}$$

where $\mathbf{F}$ is the net force per unit volume acting on the fluid parcel. Let again the volume parcel be very small such that its density and net force acting on it are uniform. We have:

$$\rho\frac{d\mathbf{u}}{dt}\Delta V = \mathbf{F}\Delta V \tag{4.13}$$

$$\rho\frac{d\mathbf{u}}{dt} = \mathbf{F} \tag{4.14}$$

Recall the Lagrangian derivative operator from Eq. 3.5 to obtain:

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = \frac{\mathbf{F}}{\rho} \tag{4.15}$$

This equation states that the acceleration of a fluid parcel at any fixed point in space is equal to the net force per unit mass acting on it, divided by the fluid density. The second term on the left-hand side is the *advection term*. It represents the local acceleration of the fluid parcel due to the properties of the fluid flow itself. Consider for example a 1-dimensional flow such that the advection term reduces to $u\frac{\partial u}{\partial x}$. Notice that the advection term is zero only in two special cases: when the velocity is zero or when the velocity is spatially uniform. In all other cases the advection term is non-zero and contributes to the local acceleration.

Because the advection term is velocity multiplied by its gradient, it is *nonlinear*. This single property of this term makes accurate analysis and prediction of fluid flows difficult. For example, the nonlinear advection term is responsible for the existence of *chaos* in fluid flows, where small differences in initial conditions lead to vastly different outcomes (in popular culture known as the *butterfly effect*). One consequence of this in our daily lives is that weather predictability is limited to a finite lead time horizon, for example one to two weeks depending on the weather patterns of interest. If, however, we could assume that either the velocity or its gradient are so small that they could be neglected, the equation simplifies significantly and often allows for analytical solutions. $\mathbf{u} \cdot \nabla \mathbf{u}$ is the most important term for turbulence, weather prediction and predictability, and a major obstacle toward analytical solutions of Eq. 4.15 and its variants. Remember this now.

Back to our equation. For a 3-dimensional Cartesian flow where the velocity field is $\mathbf{u} = (u, v, w)$ and net forces are $\mathbf{F} = (F_x, F_y, F_z)$, Eq. 4.15 becomes a system of three equations, one for each component of the velocity field. Recall from the Lagrangian derivative operator that $\mathbf{u} \cdot \nabla \mathbf{u}$ is an operator acting on $\mathbf{u}$ (as opposed to divergence of a gradient). The $\mathbf{u} \cdot \nabla$ operator then expands to $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$. Our vector equations becomes a system of three scalar equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{F_x}{\rho} \tag{4.16}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{F_y}{\rho} \tag{4.17}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{F_z}{\rho} \tag{4.18}$$

Each of the prognostic equations for the velocity components thus has exactly three advective components that correspond to the gradients of the velocity in each respective direction.

### 4.2.2 Incorporating the forces

Now we should consider what forces may be acting on the fluid. We distinguish between two types of forces: surface forces and body forces. Surface forces act on the surface of the fluid parcel due to the motion of the fluid molecules, in all directions at that surface. For example, organized motion of molecules into the surface may cause pressure on that surface, and the sheared motion of molecules (e.g. if flow is antiparallel to the surface) may cause shear stress on the surface, leading to the deformation of the fluid parcel. In contrast, body forces act remotely (meaning, from a distance) on the entire volume of the fluid parcel because that parcel is immersed in one or more force fields. Gravity is one such body force, and it's the only one we'll consider here. Although in Eq. 4.15 we wrote the net force as $\mathbf{F}$, it's useful to write it as the sum of body forces $\mathbf{F}_b$ and surface forces $\mathbf{F}_s$:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\rho}(\mathbf{F}_s + \mathbf{F}_b) \tag{4.19}$$

Let's derive the surface forces first. We want to find out the local change of momentum only due to the surface forces. Analogous to how the flow through the volume determined the rate of change of density inside that volume, as we saw in the continuity equation (Eq. 4.4), the change in momentum inside the volume is determined by the surface forces acting on the volume (Fig. 4.3).

Mathematically, we can express this change as:

$$\int_V \mathbf{F_s} dV = \int_S \boldsymbol{\sigma} \cdot d\mathbf{S} \tag{4.20}$$

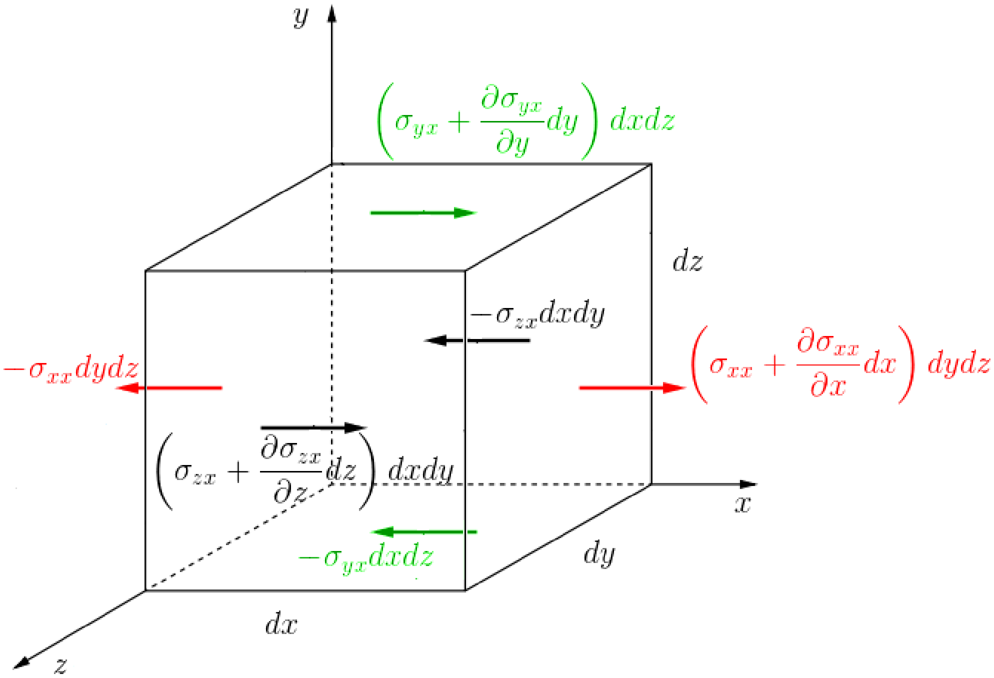


Figure 4.3: Normal components of the stress tensor $\sigma$ acting on a fluid parcel. Reproduced from https://en.wikipedia.org/wiki/Cauchy_momentum_equation under the CC BY-SA 4.0 license.

where $\boldsymbol{\sigma}$ is the second-order stress tensor acting on the surface $S$ of the fluid parcel. As before, recall the divergence theorem (Eq. 2.28) to obtain:

$$\int_V \mathbf{F_s} dV = \int_V \nabla \cdot \boldsymbol{\sigma} dV \tag{4.21}$$

$$\mathbf{F_s} = \nabla \cdot \boldsymbol{\sigma} \tag{4.22}$$

The surface force thus equals the divergence of the stress tensor. Insert this into Eq. 4.15 to get our new form of the momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + \frac{\mathbf{F}_b}{\rho} \tag{4.23}$$

This form of the momentum equation is often called the *Cauchy momentum equation*.

Let's now look at what this stress tensor divergence term $\nabla \cdot \boldsymbol{\sigma}$ is.

### 4.2.3 Pressure gradient

There is a fundamental difference in the meaning of the diagonal and off-diagonal components of the stress tensor. The diagonal components of the stress tensor, $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{zz}$, represent the normal stress components, i.e. the force per unit area acting on a surface element that is oriented in the $x$, $y$, and $z$ directions, respectively. The off-diagonal components of the stress tensor represent the shear stress components, each acting on all three surfaces. For example, $\sigma_{xy}$ represents the $x$-component of the stress tensor acting on the surface that is perpendicular to the $y$-axis. Let's write out the stress tensor in Cartesian coordinates:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{4.24}$$

This tensor can be decomposed into its normal and shear components:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \tag{4.25}$$

where $p$ is the pressure, $\mathbf{I}$ is the identity tensor, and $\boldsymbol{\tau}$ is the deviatoric stress tensor, or, the viscous shear stress tensor. Written out explicitly in Cartesian coordinates and using Eq. 4.25, the stress tensor is:

$$\boldsymbol{\sigma} = \begin{bmatrix} -p + \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -p + \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p + \tau_{zz} \end{bmatrix} \tag{4.26}$$

The divergence of the stress tensor is then:

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla p + \nabla \cdot \boldsymbol{\tau} \tag{4.27}$$

Let's insert this into Eq. 4.15 to get our new form of the momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \boldsymbol{\tau} + \frac{\mathbf{F}_b}{\rho} \tag{4.28}$$

Pressure is one of the fluid properties that determine its state. Collective, organized, motion of molecules at a macroscopic scale induces pressure on a surface and an associated force acting normal to that surface. Recall that the surface vector is normal to the surface and pointing outward, and the force acting on the fluid surface is oriented inward, thus the minus sign.

In an ideal, *inviscid* fluid, that is, a fluid that exhibits no viscous forces, the stress tensor $\boldsymbol{\sigma}$ is only composed of the diagonal terms (pressure), and the divergence of the stress tensor is zero. Dropping $\nabla \cdot \boldsymbol{\tau}$ and the body forces $\mathbf{F}_b$ for now, the Cauchy momentum equation simplifies to:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p \tag{4.29}$$

This form of the momentum equation is often called the *Euler equation*.

### 4.2.4 Viscous forces

Now, let's look at the shear stress tensor divergence $\nabla \cdot \boldsymbol{\tau}$. Written out explicitly as a matrix of all its components, $\boldsymbol{\tau}$ is:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \tag{4.30}$$

The diagonal components of the deviatoric stress tensor are the normal stresses, while the off-diagonal components are the shear stresses. The normal stresses are non-zero only in compressible fluids ($\nabla \cdot \mathbf{u} \neq 0$), while the shear stresses are zero in non-viscous flows. The divergence of this tensor, written out explicitly as a matrix of all its components, is:

$$\nabla \cdot \boldsymbol{\tau} = \begin{bmatrix} \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \end{bmatrix} \tag{4.31}$$

Now, write out 4.28 as a system of three scalar equations, one for each component of the velocity field, and insert the shear stress divergence terms to get:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \frac{F_x}{\rho} \tag{4.32}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \frac{F_y}{\rho} \tag{4.33}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \frac{F_z}{\rho} \tag{4.34}$$

Each of the prognostic equations for the velocity components thus has exactly one pressure gradient and two shear stress gradient terms, all arising from the surface forces.

Experimentally, it was found that the viscous shear stress tensor $\boldsymbol{\tau}$ is proportional to the gradient of the velocity field, i.e. $\boldsymbol{\tau} = \mu \nabla \mathbf{u}$. This property of the fluid makes it a so-called *Newtonian fluid*. The proportionality constant $\mu$ is the dynamic viscosity and depends on the fluid properties and temperature. Inserting this into Eq. 4.28 and assuming incompressibility, we get:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot (\mu \nabla \mathbf{u}) + \frac{\mathbf{F}_b}{\rho} \tag{4.35}$$

We can further simplify this equation by assuming that the viscosity is constant and that the flow is incompressible. This allows us to neglect the viscous stress gradient term, leading to the *Navier-Stokes equation*.

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \frac{\mathbf{F}_b}{\rho} \tag{4.36}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. The operator $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ is the *Laplacian*. It is a second-order differential operator that appears in many partial differential equations, including the heat equation, the wave equation, and the Laplace equation. More on these later.

Let's now look at the body forces to conclude our derivation.

### 4.2.5 Gravity

As we mentioned earlier, gravity is the only body force we'll consider here. The force of gravity per unit mass is given by $\mathbf{g} = (0, 0, -g)$, where $g$ is the gravitational acceleration. Here we assume that the gravitational acceleration is constant and points downward. Insert this into Eq. 4.36, and assuming incompressibility, we get:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u} \tag{4.37}$$

Written out explicitly for each of the three spatial dimensions ($x$, $y$, and $z$), we get:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{4.38}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \tag{4.39}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \tag{4.40}$$

This completes the full system of momentum conservation equations in the Cartesian coordinate system.

## 4.3 Hydrostatic balance

Take Eq. 4.40 and assume that the vertical acceleration $\frac{dw}{dt}$ is small compared to $g$, and that the spatial variations of $w$ are small. We can then drop the $\frac{dw}{dt}$ and $\nu\nabla^2 w$ terms to get the *hydrostatic approximation*:

$$\frac{\partial p}{\partial z} = -\rho g \tag{4.41}$$

which states that the vertical pressure gradient is governed by the density of the fluid and the gravitational acceleration. It's often a good approximation for large-scale atmospheric and oceanic flows, where the vertical variations of the horizontal velocity components are much smaller than the horizontal variations of the vertical velocity component. Notice however that the hydrostatic approximation does not imply that there is no vertical motion or that it does not vary over time. Instead, according to the continuity equation (Eq. 4.10), it means that the vertical motion is completely governed by the change in density and the divergence of the horizontal velocity field. Further, if the flow is incompressible ($\nabla \cdot \mathbf{u} = 0$), we get:

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \tag{4.42}$$

which relates the vertical acceleration to the horizontal divergence. Integrating this equation vertically allows us to calculate the vertical velocity anywhere in the fluid column provided bottom and top boundary conditions:

$$w(z) = w(z + \Delta z) - \int_{z}^{z+\Delta z} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz' \tag{4.43}$$

This relationship will prove to be extremely useful in ocean applications where only the horizontal velocity field is resolved. For example, a group of ocean surface drifters converging towards a region is indicative of downwelling (downward motion in the ocean) in that region. Another example is that of ocean circulation models, which are typically designed as hydrostatic. In the case of such models, the horizontal components of the velocity are prognostic variables, and the vertical velocity is diagnosed using Eq. 4.42.

## 4.4 Equation of state

Now that we have derived the mass and momentum conservation equations, let's write them out together in vector form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u} \tag{4.44}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{4.45}$$

Momentum and mass conservation equations are prognostic equations for the vector velocity field $\mathbf{u}$ and the scalar density field $\rho$, respectively. Notice the one remaining unknown: the scalar pressure field $p$. As of now, we have a system of two independent equations for the three unknowns: $\mathbf{u}$, $p$, and $\rho$. We need one more equation to close the system—the *equation of state*—to relate the pressure to the other properties of the fluid, such as temperature, density, and composition.

The prognostic equations that we have derived so far describe equally well the evolution of both the atmosphere and the ocean, despite their significant differences. The equation of state is where our systems of governing equations for the ocean and the atmosphere begin to diverge. Namely, the atmosphere is a mixture of dry air and water vapor, and the ocean is composed of liquid water with varying amounts of dissolved salts. These differences will reflect in the choice of the equation of state to use in each of these systems.

### 4.4.1 In the atmosphere

In the atmospheres, *ideal gas law* is often used as the equation of state:

$$p = \rho R T \tag{4.46}$$

where $R$ is the specific gas constant for the gas in question, and $T$ is the temperature. For the moist air, we need to account for both the properties of dry air ($R_d \approx\ 287\ Jkg^{-1}K^{-1}$) and those of water vapor ($R_v \approx\ 461\ Jkg^{-1}K^{-1}$). The equation of state for moist air relies on the so-called *virtual temperature* to account for the moisture in the air:

$$p = \rho R_d T_v \tag{4.47}$$

where:

$$T_v = T\left[1 + q\left(\frac{R_v}{R_d} - 1\right)\right] \tag{4.48}$$

where $q$ is the specific humidity of the air. So, the equation of state for moist air is:

$$p = \rho R_d T\left[1 + q\left(\frac{R_v}{R_d} - 1\right)\right] \tag{4.49}$$

Recall that we intended to close our system of equations by finding the equation for pressure. Although we did solve for pressure, it seems that we introduced two new unknown variables: the temperature $T$ and the specific humidity $q$. Each of these variables are governed by their own conservation equations, akin to that for density, but with the addition of source and sink terms that control their production and loss:

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \dot{S}_T \tag{4.50}$$

$$\frac{\partial q}{\partial t} + (\mathbf{u} \cdot \nabla)q = \dot{S}_q \tag{4.51}$$

where $\dot{S}_T$ and $\dot{S}_q$ are the sources and sinks of temperature and specific humidity, respectively. They are governed by a plethora of thermodynamic processes such as radiation, evaporation, condensation, etc.

Although we won't delve further into the details behind the sources and sinks of temperature and specific humidity in the atmosphere, we can denote these equations as completing the full system of prognostic equations for the atmosphere: Eqs. 4.44, 4.45, 4.49, 4.50, and 4.51. These equations form the basis of many weather and climate prediction models.

### 4.4.2 In the ocean

Ideal gas law (Eq. 4.46) doesn't apply to liquids and the equation of state for seawater is not easily derived. Instead, we assume that the ocean is a single-component fluid, and we use the density field $\rho$ as the equation of state.

$$\rho = \rho(T, S, p) = \rho_0\left[1 - \beta_T(T - T_0) + \beta_S(S - S_0) + \beta_p(p - p_0)\right] \tag{4.52}$$

where $\rho_0$ is the reference density at the reference temperature $T_0$, salinity $S_0$, and pressure $p_0$. The coefficients $\beta_T$, $\beta_S$, and $\beta_p$ are the thermal expansion coefficient, the saline contraction coefficient, and the pressure coefficient, respectively. This form of the equation of state is a linear equation of state (as in, the dependence of density on temperature, salinity, and pressure each is linear). Dependence of density on temperature and salinity at two different pressure levels is shown in Fig. 4.4. Higher order equations of state are often used for higher accuracy, however they're out of scope for this course.

Like we did in the case of atmosphere, here we need to additional prognostic equations, one of temperature and another for salinity:

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \dot{S}_T \tag{4.53}$$

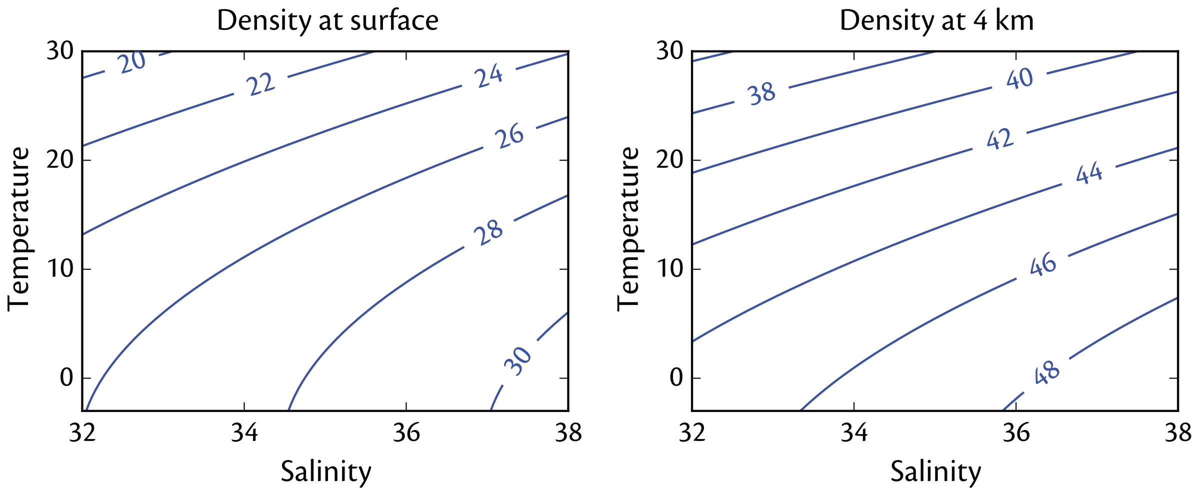


Fig. 1.3

Figure 4.4: Contours of density as a function of temperature and salinity for seawater. Contour labels are (density - 1000) kg m$^{-3}$. Left panel: at sea-level ($p = 10^5$ $Pa$, or 1000 mb). Right panel: at $p = 4 \times 10^7$ $Pa$ (about 4 km depth). In both cases the contours are slightly convex, so that if two parcels at the same density but different temperatures and salinities are mixed, the resulting parcel is of higher density. (The average temperature is not exactly conserved on mixing, but it very nearly is.) This is Fig. 1.3 in AOFD (Vallis, 2017).

$$\frac{\partial S}{\partial t} + (\mathbf{u} \cdot \nabla)S = \dot{S}_S \tag{4.54}$$

where $\dot{S}_T$ and $\dot{S}_S$ are the sources and sinks of water temperature and salinity, respectively.

Equations 4.44, 4.45, 4.52, 4.53, and 4.54 are the governing equations used in most numerical ocean circulation models.

## 4.5 Nondimensionalization and scaling

A useful technique to simplify the analysis of the governing equations is to scale the variables using characteristic values for each of the variables. This is known as *nondimensionalization* or *scaling the equations*. In practice, for each (dependent or independent) variable $x$ in the equations, we define a characteristic value $X$. For example, for the velocity $\mathbf{u}$, we may pick the characteristic value of $U = 1$ $m$ $s^{-1}$ or $U = 10$ $m$ $s^{-1}$ for the ocean or atmosphere, respectively. We then divide each term in the equations by the characteristic value to obtain a nondimensional (unitless) number. This helps us identify the important parameters that govern the behavior of the system and to group terms in the equations that are of similar magnitudes. This is especially useful for large-scale flows, where the length and time scales can vary over several orders of magnitude.

Let's look, for example, at the vector equation for horizontal momentum (thus, ignoring **g** for now):

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \tag{4.55}$$

The characteristic scales for each term are:

$$\frac{\partial \mathbf{u}}{\partial t} \sim \frac{U}{T} \tag{4.56}$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} \sim \frac{U^2}{L} \tag{4.57}$$

$$-\frac{1}{\rho}\nabla p \sim \frac{1}{\rho}\frac{P}{L} \tag{4.58}$$

$$\nu \nabla^2 \mathbf{u} \sim \nu \frac{U}{L^2} \tag{4.59}$$

where $U$, $T$, $L$, and $P$ are the characteristic scales for the velocity, time, length, and pressure, respectively. So, if for a given flow we can estimate these characteristic values, we can easily determine which terms are important and which can be neglected. This is the basis of scaling arguments in fluid mechanics.

This approach also enables characterizing the flows in terms of nondimensional numbers. For example, to describe how turbulent or laminar a flow is, it's useful to relate the inertial to the viscous terms in the momentum equation. Their ratio is called the *Reynolds number*:

$$\frac{(\mathbf{u} \cdot \nabla)\mathbf{u}}{\nu \nabla^2 \mathbf{u}} \sim \frac{\frac{U^2}{L}}{\frac{\nu U}{L^2}} = \frac{UL}{\nu} \equiv \mathrm{Re} \tag{4.60}$$

You see that the Reynolds number is proportional to the velocity and length scales each, and inversely proportional to the viscosity. A larger Reynolds number corresponds to a more turbulent flow.

## Exercises

1. Derive the Lagrangian form of the continuity equation from the Eulerian form and vice versa. What is the key equation that relates the two forms?

2. Consider two opposing, horizontal, surface currents along the $x$-axis. In the vertical they uniformly span a mixed layer that extends from the surface to the depth of 20 meters, with a magnitude of 1 m s$^{-1}$. The two currents meet at a stagnation zone that is 100 meters wide. Calculate the downwelling velocity at the bottom of the mixed layer. Assume $\nabla \cdot \mathbf{u} = 0$, no change in mean sea level, and no flow in the $y$-direction.

3. Write out the Cauchy, Euler, and Navier-Stokes equations in vector form and discuss their similarities and differences. Give examples of flows that are well described by each of these equations.

4. Write a computer program that calculates the divergence of a second-order tensor in a Cartesian, 3-dimensional coordinate system.

5. Write a function in your favorite programming language that takes a value of temperature, salinity, and pressure and returns the density of seawater. Assume linear dependence of density on temperature, salinity, and pressure. Take the thermal expansion coefficient to be $\beta_T = 1.67 \times 10^{-4} K^{-1}$, the Haline contraction coefficient to be $\beta_S = 7.8 \times 10^{-4} g\ kg^{-1}$, and the compressibility coefficient to be $\beta_p = 4.4 \times 10^{-10} Pa^{-1}$. Take the reference density to be $\rho_0 = 1027\ kg\ m^{-3}$, the reference temperature to be $T_0 = 283\ K$, the reference salinity to be $S_0 = 35 g\ kg^{-1}$, and the reference pressure to be $p_0 = 10^5\ Pa$. When you implement your function, calculate the density of seawater for the range of temperatures from -2 to 30 degrees Celsius, and salinities from 20 to 40 g/kg, and plot it as a contour plot as a function of temperature and salinity. Make such plots for pressure values of $10^5$, $10^6$, and $10^7$ Pa.

6. Calculate the Reynolds number for: (a) a synoptic-scale mid-latitude cyclone in the atmosphere; (b) an mesoscale ocean eddy; (c) a river inflow into the ocean; (d) a breaking ocean surface wave; (e) water flowing through a pipe with a diameter of 0.1 m and flow speed of 1 m s$^{-1}$. Assume $\nu = 10^{-5} m^2 s^{-1}$ for air and $\nu = 10^{-6} m^2 s^{-1}$ for water.

## Summary

In this chapter, we covered:

- Conservation of mass (continuity equation) from both Eulerian and Lagrangian perspectives;
- Conservation of momentum equations: Cauchy, Euler, and Navier-Stokes;
- The Reynolds number as a measure of the relative importance of inertial to viscous forces in a flow;
- The equation of state for seawater, relating density to temperature, salinity, and pressure;
- Examples of flows with different Reynolds numbers, from laminar pipe flow to turbulent geophysical flows.

# 5 Rotating flows

Fluids behave somewhat differently when in a rotating reference frame, for example on the surface of a rotating planet while being observed from a fixed location on that surface. In this chapter we explore the effects of rotation on the flow. We begin by deriving the temporal derivative of a general vector in a rotating reference frame, and then apply it to find the velocity and acceleration in such a frame. From there we derive the centrifugal and Coriolis forces, and discuss their implications for geophysical flows.

## 5.1 Rate of change of a rotating vector

Before determining what the velocity and acceleration should appear like in a rotating reference frame (*i.e.* on the surface of a rotating planet), we first need to understand how a vector that is fixed in the rotating frame appears to change over time to the observer in the inertial (fixed) frame. To do that, consider a vector $\mathbf{C}$ that rotates around an axis at a constant angular velocity $\mathbf{\Omega}$ (Fig. 5.1). The angular velocity $\mathbf{\Omega}$ is the rate of change of the angle in the plane that is perpendicular to the axis of rotation, and is thus $\frac{d\lambda}{dt}$. A unit vector $\mathbf{m}$ is oriented in the direction of the rate of change of $\mathbf{C}$, and is perpendicular to both $\mathbf{C}$ and $\mathbf{\Omega}$. We will assume that $\mathbf{\Omega}$ is constant. This is a generally good assumption for the rotation rates of planets, at least on time scales that we are interested in. A small change in $\mathbf{C}$ can then be expressed as:

$$\delta\mathbf{C} = |\mathbf{C}|\cos\theta\ \delta\lambda\ \mathbf{m} \tag{5.1}$$

The change in $\mathbf{C}$ is thus proportional to: its magnitude; the cosine of the angle between $\mathbf{C}$ and the horizontal plane (*i.e.* the plane perpendicular to $\mathbf{\Omega}$); the change in $\lambda$; and, the unit vector $\mathbf{m}$. Notice now that using the definition of the cross product (Eq. 2.12), and recalling that $\Omega = \frac{d\lambda}{dt}$, we can write the change in $\mathbf{C}$ as:

$$\delta\mathbf{C} = |\mathbf{C}||\mathbf{\Omega}|\sin(\pi/2-\theta)\ \mathbf{m}\ \delta t = \mathbf{\Omega}\times\mathbf{C}\ \delta t \tag{5.2}$$

so the rate of change of a rotating vector, when observed from a fixed, inertial frame is the cross product of the angular velocity and the vector itself:

$$\left(\frac{d\mathbf{C}}{dt}\right)_I = \mathbf{\Omega}\times\mathbf{C} \tag{5.3}$$

Going forward, we will use the subscript $I$ to denote the inertial frame, non-rotating reference frame.

Imagine now that you're standing on top of the rotating vector $\mathbf{C}$, and are still relative to that rotating reference frame, much like standing still on the surface of a rotating planet. To you as the observer in the rotating frame, the vector $\mathbf{C}$ appears to not change in any way. Consider now another vector $\mathbf{B}$ that may change (in direction or magnitude, or both) in the rotating reference frame. We can then say that the rate of change of $\mathbf{B}$ in the inertial frame is the vector sum of its two rates of change: The rate of change of $\mathbf{B}$ in the rotating frame, and the rate of change of the rotating frame itself:

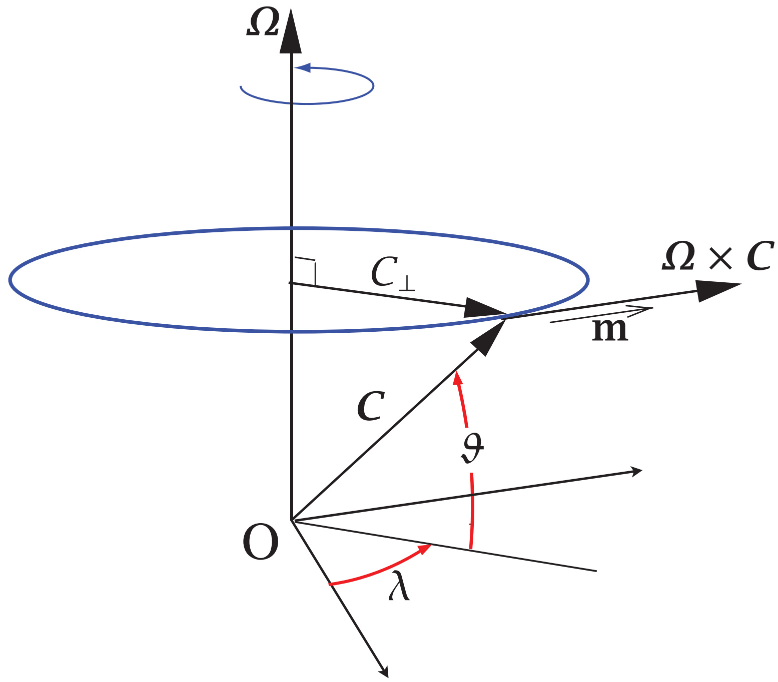


Figure 5.1: A vector $\mathbf{C}$ rotating at an angular velocity $\mathbf{\Omega}$. It appears to be a constant vector in the rotating frame, whereas in the inertial frame it rotates according to $(d\mathbf{C}/dt)_I = \mathbf{\Omega} \times \mathbf{C}$. This is Fig. 2.1 in AOFD (Vallis, 2017).

$$\left(\frac{d\mathbf{B}}{dt}\right)_I = \left(\frac{d\mathbf{B}}{dt}\right)_R + \mathbf{\Omega} \times \mathbf{B} \tag{5.4}$$

We now have a useful tool to use to determine the velocity and acceleration in a rotating frame, such as that of of the surface of a rotating planet.

## 5.2 Velocity and acceleration in a rotating frame

Consider now a position vector $\mathbf{r}$ that locates a parcel in the rotating frame. The velocity of the parcel in the inertial frame is then given by the rate of change of the position vector. Apply Eq. 5.4 to $\mathbf{r}$ to get:

$$\left(\frac{d\mathbf{r}}{dt}\right)_I = \left(\frac{d\mathbf{r}}{dt}\right)_R + \mathbf{\Omega} \times \mathbf{r} \tag{5.5}$$

As the time derivative of a position vector is velocity by definition, we can write this as:

$$\mathbf{u}_I = \mathbf{u}_R + \mathbf{\Omega} \times \mathbf{r} \tag{5.6}$$

This relates the inertial and rotating velocities. Recall that we are interested in accelerations, as it's the acceleration that we solve for in the Navier-Stokes equations and relate to the forces that act on the fluid. We know that the acceleration is the rate of change of velocity, so let's apply Eq. 5.4 to the rotating velocity:

$$\left(\frac{d\mathbf{u}_R}{dt}\right)_I = \left(\frac{d\mathbf{u}_R}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{u}_R \tag{5.7}$$

Now, use Eq. 5.6 to substitute for $\mathbf{u}_I$ in Eq. 5.7:

$$\left(\frac{d\left(\mathbf{u}_I - \boldsymbol{\Omega} \times \mathbf{r}\right)}{dt}\right)_I = \left(\frac{d\mathbf{u}_R}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{u}_R \tag{5.8}$$

$$\left(\frac{d\mathbf{u}_I}{dt}\right)_I = \left(\frac{d\mathbf{u}_R}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{u}_R + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} + \boldsymbol{\Omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_I \tag{5.9}$$

Recall that:

$$\left(\frac{d\mathbf{r}}{dt}\right)_I = \left(\frac{d\mathbf{r}}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{r} = \mathbf{u}_R + \boldsymbol{\Omega} \times \mathbf{r} \tag{5.10}$$

If $\boldsymbol{\Omega}$ is constant, as we have assumed at the beginning, inserting Eq. 5.10 into Eq. 5.9 yields:

$$\left(\frac{d\mathbf{u}_R}{dt}\right)_R = \left(\frac{d\mathbf{u}_I}{dt}\right)_I - 2\boldsymbol{\Omega} \times \mathbf{u}_R - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{5.11}$$

The interpretation of the terms in Eq. 5.11 is:

- $\left(\frac{d\mathbf{u}_R}{dt}\right)_R$, is the rate of change of the relative velocity as observed in the rotating frame. This is the rate of change of the velocity that you would measure with an anemometer or current meter if position fixed relative to the rotating planet's surface.

- $\left(\frac{d\mathbf{u}_I}{dt}\right)_I$, is the rate of change of the inertial velocity, *i.e.* the velocity as observed in the inertial frame.

- $-2\boldsymbol{\Omega} \times \mathbf{u}_R$, is the *Coriolis acceleration* . The Coriolis acceleration (and correspondingly, the Coriolis force) is responsible for the organized rotation of large-scale atmospheric and oceanic flows. Notice that the Coriolis acceleration is always perpendicular to the relative velocity $\mathbf{u}_R$. This means that whenever we have a flow in a rotating frame, the Coriolis force deflects the flow to the right or the left depending on the orientation of $\Omega$ relative to the plane of the flow (i.e. the deflection is to the right on the northern hemisphere and to the left on the southern hemisphere).

- $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$, is the *centrifugal acceleration*. It's always antiparallel to the position vector $\mathbf{r}$ by definition. Notice also that the centrifugal acceleration is not dependent on the velocity of the parcel, but only on its position and the angular velocity of the rotating frame. This force could then be considered a body force, much like gravity. Indeed, for practical reasons, centrifugal force if often bundled together with gravitational force and expressed as a gradient of the scalar potential $\Phi$:

$$\mathbf{g} - \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{r} \equiv -\nabla \Phi \tag{5.12}$$

Effects of the centrifugal force on the effective gravity is illustrated in Fig. 5.2.

If we bundle the centrifugal and the gravitational accelerations together and express them as a geopotential gradient, we can write our momentum balance with the effects of rotation as:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Phi - 2\boldsymbol{\Omega} \times \mathbf{u} + \nu \nabla^2 \mathbf{u} \tag{5.13}$$

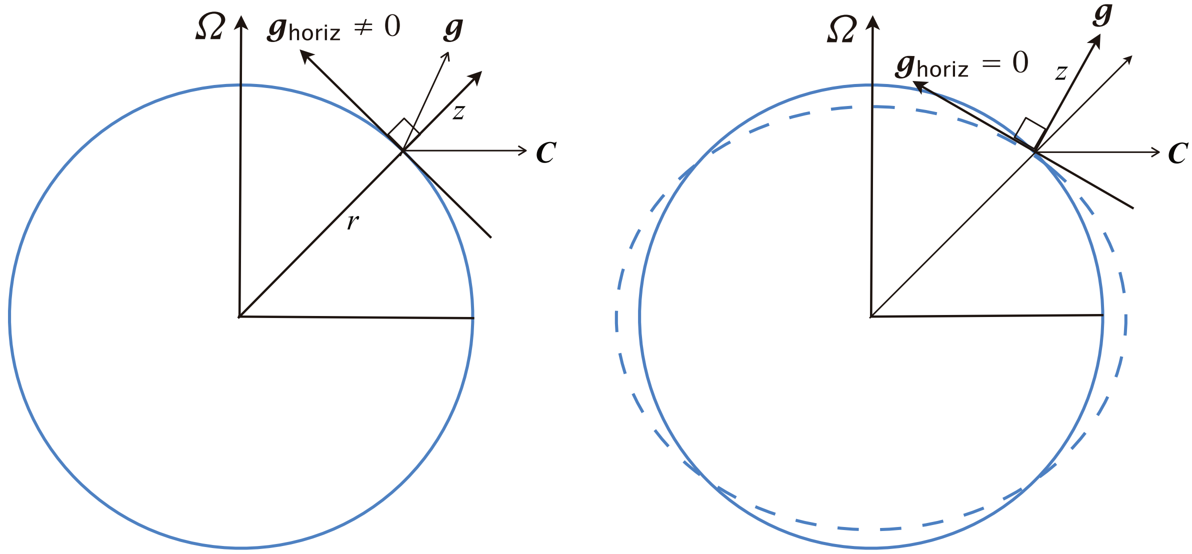


Fig. 2.2

Figure 5.2: Left: directions of forces and coordinates in true spherical geometry. $\mathbf{g}$ is the effective gravity (including the centrifugal force, $\mathbf{C}$) and its horizontal component is evidently non-zero. Right: a modified coordinate system, in which the vertical direction is defined by the direction of $\mathbf{g}$, and so the horizontal component of $\mathbf{g}$ is identically zero. The dashed line schematically indicates a surface of constant geopotential. The differences between the direction of $\mathbf{g}$ and the direction of the radial coordinate, and between the sphere and the geopotential surface, are much exaggerated and in reality are similar to the thickness of the lines themselves. This is Fig. 2.2 in AOFD (Vallis, 2017).

## 5.3 Coriolis force components

Let's now examine in more detail the effects the Coriolis force on the flow. The angular velocity $\boldsymbol{\Omega}$ is a vector that points in the direction oriented from the center of the Earth toward the North Pole (see Fig. 5.3). On the surface of the planet, thus, it has two components: A locally vertical one, $\Omega_z$, and a meridional one, $\Omega_y$:

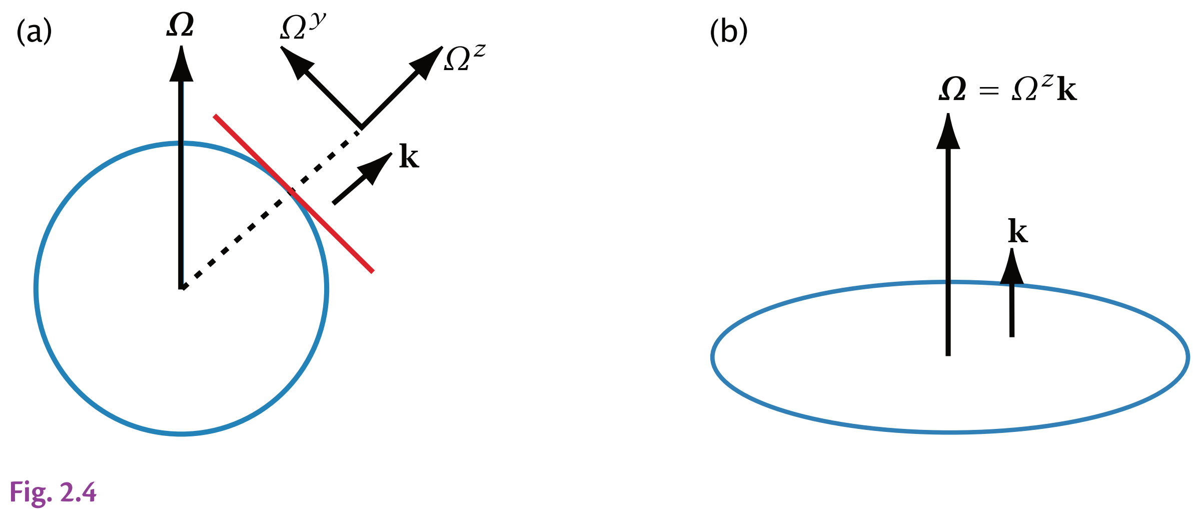


Figure 5.3: (a) On the sphere the rotation vector $\mathbf{\Omega}$ can be decomposed into two components, one in the local vertical and one in the local horizontal, pointing toward the pole. That is, $\mathbf{\Omega} = \Omega_y \mathbf{j} + \Omega_z \mathbf{k}$ where $\Omega_y = \Omega \cos\theta$ and $\Omega_z = \Omega \sin\theta$. In geophysical fluid dynamics, the rotation vector in the local vertical is often the more important component in the horizontal momentum equations. On a rotating disk, (b), the rotation vector $\mathbf{\Omega}$ is parallel to the local vertical $\mathbf{k}$. This is Fig. 2.4 in AOFD (Vallis, 2017).

$$\mathbf{\Omega} = \begin{bmatrix} 0 \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ \Omega\cos\theta \\ \Omega\sin\theta \end{bmatrix} \tag{5.14}$$

where $\theta$ is the latitude.

The Coriolis force is then:

$$-2\mathbf{\Omega} \times \mathbf{u} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2\Omega\cos\theta & -2\Omega\sin\theta \\ u & v & w \end{bmatrix} = \begin{bmatrix} -2\Omega w\cos\theta + 2\Omega v\sin\theta \\ -2\Omega u\sin\theta \\ 2\Omega u\cos\theta \end{bmatrix} \tag{5.15}$$

The Coriolis term thus contributes to all three components of the flow, and their components vary with latitude. Let's look at the horizontal components first. On geophysical scales, generally $w \ll u$ and so $2\Omega w\cos\theta$ can often be neglected. The two dominant horizontal components of the Coriolis force then become $(-2\Omega v\sin\theta, 2\Omega u\sin\theta)$. These components are zero at the Equator and increase poleward. The vertical component, $-2\Omega u\cos\theta$, is negligible as well compared to the other terms in the momentum equation, most notably the gravitational acceleration $\mathbf{g}$ and the vertical pressure gradient that balances it. The horizontal effect is thus significantly more relevant for the horizontal motion than the vertical one.

The practical implications of the Coriolis force on the flow is that it deflects it toward the right on the Northern hemisphere and toward the left on the Southern hemisphere. If a parcel or a particle with some initial velocity on a rotating planet is let undisturbed by other forces, it will appear to the observer standing on the surface of the planet to move in circles with some radius. We will calculate soon exactly how big this radius is depending on where on the planet we are and the initial velocity of the parcel. Let's now incorporate the Coriolis force components into the vector-component form of the momentum equation and apply some convenient approximations, namely the $f$-plane and the $\beta$-plane approximations.

## 5.4 $f$-plane and $\beta$-plane approximations

Although geophysical fluids flow in a thin layer on a sphere, the curvature of the surface of the planet is negligible for many applications. Here we will make the so-called $f$-plane approximation in which the flow is assumed to be on a flat plane tangent to the surface of a curved planet. The main assumption of the $f$-plane approximation is that the planet's rotation exhibits only a locally vertical component anywhere on that planet's surface. In other words, we'll neglect the horizontal component (*i.e.* $\Omega_y$). With that assumption, the Coriolis force becomes strictly horizontal:

$$-2\mathbf{\Omega} \times \mathbf{u} = \begin{bmatrix} 2\Omega v \sin\theta \\ -2\Omega u \sin\theta \\ 0 \end{bmatrix} \tag{5.16}$$

Let's now define the so-called *Coriolis parameter* $f_0 = 2\Omega_z = 2\Omega \sin\theta$, so we can write the Coriolis force more concisely as:

$$-f_0 \mathbf{k} \times \mathbf{u} = \begin{bmatrix} f_0 v \\ -f_0 u \\ 0 \end{bmatrix} \tag{5.17}$$

The effect of the Coriolis force on the flow is now even more apparent: A positive meridional flow causes a positive zonal acceleration, and a positive zonal flow causes a negative meridional acceleration. The implication of this is that the Coriolis force induces clockwise and counterclockwise rotations in the Northern and Southern hemispheres, respectively.

Ignoring viscosity for brevity, we can re-write our system of momentum equations as:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_0 v \tag{5.18}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - f_0 u \tag{5.19}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{5.20}$$

While on the small plane tangential to the planet's surface the local rotation may be uniform in space, in reality it does vary with latitude:

$$f = 2\Omega \sin\theta \approx 2\Omega \sin\theta_0 + 2\Omega(\theta - \theta_0)\cos\theta_0 \tag{5.21}$$

for small deviations in $\theta$. We obtained this expression by expanding $f$ in a Taylor series to the first order around $\theta_0$. On a plane, the above can be expressed as:

$$f = f_0 + \beta y \tag{5.22}$$

where $f_0 = 2\Omega \sin\theta_0$ and $\beta = \partial f/\partial y = (2\Omega \cos\theta_0)/R_E$ (where $R_E$ is the radius of the Earth).

## 5.5 Geostrophic balance

Now that we have incorporated the effects of rotation into our equations of motion, let's evaluate the scales of the terms in the horizontal momentum equations. We will start from Eq. 5.13, use the f-plane notation for the Coriolis term, ignore the viscous terms, and drop the gravity term as we're looking at the flow in the horizontal plane:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho}\nabla p \tag{5.23}$$

As we did in Section 4.5, let's scale each term on the left-hand side with their characteristic scales for mesoscale ocean flow ($L \sim 10^5\ m$, $T \sim 10^6\ s$, $U \sim 10^{-1}\ m/s$):

- $\frac{\partial \mathbf{u}}{\partial t} \sim \frac{U}{T} \sim 10^{-7}$
- $(\mathbf{u} \cdot \nabla)\mathbf{u} \sim \frac{U^2}{L} \sim 10^{-7}$
- $\mathbf{f} \times \mathbf{u} \sim f_0 U \sim 10^{-6}$

This means that on these oceanic scales ($L \sim 100\ km$, $T \sim 1\ day$), the inertial terms are of the same order of magnitude as the Coriolis term. In other words, rotation here is much more important than the local rate of change or advection. Also, whatever the scale of the pressure gradient term is, it is the only term that can balance the rotation. Thus, if we can state that the inertial terms can be neglected, we can also state:

$$\mathbf{f} \times \mathbf{u} \approx -\frac{1}{\rho}\nabla p \tag{5.24}$$

or, in scalar component form:

$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{5.25}$$

$$fv \approx \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{5.26}$$

This balance is called the *geostrophic balance*, and it is a key concept in geophysical fluid dynamics. It states that the flow is governed by the balance between the rotation and the pressure gradient force. Although the geostrophic balance is strictly an approximation and it never holds exactly, large scale oceanic ($L \sim 100\ km$ and larger) and atmospheric ($L \sim 1000\ km$ and larger) flows are often in geostrophic balance. For the analysis of geophysical flows at such scales, it is then useful to define the *geostrophic velocity* as:

$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \tag{5.27}$$

$$v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x} \tag{5.28}$$

Notice that the geostrophic flow is always perpendicular to the pressure gradient, which means it is parallel to the isobars (lines of constant pressure). This also means that the isobars are streamlines of the geostrophic flow. In the northern hemisphere ($f > 0$), the geostrophic flow is cyclonic (counter-clockwise) around the low-pressure region and anti-cyclonic (clockwise) around the high-pressure region. In the southern hemisphere ($f < 0$), it is the opposite. A nearly geostrophic flow is illustrated in Fig. 5.4.

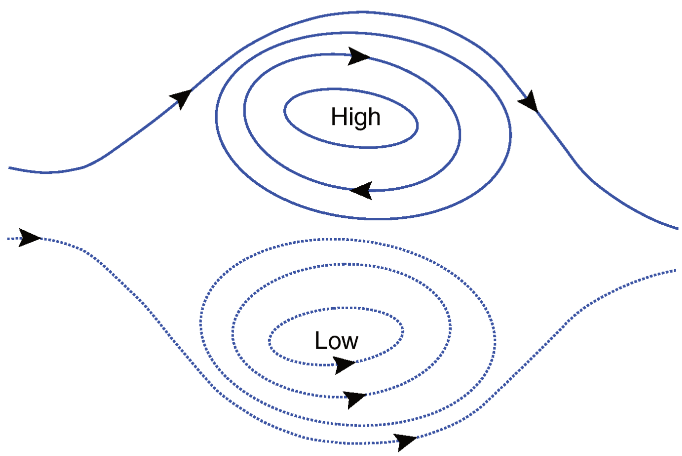


Fig. 2.5

Figure 5.4: Geostrophic flow with a positive value of the Coriolis parameter $f$. Flow is parallel to the lines of constant pressure (isobars). Cyclonic flow is anticlockwise around a low pressure region and anticyclonic flow is clockwise around a high. If $f$ were negative, as in the Southern Hemisphere, (anti)cyclonic flow would be (anti)clockwise. This is Fig. 2.5 in AOFD (Vallis, 2017).

## 5.6 Rossby number

Recall that we required the inertial terms to be much smaller than the Coriolis term for the geostrophic approximation to hold. Like we did earlier with the Reynolds number to quantify how turbulent a flow is, we can define the *Rossby number* as:

$$\mathrm{Ro} \equiv \frac{\text{Advection}}{\text{Rotation}} = \frac{(\mathbf{u}\cdot\nabla)\,\mathbf{u}}{\mathbf{f}\times\mathbf{u}} \approx \frac{\frac{U^2}{L}}{fU} \approx \frac{U}{fL} \tag{5.29}$$

Although the Rossby number characterizes the relative importance of rotation in the flow, notice that the rotation term is in the denominator. The Rossby number is thus small for flows in which rotation dominates over advection. In general, flows with a Rossby number of 0.1 or smaller are considered approximately geostrophically balanced.

## 5.7 Inertial oscillations

An analytical solution to the linearized horizontal momentum equations with rotation gives rise to a steady circular motion called the *inertial oscillation*. Start from the linearized horizontal momentum equations with rotation and with the pressure gradient force neglected:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f} \times \mathbf{u} = 0 \tag{5.30}$$

In scalar component form, this is:

$$\frac{\partial u}{\partial t} + fv = 0 \tag{5.31}$$

$$\frac{\partial v}{\partial t} - fu = 0 \tag{5.32}$$

We are now looking for a solution for $(u(t), v(t))$. These two equations are linear but coupled, so we need to decouple them first to obtain the equations with one unknown variable each. Differentiate each equation with respect to time to get:

$$\frac{\partial^2 u}{\partial t^2} + f\frac{\partial v}{\partial t} = 0 \tag{5.33}$$

$$\frac{\partial^2 v}{\partial t^2} - f\frac{\partial u}{\partial t} = 0 \tag{5.34}$$

and then insert Eqs. (5.31)-(5.32) into the above to get:

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = 0 \tag{5.35}$$

$$\frac{\partial^2 v}{\partial t^2} - f^2 v = 0 \tag{5.36}$$

The equations are now decoupled and each is a second-order, linear, homogeneous, ordinary differential equation with constant coefficients. The general solution to these equations is:

$$u = A\cos(ft) + B\sin(ft) \tag{5.37}$$

$$v = C\cos(ft) + D\sin(ft) \tag{5.38}$$

To find the constants $A$, $B$, $C$, and $D$, take the initial conditions for the velocity to be $\mathbf{u}(t = 0) = [u_0, v_0]$. This results in:

$$u = u_0 \cos(ft) + v_0 \sin(ft) \tag{5.39}$$

$$v = v_0 \cos(ft) - u_0 \sin(ft) \tag{5.40}$$

These equations describe a circular motion in the horizontal plane with a radius of $r_0 = \sqrt{u_0^2 + v_0^2}/f$ and a period of $2\pi/f$. It can be demonstrated that the motion is circular by integrating the velocities (Eqs. 5.39-5.40) over time to obtain displacements $x(t)$ and $y(t)$ and showing that the displacement radius $r = \sqrt{x^2 + y^2}$ is constant, which can only be true for a circular motion. As the inertial oscillations scale with $1/f$, they are larger and slower near the Equator and smaller and faster near the poles. For example, at 45 degrees latitude, $f \approx 10^{-4}\ s^{-1}$, and so the period of the inertial oscillation is $2\pi/f \approx 17.5$ hours.

## Exercises

1. Calculate the effective gravity at the Earth's Equator, poles, and 45 degrees latitude, taking into effect centrifugal acceleration.

2. Using scale analysis, show that on geophysical scales the vertical component of the Coriolis force is negligible compared to the other terms in the momentum equation.

3. Show that the kinetic energy of an inertial oscillation is constant.

## Summary

In this chapter, we covered:

- The effects of rotation on fluid motion, including centrifugal and Coriolis forces;

- Derivation of velocity and acceleration in a rotating reference frame;

- The Coriolis parameter $f$ and its variation with latitude;

- Inertial oscillations - circular motions that arise from the balance between inertia and Coriolis force;

- The solution for inertial oscillations showing circular motion with period $2\pi/f$ and radius $r_0 = \sqrt{u_0^2 + v_0^2}/f$.

# 6 Stratified flows

We derived our equations for mass and momentum conservation and we incorporated the effects of rotation. We also explored how the density may vary in the vertical according to the ideas gas law (in the atmosphere) or the equation of state for seawater. We now explore the effects of stratification on the flow and examine the common approximations used for large-scale oceanic flows.

## 6.1 The Boussinesq equations

We will now explore within our framework the density perturbations that will allow for buoyancy effects in a flow. The *Boussinesq approximation* is an approximation to the full equations of motion. It assumes that the density and pressure perturbations are much smaller than their means, and when applied to the Navier-Stokes equations, results in the *Boussinesq equations*.

To start, we will allow the density to have small variations around its mean value. Decompose the density into the mean and the perturbation components:

$$\rho = \rho_0 + \delta\rho(x, y, z, t) \tag{6.1}$$

where $\rho_0$ is the mean density and $\delta\rho$ is its perturbation. Further, we decompose the pressure as:

$$p = p_0(z) + \delta p(x, y, z, t) \tag{6.2}$$

where $p_0$ is the horizontally and temporally averaged pressure and $\delta p$ is its perturbation. Unlike in the density decomposition, the mean pressure component is allowed to vary in $z$. For both quantities, we require that their perturbations are much smaller than their respective means, *i.e.* $\delta\rho \ll \rho_0$, $\delta p \ll p_0$. In other words, the pressure vary much more in the vertical than in the horizontal or over time, and any perturbations in density, including those in the vertical, are much smaller than the mean density. This approximation can be demonstrated to hold well by using the equation of state for seawater (Eq. 4.52), for example. The hydrostatic approximation in this framework is trivially satisfied:

$$\frac{dp_0}{dz} = -\rho_0 g \tag{6.3}$$

Now that we've established the approximation we need, let's proceed to apply it to our momentum and continuity equations.

### 6.1.1 Momentum balance

Let's first apply the Boussinesq approximation to the momentum balance. Recall the Navier-Stokes equation with rotation (Eq. 5.13), while neglecting the viscosity term:

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla p - f\mathbf{k} \times \mathbf{u} + \mathbf{g} \tag{6.4}$$

Apply Eqs. 6.1-6.2 to the above equation to get:

$$(\rho_0 + \delta\rho)\left(\frac{d\mathbf{u}}{dt} + \mathbf{f}\times\mathbf{u}\right) = -\nabla\left(p_0 + \delta p\right) + \left(\rho_0 + \delta\rho\right)\mathbf{g} \tag{6.5}$$

$$-\nabla\left(p_0 + \delta p\right) = -\nabla\delta p - \frac{\partial p_0}{\partial z}\mathbf{k} = -\nabla\delta p - \rho_0\mathbf{g} \tag{6.6}$$

Now, recall that $\delta\rho \ll \rho_0$, so we can drop the $\delta\rho$ on the left-hand side:

$$\rho_0\left(\frac{d\mathbf{u}}{dt} + \mathbf{f}\times\mathbf{u}\right) = -\nabla\delta p + \delta\rho\ \mathbf{g} \tag{6.7}$$

$$\frac{d\mathbf{u}}{dt} + \mathbf{f}\times\mathbf{u} = -\frac{1}{\rho_0}\nabla\delta p + \frac{\delta\rho}{\rho_0}\mathbf{g} \tag{6.8}$$

For convenience of notation, let's now define *buoyancy* as $b = -g\delta\rho/\rho_0$, and re-write the above to obtain the Boussinesq momentum equation:

$$\frac{d\mathbf{u}}{dt} + \mathbf{f}\times\mathbf{u} = -\frac{1}{\rho_0}\nabla\delta p + b\mathbf{k} \tag{6.9}$$

This equation states that now that we are in a gradually stratified fluid, the gravity term is scaled by $\delta\rho/\rho_0$ to yield the appropriate vertical acceleration, and the pressure gradient is due to the relatively small perturbations in density $\delta\rho$ around the mean density $\rho_0$.

### 6.1.2 Continuity

As we did for the momentum equation, we'll now apply the Boussinesq approximation (*i.e.* $\rho = \rho_0 + \delta\rho$, $\delta\rho \ll \rho_0$) to the continuity equation. Recall the continuity equation in its complete form:

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{u} = 0 \tag{6.10}$$

Insert Eq. 6.1 to get:

$$\frac{d\delta\rho}{dt} + \left(\rho_0 + \delta\rho\right)\nabla\cdot\mathbf{u} = 0 \tag{6.11}$$

Then, if we can state that that $d\delta\rho/dt \ll \rho_0\nabla\cdot\mathbf{u}$, which we will for the Boussinesq approximation, we recover the original continuity equation for incompressible flows:

$$\nabla\cdot\mathbf{u} = 0 \tag{6.12}$$

Note that we do not say that strictly $d\delta\rho/dt = 0$, but rather that we can neglect it in this equation in favor of the velocity divergence term. The evolution of $\delta\rho$ is still governed by the evolution of buoyancy, which in turn is governed by the evolution of the temperature and salinity fields and the equation of state. The buoyancy $b = -g\delta\rho/\rho_0$ evolves as:

$$\frac{db}{dt} = \dot{b} \tag{6.13}$$

and the equation of state can be expressed in terms of buoyancy:

$$b = b(T, S, p) \tag{6.14}$$

which is just another form of Eq. 4.52.

Finally the temperature and salinity evolve as before, following Eqs. 4.53 and 4.54, respectively.

### 6.1.3 Complete system of equations

The full system of Boussinesq equations for the ocean are then:

$$\frac{d\mathbf{u}}{dt} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho_0}\nabla\delta p + b\mathbf{k} \tag{6.15}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{6.16}$$

$$\frac{dT}{dt} = \dot{T} \tag{6.17}$$

$$\frac{dS}{dt} = \dot{S} \tag{6.18}$$

$$b = b(T, S, p) \tag{6.19}$$

## 6.2 Thermal wind balance

Now that we regard the ocean as a stratified and rotating fluid with a buoyancy defined as $b = -g\delta\rho/\rho_0$, an emerging property of the flow appears if we combine this fact with the geostrophic balance (see Section 5.5). Recall the components of geostrophic velocity (Eqs. 6.20-6.21):

$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \tag{6.20}$$

$$v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x} \tag{6.21}$$

Differentiate each with respect to $z$ to get:

$$\frac{\partial u_g}{\partial z} = -\frac{1}{\rho f}\frac{\partial^2 p}{\partial z \partial y} = -\frac{1}{\rho f}\frac{\partial}{\partial y}\left(\frac{\partial p_0}{\partial z} + \frac{\partial \delta p}{\partial z}\right) \tag{6.22}$$

$$\frac{\partial v_g}{\partial z} = \frac{1}{\rho f}\frac{\partial^2 p}{\partial z \partial x} = \frac{1}{\rho f}\frac{\partial}{\partial x}\left(\frac{\partial p_0}{\partial z} + \frac{\partial \delta p}{\partial z}\right) \tag{6.23}$$

Applying the hydrostatic approximation (Eq. 6.3) to the above equations, and recalling the definition of buoyancy, we get:

$$\frac{\partial u_g}{\partial z} = -\frac{1}{f}\frac{\partial b}{\partial y} \tag{6.24}$$

$$\frac{\partial v_g}{\partial z} = \frac{1}{f}\frac{\partial b}{\partial x} \tag{6.25}$$

Equations 6.24-6.25 are known as the *thermal wind balance* (despite the name, it applies to oceans and atmopsheres alike!). It states that the geostrophic velocity must be vertically sheared in the presence of a horizontal buoyancy (density) gradient. This is illustrated in Fig. 6.1. Warm and light air means $\delta\rho < 0$ and thus $b > 0$, while cold and dense air means $\delta\rho > 0$ and thus $b < 0$. By hydrostasy, the vertical gradient of the pressure anomaly $\partial\delta p/\partial z$ is positive on the left and negative on the right. This establishes negative horizontal pressure gradient aloft and a positive one near the ground. As the Coriolis force balances the horizontal pressure gradients, the geostrophic wind is positive aloft (out of the page) and negative near the ground (into the page). Thus, in a geostrophic balanced flow alone, introducing a horizontal buoyancy gradient results in a vertical shear of the horizontal velocity.

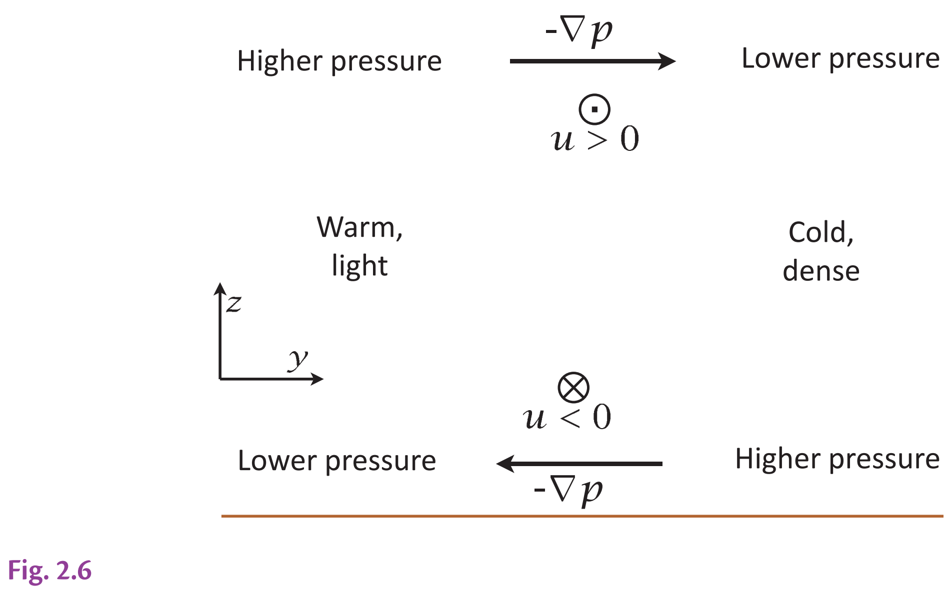


Figure 6.1: The mechanism of thermal wind. A cold fluid is denser than a warm fluid, so by hydrostasy the vertical pressure gradient is greater where the fluid is cold. Thus, pressure gradients form as shown, where ”higher” and ”lower” mean relative to the average at that height. The horizontal pressure gradients are balanced by the Coriolis force, producing (for $f > 0$) the horizontal winds shown. Only the wind shear is given by the thermal wind. This is Fig. 2.6 in AOFD (Vallis, 2017).

## 6.3 Static instability

We now consider how a fluid parcel may oscillate when its density is perturbed from its resting state and in absence of horizontal flow. This allows us to study the vertical motions due to the vertical differences in density and in isolation from other processes. We will approach this problem by displacing a fluid parcel *adiabatically* (i.e. without exchange of heat or mass with the environment) by a small distance $\delta z$ and examining how the pressure and gravity forces act on it in response (Fig. 6.2). Recall that in Eq. 6.1 we allowed for the density variations to be much smaller than the mean density, *i.e.* $\delta\rho \ll \rho_0$. Here we expand the density decomposition to a finer detail, specifically:

$$\rho = \rho_0 + \widetilde{\rho}(z) + \delta\rho(x, y, z, t) \tag{6.26}$$

where we now differentiate between the mean density $\rho_0$ and the vertically-varying environmental density $\widetilde{\rho}(z)$, while the perturbation $\delta\rho$ includes the vertical, horizontal, and temporal density variations.

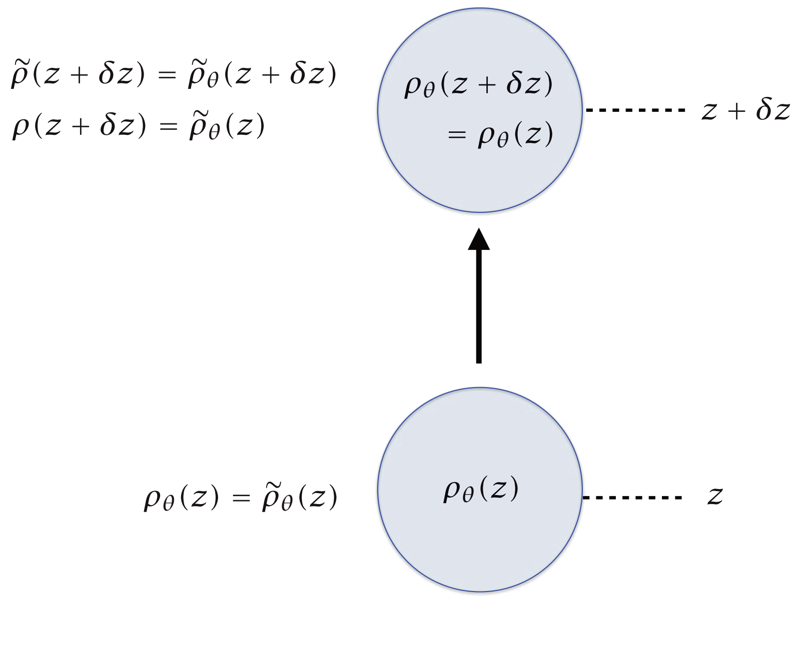


Figure 6.2: A parcel is adiabatically displaced upward from level $z$ to $z+\delta z$. A tilde denotes the value in the environment, and variables without tildes are those in the parcel. The parcel preserves its potential density, $\rho_\theta$, which it takes from the environment at level $z$. If $z+\delta z$ is the reference level, the potential density there is equal to the actual density. The parcel's stability is determined by the difference between its density and the environmental density. If the difference is positive, the displacement is stable, and if negative the displacement is unstable. This is Fig. 2.8 in AOFD (Vallis, 2017).

As the fluid parcel is displaced adiabatically, its pressure changes instantaneously to assume the same pressure as the environment. However, its temperature and salinity do not change instantaneously, resulting in a density change. To account for the instantaneous change in pressure as the parcel is displaced in height, rather than the actual density we need to consider the parcel's *potential density*, $\rho_\theta$. The potential density is the density the parcel would have if it were returned to the level where the initial pressure was $p_0$:

$$\rho_\theta = \rho + \frac{p_0}{c_s^2} = \rho + \frac{\rho_0 g z}{c_s^2} \tag{6.27}$$

where $c_s^2 = |\partial p/\partial \rho|_\theta$ is the square of the speed of sound in the fluid, which we here assume to be constant and equal to $\approx 1500\,\mathrm{m/s}$. $c_s^2$ is also related to the pressure compressibility of the fluid in the equation of state for seawater (Eq. 4.52), $\beta_p = 1/(\rho_0 c_s^2)$. Thus, if the parcel ascends or descends adiabatically, without a change in temperature or salinity, but allowing it to assume environmental pressure, its density will change but its potential density will remain constant. Potential density is thus a useful concept for understanding the static stability of the fluid.

Our goal now is to express a small change in density of the parcel relative to the environment solely in terms of the vertical gradient of the potential density. From Fig. 6.2, we start from:

$$\delta\rho = \rho(z + \delta z) - \widetilde{\rho}(z + \delta z) \tag{6.28}$$

which is the difference between the parcel's density and the environmental density at the new level. Taking the reference level to be $z + \delta z$ means that:

$$\rho(z + \delta z) = \rho_\theta(z + \delta z) \tag{6.29}$$

so we can re-write the above as:

$$\delta\rho = \rho_\theta(z + \delta z) - \widetilde{\rho}_\theta(z + \delta z) \tag{6.30}$$

Since the parcel's potential density is conserved during the adiabatic displacement, $\rho_\theta(z) = \rho_\theta(z + \delta z)$, and recall that at the starting level the parcel's potential density equals the environmental potential density, *i.e.* $\rho_\theta(z) = \widetilde{\rho}_\theta(z)$, we can write:

$$\delta\rho = \widetilde{\rho}_\theta(z) - \widetilde{\rho}_\theta(z + \delta z) \tag{6.31}$$

Then, for small $\delta z$:

$$\delta\rho = -\frac{\partial \widetilde{\rho}_\theta}{\partial z}\delta z \tag{6.32}$$

The parcel's static stability is thus determined by the vertical gradient of the locally-referenced potential density of the environment, $\widetilde{\rho}_\theta$:

$$\frac{\partial \widetilde{\rho}_\theta}{\partial z} < 0 \quad \text{(statically stable)} \tag{6.33}$$

$$\frac{\partial \widetilde{\rho}_\theta}{\partial z} > 0 \quad \text{(statically unstable)} \tag{6.34}$$

Now, to determine the oscillatory motion of the parcel, we apply Newton's second law and balance the acceleration of the parcel with the buoyancy force:

$$\frac{\partial^2 \delta z}{\partial t^2} = \frac{g}{\rho}\left(\frac{\partial \widetilde{\rho}_\theta}{\partial z}\right)\delta z = -N^2 \delta z \tag{6.35}$$

where we have defined the *Brunt-Väisälä frequency* (or buoyancy frequency) as:

$$N^2 = -\frac{g}{\widetilde{\rho}_\theta}\frac{\partial \widetilde{\rho}_\theta}{\partial z} = \frac{d\widetilde{b}}{dz} \tag{6.36}$$

while noting that $\rho(z) = \widetilde{\rho}_\theta(z)$ within $\mathcal{O}(\delta z)$. A parcel displaced from its equilibrium position will oscillate with angular frequency $N$ if $N^2 > 0$ (statically stable) and will freely accelerate upward if $N^2 < 0$ (statically unstable).

## Exercises

1. Use the linear equation of state for seawater to demonstrate that the variations of density in the ocean are very small compared to the mean density. How large (in percent relative change) are these variations with respect to the changes in temperature, salinity, and pressure in the ocean?

2. Calculate the Brunt-Väisälä frequency for: (a) a typical mid-latitude thermocline with temperature decreasing from 20°C to 5°C over 500 m depth; (b) the deep ocean where potential temperature decreases from 4°C to 2°C over 2000 m depth. Assume constant salinity of 35 g/kg.

## Summary

In this chapter, we covered:

- The Boussinesq approximation, which assumes density variations are small compared to the mean density;

- Decomposition of density and pressure into mean and perturbation components;

- Static stability and its relationship to the vertical density gradient;

- The Brunt-Väisälä frequency as a measure of stratification strength and the natural frequency of vertical oscillations in a stratified fluid.

# 7 Shallow water systems

In this chapter we move away from the continuously stratified ocean and approximate it to a single layer of incompressible fluid that is also in hydrostatic balance. It turns out that this seemingly drastic approximation still allows the reduced equation set to reproduce many observed large scale oceanic and atmospheric phenomena. In fact, the shallow water system of equations is the basis for some operational ocean prediction models, especially if applied to the nearshore and coastal ocean. We begin by introducing the key assumptions that allow the derivation of the shallow water equations, and after that we derive the general solutions to the equations.

## 7.1 Key assumptions

The name ”shallow water” may be revealing about the approximations that we will make about the flow:

1. **Shallow:** The vertical scale of the flow is much smaller than the horizontal scale. Although this doesn't mean that there is no vertical flow, it does mean that the horizontal flow is much larger, *i.e.* $u, v \gg w$.

2. **Water:** The flow is incompressible, *i.e.* $\nabla \cdot \mathbf{u} = 0$.

As a consequence of the above approximations, our flow will also be hydrostatic, *i.e.* $\partial p/\partial z = -\rho g$. This approximation will show to be instrumental in allowing us to cast the horizontal pressure gradient in terms of the surface elevation only.

The flow can then be described as a thin layer of fluid over a rigid bottom that may vary in the horizontal, and with a free surface that can freely move in the vertical in response to the flow and incompressibility (Fig. 7.1). This layer of fluid may or may not be covered on top by another layer of fluid, with its own hydrostatic pressure imposed on the surface.

## 7.2 Shallow water equations

The shallow water equations consist of the momentum and the continuity equations. For 2-dimensional horizontal flow, the momentum equation can be expressed as a single equation in vector form, or as two scalar equations in $x$ and $y$.

### 7.2.1 Momentum equation

We begin from the vector momentum equation with rotation:

$$\frac{d\mathbf{u}}{dt} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g} \tag{7.1}$$

In the vertical component of this equation we will neglect the vertical acceleration to obtain the hydrostatic balance, as we did previously:

$$\frac{\partial p}{\partial z} = -\rho g \tag{7.2}$$

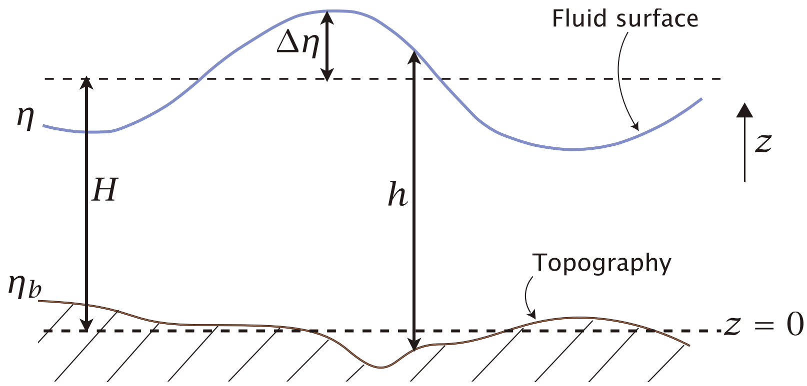


Fig. 3.1

Figure 7.1: A shallow water system. $h$ is the thickness of a water column, $H$ its mean thickness, $\eta$ the height of the free surface and $\eta_b$ is the height of the lower, rigid surface above some arbitrary origin, typically chosen such that the average of $\eta_b$ is zero. $\Delta\eta$ is the deviation free surface height, so we have $\eta = \eta_b + h = H + \Delta\eta$. This is Fig. 3.1 in AOFD (Vallis, 2017).

We can integrate the hydrostatic balance in $z$ to obtain the pressure as a function of height:

$$\int_{p(z)}^{p_\eta} dp = -\int_z^\eta \rho g \, dz \tag{7.3}$$

where $p_\eta$ is the pressure at $z = \eta$. Rearranging the terms after integration yields:

$$p(z) = p_\eta + \rho g \eta - \rho g z \tag{7.4}$$

We will now take a horizontal gradient to both sides and assume for simplicity that the horizontal gradient of $p_\eta$ is negligible compared to the other terms. Further, taking that neither the density nor gravity vary in the horizontal, and noting that $z$ as a vertical coordinate cannot vary in the horizontal, we get:

$$\nabla p = \rho g \nabla \eta \tag{7.5}$$

Inserting Eq. 7.5 into Eq. 7.1, and taking $\nabla$ to be the horizontal divergence going forward, we get:

$$\frac{d\mathbf{u}}{dt} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta \tag{7.6}$$

which is the horizontal shallow water momentum equation with rotation. Let's now proceed to derive the shallow water continuity equation and complete the system of equations.

### 7.2.2 Continuity equation

An intuitive approach to deriving the shallow water continuity is to consider a column of fluid in a one-dimensional horizontal flow whose spatial variations would cause a change in the surface elevation of that column due to the incompressibility (Fig. 7.2). Although the bottom surface here is shown to be flat, recall from Fig. 7.1 that it doesn't need to be, and the water column height $h$ comprises of the water depth (as measured from the rigid bottom to the mean water level) plus the deviation of the free surface from the mean water level, $\Delta\eta$.

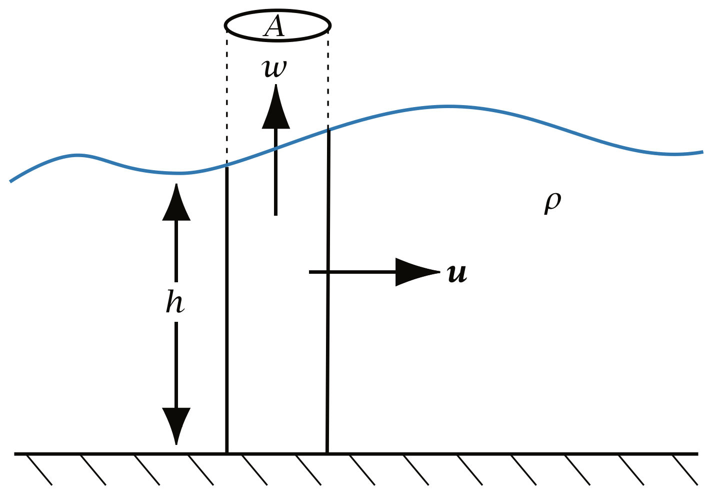


Fig. 3.2

Figure 7.2: The mass budget for a column of area $A$ in a shallow water system. There is a non-zero vertical velocity at the top of the column if the mass convergence into the column is non-zero. This is Fig. 3.2 in AOFD (Vallis, 2017).

The difference between the amount of liquid flowing into and out of the column thus must be balanced by a change in the surface elevation of the column:

$$u_2 h_2 - u_1 h_1 = \frac{\partial \eta}{\partial t} \Delta x \tag{7.7}$$

Rearranging the terms leads to:

$$\frac{\partial \eta}{\partial t} = \frac{u_2 h_2 - u_1 h_1}{\Delta x} \approx \frac{\partial (uh)}{\partial x} \tag{7.8}$$

Generalized in vector form, this becomes the Eulerian form of the shallow water continuity equation:

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0 \tag{7.9}$$

and from there, while noting that $h - \eta$, the mean water depth, does not vary in time, the Lagrangian form is:

$$\frac{d\eta}{dt} + h\nabla \cdot \mathbf{u} = 0 \tag{7.10}$$

Alternatively, we can derive the shallow water continuity from the incompressibility of the flow:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{7.11}$$

$$\frac{\partial w}{\partial z} \approx \frac{w_\eta - w_b}{h} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \tag{7.12}$$

where $w_\eta$ and $w_b$ are the vertical velocities at the free surface and the rigid bottom surface, respectively, and $h$ is, again, the distance of the free surface from the bottom. $w_b$ must be zero, of course, and $w_\eta = \partial\eta/\partial t$, so we get:

$$\frac{d\eta}{dt} + h\nabla \cdot \mathbf{u} = 0 \tag{7.13}$$

which is the Lagrangian form of the shallow water continuity and the same equation as Eq. 7.10. To get the Eulerian form from here, we first need to recognize that $d\eta/dt = dh/dt$ because $h = \overline{h} + \eta$, where $\overline{h}$ is the mean water depth. Then, expanding the Lagrangian derivative, we recover Eq. 7.9.

### 7.2.3 The complete equation set

The momentum and continuity equations that we derived above form the complete set of shallow water equations. In vector form, they are:

$$\frac{d\mathbf{u}}{dt} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta \tag{7.14}$$

$$\frac{\partial\eta}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0 \tag{7.15}$$

And in scalar form, in two dimensions:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x} \tag{7.16}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y} \tag{7.17}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0 \tag{7.18}$$

which closes our system of equations. In two dimensions, we thus have three equations for the three unknown variables $u$, $v$, and $\eta$. The flow is inviscid (no friction) but nonlinear (advective term $\mathbf{u} \cdot \nabla\mathbf{u}$ is present), so this system of equations allows for turbulence but does not dissipate energy. Also, notice that the Coriolis force is present but has seamlessly percolated from the starting equation without breaking any of the assumptions. Thus, to consider shallow water systems in a non-rotating frame, simply drop the Coriolis term.

We now proceed to further simplify this equation set to derive a general solution for the shallow water equations.

## 7.3 Poincaré waves

As we proceed without our intention to derive a solution to the equations (7.16-7.18), notice that the nonlinear terms get in the way of an analytical solution. To work around this, we will linearize the equations by decomposing the flow into a mean and a perturbation:

$$h(x, y, t) = H + \eta'(x, y, t) \tag{7.19}$$

$$\mathbf{u}(x, y, t) = \mathbf{U} + \mathbf{u}'(x, y, t) \tag{7.20}$$

and since the mean flow in space and time does not vary, and it is by definition zero for the fluid at rest, then the velocity field is equal to its perturbation:

$$\mathbf{u}(x, y, t) = \mathbf{u}'(x, y, t) \tag{7.21}$$

Insert these decompositions Eqs. (7.14) and (7.15) to get:

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}' \cdot \nabla \mathbf{u}' + \mathbf{f} \times \mathbf{u}' = -g\nabla\left(H + \eta'\right) \tag{7.22}$$

$$\frac{\partial \eta'}{\partial t} + \nabla \cdot [(H + \eta')\mathbf{u}'] = 0 \tag{7.23}$$

Although we do not (and cannot) require that the perturbations on their own are small enough to neglect, the products of two perturbations are assumed to be. This allows us to linearize the equations and obtain:

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{f} \times \mathbf{u}' + g\nabla\eta' = 0 \tag{7.24}$$

$$\frac{\partial \eta'}{\partial t} + H\nabla \cdot \mathbf{u}' = 0 \tag{7.25}$$

Or, in scalar form:

$$\frac{\partial u'}{\partial t} - fv' + g\frac{\partial \eta'}{\partial x} = 0 \tag{7.26}$$

$$\frac{\partial v'}{\partial t} + fu' + g\frac{\partial \eta'}{\partial y} = 0 \tag{7.27}$$

$$\frac{\partial \eta'}{\partial t} + H\frac{\partial u'}{\partial x} + H\frac{\partial v'}{\partial y} = 0 \tag{7.28}$$

Assume the general solution to have a wave-like form:

$$(u, v, \eta) = (\widehat{u}, \widehat{v}, \widehat{\eta})e^{i(kx+ly-\omega t)} \tag{7.29}$$

where $\widehat{u}$, $\widehat{v}$, and $\widehat{\eta}$ are the amplitudes of the wave-like perturbations, $k$ and $l$ are the wavenumbers, and $\omega$ is the angular frequency. Insert the wave form into Eqs. (7.26-7.28) to get:

$$-i\omega\widehat{u} - f\widehat{v} + igk\widehat{\eta} = 0 \tag{7.30}$$

$$-i\omega\widehat{v} + f\widehat{u} + igl\widehat{\eta} = 0 \tag{7.31}$$

$$-i\omega\widehat{\eta} + iHk\widehat{u} + iHl\widehat{v} = 0 \tag{7.32}$$

or, in matrix form:

$$\begin{bmatrix} -i\omega & -f & igk \\ f & -i\omega & igl \\ iHk & iHl & -i\omega \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \widehat{v} \\ \widehat{\eta} \end{bmatrix} = 0 \tag{7.33}$$

The solution to this system requires that the determinant of the matrix be zero, which yields:

$$\omega[\omega^2 - f^2 - gH(k^2 + l^2)] = 0 \tag{7.34}$$

A trivial solution to this equation is $\omega = 0$, which corresponds to an unperturbed, constant flow. The other, non-trivial solution is the dispersion relation for shallow water gravity waves in a rotating frame:

$$\omega = \sqrt{f^2 + gH(k^2 + l^2)} \tag{7.35}$$

This dispersion relationship connects the frequency to the wavenumber, and we see that it scales with the Coriolis frequency $f$ and the gravity wave phase speed $\sqrt{gH}$. This general solution corresponds to the so-called *Poincaré waves*, which are the surface gravity waves with effects of rotation. They are also commonly referred to as *inertial-gravity waves*. Increasing the Coriolis parameter $f$ while keeping the other parameters fixed increases the frequency of the waves by enhancing the rotation. Similarly, increasing the gravitational acceleration $g$ or the mean water depth $H$ increases the frequency of the waves by enhancing the gravity wave phase speed. Notice also that the frequency $\omega$ scales linearly with the wavenumber $k^2 + l^2$, their ratio $\omega/(k^2 + l^2)$ being the phase speed of the wave:

$$c_p = \frac{\omega}{\sqrt{k^2 + l^2}} = \sqrt{\frac{f^2}{k^2 + l^2} + gH} \tag{7.36}$$

As there are two independent parameters in Eq. 7.35 that originate from different terms in the shallow water equations, we can turn the knobs on each to explore some limiting cases of the general solution.

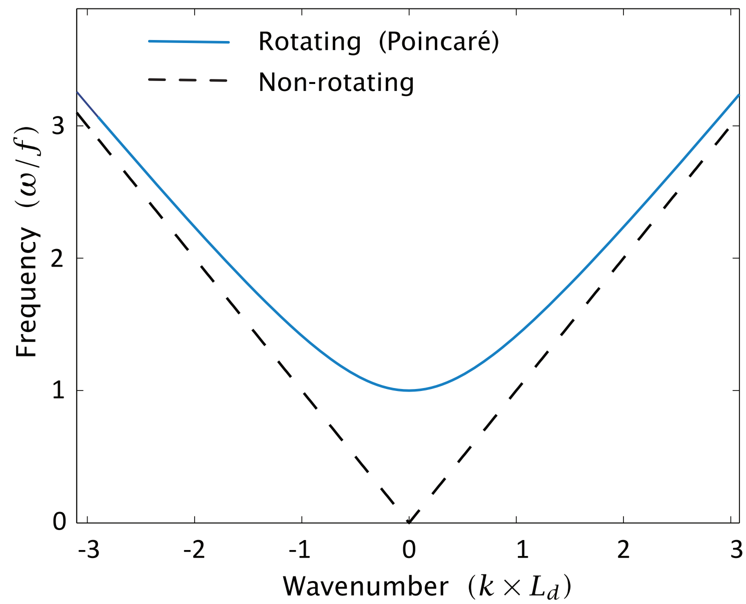


Fig. 3.8

Figure 7.3: Dispersion relation for Poincaré waves and non-rotating shallow water waves. Frequency is scaled by the Coriolis frequency $f$, and wavenumber by the inverse deformation radius $\sqrt{gH}/f$. For small wavenumbers the frequency of the Poincaré waves is approximately $f$, and for high wavenumbers is asymptotes to that of non-rotating waves. This is Fig. 3.8 in AOFD (Vallis, 2017).

### 7.3.1 Short gravity waves

In the case of short gravity waves, the pressure gradient terms (and thus, gravity) dominate the Coriolis term (rotation):

$$gH(k^2 + l^2) \gg f^2 \tag{7.37}$$

In this case, the dispersion relation simplifies to:

$$\omega = \sqrt{gH(k^2 + l^2)} \tag{7.38}$$

which is the dispersion relation for (non-rotating) shallow water gravity waves. Notice, however, that we don't require there to be no rotation at all to obtain the non-rotating gravity waves. Rather, we simply require that the waves are so short (high wavenumber) that the Coriolis force is negligible compared to the gravity force. The phase speed of these waves, that is, the speed at which they propagate, is:

$$C_p = \frac{\omega}{k} = \sqrt{gh} \tag{7.39}$$

Real-life examples of this solution include tsunamis, wind-generated swell waves on the ocean surface, or small ripples that propagate radially outward when throwing a stone into a pond.

### 7.3.2 Inertial oscillations

If the wavenumber is so small (large wavelength) that the gravity term can be neglected in favor of the Coriolis term, we recover a class of motion that we explored earlier, the inertial oscillations. In this case, the rotation dominates over the gravity:

$$f^2 \gg gH(k^2 + l^2) \tag{7.40}$$

and the dispersion relation simplifies to:

$$\omega = f \tag{7.41}$$

which corresponds to a circular motion with the frequency that exactly equals the Coriolis frequency (because $(u, v) = (\widehat{u}, \widehat{v})e^{-ift}$). Recall that we already explored this solution by dropping the pressure gradient terms in the rotating momentum equations back in Section 5.7. Here, it comes out as a limiting case from the general solution which we couldn't obtain prior to the shallow water approximations and linearization.

## 7.4 Kelvin waves

A special case of the general solution that is particularly relevant to the atmospheric and oceanic dynamics is that of a linearized shallow water flow that is bounded on one side by a solid boundary, such as a coastline. The resulting solution is a special class of gravity waves called *Kelvin waves*, which propagate as a shallow water gravity wave along the solid boundary and whose propagation direction, as well as the perturbation scale in the direction away from the boundary, are governed by the planetary rotation rate. Kelvin waves appear in both the atmosphere and the ocean.

To derive the Kelvin waves, we start from the linearized shallow water equations (where we drop the primes for brevity):

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{7.42}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{7.43}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{7.44}$$

Now, suppose that our solid boundary is along the $x$-axis at $y = 0$, which allows us to neglect the meridional flow ($v = 0$):

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{7.45}$$

$$fu = -g\frac{\partial \eta}{\partial x} \tag{7.46}$$

$$\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0 \tag{7.47}$$

Differentiate Eq. 7.45 with respect to time and Eq. 7.47 with respect to $x$, and combine them to get:

$$\frac{\partial^2 u}{\partial t^2} - gH\frac{\partial^2 u}{\partial x^2} = 0 \tag{7.48}$$

which is the standard wave equation, whose solution is a wave that propagates with the phase speed $c = \sqrt{gH}$. We will thus assume a wave-like solution for $u$, like we did for the Poincaré waves in Section 7.3. However, since we now have a solid boundary at $y = 0$, we should also assume that the solution should vary in the $y$ direction (because it must be zero at the boundary, and non-zero elsewhere). The general solution for $u$ may be:

$$u = \widehat{u}(y)e^{i(k-ct)} \tag{7.49}$$

As for the elevation $\eta$, insert Eq. 7.49 into Eq. 7.47 to get:

$$\eta = \sqrt{\frac{g}{H}}\widehat{u}(y)e^{i(k-ct)} \tag{7.50}$$

We still need to solve for $\widehat{u}(y)$, so we look for the equation that has a derivative with respect to $y$. So, insert Eqs. 7.49 and 7.50 into Eq. 7.46 to get:

$$f\widehat{u}(y) = -\sqrt{\frac{H}{g}}\frac{\partial \widehat{u}(y)}{\partial y} \tag{7.51}$$

which integrates to:

$$\widehat{u}(y) = \widehat{u}_0 e^{-\frac{y}{L_d}} \tag{7.52}$$

where

$$L_d = \frac{\sqrt{gH}}{f} \tag{7.53}$$

is the *Rossby radius of deformation* , which is the length scale at which planetary rotation becomes important relative to the effects of gravity (or buoyancy, in stratified flows). The complete solutions for the shallow water Kelvin waves are then:

$$u = \widehat{u}_0 e^{-\frac{y}{L_d}} e^{i(k-ct)} \tag{7.54}$$

$$\eta = \sqrt{\frac{H}{g}}\widehat{u}_0 e^{-\frac{y}{L_d}} e^{i(k-ct)} \tag{7.55}$$

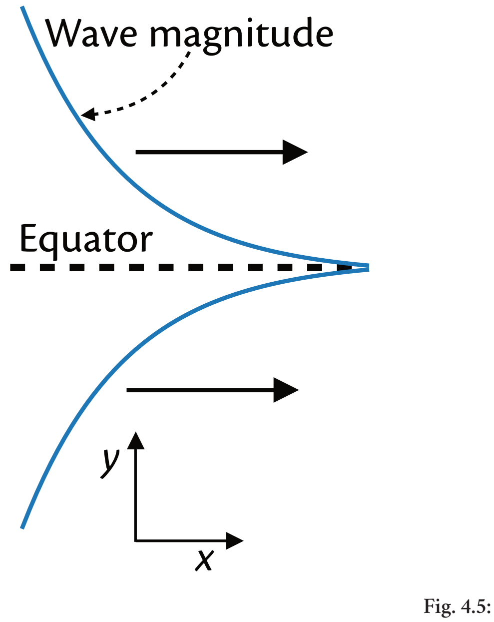


Figure 7.4: Kelvin waves propagating eastward along the equator and decaying rapidly away to either side. This is Fig. 4.5 in Vallis (EAOD).

## 7.5 Conservative properties

We now look at some conservative properties of the shallow water equations, namely the potential vorticity conservation and the conservation of energy. The former is a material conservative property, meaning that it is conserved along a fluid parcel as it moves and deforms. The latter is a volume-integrated conservative property, meaning that it is conserved in a control volume as the fluid evolves in time. The conservation of potential vorticity yields some interesting emerging properties of the flow, such as the vortex stretching due to the change in the fluid depth, and the planetary waves due to the meridional variation of the planetary vorticity (Coriolis parameter $f$).

### 7.5.1 Potential vorticity

*Potential vorticity* (PV) describes the rate of rotation of a fluid parcel scaled by the fluid depth. It is a material property, meaning that it is conserved along a fluid parcel as it moves and deforms. Start from the momentum equation with effects of rotation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{f} \times \mathbf{u} = -g \nabla \eta \tag{7.56}$$

We will rely on the following vector identity to rewrite the advective term:

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla (\mathbf{u}^2) - \mathbf{u} \times (\nabla \times \mathbf{u}) \tag{7.57}$$

and recognize $\nabla \times \mathbf{u} = \boldsymbol{\omega}$ as the vorticity to rewrite the above as:

$$\frac{\partial \mathbf{u}}{\partial t} + (\boldsymbol{\omega} + \mathbf{f}) \times \mathbf{u} = -g\nabla \left(\eta + \frac{1}{2}\mathbf{u}^2\right) \tag{7.58}$$

Take a curl of this equation to get:

$$\frac{\partial(\nabla \times \mathbf{u})}{\partial t} + \nabla \times [(\boldsymbol{\omega} + \mathbf{f}) \times \mathbf{u}] = -\nabla \times \nabla \left(\eta + \frac{1}{2}\mathbf{u}^2\right) \tag{7.59}$$

Next, we use the vector triple product identity:

$$\nabla \times \boldsymbol{\omega} \times \mathbf{u} = (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \boldsymbol{\omega}\nabla \cdot \mathbf{u} - \mathbf{u}\nabla \cdot \boldsymbol{\omega} \tag{7.60}$$

Since vorticity must be divergence free ($\nabla \cdot \boldsymbol{\omega} = 0$), and it's perpendicular to the velocity vector ($\boldsymbol{\omega} \cdot \mathbf{u} = 0$), the second and the fourth terms vanish. Define the vertical component of the vorticity to be:

$$\zeta = \mathbf{k} \cdot \boldsymbol{\omega} \tag{7.61}$$

to get:

$$\frac{\partial \zeta}{\partial t} + (\mathbf{u} \cdot \nabla)(\zeta + f) = -(\zeta + f)\nabla \cdot \mathbf{u} \tag{7.62}$$

Recall the continuity equation:

$$\frac{dh}{dt} = -h\nabla \cdot \mathbf{u} \tag{7.63}$$

Multiply both sides by $(\zeta + f)$ to write:

$$\frac{dh}{dt}\frac{\zeta + f}{h} = -(\zeta + f)\nabla \cdot \mathbf{u} \tag{7.64}$$

$$\frac{d(\zeta + f)}{dt} = \frac{\zeta + f}{h}\frac{dh}{dt} \tag{7.65}$$

Then, notice that:

$$\frac{d}{dt}\left(\frac{\zeta}{h}\right) = \frac{1}{h}\frac{d\zeta}{dt} + \zeta\frac{d}{dt}\left(\frac{1}{h}\right) = \frac{1}{h}\frac{d\zeta}{dt} - \frac{\zeta}{h^2}\frac{dh}{dt} = \frac{1}{h}\left(\frac{d\zeta}{dt} - \frac{\zeta}{h}\frac{dh}{dt}\right) \tag{7.66}$$

which leads to:

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0 \tag{7.67}$$

where $(\zeta + f)/h$ is the *potential vorticity*, and Eq. 7.67 is the conservation of potential vorticity.

Let's consider some implications of it. First, without planetary rotation ($f = 0$), potential vorticity is $\zeta/h$. Imagine a parcel of fluid with some vorticity $\zeta$(for example, a small eddy). The eddy propagates zonally over a seamount such that the mean water depth gradually decreases. As the eddy enters progressively shallower water, its vorticity must increase so that the potential vorticity is conserved. An cold eddy (with $\zeta > 0$) will thus rotate more rapidly (cyclonically, or counter-clockwise in the Northern Hemisphere) as it approaches the tip of the seamount where the water is shallowest, and then decrease again as it moves away from the tip of the seamount into deeper water. Similarly, a warm eddy (with $\zeta < 0$) will weaken its anticyclonic (clockwise) rotation as it moves toward the tip of the seamount, and then strengthen it again as it moves away from the tip of the seamount into deeper water. Another consequence of the conservation of potential vorticity is that on a $\beta$-plane, or more generally, a rotating sphere, where the Coriolis parameter $f$ varies with latitude, the vorticity of a parcel will adjust to meridional displacements and changes in $f$ to conserve potential vorticity. The latter mechanism yields the so-called *Rossby waves*, a key feature of mid-latitude weather dynamics.

### 7.5.2 Energy

Start from the definitions of potential and kinetic energy:

$$PE = \int_0^h \rho g z \; dz = \frac{1}{2}\rho g h^2 \tag{7.68}$$

$$KE = \int_0^h \frac{1}{2}\rho \mathbf{u}^2 \; dz = \frac{1}{2}\rho \mathbf{u}^2 h \tag{7.69}$$

The total energy is the sum of potential and kinetic energy:

$$E = PE + KE = \frac{1}{2}\rho g h^2 + \frac{1}{2}\rho \mathbf{u}^2 h \tag{7.70}$$

Let's now proceed to derive the PE and KE equations for the shallow water systems. Recall the shallow water continuity equation:

$$\frac{dh}{dt} + h\nabla \cdot \mathbf{u} = 0 \tag{7.71}$$

Multiply it by $gh$ to get:

$$\frac{d}{dt}\left(\frac{gh^2}{2}\right) + gh^2\nabla \cdot \mathbf{u} = 0 \tag{7.72}$$

Expand the Lagrangian derivative:

$$\frac{\partial}{\partial t}\left(\frac{gh^2}{2}\right) + \mathbf{u} \cdot \nabla\left(\frac{gh^2}{2}\right) + gh^2\nabla \cdot \mathbf{u} = 0 \tag{7.73}$$

Then, we borrow a half of the third term to combine it with the second term:

$$\frac{\partial}{\partial t}\left(\frac{gh^2}{2}\right) + \nabla\left(\mathbf{u}\frac{gh^2}{2}\right) + \frac{gh^2}{2}\nabla \cdot \mathbf{u} = 0 \tag{7.74}$$

which is the equation for the evolution of potential energy. Note that the density $\rho$ is assumed constant and is omitted here for brevity.

Next, recall the momentum equation, assuming uniform mean water depth for simplicity:

$$\frac{d\mathbf{u}}{dt} = -g\nabla h \tag{7.75}$$

Multiply this by $\mathbf{u}$ and re-arrange to get:

$$\mathbf{u}h\frac{d\mathbf{u}}{dt} + g\mathbf{u}h\nabla h = 0 \tag{7.76}$$

$$\frac{d}{dt}\left(\frac{h\mathbf{u}^2}{2}\right) - \frac{\mathbf{u}^2}{2}\frac{dh}{dt} + g\mathbf{u}\nabla\left(\frac{h^2}{2}\right) = 0 \tag{7.77}$$

Recall the shallow water continuity to write:

$$\frac{d}{dt}\left(\frac{h\mathbf{u}^2}{2}\right) + \frac{h\mathbf{u}^2}{2}\nabla\cdot\mathbf{u} + g\mathbf{u}\nabla\left(\frac{h^2}{2}\right) = 0 \tag{7.78}$$

Expand the Lagrangian derivative:

$$\frac{\partial}{\partial t}\left(\frac{h\mathbf{u}^2}{2}\right) + \mathbf{u}\cdot\nabla\left(\frac{h\mathbf{u}^2}{2}\right) + \frac{h\mathbf{u}^2}{2}\nabla\cdot\mathbf{u} + g\mathbf{u}\nabla\left(\frac{h^2}{2}\right) = 0 \tag{7.79}$$

and combine the second and third terms to write:

$$\frac{\partial}{\partial t}\left(\frac{h\mathbf{u}^2}{2}\right) + \nabla\cdot\left(\mathbf{u}\frac{h\mathbf{u}^2}{2}\right) + g\mathbf{u}\nabla\left(\frac{h^2}{2}\right) = 0 \tag{7.80}$$

which is the equation for the evolution of kinetic energy.

Now, combine Eqs. 7.74 and 7.80 to get:

$$\frac{\partial}{\partial t}\frac{1}{2}\left(h\mathbf{u}^2 + gh^2\right) + \nabla\cdot\left[\mathbf{u}\left(\frac{1}{2}h\mathbf{u}^2 + gh^2\right)\right] = 0 \tag{7.81}$$

which is the conservation of total energy $E = PE + KE$, and $\mathbf{F} = \mathbf{u}\left(\frac{1}{2}h\mathbf{u}^2 + gh^2\right)$ is the energy flux such that we can write:

$$\frac{\partial E}{\partial t} + \nabla\cdot\mathbf{F} = 0 \tag{7.82}$$

The total energy of the system $E$ is thus conserved and entirely governed by the divergence of the energy flux $\mathbf{F}$.

## 7.6 Rossby waves

One emerging pattern from the conservation of potential vorticity arises if the planetary vorticity $f$ is allowed to vary with latitude. This is true on a sphere where $f = 2\Omega\sin(\theta)$, or on a $\beta$-plane where $f = f_0 + \beta y$. This pattern is called *Rossby waves* (also called *planetary waves*) and is among the most important class of motions in both the ocean and the atmosphere.

To derive the solution for Rossby waves, we start from the shallow-water potential vorticity conservation equation:

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0 \tag{7.83}$$

To simplify the derivation, we will assume a flat bottom so that

$$\frac{d(\zeta + f)}{dt} = 0 \tag{7.84}$$

Expand the Lagrangian derivative to get:

$$\frac{\partial \zeta}{\partial t} + (\mathbf{u} \cdot \nabla)\,\zeta + v\beta = 0 \tag{7.85}$$

which is the potential vorticity conservation equation on a $\beta$-plane.

We still have only one equation with two unknowns (relative vorticity $\zeta$ and velocity $\mathbf{u}$), so we somehow need to reduce them to one unknown variable. One approach is to introduce a streamfunction $\psi$ such that:

$$(u, v) = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right) \tag{7.86}$$

We can then express the relative vorticity as:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi \tag{7.87}$$

Insert Eqs. 7.86 and 7.87 into Eq. 7.85 to get:

$$\frac{\partial}{\partial t}\nabla^2 \psi + U\frac{\partial}{\partial x}\nabla^2 \psi + \beta\frac{\partial \psi}{\partial x} = 0 \tag{7.88}$$

which is the potential vorticity equation on a $\beta$-plane in terms of the streamfunction.

As before, assume a wave-like solution but this time for the streamfunction:

$$\psi = \widehat{\psi} e^{i(kx - \omega t)} \tag{7.89}$$

and insert it into Eq. 7.88 to get the dispersion relation for Rossby waves:

$$\omega = Uk - \frac{\beta}{k} \tag{7.90}$$

The phase speed of Rossby waves is:

$$c = \frac{\omega}{k} = U - \frac{\beta}{k^2} \tag{7.91}$$

## Exercises

1. Assuming shallow water approximation and mid-latitudes, quantify the relative importance of planetary rotation in the flow for (a) wind-generated swell waves, (b) a submesoscale eddy, (c) Gulf Stream, and (d) a synoptic-scale cyclone in the atmosphere.

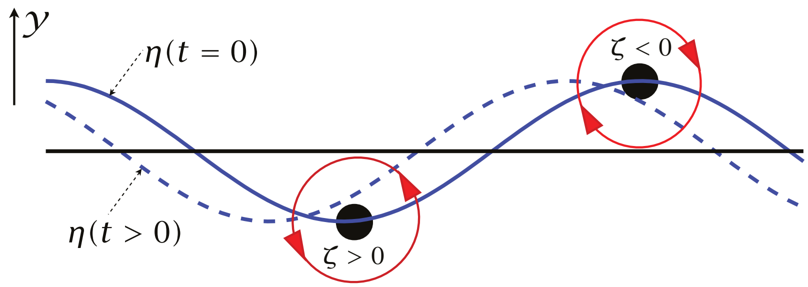


Figure 7.5: A two-dimensional (x-y) Rossby wave. An initial disturbance displaces a material line at constant latitude (the straight horizontal line) to the solid line marked $\eta(t = 0)$. Conservation of potential vorticity, $\zeta + \beta y$, leads to the production of relative vorticity, $\zeta$, as shown. The associated velocity field (arrows on the circles) then advects the fluid parcels, and the material line evolves into the dashed line with the phase propagating westward. This is Fig. 6.3 in Vallis (EAOD).

2. Consider characteristic mid-latitude flows on Earth, Jupiter, and Titan. At what spatial scales does the gravity play equal role as the rotation? Assume the shallow water dispersion relationship for your analysis.

3. Show that the vector identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a} \tag{7.92}$$

holds for arbitrary vector fields $\mathbf{a}$ and $\mathbf{b}$.

4. An ocean eddy with initial relative vorticity $\zeta_0$ begins its journey northward at 30°N and depth of 2000 m and travels with the mean flow to 40°N and depth of 1000 m. Assuming the potential vorticity of the eddy is conserved, calculate the its final relative vorticity.

## Summary

In this chapter, we covered:

- The shallow water equations as a simplified model for large-scale ocean and atmospheric flows;
- Key assumptions of the shallow water system: horizontal scales much larger than vertical scales, incompressible flow, and hydrostatic balance;
- Rossby waves - westward propagating planetary waves that arise from the variation of the Coriolis parameter with latitude;
- The dispersion relationship and phase speed of Rossby waves;
- Conservation of potential vorticity and its role in generating relative vorticity as fluid parcels move meridionally.

## Further reading

- Chapter 4 (Shallow water systems) of EAOD by Vallis.
- Section 6.3 (Rossby wave essentials) of EAOD by Vallis.

# 8 Turbulence

Turbulence is the nonlinear and chaotic fluid motion that occurs when a fluid is driven by sufficiently strong forces. It is characterized by large fluctuations in time and space and over a broad range of scales. Fluid elements with high vorticity of either sign move and interact with each other, transferring energy and vorticity across scales. In this chapter, we investigate turbulence from the point of view of the governing equations of fluid motion. To do that, we begin by introducing the Reynolds decomposition, a fundamental tool in the study of turbulence. By applying it to the Navier-Stokes equation, we will derive the Reynolds-averaged Navier-Stokes (RANS) equation, which describes and predicts the evolution of the mean flow while accounting for the effects of turbulence. We will discuss the so-called closure problem of turbulence, which is the challenge of representing the effects of the smallest scales on the larger scales. Using the RANS equation, we will derive the turbulent kinetic energy budget equation, and investigate the turbulent cascade in both 2D and 3D flows. The new understanding from this chapter will allow us to study the boundary layers in the atmosphere and the ocean alike.

## 8.1 Reynolds decomposition

We begin by introducing the decomposition of a field (in this case velocity $\mathbf{u}$, a vector field) into the mean and fluctuating parts:

$$\mathbf{u}(x,t) = \overline{\mathbf{u}}(x) + \mathbf{u}'(x,t) \tag{8.1}$$

If we take the time average of the above equation, we get:

$$\overline{\mathbf{u}(x,t)} = \overline{\overline{\mathbf{u}}(x) + \mathbf{u}'(x,t)} \tag{8.2}$$

which leads to:

$$\overline{\mathbf{u}'(x,t)} = 0 \tag{8.3}$$

The averaging operation is commutative with respect to derivatives and integrals, over space or time alike:

$$\overline{\frac{\partial \mathbf{u}}{\partial t}} = \frac{\partial \overline{\mathbf{u}}}{\partial t} \tag{8.4}$$

$$\overline{\nabla \cdot \mathbf{u}} = \nabla \cdot \overline{\mathbf{u}} \tag{8.5}$$

$$\overline{\int \mathbf{u} dt} = \int \overline{\mathbf{u}} dt \tag{8.6}$$

Although here we have defined the Reynolds decomposition using the velocity field, it can be applied to any field variable, vector or scalar alike.

If the flow is incompressible ($\nabla \cdot \mathbf{u} = 0$), then the mean flow is incompressible as well:

$$\nabla \cdot \overline{\mathbf{u}} = 0 \tag{8.7}$$

and by definition the fluctuating field must also be divergence-free:

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\overline{\mathbf{u}} + \mathbf{u}') = \nabla \cdot \overline{\mathbf{u}} + \nabla \cdot \mathbf{u}' = 0 \tag{8.8}$$

$$\nabla \cdot \mathbf{u}' = 0 \tag{8.9}$$

We seek the governing equations for the mean flow that include the effects of the fluctuating field (turbulence). To do that, let's apply the Reynolds decomposition to the Navier-Stokes equation and take the time average of the resulting equation. We begin by writing out Eq. (4.36) without the body forces, for simplicity (as the body forces won't be affected by the Reynolds decomposition):

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \tag{8.10}$$

It's at this time useful to re-cast this equation in the momentum-conservative form that is prognostic for the momentum $\rho\mathbf{u}$ rather than just the velocity $\mathbf{u}$. To do that, multiply Eq. 8.10 by $\rho$ to get:

$$\rho\frac{\partial \mathbf{u}}{\partial t} + (\rho\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} \tag{8.11}$$

while recalling that the kinematic viscosity $\nu$ is defined as $\nu = \mu/\rho$. Now, we will use the Eulerian form of the continuity equation (Eq. 4.4) to reframe the left-hand side of Eq. 8.11 in terms of the momentum $\rho\mathbf{u}$:

$$\begin{aligned}
&\rho\frac{\partial \mathbf{u}}{\partial t} + (\rho\mathbf{u} \cdot \nabla)\mathbf{u} \\
&= \frac{\partial (\rho\mathbf{u})}{\partial t} - \mathbf{u}\frac{\partial \rho}{\partial t} + (\rho\mathbf{u} \cdot \nabla)\mathbf{u} \\
&= \frac{\partial (\rho\mathbf{u})}{\partial t} + \mathbf{u}\nabla \cdot (\rho\mathbf{u}) + (\rho\mathbf{u} \cdot \nabla)\mathbf{u} \\
&= \frac{\partial (\rho\mathbf{u})}{\partial t} + \nabla \cdot (\rho\mathbf{u}\mathbf{u})
\end{aligned} \tag{8.12}$$

Our momentum equation (Eq. 8.11) can then be written as:

$$\frac{\partial (\rho\mathbf{u})}{\partial t} + \nabla \cdot (\rho\mathbf{u}\mathbf{u}) = -\nabla p + \mu \nabla^2 \mathbf{u} \tag{8.13}$$

and in case of incompressible flows ($\nabla \cdot \mathbf{u} = 0$):

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \tag{8.14}$$

We are still describing the full flow with all it turbulent fluctuations. Remember that we are interested in the solution for the mean flow that accounts for the effects of turbulence, so we need apply the Reynolds decomposition to $\mathbf{u}$ and $p$, time average the resulting equation, and notice that $\overline{\mathbf{u}'}$ and $\overline{p'}$ are both zero:

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}\mathbf{u}}) = -\frac{1}{\rho}\nabla \overline{p} + \nu \nabla^2 \overline{\mathbf{u}} \tag{8.15}$$

Let's expand the advective term:

$$\nabla \cdot (\overline{\mathbf{u}\mathbf{u}}) = \nabla \cdot [\overline{(\overline{\mathbf{u}} + \mathbf{u}')(\overline{\mathbf{u}} + \mathbf{u}')}] = \nabla \cdot (\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} + \overline{\mathbf{u}'\overline{\mathbf{u}}} + \overline{\overline{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\mathbf{u}'}) \tag{8.16}$$

which reduces to:

$$\nabla \cdot (\overline{\mathbf{u}\mathbf{u}}) = \nabla \cdot (\overline{\mathbf{u}}\,\overline{\mathbf{u}}) + \nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'}) \tag{8.17}$$

Insert Eq. (8.17) into Eq. (8.14) to get:

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}}\,\overline{\mathbf{u}}) = -\frac{1}{\rho}\nabla \overline{p} + \nu \nabla^2 \overline{\mathbf{u}} - \nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'}) \tag{8.18}$$

which is the *Reynolds-Averaged Navier-Stokes (RANS) equation* , the term $\overline{\mathbf{u}'\mathbf{u}'}$ is called the *Reynolds stress tensor* , and $\nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'})$ is the *Reynolds stress divergence*.

The Reynolds-averaged continuity equation is much simpler to derive and is just:

$$\nabla \cdot \overline{\mathbf{u}} = 0 \tag{8.19}$$

Between Eqs. (8.18) and (8.19) we have two equations with three unknowns: $\overline{\mathbf{u}}$, $\overline{p}$, and $\overline{\mathbf{u}'\mathbf{u}'}$. To close the system, we need to find an equation for the Reynolds stress tensor, which brings us to the closure problem of turbulence.

## 8.2 Closure problem

To illustrate the closure problem of turbulence, let's try to derive the equation for the evolution of the Reynolds stress $\overline{\mathbf{u}'\mathbf{u}'}$. Suppose that the Reynolds stress evolves according to the yet to be determined sources and sinks of the Reynolds stress:

$$\frac{d\,(\mathbf{u}'\mathbf{u}')}{dt} = \text{sources} - \text{sinks} \tag{8.20}$$

Expanding the time derivative in a momentum-conservative form and time averaging yields similar to Eq. (8.17):

$$\frac{d\left(\overline{\mathbf{u}'\mathbf{u}'}\right)}{dt} = \frac{\partial\left(\overline{\mathbf{u}'\mathbf{u}'}\right)}{\partial t} + \nabla \cdot \left(\overline{\mathbf{u}}\overline{\mathbf{u}'\mathbf{u}'}\right) + \nabla \cdot \left(\overline{\mathbf{u}'\mathbf{u}'\mathbf{u}'}\right) \tag{8.21}$$

See, if we try to seek the equation for the evolution of the Reynolds stress, we end up with the flux of the flux itself as a new unknown. Further, if we tried to seek the equation for this new cubic term, we would end up with an equation that includes a quartic term of $\mathbf{u}'$:

$$\frac{d\left(\overline{\mathbf{u}'\mathbf{u}'\mathbf{u}'}\right)}{dt} = \frac{\partial\left(\overline{\mathbf{u}'\mathbf{u}'\mathbf{u}'}\right)}{\partial t} + \nabla \cdot \left(\overline{\mathbf{u}}\overline{\mathbf{u}'\mathbf{u}'\mathbf{u}'}\right) + \nabla \cdot \left(\overline{\mathbf{u}'\mathbf{u}'\mathbf{u}'\mathbf{u}'}\right) \tag{8.22}$$

The fact that we cannot close the RANS equations unless we somehow approximate the Reynolds stress tensor is known as the closure problem of turbulence. On one hand, it's relieving that we don't have to figure out the sources and sinks for the Reynolds stress tensor in Eq. (8.20). On the other hand, we still need to come up with some model or approximation for the Reynolds stress tensor to solve the RANS equations.

## 8.3 Reynolds stress

Recall from Chapter 4 where we first derived the Cauchy momentum equation (Eq. 4.23), ignoring the body forces for brevity:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} \tag{8.23}$$

and the associated stress tensor (Eq. 4.25):

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \tag{8.24}$$

where we had described the stress tensor $\boldsymbol{\sigma}$ as a combination of the normal stresses (pressure) on the diagonal and the deviatoric stresses off the diagonal:

$$\begin{bmatrix} -p + \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -p + \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p + \tau_{zz} \end{bmatrix} \tag{8.25}$$

Then, in Section 4.2.4, we stated that for a Newtonian fluid the deviatoric stresses can be approximated with the velocity gradients, an approximation that was established in the laboratory:

$$\nabla \cdot \boldsymbol{\tau} = \nu \nabla^2 \mathbf{u} \tag{8.26}$$

Now, in addition to the viscous stresses, we have the turbulent Reynolds stresses introduced in Eq. (8.18). The turbulent Reynolds stresses arise due to the scale separation between the large-scale mean flow and the turbulent fluctuations, which we introduced when we applied the Reynolds decomposition to the velocity field.

Eq. (8.18) can be rewritten more concisely by applying the divergence operator to the pressure and Reynolds and viscous stresses as a whole:

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}}\,\overline{\mathbf{u}}) = \frac{1}{\rho}\nabla \cdot \left(\mu \nabla \cdot \overline{\mathbf{u}} - p - \rho \overline{\mathbf{u}'\mathbf{u}'}\right) \tag{8.27}$$

If it's not obvious already, notice that $\rho\overline{\mathbf{u}'\mathbf{u}'}$ is the only term that makes Eq. (8.27) different from the original Navier-Stokes equation (Eq. 8.14). Thus, if we apply a scale separation (*i.e.* the Reynolds decomposition) to the velocity field such that we distinguish between the mean flow and the fluctuations, the equation for the mean flow contains an additional term that quantifies the contribution of the turbulent fluctuations to the mean. Note that, strictly speaking, $\rho\overline{\mathbf{u}'\mathbf{u}'}$ is a stress (as in, momentum flux), however it's common to refer to $\overline{\mathbf{u}'\mathbf{u}'}$ as the Reynolds stress as well, even when the density is omitted.

Let's look at this Reynolds stress tensor in more detail. Using our usual notation for the velocity vector to be $\mathbf{u} = (u, v, w)$, the components of the Reynolds stress tensor are:

$$\overline{u'u'} = \begin{bmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'v'} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'w'} \end{bmatrix} \tag{8.28}$$

The diagonal components of this tensor ($\overline{u'u'}$, $\overline{v'v'}$, and $\overline{w'w'}$) are called the *normal stresses*, and the off-diagonal components ($\overline{u'v'}$, $\overline{u'w'}$, $\overline{v'w'}$) are called the *shear stresses*. The Reynolds stress tensor is symmetric, which means that $\overline{u'v'} = \overline{v'u'}$, $\overline{u'w'} = \overline{w'u'}$, and $\overline{v'w'} = \overline{w'v'}$. It is only the shear stresses that contribute to the turbulent transport of momentum. An important property of boundary layer physics, the *Turbulent Kinetic Energy* (TKE) is half the sum of the diagonal components of the Reynolds stress tensor:

$$k = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right) \tag{8.29}$$

From the point of view of the Reynolds decomposition into the mean and fluctuations from the mean, TKE is the sum of velocity variances. TKE plays an important role in parameterizing the subgrid-scale turbulent processes in the boundary layer components of weather and ocean prediction models. $\overline{u'w'}$ and $\overline{v'w'}$ are also very important quantities in the study of air-sea interaction, as they govern the momentum exchange between the atmospheric surface layer, the ocean surface waves, and the upper-ocean boundary layer.

In numerical models, the vector equations must be written out explicitly in scalar component form. It's thus a useful exercise to write out the RANS equation (Eq. 8.27) as a system of scalar equations, one for each component of the mean velocity vector:

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial \overline{u}\overline{u}}{\partial x} + \frac{\partial \overline{v}\overline{u}}{\partial y} + \frac{\partial \overline{w}\overline{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} - \frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{v'u'}}{\partial y} - \frac{\partial \overline{w'u'}}{\partial z} + \nu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2}\right) \tag{8.30}$$

$$\frac{\partial \overline{v}}{\partial t} + \frac{\partial \overline{u}\overline{v}}{\partial x} + \frac{\partial \overline{v}\overline{v}}{\partial y} + \frac{\partial \overline{w}\overline{v}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} - \frac{\partial \overline{u'v'}}{\partial x} - \frac{\partial \overline{v'v'}}{\partial y} - \frac{\partial \overline{w'v'}}{\partial z} + \nu\left(\frac{\partial^2 \overline{v}}{\partial x^2} + \frac{\partial^2 \overline{v}}{\partial y^2} + \frac{\partial^2 \overline{v}}{\partial z^2}\right) \tag{8.31}$$

$$\frac{\partial \overline{w}}{\partial t} + \frac{\partial \overline{u}\overline{w}}{\partial x} + \frac{\partial \overline{v}\overline{w}}{\partial y} + \frac{\partial \overline{w}\overline{w}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} - \frac{\partial \overline{u'w'}}{\partial x} - \frac{\partial \overline{v'w'}}{\partial y} - \frac{\partial \overline{w'w'}}{\partial z} + \nu\left(\frac{\partial^2 \overline{w}}{\partial x^2} + \frac{\partial^2 \overline{w}}{\partial y^2} + \frac{\partial^2 \overline{w}}{\partial z^2}\right) \tag{8.32}$$

## 8.4 Turbulent kinetic energy budget

Turbulent kinetic energy (TKE) is a fundamental quantity in the study of turbulence. It's a prognostic variable in many subgrid-scale parametric models of atmospheric and oceanic boundary layers. Here we derive the prognostic equation for TKE from the fundamental equations with Reynolds decomposition, often referred to as the TKE budget equation.

The derivation of the TKE budget equation involves the following steps:

1. Start from the Navier-Stokes equation (Eq. 8.10) and apply the Reynolds decomposition to the velocity field.

2. Subtract the RANS equation from the original Navier-Stokes equation with Reynolds decomposition to obtain the equation for the velocity fluctuations.

3. Multiply the equation for the velocity fluctuations by the fluctuating velocity components and time-average to obtain the equation for the TKE.

For completeness, we will also consider the buoyancy term that we derived in the Boussinesq approximation, as it will turn out that this term plays a role in the TKE budget. We start from the Navier-Stokes equation but in the advective (non-conservative) form, rather than the flux (conservative) form, as the advective form makes the TKE budget derivation more straightforward (they are equivalent for incompressible flows, $\nabla \cdot \mathbf{u} = 0$).

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{\delta\rho}{\rho}\mathbf{g} + \nu\nabla^2\mathbf{u} \tag{8.33}$$

Apply the Reynolds decomposition to $\mathbf{u}$, $p$, and $\delta\rho$ to get:

$$\begin{aligned} \frac{\partial \overline{\mathbf{u}}}{\partial t} + \frac{\partial \mathbf{u}'}{\partial t} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}} + (\mathbf{u}' \cdot \nabla)\overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}' + (\mathbf{u}' \cdot \nabla)\mathbf{u}' = \\ -\frac{1}{\rho}\nabla\overline{p} - \frac{1}{\rho}\nabla p' + \frac{\overline{\delta\rho}}{\rho}\mathbf{g} + \frac{\delta\rho'}{\rho}\mathbf{g}' + \nu\nabla^2\overline{\mathbf{u}} + \nu\nabla^2\mathbf{u}' \end{aligned} \tag{8.34}$$

The RANS equation in the advective form is:

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}} + \overline{(\mathbf{u}' \cdot \nabla)\mathbf{u}'} = -\frac{1}{\rho}\nabla\overline{p} + \frac{\overline{\delta\rho}}{\rho}\mathbf{g} + \nu\nabla^2\overline{\mathbf{u}} \tag{8.35}$$

Subtract Eq. (8.35) from Eq. (8.34) to obtain the equation for the velocity fluctuations:

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}' \cdot \nabla)\overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}' + (\mathbf{u}' \cdot \nabla)\mathbf{u}' - \overline{(\mathbf{u}' \cdot \nabla)\mathbf{u}'} = -\frac{1}{\rho}\nabla p' + \frac{\delta\rho'}{\rho}\mathbf{g} + \nu\nabla^2\mathbf{u}' \tag{8.36}$$

Multiply by $\mathbf{u}'$ to get:

$$\mathbf{u}'\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}'(\mathbf{u}' \cdot \nabla)\overline{\mathbf{u}} + \mathbf{u}'(\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}' + \mathbf{u}'(\mathbf{u}' \cdot \nabla)\mathbf{u}' - \mathbf{u}'\overline{(\mathbf{u}' \cdot \nabla)\mathbf{u}'} = -\frac{1}{\rho}\mathbf{u}'\nabla p' + \frac{\delta\rho'}{\rho}\mathbf{u}'\mathbf{g} + \nu\mathbf{u}'\nabla^2\mathbf{u}' \tag{8.37}$$

Rearrange the terms:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\mathbf{u}'^2}{2}\right) + (\overline{\mathbf{u}} \cdot \nabla)\left(\frac{\mathbf{u}'^2}{2}\right) + (\mathbf{u}'\mathbf{u}' \cdot \nabla)\overline{\mathbf{u}} + \frac{1}{2}\nabla \cdot (\mathbf{u}'\mathbf{u}'\mathbf{u}') - \mathbf{u}'\overline{(\mathbf{u}' \cdot \nabla)\mathbf{u}'} = \\ -\frac{1}{\rho}\mathbf{u}'\nabla p' + \frac{\delta\rho'}{\rho}\mathbf{u}' \cdot \mathbf{g} + \nu\mathbf{u}'\nabla^2\mathbf{u}' \end{aligned} \tag{8.38}$$

Finally, time-average to get the TKE budget equation, noting that the last term on the left-hand side drops out due to time-averaging, and that $k \equiv \frac{1}{2}\overline{\mathbf{u}'^2}$:

$$\frac{\partial k}{\partial t} + \overline{\mathbf{u}} \cdot \nabla k = -\frac{1}{2}\nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'\mathbf{u}'}) - (\mathbf{u}'\mathbf{u}' \cdot \nabla)\overline{\mathbf{u}} - \frac{1}{\rho}\overline{\mathbf{u}'\nabla p'} + \overline{\frac{\delta\rho'}{\rho}\mathbf{u}' \cdot \mathbf{g}} + \nu\overline{\mathbf{u}'\nabla^2\mathbf{u}'} \tag{8.39}$$

So far we broke down the advective term from the original Navier-Stokes equation to produce three new terms. We're still left with the viscous term, which can be rearranged into two terms for a more intuitive physical interpretation. Here we'll use the following identity to expand the Laplacian:

$$\overline{\nu\mathbf{u}'\nabla^2\mathbf{u}'} = \nu\nabla \cdot (\overline{\mathbf{u}'\nabla\mathbf{u}'}) - \nu\overline{\nabla\mathbf{u}' \cdot \nabla\mathbf{u}'} = \nu\nabla^2 k - \nu\overline{(\nabla\mathbf{u}' \cdot \nabla\mathbf{u}')} \tag{8.40}$$

Inserting Eq. (8.40) into Eq. (8.39) gives us our final form of the TKE budget equation:

$$\frac{\partial k}{\partial t} + \overline{\mathbf{u}} \cdot \nabla k = -\frac{1}{2}\nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'\mathbf{u}'}) - (\overline{\mathbf{u}'\mathbf{u}'} \cdot \nabla)\overline{\mathbf{u}} - \frac{1}{\rho}\overline{\mathbf{u}'\nabla p'} + \overline{\frac{\delta\rho'}{\rho}\mathbf{u}' \cdot \mathbf{g}} + \nu\nabla^2 k - \nu\overline{\nabla\mathbf{u}' \cdot \nabla\mathbf{u}'} \tag{8.41}$$

Let's look at each term in Eq. (8.41) and discuss its physical meaning:

- $\frac{\partial k}{\partial t}$: Eulerian rate of change of TKE in a fixed point in space.
- $\overline{\mathbf{u}} \cdot \nabla k$: Advection of TKE by the mean flow. Like any other fluid property, TKE as well is subject to advection by the mean flow, *i.e.* $dk/dt = \partial k/\partial t + \overline{\mathbf{u}} \cdot \nabla k$.
- $-\frac{1}{2}\nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'\mathbf{u}'})$ is the turbulent transport of TKE. In other words, this term quantifies how much turbulent eddies are transported by the turbulent eddies themselves.
- $-(\overline{\mathbf{u}'\mathbf{u}'} \cdot \nabla)\overline{\mathbf{u}}$ is the production of TKE by the mean flow, also known as the shear production.
- $-\frac{1}{\rho}\overline{\mathbf{u}'\nabla p'}$ is the production of TKE by the turbulent fluctuations of the pressure gradient, also known as pressure diffusion.
- $\overline{\frac{\delta\rho'}{\rho}\mathbf{u}' \cdot \mathbf{g}}$ is the production of TKE by buoyancy. Notice the dot product between the velocity vector and the gravitational acceleration, which means that the buoyancy production occurs only by the vertical velocity component, and is scaled by the buoyancy anomaly $\delta\rho'$. The stronger the stratification of the fluid, the larger the buoyancy production (or dissipation, depending on the sign of stratification) of TKE.
- $\nu\nabla^2 k$ is the dissipation of TKE by molecular diffusion.
- $-\nu\overline{\nabla\mathbf{u}' \cdot \nabla\mathbf{u}'}$ is the turbulent eddy dissipation of TKE. Note that $\nabla\mathbf{u}'$ are rank-2 tensors, so the inner product $\nabla\mathbf{u}' \cdot \nabla\mathbf{u}'$ is a rank-4 tensor, which is averaged to get a scalar.

Given Eq. (8.41) and the interpretation of its terms, we can proceed to apply dimensional analysis in an attempt to learn the distribution and transfer of turbulence across spatial scales.

## 8.5 Turbulent cascade

The two most common sources of turbulence are shear (mechanical) and buoyancy (thermodynamic). As such, the turbulent energy is predominantly generated at the larger scales, where the largest coherent eddies tend to be of the same scale as the flow itself. For example, the largest eddies that the Gulf Stream sheds are of similar diameter as the width of the Gulf Stream itself. Similarly, the largest eddies in a coffee cup are of similar size as the spoon that does the stirring. An example of buoyancy generation of turbulence is the convection in the atmospheric boundary layer due to cool air over warm land or ocean surface. So, most turbulence tends to be produced at the scales many orders of magnitude that of the viscous scales. At the smallest scales, we know that viscosity does the work to dissipate mechanical energy into heat. What happens between the largest and the smallest scales is less clear and is the subject of this section. A concept of *turbulent energy cascade*, first introduced by Richardson (1922), suggests that the energy is transferred from the large to the small scales, and that this transfer is a cascade. He put it succinctly as:

*Big whirls have little whirls,*
*Which feed on their velocity;*
*And little whirls have lesser whirls,*
*And so on to viscosity.*

To answer how the velocity statistics are distributed from the largest to the smallest scales, we evaluate the TKE budget equation for a very turbulent flow in which $Re = UL/\nu$ is very large. The turbulent cascade is illustrated in Fig. 8.1.

We may first ask at what length scale does the viscosity become a dominant player. As useful tools we will recall dimensional analysis and the Reynolds number, which quantified the relative importance of inertial over viscous forces.

$$Re = \frac{UL}{\nu} \tag{8.42}$$

If we know that at the largest (think, geophysical) scales the viscosity is negligible (large $Re$), we could say that the viscosity becomes more important than turbulent motion at the scale at which $Re \approx 1$. From there, we can define the viscous length scale as:

$$L_\nu = \frac{\nu}{U} \tag{8.43}$$

What are some characteristic values of $L_\nu$ in the ocean and in the atmosphere? An ocean flow with $U \approx 10^{-1}$ m/s and viscosity of $\nu \approx 10^{-6}$ m$^2$/s gives $L_\nu \approx 10^{-5}$ m, or, one hundredth of a millimeter. In the atmosphere with $U \approx 10$ m/s and viscosity of $\nu \approx 10^{-5}$ m$^2$/s, we get $L_\nu \approx 10^{-6}$ m, or, one micron. These are obviously very small scales.

To answer what happens to the flow statistics between the largest scales at which the turbulence is generated and the smalles scales at which viscosity dissipates all mechanical energy into heat, Kolmogorov (1941) proposed a new Kolmogorov's three turbulence hypotheses are:

1. **Hypothesis of local isotropy:** At sufficiently high $Re$ and sufficiently small $L$, the turbulence is *locally isotropic*, *i.e.* the flow statistics at a point are the same in all directions.

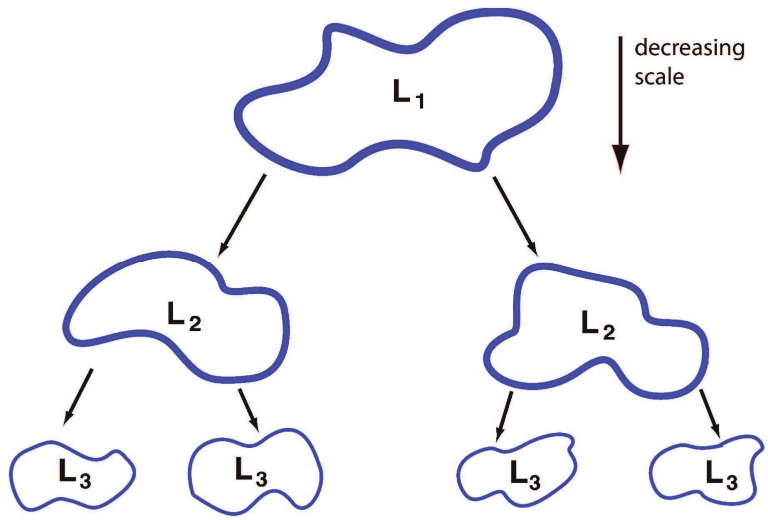


Fig. 11.2

Figure 8.1: The passage of energy to smaller scales: eddies at large scale break up into ones at smaller scale, thereby transferring energy to smaller scales. The eddies in reality are embedded within each other. If the passage occurs between eddies of similar sizes (*i.e.*, if it is spectrally local), the transfer is said to be a cascade. This is Figure 11.2 from Vallis (AOFD).

2. **First similarity hypothesis:** At sufficiently high $Re$ and sufficiently small $L$, the flow statistics have a universal form that is uniquely determined by the viscosity $\nu$ and the energy dissipation rate $\varepsilon$.

3. **Second similarity hypothesis:** At sufficiently high $Re$ and and sufficiently large $L$, the flow statistics have a universal form that is uniquely determined by the energy dissipation rate $\varepsilon$, and that is independent of viscosity $\nu$.

The energy dissipation rate $\varepsilon$ comes straight from the TKE budget equation (Eq. 8.41) and is defined as:

$$\varepsilon = \nu \overline{\nabla \mathbf{u}' \cdot \nabla \mathbf{u}'} \tag{8.44}$$

In a nutshell, Kolmogorov's three hypotheses state that a turbulent flow at sufficiently small scales is the same looking in all directions, that statistically all such turbulent flows are the same, and that they are uniquely determined by either by energy dissipation rate alone, or by the energy dissipation rate and viscosity, depending on the scale. Through dimensional analysis, Kolmogorov also introduced the fundamental turbulent scales, now commonly known as Kolmogorov scales:

- $\eta \equiv \left(\nu^3/\varepsilon\right)^{1/4}$, or, Kolmogorov length scale, is the scale at which the energy dissipation by molecular diffusion balances the energy input by the mean flow.

- $u_\eta \equiv (\varepsilon\nu)^{1/4}$ is the velocity at the Kolmogorov length scale.
- $\tau_\eta \equiv (\nu/\varepsilon)^{1/2}$ is the Kolmogorov time scale at which the smallest coherent eddy can exist.

Any flow feature at scales smaller than these is governed by viscous dissipation of kinetic energy into heat.

Examining the Reynolds number using the Kolmogorov scales indeed shows that it reduces to unity, consistent with Eq. (8.43):

$$Re_\eta = \frac{u_\eta \eta}{\nu} = \frac{(\varepsilon\nu)^{1/4}\,(\nu^3/\varepsilon)^{1/4}}{\nu} = 1 \tag{8.45}$$

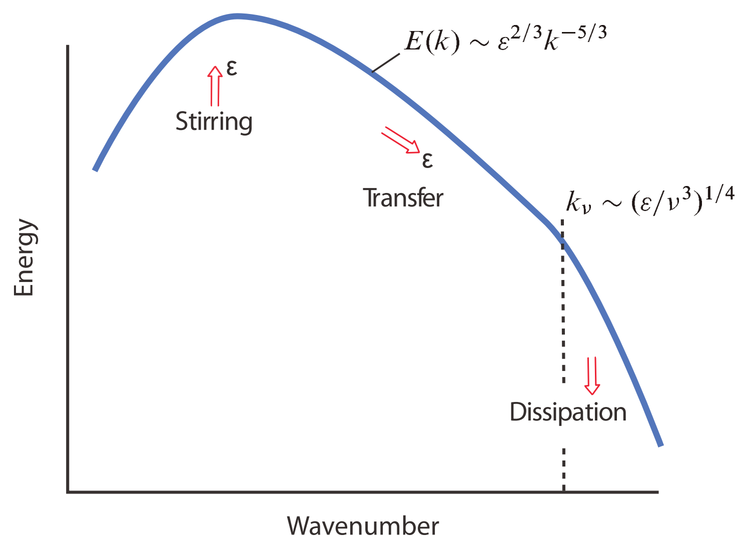


Fig. 11.3

Figure 8.2: The energy spectrum in three-dimensional turbulence, in the theory of Kolmogorov (1941). Energy is supplied at some rate $\varepsilon$; it is cascaded to small scales, where it is ultimately dissipated by viscosity. There is no systematic energy transfer to scales larger than the forcing scale, so here the energy falls off. This is Figure 11.3 from Vallis (AOFD).

Now, we may ask, how is the turbulent energy distributed across the scales? Define the energy spectrum $E(k)$ as the energy per unit mass per unit wavenumber:

$$E = \frac{1}{2}\int \mathbf{u}'^2(k)\ dk = \int E(k)\ dk \tag{8.46}$$

What is the form of the energy spectrum $E(k)$? Kolmogorov's second similarity hypothesis states that the energy spectrum is universal and uniquely determined by the energy dissipation rate $\varepsilon$. If that is true, then it must be some function of $\varepsilon$ and $k$:

$$E(k) = F(\varepsilon, k) \tag{8.47}$$

The dimensions of $E(k)$ are $L^3T^{-2}$. Since the wavenumber $k$ has dimensions of $L^{-1}$ and thus no temporal dependence, the only way it can match the dimensions of $E(k)$ is if the energy spectrum scales with $\varepsilon^{2/3}$:

$$E(k) = \varepsilon^{2/3}G(k) \tag{8.48}$$

$$\frac{L^3}{T^2} \sim \frac{L^{4/3}}{T^2}G(k) \tag{8.49}$$

where $G(k)$ is some yet to be determined function of $k$. Then, by dimensional analysis, $g(k)$ must have dimensions of $L^{5/3}$, making the energy spectrum:

$$E(k) = \mathcal{K}\varepsilon^{2/3}k^{-5/3} \tag{8.50}$$

where $\mathcal{K}$ is a constant not determined by Kolmogorov's theory. The functional form of $E(k)$ is known as the Kolmogorov 5/3 law and is illustrated in Figure 8.2.

## Summary

In this chapter, we covered:

- Reynolds decomposition of turbulent flows into mean and fluctuating components;
- The turbulent energy spectrum and its distribution across scales;
- Kolmogorov's similarity hypotheses and dimensional analysis leading to the -5/3 law;
- The turbulent energy cascade from large to small scales in 3D turbulence;
- The role of energy dissipation rate $\varepsilon$ in determining the energy spectrum.

## Further reading

- Chapter 11 of *AOFD* by Vallis
- Chapter 6 of *Turbulent Flows* by Pope

# 9 Boundary layers

Boundary layers occur when a fluid flows over some kind of boundary, whether rigid or free, stationary or moving. They are both interesting and convenient because they constrain the flow near the boundary and thus allow simplifications that may lead to analytical solutions. They are important because they are often the dominant flow structure in geophysical flows. For example, a planetary boundary layer separates the atmosphere from the surface of the Earth. The surface beneath the planetary boundary layer may be rigid (land or sea ice) or free (ocean), and its roughness and thermodynamic properties may vary greatly from place to place. In this chapter, we start from the simplest boundary layer, a channel flow, and derive the stress and mean velocity profiles in laminar flows. Then, we zoom into the vertical structure of the boundary layer in turbulent flows, and examine different regimes that occur depending on the distance from the boundary.

## 9.1 Channel flow

A channel flow is a classic problem in fluid mechanics that is both relevant to engineering applications, and analogous to larger-scale geophysical flows. We begin by setting up the problem and establishing the governing equations and the notation that we will use. Then, we explore some analytical and numerical solutions for the time-mean flow structure within the channel.

### 9.1.1 Governing equations and boundary conditions

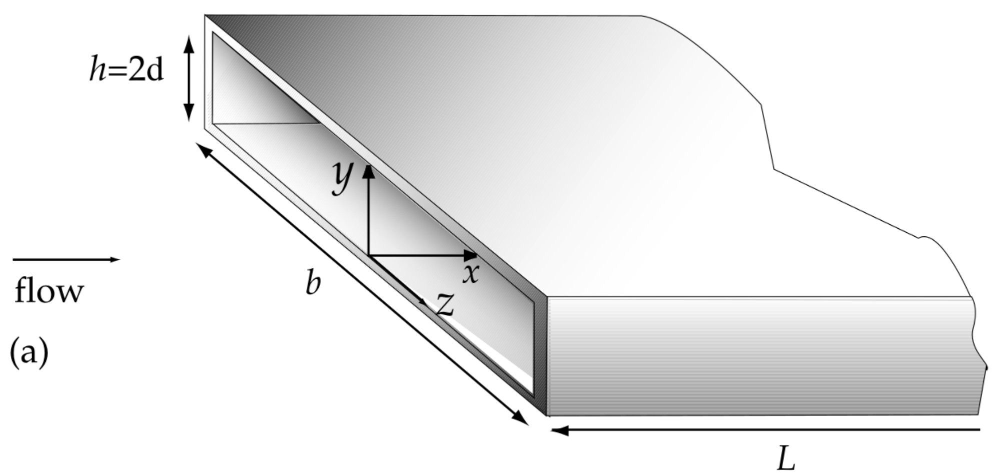


Figure 9.1: Sketch of a channel flow. The height of the channel is $h$ and the flow is in the $x$ direction. Although the vertical and the cross-stream coordinates are denoted as $y$ and $z$ here, respectively, we will be using the opposite notation with $z$ being the vertical coordinate and $y$ the cross-stream coordinate. This is Figure 7.1a from Turbulent Flows by Pope.

Let's examine a flow in a channel between two flat plates, spaced apart by a a distance $h = 2\delta$, such that $\delta$ represents the centerline distance between the plates (Fig. 9.1). The channel is long ($L \gg \delta$) and wide ($W \gg \delta$), so there is no variability in the $x$ and $y$ directions. The mean flow is predominantly in the $x$ direction, so if the velocity is defined as having components $u$, $v$, and $w$ in the streamwise, spanwise, and vertical directions, respectively, then:

$$\overline{u}(z) > 0 \tag{9.1}$$

$$\overline{v} = 0 \tag{9.2}$$

For simplicity, we won't consider what happens at the very entrance into the channel where the flow develops, and we'll only consider the fully developed flow well into the channel such that $\partial \overline{u}/\partial x = 0$. Thus, from a statistical point of view, this is a stationary, one-dimensional flow that varies only in the $z$ direction.

The flow can be characterized using two different Reynolds numbers:

$$Re \equiv \frac{\langle \overline{u} \rangle \, 2\delta}{\nu} \tag{9.3}$$

where $\langle \overline{u} \rangle$ is the mean velocity in the channel (often also called *bulk velocity*):

$$\langle \overline{u} \rangle = \frac{1}{\delta} \int_0^\delta \overline{u}(z) \; dz \tag{9.4}$$

Another useful Reynolds number is the one based on the centerline distance between the plates:

$$Re_0 \equiv \frac{u_0 \delta}{\nu} \tag{9.5}$$

where $u_0$ is the centerline velocity $u(z = \delta)$. Based on laboratory experiments, we know that the channel flow is laminar for $Re < 1350$ and turbulent for $Re > 1800$, with transitional effects observable up to $Re \approx 3000$.

Start from the Reynolds-averaged Navier-Stokes equation for $\overline{u}$:

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2}\right) - \frac{\partial}{\partial x}\overline{u'u'} - \frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'} \tag{9.6}$$

For an incompressible flow, the continuity is $\nabla \cdot \overline{\mathbf{u}} = 0$, which is effectively $\partial \overline{w}/\partial z = 0$ since the flow doesn't vary in the $x$ and $y$ directions. $\overline{w}$ must be zero as we can't have any flow through the walls of the channel, and so continuity requires that $\overline{w}$ is zero everywhere. Accounting for stationarity ($\partial \overline{u}/\partial t = 0$), homogeneity in the $x$ and $y$ directions ($\partial \overline{u}/\partial x = \partial \overline{u}/\partial y = 0$), and the fact that $\overline{w} = 0$, Eq. (9.6) greatly simplifies to:

$$\frac{\partial \overline{p}}{\partial x} = \rho\nu\frac{\partial^2 \overline{u}}{\partial z^2} - \rho\frac{\partial}{\partial z}\overline{u'w'} \tag{9.7}$$

This stationary, one-dimensional flow is thus driven by the streamwise pressure gradient that is balanced by the normal viscous stress and the cross-stream Reynolds stress. The above can be further simplified to:

$$\frac{\partial \overline{p}}{\partial x} = \frac{\partial \tau}{\partial z} \tag{9.8}$$

where stress $\tau$ is the sum of the viscous and the turbulent Reynolds stresses:

$$\tau = \rho \left( \nu \frac{\partial \overline{u}}{\partial z} - \overline{u'w'} \right) \tag{9.9}$$

Since the flow is stationary, the streamwise pressure gradient that drives it must be constant, and so does the vertical stress gradient as well:

$$\frac{\partial \tau}{\partial z} = \text{constant} \tag{9.10}$$

Assuming symmetry around the centerline of the channel requires that the stress there is zero (as there should not be any mean transport through the centerline). Integrating the above from $z = 0$ to $z = \delta$ we get:

$$\tau(z) = az + b \tag{9.11}$$

where $a$ and $b$ are constants. Use the boundary conditions $\tau(z = 0) = \tau_w$ and $\tau(z = \delta) = 0$ to get:

$$\tau(z) = \tau_w \left(1 - \frac{z}{\delta}\right) \tag{9.12}$$

where $\tau_w$ is the so-called wall stress whose value is yet to be determined. The stress thus decreases linearly from $\tau_w$ at the bottom wall to zero at the centerline, reaching $-\tau_w$ at the top wall.

## 9.2 Laminar flow

What does the velocity profile look like in the case of laminar flow? We can drop the Reynolds stress term in Eq. (9.9) and combine it with Eq. (9.12) to get:

$$\frac{\partial \overline{u}}{\partial z} = \frac{\tau_w}{\rho \nu} \left(1 - \frac{z}{\delta}\right) \tag{9.13}$$

Integrate the above with respect to $z$ to get:

$$\overline{u}(z) = \frac{\tau_w z}{\rho \nu} \left(1 - \frac{z}{2\delta}\right) \tag{9.14}$$

The velocity profile thus has a quadratic form that reaches zero at either wall, and that has a centerline value of:

$$u_0 = \overline{u}(z = \delta) = \frac{\tau_w \delta}{2\rho\nu} \tag{9.15}$$

The preceding equations determine the stress and velocity profiles strictly in laminar flows, *i.e.* for relatively small Reynolds numbers. Let's see what the profiles may look like in turbulent flows.

## 9.3 Turbulent flow

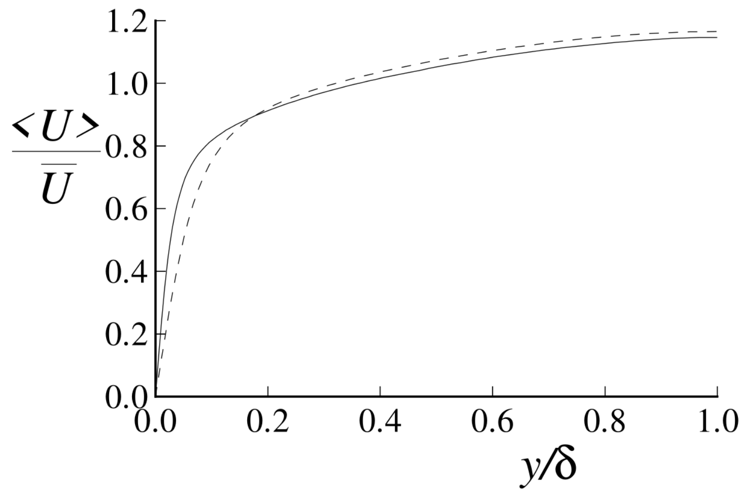


Figure 9.2: Mean velocity profile normalized by the bulk velocity in a fully developed turbulent channel flow, from the DNS of Kim et al. (1987). Dashed and solid lines are for $Re = 5,600$ and $Re = 13,750$, respectively. Note that in the axis labels, $y$ is the vertical coordinate and the angle brackets and overline denote averaging in the opposite sense from our notation in the main text. This is Figure 7.2 from Turbulent Flows by Pope.

In the laminar case, we were able to analytically derive the velocity and stress profiles. However, in the turbulent case, the problem is much more complex and analytical solutions are not feasible due to the presence of the turbulent Reynolds stress term. Direct Numerical Simulations (DNS) reveal what a turbulent velocity profile in a channel may look like (Fig. 9.2).

At the boundaries, we can't have any flow through the walls of the channel, the velocity and thus the turbulent Reynolds stresses must be zero, and so the wall shear stress must be entirely due to the viscosity:

$$\tau_w = \rho\nu \left(\frac{\partial \overline{u}}{\partial z}\right)_{z=0} \tag{9.16}$$

Recall from Eq. (9.9) that the stress $\tau$ is always composed of the viscous and turbulent parts. However, in turbulent flows, the relative contributions of the viscous and turbulent parts vary greatly as we move away from the wall. Fig. 9.2 shows how, in a well developed turbulent flow, the mean velocity increases as we move further away from the wall. At about 0.4 of the way toward the centerline, the time-mean velocity approximately equals the bulk velocity, and exceeds it as we approach the centerline. The profiles are also somewhat different depending on the Reynolds number, with the velocity profile being gentler for a smaller Reynolds number flow. This is somewhat intuitive, as we know that the turbulent Reynolds stresses are much more effective at mixing than the molecular viscosity. A somewhat less turbulent flow is thus expected to have a gentler velocity, as its momentum is being mixed more by viscosity and less by turbulence.

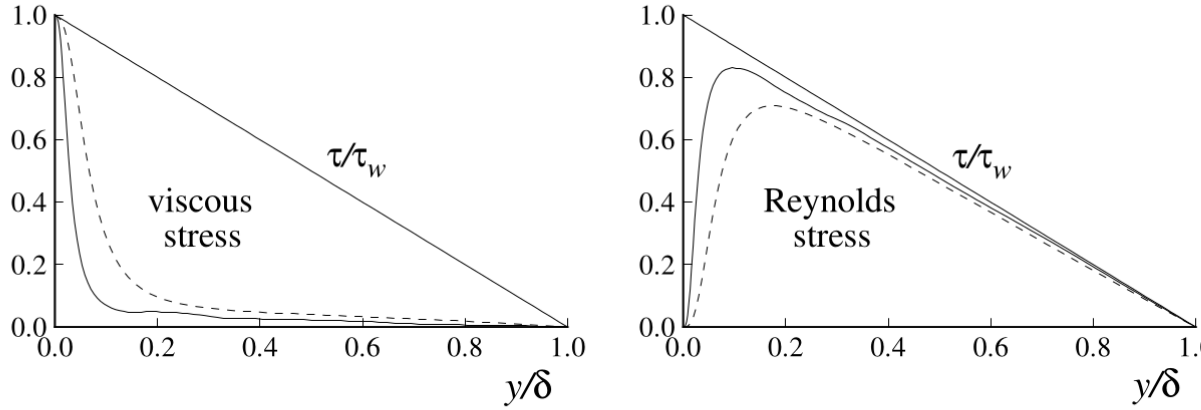


Figure 9.3: As in Fig. 9.2, but for the vertical profiles of the viscous and turbulent Reynolds stresses. This is Figure 7.3 from Turbulent Flows by Pope.

What is the vertical structure of the viscous and turbulent Reynolds stresses then? We don't have an analytical solution for the stress profiles, like we did in the laminar case, but we can look at the DNS data to see what the profiles look like. Fig. 9.3 shows the vertical profiles of the viscous and turbulent Reynolds stresses based on the DNS data of Kim et al. (1987). Consistent with Eq. (9.12), the total stress decreases linearly from $\tau_w$ at the wall to zero at the centerline. However, the stress components vary differently between one another. The viscous stress makes up all of the stress at the very wall, and rapidly decreases as we move away from the wall. The turbulent stress, on the other hand, is zero at the wall, and rapidly increases as we move away from the wall. At a lower Reynolds number, the turbulent stress reaches a lower peak value, with the peak being further away from the wall, compared to the higher Reynolds number case.

It is clear from Figs. 9.2 and 9.3 that viscosity (via the Reynolds number) and the wall stress $\tau_w$ are important parameters for the vertical structure of the flow. These quantities, alongside the fluid density $\rho$, allow us to define the *viscous scales* (length and velocity) that govern the the flow near the wall. These are the *friction velocity*:

$$u_* \equiv \sqrt{\tau_w/\rho} \tag{9.17}$$

and the *viscous length scale*:

$$\delta_\nu \equiv \nu/u_* \tag{9.18}$$

The viscous length scale, also known as the *wall unit*, quantifies the distance from the wall at which the smallest turbulent motions are felt, and within which all dissipation of kinetic energy is done by viscosity. On the other hand, the friction velocity $u_*$ is not a physical velocity of the flow at any single location, but rather a scaling parameter with the units of velocity that characterizes the flow near the wall. Mathematically, you can think of it as the wall shear stress expressed in units of velocity.

It's useful to also distinguish between the viscous Reynolds number:

$$Re_{\nu} \equiv \frac{u_* \delta_{\nu}}{\nu} \tag{9.19}$$

which, as we saw before, is identically unity, and the *friction Reynolds number*, defined as:

$$Re_{\tau} \equiv \frac{u_* \delta}{\nu} \tag{9.20}$$

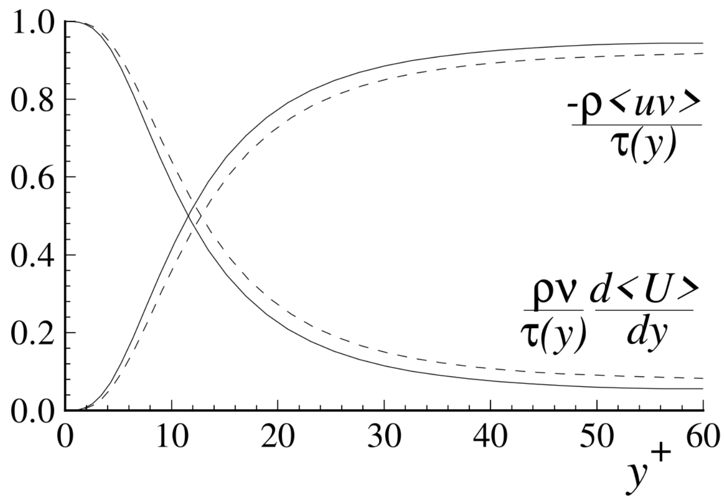


Figure 9.4: Profiles of the fractional contributions of the viscous and turbulent Reynolds stresses to the total stress, based on the DNS data of Kim et al. (1987), as in Figs. 9.2 and 9.3. This is Figure 7.4 from Turbulent Flows by Pope.

Based on the viscous length scale, we define a new non-dimensional coordinate $z^+$ as:

$$z^+ \equiv \frac{z}{\delta_{\nu}} = \frac{u_* z}{\nu} \tag{9.21}$$

which is the physical vertical distance normalized by the viscous length scale. This quantity thus allows us to see how the flow properties vary with the distance expressed as a number of wall units. One example of that is the fractional contribution of the viscous and turbulent stresses to the total stress, shown in Fig. 9.4. The fact that the stress contribution profiles between the lower and higher Reynolds number cases almost collapse on one another when plotted against $z^+$ (compare with the two cases in Fig. 9.3) provides a hint into the usefulness of this non-dimensionalization. It demonstrates the universality of the turbulent flow structure, and allows us to make some general statements about the flow structure that are independent of the Reynolds number. This figure shows that the viscous and turbulent stresses become approximately equal at about $z^+ \approx 12$. Some useful criteria for $z^+$ in characterizing the flow regimes are:

$$z^+ \lesssim 5 \quad \text{(viscous sublayer)} \tag{9.22}$$

$$5 \lesssim z^+ \lesssim 50 \quad \text{(viscous wall region)} \tag{9.23}$$

$$z^+ \gtrsim 50 \quad \text{(outer region)} \tag{9.24}$$

Let's now examine in more detail each of these regions and see if flow structure varies significantly between them.

## 9.4 Velocity structure in various wall regions

Now, let's look at the time-mean velocity profiles in the turbulent channel flow, and in various regions near and away from the wall. When fully developed, such flow is completely determined by the fluid density $\rho$, the kinematic viscosity $\nu$, the channel half-height $\delta$, and the friction velocity $u_*$, because:

$$u_* = \sqrt{-\frac{\delta}{\rho}\frac{\partial \overline{p}}{\partial x}} \tag{9.25}$$

Between these parameters, there are only two independent non-dimensional groups that can be formed: $z/\delta$ and $Re_\tau = u_*\delta/\nu$. It should then be possible to express the velocity profile as a function of these parameters:

$$\overline{u}(z) = u_* F\left(\frac{z}{\delta}, Re_\tau\right) \tag{9.26}$$

where $F$ is some yet-to-be-determined non-dimensional function. However, since both the viscous stress and the turbulent production are determined by the mean shear $\partial\overline{u}/\partial z$, it may be more useful to seek the form of the velocity profile in terms of the mean shear:

$$\frac{\partial \overline{u}}{\partial z} = \frac{u_*}{z}\Phi\left(\frac{z}{\delta}, \frac{z}{\delta_\nu}\right) \tag{9.27}$$

where $\Phi$ is, like $F$ before, some yet-to-be-determined non-dimensional function, and the proportionality to $u_*/z$ is proposed on dimensional grounds. Notice that the second argument of $G$, $z/\delta_\nu$ (which we also defined earlier as $z^+$), is equivalent to $Re_\tau z/\delta$, so it is useful to see $G$ as a function of two non-dimensional heights, one characteristic of the boundary layer and another of the viscous sublayer.

Let's focus for now on the region closest to the wall, which may include the viscous sublayer and extend somewhat beyond it. Prandtl (1925) hypothesized that at high Reynolds numbers, there is a region very near the wall ($z \ll \delta$), called the *inner layer*, in which the mean velocity profile is entirely governed by viscosity, and is independent of the boundary layer size $\delta$ and the centerline velocity $u_0$. Thus, as $z/\delta \to 0$, $\Phi(z/\delta, z/\delta_\nu) \to \Phi_I(z/\delta_\nu)$, so in this region Eq. (9.27) reduces to:

$$\frac{\partial \overline{u}}{\partial z} = \frac{u_*}{z}\Phi_I\left(\frac{z}{\delta_\nu}\right) = \frac{u_*}{z}\Phi_I\left(z^+\right) \tag{9.28}$$

Since $\Phi_I$ is a function of $z^+$ and it's the function that we want to determine, let's express the other variables in Eq. (9.28) in terms of $z^+$ as well. To do that, we introduce the non-dimensional velocity which is the velocity normalized by the friction velocity:

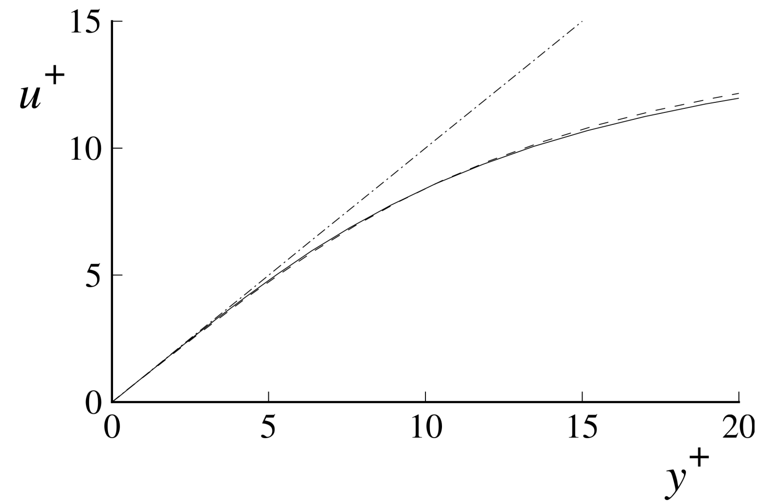


Figure 9.5: Near-wall profiles of mean velocity from the DNS data of Kim et al. (1987): dashed line, $Re = 5,600$; solid line, $Re = 13,750$; dot-dashed line, $u^+ = z^+$. This is Figure 7.5 from Turbulent Flows by Pope.

$$u^+ \equiv \frac{\overline{u}}{u_*} \tag{9.29}$$

Recalling that $u_* = \nu/\delta_\nu$ and that $z^+ = z/\delta_\nu$, we can express Eq. (9.28) as:

$$\frac{\partial u^+}{\partial z^+} = \frac{1}{z^+}\Phi_I\left(z^+\right) \tag{9.30}$$

The non-dimensional velocity $u^+$ is thus a function of $z^+$ alone:

$$u^+ = f_w(z^+) \tag{9.31}$$

where $f_w$ is the *wall function*, expressed in terms of $z^+$ as:

$$f_w(z^+) = \int_0^{z^+} \Phi_I(z)dz \tag{9.32}$$

Equations (9.31)-(9.32) make the so-called *law of the wall*. There is copious experimental and DNS evidence that $f_w(z^+)$ is a universal function for boundary layers in general. Let's find the form of this function for small and large values of $z^+$.

In the viscous sublayer, we can establish from Eq. (9.16) and the no slip boundary condition that:

$$f_w(0) = 0 \tag{9.33}$$

$$f_w'(0) = 1 \tag{9.34}$$

which implies that for very small values of $z^+$, the wall function is:

$$f_w(z^+) \approx z^+ \tag{9.35}$$

The validity of the linear scaling of the velocity with $z^+$ in the inner layer is shown based on DNS data in Fig. 9.5. Up to about $z^+ \approx 5$, the velocity scales linearly with $z^+$, as expected from the viscous sublayer. However, beyond $z^+ \approx 5$, the velocity scales differently and we need to seek a different functional form for $f_w(z^+)$. Based on the data, it seems like the function may have a logarithmic dependence on $z^+$.

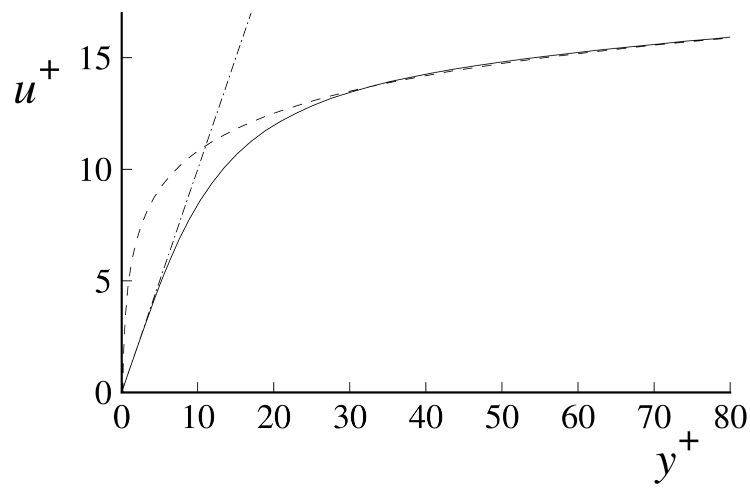


Figure 9.6: Near-wall profiles of mean velocity: Solid line, DNS data of Kim et al. (1987), $Re = 13,750$; dot-dashed line, $u^+ = z^+$; dashed line, the log-law. This is Figure 7.6 from Turbulent Flows by Pope.

Away from the wall, we can suppose that the viscosity plays smaller role, and thus $\Phi_I(z^+)$ reduces to a constant, experimentally determined to be $1/\kappa$, where $\kappa$ is the von Kármán constant and approximately equal to 0.41:

$$\Phi_I(z^+) = \frac{1}{\kappa}, \quad \text{for } \frac{z}{\delta} \ll 50 \text{ and } z^+ \gg 1 \tag{9.36}$$

In this region, the velocity shear is then:

$$\frac{\partial u^+}{\partial z^+} = \frac{1}{\kappa z^+} \tag{9.37}$$

which integrates to:

$$u^+ = \frac{1}{\kappa}\ln(z^+) + C \tag{9.38}$$

where $C$ is an integration constant, experimentally determined to be about 5.2. Returning back to our dimensional variables, we can express the velocity profile as:

$$\overline{u}(z) = u_* \left[\frac{1}{\kappa}\ln\left(\frac{z}{\delta_\nu}\right) + 5.2\right] \tag{9.39}$$

The log-law is demonstrated based on DNS data in Fig. 9.6, and its universality (*i.e.* independence of the Reynolds number) is demonstrated based on experimental data in Fig. 9.7.

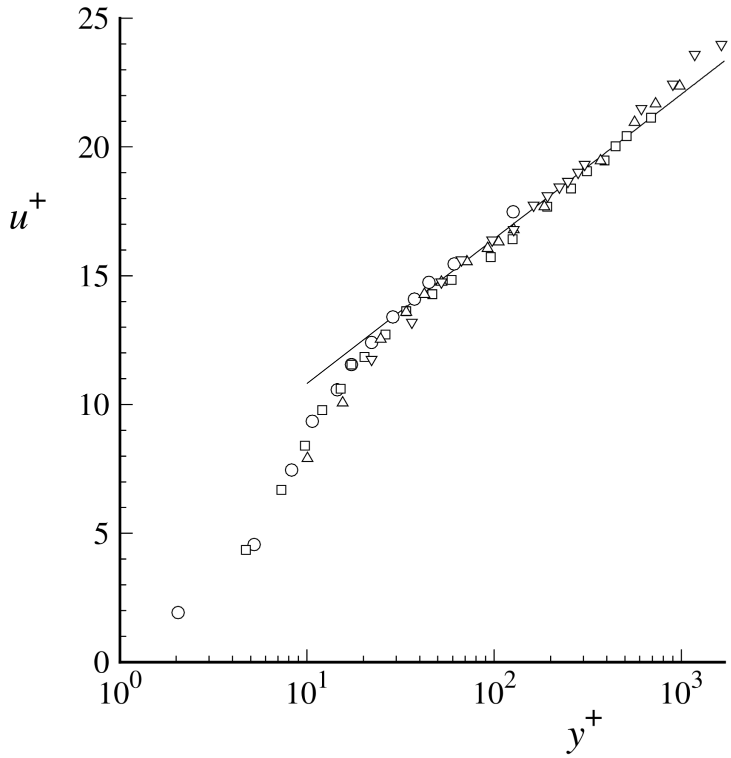


Figure 9.7: Mean velocity profiles in fully developed turbulent channel flow measured by Wei and Willmarth (1989): Circles, $Re_0 = 2,970$; squares, $Re_0 = 14,914$; upward triangles, $Re_0 = 22,776$; downward triangles, $Re_0 = 39,582$; line, the log-law. This is Figure 7.7 from Turbulent Flows by Pope.

## Exercises

1. Find the expression for the bulk velocity (see Eq. 9.4) of a laminar channel flow. How large is it compared to the centerline velocity $u_0$? How about the bulk Reynolds number relative to the centerline Reynolds number $Re_0$?

2. Consider a fully developed turbulent channel flow at $Re = 10^5$ (Eq. 9.3). The fluid is water with $\rho = 10^3$ kg/m$^3$ and viscosity $\nu = 10^{-6}$ m$^2$/s. The channel half-height is $\delta = 0.1$ m, and the skin-friction coefficient is $C_f = u_*^2/U_0^2 = 4 \times 10^{-3}$. Find the bulk velocity $\langle \overline{u} \rangle$, the friction velocity $u_*$, the friction Reynolds number $Re_\tau$, and the viscous length scale $\delta_\nu$.

## Summary

In this chapter, we covered:

- The structure of turbulent boundary layers and channel flows;
- The law of the wall and its different regimes (viscous sublayer, buffer layer, log-law region);
- Non-dimensional velocity and length scales based on the friction velocity $u_*$;
- The universality of the log-law across different Reynolds numbers;
- The relationship between mean velocity profiles and wall stress through the friction coefficient $C_f$;
- Experimental validation of boundary layer theory using DNS and laboratory measurements.

## Further reading

- Chapter 7 of Turbulent Flows by Stephen Pope.

# 10 Surface gravity waves

In this chapter, we examine in detail the boundary between the atmosphere and the ocean, that is, the surface waves. The key restoring force for the surface waves, as we will soon see, is gravity, and so these waves are often called *gravity waves*, much like the waves we explored as a solution of the shallow water equations in Chapter 7. In the first part of this chapter, we derive the solution for the small-amplitude (also often called linear) waves, which are valid when the wave amplitude is much smaller than the wavelength and the water depth. This assumption allows for a relatively straightforward solution of the flow anywhere below the free wavy surface. Although simplistic in its approximations, the linear wave theory has been surprisingly successful in predicting the behavior of the waves even when the assumptions behind it are clearly violated. The linear wave theory remains the basis of modern wave prediction models that are used in operational weather and ocean forecasting. After deriving the linear wave solutions, we will explore their properties and derive some second-order quantities with implication on mean ocean circulation.

## 10.1 Small-amplitude wave derivation

### 10.1.1 Governing equations

Key assumptions are that the fluid is incompressible ($\nabla \cdot \mathbf{u} = 0$), inviscid ($\nu \nabla^2 \mathbf{u} = 0$), and irrotational ($\nabla \times \mathbf{u} = 0$). Velocity $\mathbf{u}$ then has a scalar potential $\phi$ such that:

$$\mathbf{u} = \nabla \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} \tag{10.1}$$

Incompressibility then dictates that:

$$\nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \tag{10.2}$$

This is called the Laplace equation, and it holds throughout the fluid. In two dimensions, horizontal and vertical, Eq. (10.2) is:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{10.3}$$

which is sufficient if we consider surface waves that propagate in the $x$-direction and that are otherwise uniform in the $y$-direction.

Although $\phi$ is allowed to vary in both space and time, the Laplace equation states that at any given time, $\phi$ anywhere in the interior of the fluid is determined by its values at the boundary (*i.e.* the boundary conditions). It does not, however, determine how $\phi$ evolves in time. One important property of the velocity potential is that it is not unique, *i.e.* there are infinitely many functions that satisfy the Laplace equation. For example, if $\phi$ is a velocity potential, then so is $\phi + C$, where $C$ is a scalar constant, and so is $\phi + f(t)$, where $f(t)$ is an arbitrary function of time. Another one is that a sum of any number of velocity potentials is also a velocity potential.

Now, to determine the time dependence of $\phi$, we can integrate the Euler equations of motion (introduced back in Section 4.2, see Eq. 4.29) to obtain a steady-state relationship between the pressure and the velocity of the fluid. The Euler equations in $x$-$z$ plane are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{10.4}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{10.5}$$

Now, recall that we require the flow to be irrotational, so:

$$\omega = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0 \tag{10.6}$$

which leads to:

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} \tag{10.7}$$

We can use this to rewrite Eqs. (10.4)-(10.5) as:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u^2}{\partial x} + \frac{\partial w^2}{\partial x}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{10.8}$$

$$\frac{\partial w}{\partial t} + \frac{1}{2}\left(\frac{\partial u^2}{\partial z} + \frac{\partial w^2}{\partial z}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{10.9}$$

Now, express the velocity components in the time derivatives as gradients of the velocity potential:

$$\frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + \frac{p}{\rho}\right] = 0 \tag{10.10}$$

$$\frac{\partial}{\partial z}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + \frac{p}{\rho}\right] = -g \tag{10.11}$$

Integrating these equations with respect to $x$ and $z$ respectively, we obtain:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + \frac{p}{\rho} = C'(z,t) \tag{10.12}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + \frac{p}{\rho} = C(x,t) - gz \tag{10.13}$$

where $C(x,t)$ and $C'(z,t)$ are integration constants that can vary in dimensions other than their respective dimension of integration. Since these equations have the same left-hand sides, their right-hand sides must be equal:

$$C(x,t) = C'(z,t) + gz \tag{10.14}$$

$C(x,t)$ thus can only depend on time, and we get our final equation form called the *Bernoulli equation*:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + \frac{p}{\rho} + gz = C(t) \tag{10.15}$$

The Bernoulli equation will serve as a dynamic free surface boundary condition as we proceed to derive the solutions for the surface gravity waves.

### 10.1.2 Boundary conditions

Now that we established the governing equations to solve, we need to specify the boundary conditions to determine the velocity potential in the interior. We will rely on a total of four boundary conditions:

1. **Kinematic free surface boundary condition**: This boundary condition determines the vertical velocity at the free surface $\eta(x,t)$ by exploiting the fact that the Lagrangian (material) change of the vertical position is the vertical velocity itself:

$$w = \left.\frac{dz}{dt}\right|_{z=\eta} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \tag{10.16}$$

   Expressed in terms of the velocity potential, this boundary condition becomes:

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}, \text{ at } z = \eta(x,t) \tag{10.17}$$

2. **Dynamic free surface boundary condition**: We leverage the Bernoulli equation (Eq. 10.15) at the free surface ($z = \eta$) and set the surface pressure to be zero:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + g\eta = C(t), \text{ at } z = \eta(x,t) \tag{10.18}$$

3. **Bottom boundary condition**: The bottom is rigid and impermeable, so the vertical velocity is zero at the bottom:

$$w = 0, \text{ at } z = -h \tag{10.19}$$

   where $h$ is the mean depth of the fluid.

4. **Lateral boundary condition**: At the lateral boundaries, since we're seeking a wave solution, we know that the velocity potential must be periodic in the horizontal space as well as time:

$$\phi(x,t) = \phi(x+L,t) \tag{10.20}$$

$$\phi(x,t) = \phi(x,t+T) \tag{10.21}$$

   where $L$ is the wavelength and $T$ is the period.

With these four boundary conditions, we are now equipped to solve for the velocity potential in the interior of the fluid.

### 10.1.3 Solution

Our key equation to solve is the Laplace equation (Eq. 10.2) for the velocity potential $\phi$ that varies in the horizontal and vertical direction $x$ and $z$ respectively, as well as time $t$:

$$\nabla^2\phi(x, z, t) = 0 \tag{10.22}$$

To solve this equation, we will rely on the method of separation of variables, where we assume that the solution can be written as a product of functions that depend on each coordinate separately:

$$\phi(x, z, t) = \phi_x(x)\phi_z(z)\phi_t(t) \tag{10.23}$$

We can start from the time-dependent part $\phi_t(t)$ and recall the lateral boundary condition which states that the velocity potential must be periodic in time, which is true for sines and cosines (and some combinations of them). For a sine function of a phase $\varphi$, this is true:

$$\sin(\varphi) = \sin(\varphi + 2\pi) \tag{10.24}$$

And expressing it as a function of time:

$$\sin(\omega t) = \sin(\omega t + 2\pi) \tag{10.25}$$

where $\omega$ is the angular frequency in units of radians per second, so that the phase $\varphi$ has angle units (radians). Notice that we could have picked (and soon, we will) a cosine function instead of a sine function, and the solution would be still be valid. With the choice of a sine for the time-dependent part of the potential, we write the full velocity potential as:

$$\phi(x, z, t) = \phi_x(x)\phi_z(z)\sin(\omega t) \tag{10.26}$$

Insert this into Eq. (10.22) to get:

$$\frac{\partial^2\phi_x}{\partial x^2}\phi_z\sin(\omega t) + \phi_x\frac{\partial^2\phi_z}{\partial z^2}\sin(\omega t) = 0 \tag{10.27}$$

Divide by $\phi_x\phi_z\sin(\omega t)$ to get:

$$\frac{1}{\phi_x}\frac{\partial^2\phi_x}{\partial x^2} + \frac{1}{\phi_z}\frac{\partial^2\phi_z}{\partial z^2} = 0 \tag{10.28}$$

Can we separate this even further? Recall that $\phi_x$ and $\phi_z$ are functions of $x$ and $z$ respectively. If, for example, we hold $x$ constant and consider variations in $z$, the first term would remain constant but the second term would not! You can arrive to the same conclusion by holding $z$ constant and varying $x$. This would clearly violate Eq. (10.28), and so the only way that equation can hold is if both $\phi_x$ and $\phi_z$ are equal to the same constant but with opposite signs:

$$\frac{1}{\phi_x}\frac{\partial^2\phi_x}{\partial x^2} = -k^2 \tag{10.29}$$

$$\frac{1}{\phi_z}\frac{\partial^2 \phi_z}{\partial z^2} = k^2 \tag{10.30}$$

where $k$ is the separation constant. These can also be written as:

$$\frac{\partial^2 \phi_x}{\partial x^2} + k^2 \phi_x = 0 \tag{10.31}$$

$$\frac{\partial^2 \phi_z}{\partial z^2} - k^2 \phi_z = 0 \tag{10.32}$$

For real values of $k$, the solutions to these equations are:

$$\phi_x(x) = A\sin(kx) + B\cos(kx) \tag{10.33}$$

$$\phi_z(z) = Ce^{kz} + De^{-kz} \tag{10.34}$$

where $A$, $B$, $C$, and $D$ are constants that are yet to be determined. We now write our intermediate solution for the velocity potential as:

$$\phi(x,z,t) = [A\sin(kx) + B\cos(kx)]\left[Ce^{kz} + De^{-kz}\right]\sin(\omega t) \tag{10.35}$$

Next, let's attempt to constrain the $z$-dependent part of the potential, $\phi_z(z) = Ce^{kz} + De^{-kz}$. Recall the bottom boundary condition which for a flat bottom requires $w = 0$ at $z = -h$. Then:

$$w = \frac{\partial \phi_z}{\partial z} = k\left(Ce^{kz} - De^{-kz}\right) = 0 \tag{10.36}$$

which implies $C = De^{2kh}$. Insert this back into Eq. (10.34) to get:

$$\begin{aligned}\phi_z(z) &= D\left(e^{2kh}e^{kz} + e^{-kz}\right)\\ &= 2De^{kh}\left(e^{k(z+h)} + e^{-k(z+h)}\right)\\ &= 2De^{kh}\cosh\left(k(z+h)\right)\end{aligned} \tag{10.37}$$

Inserting this back into Eq. (10.35) we get:

$$\phi(x,z,t) = [A\sin(kx) + B\cos(kx)]\, 2De^{kh}\cosh\left(k(z+h)\right)\sin(\omega t) \tag{10.38}$$

Now, how about the free surface boundary condition? Recall the Bernoulli equation at $z = \eta$ with $p = 0$:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + g\eta = C(t), \text{ at } z = \eta(x,t) \tag{10.39}$$

If we denote this equation as BE, we could evaluate it at the free surface $z = \eta$ using a Taylor expansion around $z = 0$:

$$BE_{z=\eta} = BE_{z=0} + \eta\frac{\partial BE}{\partial z} + \frac{1}{2}\eta^2\frac{\partial^2 BE}{\partial z^2} + \cdots \tag{10.40}$$

To greatly simplify the algebra, this is where we invoke the small-amplitude approximation, which effectively states that if $\eta \ll 1$, then $\eta^2 \ll \eta$, $\eta \ll u\eta$, $u \ll u^2$, and so on. This is where the *linearization* of the wave solution occurs, and it's where it is possible to find wave solutions to a higher-order, such as those of the Stokes wave theory. For brevity, we keep only the first-order terms being the largest and write:

$$\left(\frac{\partial \phi}{\partial t} + g\eta\right)_{z=0} = C(t) \tag{10.41}$$

and from here we have the expression for the free surface elevation as function of the potential and time:

$$\eta = -\frac{1}{g}\left.\frac{\partial \phi}{\partial t}\right|_{z=0} + \frac{C(t)}{g} \tag{10.42}$$

As by definition $\eta$ is a periodic displacement around the mean water level, its spatial and temporal average is zero, so $C(t)$ must be zero as well. The surface elevation is then:

$$\eta = -2D\frac{\omega}{g}e^{kh}\cosh kh\,[A\cos(kx) + B\sin kx]\cos(\omega t) \tag{10.43}$$

and so the constant $D$ must be such that the wave amplitude is:

$$a = -2D\frac{\omega}{g}e^{kh}\cosh kh \tag{10.44}$$

and the constant $D$ is:

$$D = -\frac{ag}{2\omega e^{kh}\cosh kh} \tag{10.45}$$

Insert this back to our intermediate solution for the velocity potential (Eq. 10.38) and moving the minus sign into the $z$-dependent part of the potential, we get:

$$\phi(x,z,t) = \frac{ag}{\omega}\frac{\cosh[k(h+z)]}{\cosh(kh)}\,[A\sin(kx) + B\cos(kx)]\sin(-\omega t) \tag{10.46}$$

which is our new intermediate solution for $\phi$.

Let's revisit again the lateral boundary conditions and recognize that $\phi_x = A\cos(kx) + B\sin(kx)$, $\phi_x = A\cos(kx)$ and $\phi_x = B\sin(kx)$ are all valid solutions to the Laplace equation, which means that any combination of them is a valid wave solution. Same is true for $\phi_z$ where we chose a sine form, but we could have chosen a cosine form instead, or some combination of the two. This property of the velocity potential allows us to describe both standing and progressive waves with the same functional form for $\phi$. Specifically:

$$\phi(x,z,t) = \frac{ag}{\omega}\frac{\cosh[k(h+z)]}{\cosh(kh)}\cos(kx)\sin(-\omega t) \tag{10.47}$$

is a valid velocity potential that belongs to a *standing wave* with amplitude $a$. Recognizing that both the sines and cosines are valid forms for the $x$- and $t$-dependent parts of the velocity potential, and we can thus linearly combine them to obtain a valid velocity potential that belongs to a progressive wave, specifically:

$$\sin(kx)\cos(-\omega t) - \cos(kx)\sin(-\omega t) = \sin(kx - \omega t) \tag{10.48}$$

A valid velocity potential that belongs to a *progressive wave* with amplitude $a$ is then:

$$\phi(x, z, t) = \frac{ag}{\omega} \frac{\cosh[k(h + z)]}{\cosh(kh)} \sin(kx - \omega t) \tag{10.49}$$

The elevation that corresponds to this velocity potential is:

$$\eta = a\cos(kx - \omega t) \tag{10.50}$$

Equations (10.49) and (10.50) fully describe the spatial and temporal evolution of a wave with amplitude $a$ and wavenumber $k$ over mean water depth $h$. The wave potential field for a linear, progressive gravity wave with $a = 0.1$ m and $k = 1$ rad/m, and its corresponding elevation, are shown in Fig. 10.1. The velocity potential for this wave has a positive maximum at the surface on the front face of the wave and a negative minimum at the surface on the back face of the wave. The potential is largest at the surface and decays exponentially with depth.

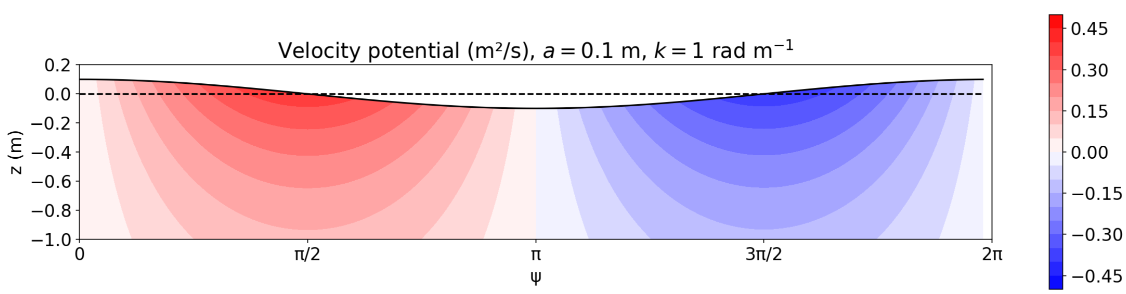


Figure 10.1: Wave elevation (black line) and velocity potential (color) for a linear wave with amplitude $a = 0.1$ m and wavenumber $k = 1$ rad m$^{-1}$, in deep water ($h = 100$ m). The mean water level ($z = 0$) is indicated by the horizontal dashed line. The wave is propagating from left to right.

In the deep water limit, the wave potential can be written more concisely as:

$$\phi(x, z, t) = \frac{ag}{\omega} e^{kz} \sin(kx - \omega t) \tag{10.51}$$

As we explore the wave kinematics, we will leverage this form for brevity.

## 10.2 Dispersion of gravity waves

An important property of gravity waves is that they disperse, meaning that waves of different wavenumbers travel at different speeds. A dispersion relationship describes how the wavenumber changes as a function of the frequency. To derive it, we leverage the kinematic free surface boundary condition (Eq. 10.16), where, to the first order (recall our small-amplitude approximation), we have:

$$w = \frac{\partial \phi}{\partial z}\bigg|_{z=0} = \frac{\partial \eta}{\partial t}\bigg|_{z=0} \quad (10.52)$$

which leads to:

$$\frac{agk}{\omega}\frac{\sinh(kh)}{\cosh(kh)}\sin(kx - \omega t) = a\omega \sin(kx - \omega t) \quad (10.53)$$

Simplifying on both sides, we arrive to:

$$\omega^2 = gk \tanh(kh) \quad (10.54)$$

which is the *dispersion relationship* for surface gravity waves. Wavenumber as function of frequency is shown in Fig. 10.2.

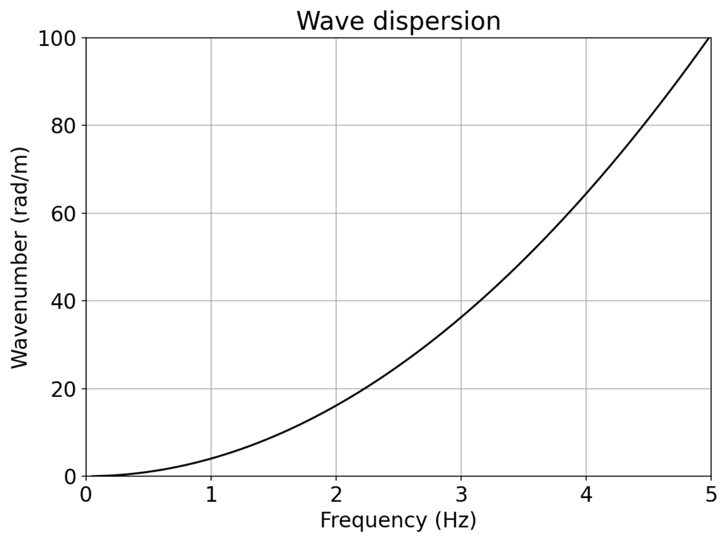


Figure 10.2: Wavenumber as function of frequency for the surface gravity waves in deep ($h$ = 1000 m) water. The wavenumber range is from $k = 0.01$ to $k = 100$ rad/m. The corresponding frequency range is from approximately 0.05 to 5 Hz.

Let's now evaluate this dispersion relationship in the limits of shallow and deep water. In deep water, $kh \to \infty$ and so $\tanh(kh) \to 1$. Then:

$$\omega^2 = gk \tanh(kh) \to gk \quad (10.55)$$

So, in deep water, the frequency is not dependend on water depth, which we expected–the rigid bottom is so far away from the free surface that the waves don't feel it at all. The frequency is dependent on the wavenumber and gravity only. In contrast, in shallow water, $kh \to 0$ and so $\tanh(kh) \to kh$. Then:

$$\omega^2 = gk \tanh(kh) \to gk^2 h \quad (10.56)$$

or:

$$\omega = \sqrt{gh}k \tag{10.57}$$

Here, the frequency depends on both the wavenumber and the water depth. So, as waves enter progressively shallower water, their frequency decreases. However, unlike in deep water where the frequency is nonlinearly dependent on wavenumber, in shallow water they are linearly correlated by $\sqrt{gh}$.

An important wave property that directly follows from the dispersion relationship is that for the *wave celerity*, or *phase speed*, which is the speed at which the wave potential field and elevation propagate in the horizontal direction:

$$C_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh(kh)} \tag{10.58}$$

Like the dispersion relationship itself, the expression for the phase speed simplifies as well in the limits of shallow and deep water. In deep water, $C_p = \sqrt{g/k}$, and in shallow water, $C_p = \sqrt{gh}$. It is no coincidence that this is the same expression that we found for the phase speed of Poincaré waves in the limit of negligible planetary rotation (Eq. 7.39).

## 10.3 Wave kinematics

With the velocity potential well defined in Eq. 10.51, the instantaneous wave-induced velocities can be readily obtained as:

$$u = \frac{\partial \phi}{\partial x} = a\omega e^{kz}\cos(kx - \omega t) \tag{10.59}$$

$$w = \frac{\partial \phi}{\partial z} = a\omega e^{kz}\sin(kx - \omega t) \tag{10.60}$$

The horizontal and vertical velocities for a linear wave with amplitude $a = 0.1$ m and wavenumber $k = 1$ rad m$^{-1}$, in deep water ($h = 100$ m), are shown in Fig. 10.3.

Now that we have the velocities, it is instructive to look at how the water particles move in the wave field. The horizontal and vertical displacements can be obtained by integrating their respective velocities over time:

$$\zeta = \int u \; dt = -ae^{kz}\sin(kx - \omega t) \tag{10.61}$$

$$\xi = \int w \; dt = ae^{kz}\cos(kx - \omega t) \tag{10.62}$$

The displacements are thus closed orbits when evaluated at any fixed depth $z$.

The wave-induced accelerations are also relevant because they govern the wave-induced forces on submerged bodies.

$$a_x = \frac{\partial u}{\partial t} = a\omega^2 e^{kz}\sin(kx - \omega t) \tag{10.63}$$

$$a_z = \frac{\partial w}{\partial t} = -a\omega^2 e^{kz}\cos(kx - \omega t) \tag{10.64}$$

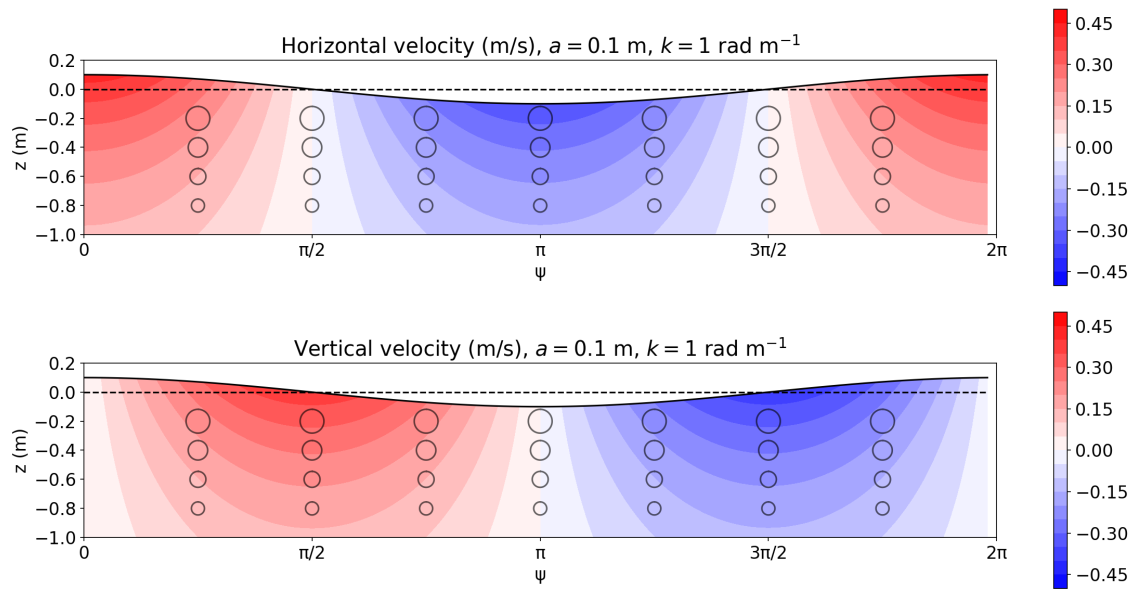


Figure 10.3: Wave elevation (black line) and horizontal (top) and vertical (bottom) velocities (color) for a linear wave with amplitude $a = 0.1$ m and wavenumber $k = 1$ rad m$^{-1}$, in deep water ($h = 100$ m). Thin lines indicate the water particle trajectories.

## 10.4 Mean Lagrangian velocity

It's clear that averaging the instantaneous orbital velocities at any given fixed depth $z$ over one or more period yields zero. However, if we follow the water particles in Lagrangian sense, their average horizontal velocity will be non-zero because the particles move forward a larger distance than backward over the course of one period. This mean Lagrangian velocity is called the *Stokes drift*. We can find the expression for this mean residual drift by first recognizing that we can approximate the velocity at a particle position $(x + \zeta, z + \xi)$ to the first order as:

$$u(x + \zeta, z + \xi) = u(x, z) + \frac{\partial u}{\partial x}\zeta + \frac{\partial u}{\partial z}\xi \tag{10.65}$$

$$u(x + \zeta, z + \xi) = u(x, z) + a^2\omega k e^{2kz}\sin^2(kx - \omega t) + a^2\omega k e^{2kz}\cos^2(kx - \omega t) \tag{10.66}$$

$$u(x + \zeta, z + \xi) = u(x, z) + a^2\omega k e^{2kz} \tag{10.67}$$

The Stokes drift can then be obtained by averaging this material velocity over one period:

$$u_{St} = \frac{1}{T}\int_0^T u(x + \zeta, z + \xi)\ dt = a^2\omega k e^{2kz} \tag{10.68}$$

It can be easily shown by following the same procedure for the vertical velocity that the vertical Stokes drift is zero.

## 10.5 Wave groups

Waves if a given wavenumber and frequency are rarely alone and in reality the ocean surface is densely populated by a spectrum of waves with different wavenumbers and frequencies. The simplest case of this is if we considered two waves and their resulting elevation:

$$\eta = \eta_1 + \eta_2 = a\cos(k_1 x - \omega_1 t) + a\cos(k_2 x - \omega_2 t) \tag{10.69}$$

Superposing two waves with different wavenumbers leads to the formation of wave groups (Fig. 10.4).

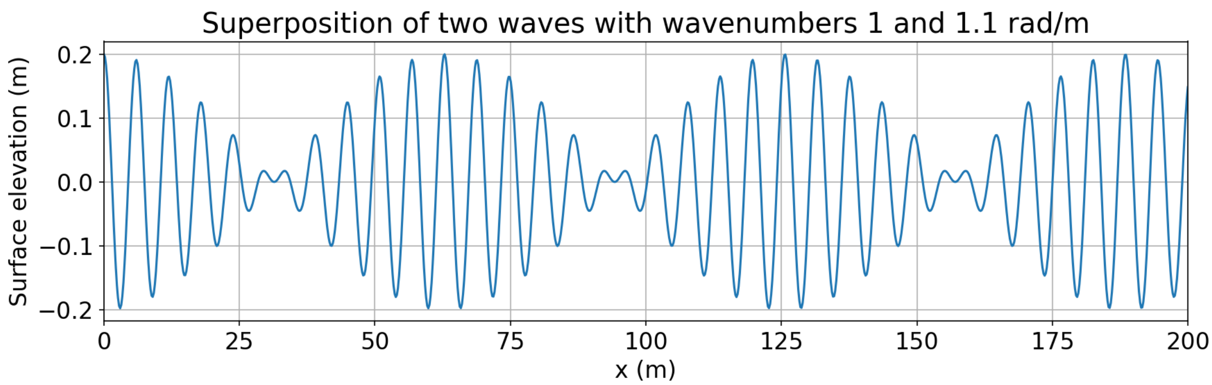


Figure 10.4: Superposition of two waves with wavenumbers 1 and 1.1 rad/m and amplitudes of 0.1 m.

Now, to see at what speed does the wave group propagate, write the wavenumbers and frequencies of independent waves as:

$$k_1 = k - \frac{\Delta k}{2} \tag{10.70}$$

$$\omega_1 = \omega - \frac{\Delta \omega}{2} \tag{10.71}$$

$$k_2 = k + \frac{\Delta k}{2} \tag{10.72}$$

$$\omega_2 = \omega + \frac{\Delta \omega}{2} \tag{10.73}$$

Then, Eq. 10.69 can be rewritten as:

$$\eta = a\cos\left[\frac{1}{2}[(k_1 + k_2)x - (\omega_1 + \omega_2)t]\right]\cos\left[\frac{1}{2}[(k_1 - k_2)x - (\omega_1 - \omega_2)t]\right] \tag{10.74}$$

which is equivalent to:

$$\eta = a\cos(kx - \omega t)\cos\left[\frac{1}{2}\Delta k\left(x - \frac{\Delta\omega}{\Delta k}t\right)\right] \tag{10.75}$$

This form corresponds to individual waves moving with phase speed $C_p = \omega/k$, and the envelope moving with the so-called *group speed*:

$$C_g = \frac{\Delta\omega}{\Delta k} \approx \frac{\partial\omega}{\partial k},\ \text{for } \Delta k \to 0 \tag{10.76}$$

Notice that the group speed of gravity waves approaches half the phase speed in the deep water limit, and is equal to the phase speed in shallow water. Group speed of gravity waves is important because it is the advective speed of wave energy as well:

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + \nabla\cdot(\mathbf{C}_g E) = 0 \tag{10.77}$$

This equation is called the wave energy balance and it is the key governing equation in most ocean wave prediction models.

## Summary

In this chapter, we covered:

- The derivation of small-amplitude (linear) wave theory;
- The dispersion relationship for surface gravity waves;
- The wave kinematics and mean Lagrangian (Stokes) drift;
- Wave groups and wave energy balance.

## Further reading

- Chapters 3 and 4 of *Water wave mechanics* by Dean and Dalrymple

# A Quick reference

This section serves a quick reference for the key equations used in this book.

**Gradient:**

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{A.1}$$

**Divergence:**

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{A.2}$$

**Curl:**

$$\nabla \times \mathbf{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k} \tag{A.3}$$

**Laplacian:**

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{A.4}$$

**Curl of a gradient:**

$$\nabla \times (\nabla T) = 0 \tag{A.5}$$

**Divergence of a curl:**

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0 \tag{A.6}$$

**Lagrangian derivative operator:**

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \tag{A.7}$$

**Velocity as a gradient of a scalar potential:**

$$\mathbf{u} = \nabla \phi \tag{A.8}$$

**Continuity, Eulerian form:**

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) = 0 \tag{A.9}$$

**Continuity, Lagrangian form:**

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0 \tag{A.10}$$

**Momentum, Cauchy:**

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} + \frac{\mathbf{F}_b}{\rho} \tag{A.11}$$

**Stress tensor as a combination of pressure and deviatoric stress:**

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \tag{A.12}$$

**Momentum, Euler:**

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p \tag{A.13}$$

**Momentum, Navier-Stokes:**

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \frac{\mathbf{F}_b}{\rho} \tag{A.14}$$

**Momentum, with body force (gravity):**

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu\nabla^2\mathbf{u} \tag{A.15}$$

**Momentum, Navier-Stokes, in scalar form:**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{A.16}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{A.17}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{A.18}$$

**Equation of state, moist air:**

$$p = \rho R_d T\left[1 + q\left(\frac{R_v}{R_d} - 1\right)\right] \tag{A.19}$$

**Equation of state, seawater:**

$$\rho = \rho_0\left[1 - \beta_T(T - T_0) + \beta_S(S - S_0) - \beta_p(p - p_0)\right] \tag{A.20}$$

**Hydrostatic approximation:**

$$\frac{dw}{dt} = 0 \tag{A.21}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{A.22}$$

**Rate of change of a rotating vector:**

$$\left(\frac{d\mathbf{C}}{dt}\right)_I = \boldsymbol{\Omega} \times \mathbf{C} \tag{A.23}$$

**Rate of change of a rotating vector in a rotating frame:**

$$\left(\frac{d\mathbf{B}}{dt}\right)_I = \left(\frac{d\mathbf{B}}{dt}\right)_R + \mathbf{\Omega} \times \mathbf{B} \tag{A.24}$$

**Rate of change of velocity in a rotating frame:**

$$\left(\frac{d\mathbf{u}_R}{dt}\right)_R = \left(\frac{d\mathbf{u}_I}{dt}\right)_I - 2\mathbf{\Omega} \times \mathbf{u}_R - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \tag{A.25}$$

**Navier-Stokes equation with rotation:**

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p - \frac{1}{\rho}\nabla \Phi - 2\mathbf{\Omega} \times \mathbf{u} + \nu \nabla^2 \mathbf{u} \tag{A.26}$$

**Coriolis parameter:**

$$f = 2\Omega \sin(\theta) \tag{A.27}$$

**$f$-plane approximation:**

$$f = f_0 = 2\Omega \sin(\theta_0) \tag{A.28}$$

**$\beta$-plane approximation:**

$$f = f_0 + \beta y \tag{A.29}$$

$$\beta = \frac{\partial f}{\partial y} = \frac{2\Omega \cos(\theta_0)}{R_E} \tag{A.30}$$

**Geostrophic balance:**

$$\mathbf{f} \times \mathbf{u} = -\frac{1}{\rho}\nabla p \tag{A.31}$$

**Geostrophic velocity:**

$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \tag{A.32}$$

$$v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x} \tag{A.33}$$

**Rossby number:**

$$\mathrm{Ro} \equiv \frac{(\mathbf{u} \cdot \nabla)\,\mathbf{u}}{\mathbf{f} \times \mathbf{u}} \approx \frac{U}{fL} \tag{A.34}$$

**Boussinesq approximation:**

$$\rho = \rho_0 + \delta\rho(x, y, z, t) \tag{A.35}$$

$$p = p_0(z) + \delta p(x, y, z, t) \tag{A.36}$$

**Boussinesq equations:**

$$\frac{d\mathbf{u}}{dt} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho_0}\nabla\delta p + b\mathbf{k} \tag{A.37}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{A.38}$$

$$\frac{dT}{dt} = \dot{T} \tag{A.39}$$

$$\frac{dS}{dt} = \dot{S} \tag{A.40}$$

$$b = b(T, S, p) \tag{A.41}$$

**Buoyancy:**

$$b = -g\frac{\delta\rho}{\rho_0} \tag{A.42}$$

**Thermal wind balance:**

$$\frac{\partial u_g}{\partial z} = -\frac{1}{f}\frac{\partial b}{\partial y} \tag{A.43}$$

$$\frac{\partial v_g}{\partial z} = \frac{1}{f}\frac{\partial b}{\partial x} \tag{A.44}$$

**Potential density:**

$$\rho_\theta = \rho + \frac{p_0 g z}{c_s^2} \tag{A.45}$$

**Brunt-Väisälä (buoyancy) frequency:**

$$N^2 = -\frac{g}{\widetilde{\rho}_\theta}\frac{\partial\widetilde{\rho}_\theta}{\partial z} \tag{A.46}$$

**Static instability:**

$$\frac{\partial\widetilde{\rho}_\theta}{\partial z} < 0 \quad \text{(stable)} \tag{A.47}$$

$$\frac{\partial\widetilde{\rho}_\theta}{\partial z} > 0 \quad \text{(unstable)} \tag{A.48}$$

**Shallow water momentum equation:**

$$\frac{d\mathbf{u}}{dt} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta \tag{A.49}$$

**Shallow water continuity equation:**

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0 \tag{A.50}$$

**Inertial-gravity wave dispersion:**

$$\omega = \sqrt{f^2 + gH(k^2 + l^2)} \tag{A.51}$$

**Gravity wave dispersion:**

$$\omega = \sqrt{gH(k^2 + l^2)} \tag{A.52}$$

**Inertial wave dispersion:**

$$\omega = f \tag{A.53}$$

**Kelvin wave:**

$$u = \widehat{u}_0 e^{\frac{y}{L_d}} e^{i(x - \sqrt{gH}t)} \tag{A.54}$$

$$\eta = \sqrt{\frac{H}{g}} \widehat{u}_0 e^{\frac{y}{L_d}} e^{i(x - \sqrt{gH}t)} \tag{A.55}$$

**Rossby radius of deformation:**

$$L_d = \frac{\sqrt{gH}}{f} \tag{A.56}$$

**Conservation of potential vorticity:**

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0 \tag{A.57}$$

**Potential vorticity:**

$$\frac{\zeta + f}{h} \tag{A.58}$$

**Conservation of potential energy:**

$$\frac{\partial}{\partial t}\left(\frac{gh^2}{2}\right) + \nabla\left(\mathbf{u}\frac{gh^2}{2}\right) + \frac{gh^2}{2}\nabla \cdot \mathbf{u} = 0 \tag{A.59}$$

**Conservation of kinetic energy:**

$$\frac{\partial}{\partial t}\left(\frac{h\mathbf{u}^2}{2}\right) + \nabla \cdot \left(\mathbf{u}\frac{h\mathbf{u}^2}{2}\right) + g\mathbf{u}\nabla\left(\frac{h^2}{2}\right) = 0 \tag{A.60}$$

**Conservation of total energy:**

$$\frac{\partial E}{\partial t} = \frac{\partial PE}{\partial t} + \frac{\partial KE}{\partial t} \tag{A.61}$$

$$\frac{\partial}{\partial t}\frac{1}{2}\left(h\mathbf{u}^2+gh^2\right)+\nabla\cdot\left[\mathbf{u}\left(\frac{1}{2}h\mathbf{u}^2+gh^2\right)\right]=0 \tag{A.62}$$

$$\frac{\partial E}{\partial t}+\nabla\cdot(\mathbf{F})=0 \tag{A.63}$$

**Energy flux:**

$$\mathbf{F}=\mathbf{u}\left(\frac{1}{2}h\mathbf{u}^2+gh^2\right) \tag{A.64}$$

**Rossby wave frequency:**

$$\omega=Uk-\frac{\beta}{k} \tag{A.65}$$

**Rossby wave phase speed:**

$$c=U-\frac{\beta}{k^2} \tag{A.66}$$

**Reynolds decomposition:**

$$\mathbf{u}=\overline{\mathbf{u}}+\mathbf{u}' \tag{A.67}$$

**Reynolds-averaged Navier-Stokes equation:**

$$\frac{\partial\overline{\mathbf{u}}}{\partial t}+\nabla\cdot(\overline{\mathbf{u}}\,\overline{\mathbf{u}})=-\frac{1}{\rho}\nabla\overline{p}+\nu\nabla^2\overline{\mathbf{u}}+\nabla\cdot\left(\overline{\mathbf{u}'\mathbf{u}'}\right) \tag{A.68}$$

**Reynolds-averaged continuity equation:**

$$\nabla\cdot\overline{\mathbf{u}}=0 \tag{A.69}$$

**Turbulent Kinetic Energy:**

$$k=\frac{1}{2}\overline{\mathbf{u}'^2} \tag{A.70}$$

**Turbulent Kinetic Energy budget:**

$$\frac{\partial k}{\partial t}+\overline{\mathbf{u}}\cdot\nabla k=-\frac{1}{2}\nabla\cdot(\overline{\mathbf{u}'\mathbf{u}'\mathbf{u}'})-(\overline{\mathbf{u}'\mathbf{u}'}\cdot\nabla)\overline{\mathbf{u}}-\frac{1}{\rho}\overline{\mathbf{u}'\nabla p'}+\overline{\frac{\delta\rho'}{\rho}\mathbf{u}'\cdot\mathbf{g}}+\nu\nabla^2k-\nu\overline{\nabla\mathbf{u}'\cdot\nabla\mathbf{u}'} \tag{A.71}$$

**Kolmogorov's turbulence spectrum:**

$$E(k)=\mathcal{K}\varepsilon^{2/3}\left(\frac{k}{\varepsilon}\right)^{5/3} \tag{A.72}$$

**Wave phase:**

$$\psi=kx-\omega t \tag{A.73}$$

**Wave elevation:**

$$\eta = a \cos \psi \tag{A.74}$$

**Wave velocity potential:**

$$\phi = \frac{ag}{\omega} \frac{\cosh[k(z+h)]}{\cosh(kh)} \sin \psi \tag{A.75}$$

**Wave orbital velocities:**

$$u = \frac{\partial \phi}{\partial x} = -\frac{a\omega}{k} \frac{\cosh[k(z+h)]}{\cosh(kh)} \cos \psi \tag{A.76}$$

$$w = \frac{\partial \phi}{\partial z} = \frac{a\omega}{k} \frac{\sinh[k(z+h)]}{\cosh(kh)} \sin \psi \tag{A.77}$$

**Wave particle displacements:**

$$\zeta = \int u \ dt = -ae^{kz} \sin \psi \tag{A.78}$$

$$\xi = \int w \ dt = ae^{kz} \cos \psi \tag{A.79}$$

**Wave orbital accelerations:**

$$a_x = \frac{\partial u}{\partial t} = a\omega^2 e^{kz} \sin \psi \tag{A.80}$$

$$a_z = \frac{\partial w}{\partial t} = -a\omega^2 e^{kz} \cos \psi \tag{A.81}$$

**Linear gravity wave dispersion:**

$$\omega = \sqrt{gk \tanh(kh)} \tag{A.82}$$

Deep water: $kh \to \infty$

$$\omega = \sqrt{gk} \tag{A.83}$$

Shallow water: $kh \to 0$

$$\omega = \sqrt{gh}k \tag{A.84}$$

**Phase speed:**

$$C_p = \frac{\omega}{k} \tag{A.85}$$

Deep water: $kh \to \infty$

$$C_p = \sqrt{\frac{g}{k}} \tag{A.86}$$

Shallow water: $kh \to 0$

$$C_p = \sqrt{gh} \tag{A.87}$$

**Group speed:**

$$C_g = \frac{\partial \omega}{\partial k} \tag{A.88}$$

Deep water: $kh \to \infty$

$$C_g = \frac{C_p}{2} \tag{A.89}$$

Shallow water: $kh \to 0$

$$C_g = C_p \tag{A.90}$$

**Stokes drift (deep water):**

$$u_{St} = a^2 \omega k e^{2kz} \tag{A.91}$$

**Wave energy balance:**

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{C_g} E) = 0 \tag{A.92}$$

# Index