1. (1) $\lim\limits_{\substack{x\to 0\\ y\to 0}}\sqrt{|xy|} = \lim\limits_{r\to 0}\sqrt{|r^2\sin\theta\cos\theta|} = \lim\limits_{r\to 0}|r|\sqrt{|\sin\theta\cos\theta|} = 0$

(2) $\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x} = \dfrac{\sqrt{|\Delta x\cdot 0|}-0}{\Delta x} = 0$

同理可得 $f_y(x,y)=0$

(3) $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{f(\Delta x,\Delta y)-0-0}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = \lim\limits_{r\to 0}\sqrt{\dfrac{r^2|\sin\theta\cos\theta|}{r^2}} = \lim\limits_{r\to 0}\sqrt{|\sin\theta\cos\theta|}$

∴ $f(x,y)$ 在 $(0,0)$ 不连续

2. 由 $y^2=4x$ $y=2\sqrt{x}$ 在 $(1,2)$ 处切线斜率为 1

即方向余弦 $\{\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\}$

$f_x(1,2) = \frac{1}{x+y}\Big|_{(1,2)} = \frac{1}{3}$

$f_y(1,2) = \frac{1}{x+y}\Big|_{(1,2)} = \frac{1}{3}$

∴ 方向导数为 $\frac{1}{3}\times\frac{\sqrt{2}}{2}+\frac{1}{3}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{3}$

3. 对两式求对 x 偏导

$1 = e^u\cdot\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\sin v+u\cos v\frac{\partial v}{\partial x}$

$0 = e^u\frac{\partial u}{\partial x}-(\frac{\partial u}{\partial x}\cos v-u\sin v\frac{\partial v}{\partial x})$

得 $\frac{\partial u}{\partial x} = \frac{\sin v}{e^u(\sin v-\cos v)+1}$

同理 对 y 求偏导.

$0 = e^u\frac{\partial u}{\partial y}+\frac{\partial u}{\partial y}\sin v+u\cos v\frac{\partial v}{\partial y}$

$1 = e^u\frac{\partial u}{\partial y}-(\frac{\partial u}{\partial y}\cos v-u\sin v\frac{\partial v}{\partial y})$

$\frac{\partial v}{\partial y} = \frac{e^u+\sin v}{u+ue^u(\sin v-\cos v)}$

4. $\frac{\partial z}{\partial x} = yf'\left(\frac{1}{y}\right) + \varphi\left(\frac{y}{x}\right) + x\varphi'\cdot\left(-\frac{y}{x^2}\right) = f' + \varphi\left(\frac{y}{x}\right) - \frac{y}{x}\cdot\varphi'$

$\frac{\partial^2 z}{\partial x^2} = f''\left(\frac{1}{y}\right) + \varphi'\left(-\frac{y}{x^2}\right) - \left(-\frac{y}{x^2}\varphi' + \frac{y}{x}\cdot\varphi''\left(-\frac{y}{x^2}\right)\right)$

$= \frac{1}{y}f'' + \frac{y^2}{x^3}\varphi''$

$\frac{\partial^2 z}{\partial x\partial y} = f'\left(-\frac{x}{y^2}\right) + \varphi'\left(\frac{1}{x}\right) - \left(\frac{1}{x}\varphi' + \frac{y}{x}\cdot\varphi''\left(\frac{1}{x}\right)\right)$

$= -\frac{x}{y^2}f'' - \frac{y}{x^2}\varphi''$

$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\partial y} = \frac{x}{y}f'' + \frac{y^2}{x^2}\varphi'' + \left(-\frac{x}{y}f'' - \frac{y^2}{x^2}\varphi''\right) = 0$

∴ 得证.

5. $f_x = 2x - 12 = 0$. $x = 6$ $(6,-8)$ 不在

$f_y = 2y + 16 = 0$ $y = -8$ 区域 D 内.

令 $L = x^2 + y^2 - 12x + 16y + \lambda(x^2 + y^2 - 25)$

$L_x = 2x - 12 + 2\lambda x$

$L_y = 2y + 16 + 2\lambda y$

$L_\lambda = x^2 + y^2 - 25$

令 $L_x = 0$ $L_y = 0$ $L_\lambda = 0$

得 $\begin{cases} x = 3 \\ y = -4 \end{cases}$ 或 $\begin{cases} x = -3 \\ y = 4 \end{cases}$

当 $x = 3$, $y = -4$ $f(3,-4) = -75$

$f(-3,4) = 125$

∴ 最大值为 125 最小值为 -75.

6. 设点 $M(x_0, y_0, z_0)$ 使得 $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = 1$

令 $F = \sqrt{x} + \sqrt{y} + \sqrt{z} - 1$

$F_x|_{M_0} = \frac{1}{2}x_0^{-\frac{1}{2}}$ $F_y|_{M_0} = \frac{1}{2}y_0^{-\frac{1}{2}}$ $F_z|_{M_0} = \frac{1}{2}z_0^{-\frac{1}{2}}$

切平面: $\frac{1}{2}x_0^{-\frac{1}{2}}(x - x_0) + \frac{1}{2}y_0^{-\frac{1}{2}}(y - y_0) + \frac{1}{2}z_0^{-\frac{1}{2}}(z - z_0) = 0$

令 $x = 0, y = 0$ 得 $z = \sqrt{z_0}$ 同理可得 $x = \sqrt{x_0}$, $y = \sqrt{y_0}$

即求 $\sqrt{x_0 y_0 z_0}$ 在 $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = 1$ 条件下最大值

令 $L=\sqrt{x_0y_0z_0}+\lambda\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}-1\right)$

$L_x=\sqrt{y_0z_0}\,\frac{1}{2}x_0^{-\frac{1}{2}}+\lambda\cdot\frac{1}{2}x_0^{-\frac{1}{2}}$

$L_y=\sqrt{x_0z_0}\,\frac{1}{2}y_0^{-\frac{1}{2}}+\lambda\cdot\frac{1}{2}y_0^{-\frac{1}{2}}$

$L_z=\sqrt{x_0y_0}\cdot\frac{1}{2}z_0^{-\frac{1}{2}}+\lambda\cdot\frac{1}{2}z_0^{-\frac{1}{2}}$

$L_\lambda=\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}-1$

令 $L_x=0$ $L_y=0$ $L_z=0$ $L_\lambda=0$

得 $x=\frac{1}{9}$ $y_0=\frac{1}{9}$ $z_0=\frac{1}{9}$

$\therefore$ 切平面方程为 $\frac{3}{2}(x-\frac{1}{9})+\frac{3}{2}(y-\frac{1}{9})+\frac{3}{2}(z-\frac{1}{9})=0$