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Parameterized Algorithms
assignment 1

Problem 1

Let

$$V(G) = \{v_1, \ldots, v_n\} \quad \text{and} \quad S_n = \left\{\sigma : \sigma \in \{1, \ldots, n\}^{\{1,\ldots,n\}} \wedge \sigma \text{ is a bijection}\right\}.$$

Define

$$\text{cutwidth}_\sigma(i) = |\{(u, v) : u \in \{v_{\sigma_1} \ldots v_{\sigma_i}\} \wedge v \in \{v_{\sigma_{i+1}}, \ldots, v_{\sigma_n}\} \wedge (u, v) \in E(G)\}|,$$

where $\sigma \in S_n$ is any permutation. The objective is to find a permutation $\sigma \in S_n$ that minimizes the value

$$\max_{i \in \{1,\ldots,n-1\}} \text{cutwidth}_\sigma(i)$$

and to calculate this minimum.

Define the function

$$\text{out}(X) = |\{(u, v) : u \in X \wedge v \in V(G) \setminus X \wedge (u, v) \in E(G)\}|,$$

where X is any subset of $V(G)$. Then

$$\text{cutwidth}_\sigma(i) = \text{out}(\{v_{\sigma_1}, \ldots, v_{\sigma_i}\}).$$

For any given X, $\text{out}(X)$ can be calculated in time $n^{\mathcal{O}(1)}$.

We will use dynamic programming over subsets. Define

$$\text{dp}(X) = \min\left\{\max_{i \in \{1,\ldots,|X|-1\}} \text{cutwidth}_\sigma(i) : \sigma \in S_n \wedge \{v_{\sigma_1}, \ldots, v_{\sigma_{|X|}}\} = X\right\},$$

where X is any subset of $V(G)$. Then

$$\begin{aligned} \text{dp}(\emptyset) &= 0, \\ \text{dp}(X) &= \min\{\max(\text{dp}(X \setminus \{x\}), \text{out}(X \setminus \{x\})) : x \in X\}. \end{aligned}$$

The answer to the problem is $\text{dp}(\{1, \ldots, n\})$. To compute dp for all subsets of $V(G)$, we iterate over subsets in non-decreasing order of size. The time complexity of this algorithm is $2^n \cdot n^{\mathcal{O}(1)}$.

Problem 2

<u>Lemma 1</u> A set of points S, in which no three points are collinear, does not form the vertices of a convex polygon if and only if there exist points $A, B, C, D \in S$ such that D lies inside $\triangle ABC$.

<u>Proof of lemma 1</u> The implication „to the left" is obvious, so we only need to prove the implication „to the right". Let $\{H_1, \ldots, H_h\}$ be the convex hull of the set S, with the assumption that these points are ordered counterclockwise along the hull. Let D be any point in S that does not belong to the hull, and let $A = H_1$. There exists exactly one $i \in \{2, \ldots, h-1\}$ such that points $H_2, \ldots, H_i$ lie on one side of the line AD, while points $H_{i+1}, \ldots, H_h$ lie on the other side. Taking $B = H_i$ and $C = H_{i+1}$, we will obtain the desired points.

According to lemma 1, if there exist points $A, B, C, D \in S$ such that D lies inside $\triangle ABC$, then at least of these points must be removed from S. This observation leads to the following algorithm:

Algorithm 1 ConvexDeletion

procedure $\textsc{ConvexDeletion}(S, k)$
  if no four points $A, B, C, D \in S$ satisfy that D lies inside $\triangle ABC$ **then**
    return true
  end if
  if $k \leqslant 0$ **then**
    return false
  end if
  Choose points $(A, B, C, D) \in S$ such that D lies inside $\triangle ABC$
  return $\textsc{ConvexDeletion}(S \setminus \{A\}, k-1)$ **or** $\textsc{ConvexDeletion}(S \setminus \{B\}, k-1)$ **or** $\textsc{ConvexDeletion}(S \setminus \{C\}, k-1)$ **or** $\textsc{ConvexDeletion}(S \setminus \{D\}, k-1)$
end procedure

Finding such quadruples of points A, B, C, D can easily be done in $\mathcal{O}(n^4)$ time by examining all quadruples of points, calculating the relevant cross products for each, and comparing their signs. This can also be done in $\mathcal{O}(n \log n)$ time by computing the convex hull using Graham's algorithm and applying the constructive proof of lemma 1.

The depth of the recursion tree fro of the algorithm 1 is at most k, since with each recursive call, the parameter k decreases by 1. Each node of this tree has at most four children, which gives us an upper bound on the number of nodes in the tree:

$$\sum_{i=0}^{k} 4^i = \frac{4^{k+1}-1}{3} = \mathcal{O}(4^k).$$

Therefore, the overall complexity of the algorithm 1 is $\mathcal{O}(4^k) \cdot n^{\mathcal{O}(1)}$.

Problem 3

Let $d = 10$. We will begin by reducing the size of $\mathcal{F}$ to a polynomial in k.

If $|\mathcal{F}| \leqslant k^{d+1}$, no reduction is necessary. Otherwise, there exists an element $a_1 \in \bigcup \mathcal{F}$ such that the subset $\mathcal{A}_1 = \{A \in \mathcal{F} \ : \ a_1 \in A\}$ has size at least $k^d + 1$. If this were not the case, a hitting set of size k would cover at most k^{d+1} sets. We either include a_1 in the hitting set or exclude it. If we exclude it, then there exists an element $a_2 \in \bigcup \mathcal{F} \setminus \{a_1\}$ such that $\mathcal{A}_2 = \{A \in \mathcal{A}_1 \ : \ a_2 \in A\}$ has size at least $k^{d-1} + 1$, and so on.

By repeating this process until it is possible, we obtain a set of l distinct elements $\{a_1, \ldots, a_l\}$ and a corresponding set of families $\{\mathcal{A}_1, \ldots, \mathcal{A}_l\}$ such that for each $i \in \{1, \ldots, l\}$, $\mathcal{A}_{i-1} \supseteq \mathcal{A}_i$ (with $\mathcal{A}_0 = \mathcal{F}$), $|\mathcal{A}_i| \geqslant k^{d+1-i} + 1$, and

$$\underset{j \in \{i, i+1, \ldots, l\}}{\forall} \ \underset{A \in \mathcal{A}_j}{\forall} \ a_i \in A.$$

Suppose, for the sake of contradiction, that $l > d$. Then $|\mathcal{A}_{d+1}| \geqslant 2$, and all sets in $\mathcal{A}_{d+1}$ would contain $\{a_1, \ldots, a_{d+1}\}$, which contradicts the assumption that for any two distinct sets $A, B \in \mathcal{F}$, $|A \cap B| \leqslant d$.

At least one element from $\{a_1, \ldots, a_l\}$ must be included in the hitting set. This can be verified, as otherwise, to cover the entire family $\mathcal{A}_l$ (which has size at least $k^{d+1-l} + 1$), there would need to be an element a_{l+1} contained in at least $k^{d-l}+1$ sets from $\mathcal{A}_l$. This would contradict the fact that the process was repeated until it was possible.

If we include any $a \in \{a_1, \dots, a_l\}$ in the hitting set, every set $A \in \mathcal{A}_1$ will be covered. Therefore, we can apply the following reduction, until it is possible:

- R1: If $|\mathcal{F}| > k^{d+1}$, find $\{a_1, \dots, a_l\}$ and $\{\mathcal{A}_1, \dots, \mathcal{A}_l\}$, then replace $\mathcal{F}$ with $(\mathcal{F} \setminus \mathcal{A}_1) \cup \{\{a_1, \dots, a_l\}\}$.

Note that after applying R1, the condition that for any two distinct sets $A, B \in \mathcal{F}$, $|A \cap B| \leqslant d$ still holds, as the newly added set is a subset of one or more sets that were originally in $\mathcal{F}$.

The sets $\{a_1, \dots, a_l\}$ and $\{\mathcal{A}_1, \dots, \mathcal{A}_l\}$ can be found in polynomial time with respect to the input size. To identify a_i, we iterate through each $a \in \bigcup \mathcal{A}_{i-1}$, checking which sets in $\mathcal{A}_{i-1}$ contain it.

Applying R1 will require polynomial time overall, as each application of R1 decreases the size of $\mathcal{F}$ by at least $k^d + 1 - 1 = k^d > 0$, so R1 will be applied at most $|\mathcal{F}|$ times.

Once the size of $|\mathcal{F}|$ has been reduced to $\mathcal{O}\big(k^{d+1}\big)$, we still need to reduce the size of $\bigcup \mathcal{F}$. We apply the following reduction as long as it is possible:

- R2: If there exists a set $A \in \mathcal{F}$ such that for every $B \in \mathcal{F} \setminus \{A\}$, $A \cap B = \emptyset$, we replace $\mathcal{F}$ with $\mathcal{F} \setminus \{A\}$, k with $k-1$, and add any $a \in A$ to the hitting set, unless A is empty, in which case we reject.

Now we assume that for any two distinct sets $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$, which enables us to apply the next reduction until it is no longer possible:

- R3: If there exists an element $a \in \bigcup \mathcal{F}$ that is contained in only one set in $\mathcal{F}$, replace $\mathcal{F}$ with $\{A \setminus \{a\} \ : \ A \in \mathcal{F}\}$.

This reduction is valid, as if a were required in the hitting set, it could be replaced with any other element from the set in $\mathcal{F}$ that contains it, particularly an element included in at least two sets in $\mathcal{F}$.

After applying R2 and R3, every element in $\bigcup \mathcal{F}$ is contained in at least two sets in $\mathcal{F}$. Therefore, the size of $\bigcup \mathcal{F}$ is bounded by:

$$\Big|\bigcup \mathcal{F}\Big| \leqslant \frac{1}{2} \cdot \sum_{A,B \in \mathcal{F} \ \wedge \ A \neq B} |A \cap B| \leqslant \frac{k^{d+1}\big(k^{d+1}-1\big)d}{2} = k^{\mathcal{O}(1)},$$

which completes the proof.