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Parameterized Algorithms
Assignment 3

Problem 1

We will show an fpt-reduction from the MULTICOLORED CLIQUE problem to the CUT WITH FORBIDDEN PAIRS problem. Let (G, k), where $V(G) = (V_1, V_2, \ldots, V_k)$, be an instance of the MULTICOLORED CLIQUE problem. For each $i \in \{1, 2, \ldots, k\}$, let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$ denote the set of vertices of color i.

We construct a directed graph D as follows:

1. Start with $V(D) = \{s, t\}$ and $E(D) = \emptyset$.
2. For each $i \in \{1, 2, \ldots, k\}$:
   2.1 Add $n_i - 1$ vertices to $V(D)$, denoted as $u_{i,1}, u_{i,2}, \ldots, u_{i,n_i-1}$.
   2.2 For every $v_{i,j} \in V_i$, add an arc $a(v_{i,j})$ to $E(D)$, where $a(v_{i,j}) = (u_{i,j-1}, u_{i,j})$. Here, $u_{i,0} = s$ and $u_{i,n_i} = t$.

Let $\mathcal{S}$ be the set of all pairs of arcs in D whose corresponding vertices are not adjacent in G. Formally:

$$\mathcal{S} \;=\; \{\{a(v_{i_1,j_1}), a(v_{i_2,j_2})\} \;:\; v_{i_1,j_1}, v_{i_2,j_2} \in V(G) \;\wedge\; \{v_{i_1,j_1}, v_{i_2,j_2}\} \notin E(G)\}.$$

The parameter k remains the same as in the original problem. Note that the size of the constructed instance is polynomial in the size of the original instance.

We now prove the following equivalence:

$$(G, k) \in \text{MULTICOLORED CLIQUE} \iff (D, \mathcal{S}, k) \in \text{CUT WITH FORBIDDEN PAIRS}.$$

First, assume there exists a multicolored clique $C = \{v_{1,j_1}, v_{2,j_2}, \ldots, v_{k,j_k}\}$ in G. Define $F = \{a(v_{1,j_1}), a(v_{2,j_2}), \ldots, a(v_{k,j_k})\}$. Since there are exactly k paths from s to t in D, and $a(v_{i,j_i})$ lies on the i-th path for each i, F intersects every path from s to t. Furthermore, for any $\{a_1, a_2\} \in \mathcal{S}$, $|F \cap \{a_1, a_2\}| \leqslant 1$, as otherwise C would not be a clique in G.

Conversely, assume there exists a set $F \subseteq E(D)$ of size k such that F intersects every path from s to t, and $|F \cap \{a_1, a_2\}| \leqslant 1$ for any $\{a_1, a_2\} \in \mathcal{S}$. Let $C = a^{-1}(F)$. Since F intersects every path from s to t, C contains exactly one vertex from each V_i. Additionally, for any two vertices $v_{i_1,j_{i_1}}, v_{i_2,j_{i_2}} \in C$, they are adjacent in G, as otherwise $\{a(v_{i_1,j_{i_1}}), a(v_{i_2,j_{i_2}})\}$ would belong to $\mathcal{S}$. Hence, C is a multicolored clique in G.

From the above, we conclude that the CUT WITH FORBIDDEN PAIRS problem is $\mathsf{W}[1]$-hard when parameterized by k.

By the corollary 14.23 from the Platypus Book, we know that there is no $f(k) \cdot n^{o(k)}$-time algorithm for the MULTICOLORED CLIQUE problem, for any computable function f, assuming the Exponential Time Hypothesis (ETH). By the observation 14.22 from the same book, this implies that no $f(k) \cdot n^{o(k)}$-time algorithm exists for the CUT WITH FORBIDDEN PAIRS problem under ETH, which completes the proof.