Dominik Wawszczak

Student ID Number: 440014

Group Number: 1

Problem 1

We will show an fpt-reduction from the MULTICOLORED CLIQUE problem to the CUT WITH FORBIDDEN PAIRS problem. Let (G, k), where $V(G) = (V_1, V_2, \ldots, V_k)$, be an instance of the MULTICOLORED CLIQUE problem. For each $i \in \{1, 2, \ldots, k\}$, let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$ denote the set of vertices of color i.

We construct a directed graph D as follows:

1. Start with $V(D) = \{s, t\}$ and $E(D) = \emptyset$.

2. For each $i \in \{1, 2, \ldots, k\}$:

   2.1 Add $n_i - 1$ vertices to $V(D)$, denoted as $u_{i,1}, u_{i,2}, \ldots, u_{i,n_i-1}$.

   2.2 For every $v_{i,j} \in V_i$, add an arc $a(v_{i,j})$ to $E(D)$, where $a(v_{i,j}) = (u_{i,j-1}, u_{i,j})$. Here, $u_{i,0} = s$ and $u_{i,n_i} = t$.

Let $\mathcal{S}$ be the set of all pairs of arcs in D whose corresponding vertices are not adjacent in G. Formally:

$$\mathcal{S} \;=\; \{\{a(v_{i_1,j_1}), a(v_{i_2,j_2})\} \;:\; v_{i_1,j_1}, v_{i_2,j_2} \in V(G) \;\wedge\; \{v_{i_1,j_1}, v_{i_2,j_2}\} \notin E(G)\}.$$

The parameter k remains the same as in the original problem. Note that the size of the constructed instance is polynomial in the size of the original instance.

We now prove the following equivalence:

$$(G, k) \in \text{MULTICOLORED CLIQUE} \iff (D, \mathcal{S}, k) \in \text{CUT WITH FORBIDDEN PAIRS}.$$

First, assume there exists a multicolored clique $C = \{v_{1,j_1}, v_{2,j_2}, \ldots, v_{k,j_k}\}$ in G. Define $F = \{a(v_{1,j_1}), a(v_{2,j_2}), \ldots, a(v_{k,j_k})\}$. Since there are exactly k paths from s to t in D, and $a(v_{i,j_i})$ lies on the i-th path for each i, F intersects every path from s to t. Furthermore, for any $\{a_1, a_2\} \in \mathcal{S}$, $|F \cap \{a_1, a_2\}| \leqslant 1$, as otherwise C would not be a clique in G.

Conversely, assume there exists a set $F \subseteq E(D)$ of size k such that F intersects every path from s to t, and $|F \cap \{a_1, a_2\}| \leqslant 1$ for any $\{a_1, a_2\} \in \mathcal{S}$. Let $C = a^{-1}(F)$. Since F intersects every path from s to t, C contains exactly one vertex from each V_i. Additionally, for any two vertices $v_{i_1,j_{i_1}}, v_{i_2,j_{i_2}} \in C$, they are adjacent in G, as otherwise $\{a(v_{i_1,j_{i_1}}), a(v_{i_2,j_{i_2}})\}$ would belong to $\mathcal{S}$. Hence, C is a multicolored clique in G.

From the above, we conclude that the CUT WITH FORBIDDEN PAIRS problem is W[1]-hard when parameterized by k.

By the corollary 14.23 from the Platypus Book, we know that there is no $f(k) \cdot n^{o(k)}$-time algorithm for the MULTICOLORED CLIQUE problem, for any computable function f, assuming the Exponential Time Hypothesis (ETH). By the observation 14.22 from the same book, this implies that no $f(k) \cdot n^{o(k)}$-time algorithm exists for the CUT WITH FORBIDDEN PAIRS problem under ETH, which completes the proof.

Problem 2

Let (D, s, t, k) be an instance of the DOUBLE CUT problem and let F be an optimal solution to this problem, meaning one with the smallest possible size. Denote P as the set of all the paths in D from s to t. Define:

$$X \;=\; \left\{ a \in F \;:\; \underset{p \in P}{\forall} \; a \text{ is not the last arc on } p \text{ that belongs to } F \right\}.$$

and

$$Y \;=\; \left\{ a \in F \;:\; \underset{p \in P}{\exists} \; a \text{ is the last arc on } p \text{ that belongs to } F \right\}$$

With these definitions, $F = X \cup Y$ and $X \cap Y = \emptyset$.

<u>Lemma 1</u> For any $p \in P$, $|p \cap X| \geqslant 1$.

<u>Proof of Lemma 1</u> Suppose, for the sake of contradiction, that there exists a path $p \in P$ such that $p \cap X = \emptyset$. Then, $p \cap F = p \cap Y$. Let a be the first arc on p that belongs to Y. Since $a \in Y$, there exists another path $p' \in P$ where a is the last arc belonging to F. By merging the prefix of p ending with a with the suffix of p' starting after a, we construct another path from s to t that contains only one arc from F. This contradiction completes the proof of Lemma 1.

<u>Lemma 2</u> For some optimal solution F, Y is an important (s, t)-separator in D.

<u>Proof of Lemma 2</u> If there exists a path $p \in P$ that does not pass through Y, then $p \cap F = p \cap X$. Consequently, the last arc on p belonging to F would also be in X, leading to a contradiction. Hence, Y is an (s, t)-separator in D.

Let $R_s(Y)$ denote the set of vertices reachable from s in $D \setminus Y$. Suppose there exists an (s, t)-separator Y' such that $|Y| \geqslant |Y'|$ and $R_s(Y) \subseteq R_s(Y')$, where $R_s(Y')$ is defined analogously. We claim that $F' = X \cup Y'$ is also a valid solution.

If there exists a path $p \in P$ containing at most one arc from F', that arc must belong to Y' because Y' is an (s, t)-separator. By Lemma 1, this arc belongs to X as well. Since F is a solution, p also contains another arc $a = (u, v) \in Y$, where $u \in R_s(Y)$. This implies $u \in R_s(Y')$, meaning there exists a path p' from s to u that avoids Y'. By merging the prefix of p' ending at u with the suffix of p starting at u, we obtain a path from s to t that contains no edges from Y'. This contradicts the assumption that Y' is an (s, t)-separator, proving that F' is another solution of size no greater than F. This concludes the proof of Lemma 2.

To find an optimal set X for a given Y, we can merge all vertices with outgoing arcs in Y into a single vertex and then run a maximum flow algorithm from s to this new vertex. The correctness of this approach follows from Lemma 1, and its time complexity is polynomial in n, where n is the number of vertices in D.

The final algorithm proceeds as follows:

1. Iterate over all possible important separators Y of size at most k.

2. For each Y, compute the optimal set X.

3. If $|X| + |Y| \leqslant k$, return `true`.

4. If no solution is found, return `false`.

The time complexity of this algorithm is $4^k \cdot n^{\mathcal{O}(1)}$, as important separators can be enumerated in time $4^k \cdot n^{\mathcal{O}(1)}$. This is fixed-parameter tractable when parameterized by k, concluding the proof.