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Problem 1

We define an ω-word u as *universal* if, and only if, for any $i \in \mathbb{N}$, there exists a $j > i$ such that $u[i] \neq u[j]$. Intuitively, this means that u contains an infinite subsequence of the form $(ab)^\infty$. Note that if $u \in \Sigma^\omega$ is universal, then for any $v \in \Sigma^\omega$, it holds that $v \sqsubseteq u$. This is because we can remove finitely or infinitely many letters from u to obtain $(ab)^\infty$, and for each i-th pair of consecutive ab occurrences, we can remove either a or b, depending on the value of $v[i]$.

If an ω-word u is not universal, then either $\#_a(u)$ or $\#_b(u)$ is finite. In the first case, u is of the form vb^∞, and we will refer to such an ω-word as *a-long*. In the second case, u is of the form va^∞, and we will call it *b-long*. In both cases, v is a word of finite length.

Let $u_1, u_2, \ldots$ be any infinite sequence of ω-words. We will prove that there exist indices $i < j$ such that $u_i \sqsubseteq u_j$. If there are at least two universal ω-words in this sequence, the proof is straightforward since u_1 can embed into the second universal word in the sequence. Otherwise, either there are infinitely many a-long words in the sequence, or there are infinitely many b-long words. Without loss of generality, assume there are infinitely many a-long words in the sequence. Let $k_1, k_2, \ldots$ be the indices of the a-long words. Denote $u_{k_l} = u'_{k_l} a^\infty$, where u'_{k_l} is finite. By Higman's lemma, there exist indices $i < j$ such that u'_{k_i} is a substring of u'_{k_j}, concluding the proof as $u_{k_i} \sqsubseteq u_{k_j}$.

Now, we will show that in the variant where only finitely many letters can be removed, the resulting relation is not a well-quasi-order. Consider the infinite sequence of ω-words $u_1, u_2, \ldots$ such that

$$u_i \;=\; a^i b a^{i+1} b a^{i+2} b \ldots$$

Clearly, for any $i \in \mathbb{N}^+$, we have $u_i \sqsupseteq u_{i+1}$ because the first $i+1$ letters of u_i can be removed to obtain u_{i+1}.

Suppose, for the sake of contradiction, that there exists $i \in \mathbb{N}^+$ such that $u_i \sqsubseteq u_{i+1}$. Let k be the smallest number such that a non expandable block of the form $a^k b$ remains intact, after removing finitely many letters from u_{i+1} to obtain u_i. The number of letters b in u_{i+1} before this block equals $k - i - 1$. However, the number of letters b in u_i before this block equals $k - i > k - i - 1$, which is a contradiction, as letters can only be removed.

Suppose, for the sake of contradiction, that there exists $i \in \mathbb{N}^+$ such that $u_i \sqsubseteq u_{i+1}$. Let k be the smallest number such that a non-expandable block of the form $a^k b$ remains intact after removing finitely many letters from u_{i+1} to obtain u_i. The number of b letters in u_{i+1} before this block is $k - i - 1$. However, the number of b letters in u_i before this block is $k - i$, which is greater than $k - i - 1$, contradicting the fact that letters can only be removed.

From the above, we conclude that $u_i \sqsupset u_{i+1}$ for any $i \in \mathbb{N}^+$. Therefore, $u_1, u_2, \ldots$ forms an infinite sequence of strictly decreasing elements, which shows that the relation is not a well-quasi-order.

Problem 2

For any set $A \subseteq \{1, \dots, d\}$, define a function $f_A : \mathbb{N} \to \mathbb{N}^d$ as follows:

$$f_A(n) \;=\; ([1 \in A] \cdot n, \dots, [d \in A] \cdot n),$$

where $[P]$ denotes the Iverson bracket, i.e., $[P] = 1$ if P is true, and $[P] = 0$ otherwise. A set A is called *good* if and only if $f_A(n) \in X$ for every $n \in \mathbb{N}$. Note that $\emptyset$ is good unless X is empty, which is a trivial case.

For any good set $A \subseteq \{1, \dots, d\}$, let $g(A)$ denote the set of tuples $(a_1, \dots, a_d)$ satisfying the following conditions:

1. $a_i = 0$, if $i \in A$;
2. $\underset{n \in \mathbb{N}}{\forall}\ (a_1, \dots, a_d) + f_A(n) \in X$;
3. $\underset{n_0 \in \mathbb{N}}{\exists}\ \underset{n \geqslant n_0}{\forall}\ \underset{i \in \{1,\dots,d\} \setminus A}{\forall}\ (a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_d) + f_A(n) \notin X,$

where $(a_1, \dots, a_d) + (b_1, \dots, b_d) = (a_1 + b_1, \dots, a_d + b_d)$, for any tuples $(a_1, \dots, a_d)$ and $(b_1, \dots, b_d)$ in $\mathbb{N}^d$.

Suppose, for the sake of contradiction, that $g(A)$ is infinite. Let

$$g(A) \;=\; \big\{ (a_1^1, \dots, a_d^1), (a_1^2, \dots, a_d^2), \dots \big\}$$

By Dickson's lemma, the structure $(\mathbb{N}^d, \leqslant)$ is a well-quasi-order. Therefore, there exist indices $i, j \in \mathbb{N}$ such that $i < j$ and $(a_1^i, \dots, a_d^i) \leqslant (a_1^j, \dots, a_d^j)$, contradicting the third condition for tuples in $g(A)$.

Define $\mathrm{down}(g(A))$ as the set of all tuples $(b_1, \dots, b_d) \in \mathbb{N}^d$ such that there exists a tuple $(a_1, \dots, a_d) \in g(A)$ with $(a_1, \dots, a_d) \geqslant (b_1, \dots, b_d)$. If $g(A)$ is empty, let $\mathrm{down}(g(A))$ be the singleton containing the zero tuple.

The set $\mathrm{down}(g(A))$ is finite, with the following upper bound on its size:

$$|\mathrm{down}(g(A))| \;\leqslant\; \sum_{(a_1,\dots,a_d) \in g(A)} \prod_{i=1}^{d} (a_i + 1).$$

Using this, define a semilinear set

$$Y \;=\; \bigcup_{A \subseteq \{1,\dots,d\} \;\wedge\; A \text{ is good}} \left(\mathrm{down}(g(A)) + \left(\bigcup_{i \in A} ([1 = i], \dots, [d = i]) \right)^{*} \right).$$

By definition of $g(A)$, we have $Y \subseteq X$. Our goal is to show that $Y = X$.

Suppose, for the sake of contradiction, that there exists a tuple $(a_1, \dots, a_d) \in X \setminus Y$. We will construct a tuple $(b_1, \dots, b_d) \in X$ such that $(b_1, \dots, b_d) \geqslant (a_1, \dots, a_d)$. Since Y is downward closed, this will imply $(b_1, \dots, b_d) \in X \setminus Y$.

Proceed through indices $i = 1, \dots, d$, choosing b_i iteratively while maintaining a set A, initially empty. Assume $b_1, \dots, b_{i-1}$ have been chosen. For all $j \in \{1, \dots, i-1\}$, define $b'_j = b_j$ if $j \in A$, and $b'_j = 0$ otherwise. If there exists a number $c_i \in \mathbb{N}$ such that

- $\underset{n \in \mathbb{N}}{\forall}\ (b'_1, \dots, b'_{i-1}, c_i, a_{i+1}, \dots, a_d) + f_A(n) \in X$, and

- $\underset{n_0 \in \mathbb{N}}{\exists} \; \underset{n \geqslant n_0}{\forall} \; (b'_1, \ldots, b'_{i-1}, c_i + 1, a_{i+1}, \ldots, a_d) + f_A(n) \notin X$,

set $b_i = c_i$. Otherwise, let $b_i = a_i$ and add i to A. Note that $b_i \geqslant a_i$.

By construction, $(b_1, \ldots, b_d) \in X$, and A is good. Furthermore,

$$\underset{n_0 \in \mathbb{N}}{\exists} \; \underset{n \geqslant n_0}{\forall} \; \underset{i \in \{1,\ldots,d\} \setminus A}{\forall} \; (b_1, \ldots, b_{i-1}, b_i + 1, b_{i+1}, \ldots, b_d) + f_A(n) \notin X.$$

Thus,

$$(b_1, \ldots, b_d) \;\in\; \mathrm{down}(g(A)) + \left(\bigcup_{i \in A} ([1 = i], \ldots, [d = i]) \right)^* .$$

This contradiction completes the proof that X is a semilinear set.