Languages, automata and computation II
Homework 2

Problems: deadline 10/01/2025

Problem 1. A *parametric finite weighted automaton* is a finite weighted automaton where the initial vector, the final vector, and the transition matrices instead of using values from the field $\mathbb{Q}$, use polynomials with one variable x. For every choice of $x \in \mathbb{Q}$, this gives us a finite weighted automaton in the usual sense, by evaluating the polynomials to get values in the field. Show that the set

$$\{x \in \mathbb{Q} \mid \text{the automaton with parameter } x \text{ has the zero semantics}\}$$

is either finite or equal to $\mathbb{Q}$. (Hint: consider short words)

Problem 2. Show that the following problem is decidable:

1. **Input.** A weighted automaton, which defines a function $f : \Sigma^* \to \mathbb{Q}$;
2. **Question.** Is the function f commutative, i.e. $f(w)$ does not depend on the order of the letters in the input word w.

Problem 3. Consider a finite field $\mathbb{F}$. Show that weighted automata and polynomial automata compute the same functions of type $\Sigma^* \to \mathbb{F}$.

Star problems

The deadline for these problems is until the last week of the lectures.

(*) Problem 4. Show that for every n there exists a finite undirected graph G with the following properties:

1. For every vertices v and w, there is an automorphism of G that maps v and w. Recall that an automorphism of a graph is a permutation of its vertices that preserves the edges.
2. Every graph with at most n vertices is an induced subgraph of G.

Observe that the infinite random graph has the two properties above; but we are looking for a finite one.

(*) Problem 5. Consider a vector addition system with states with a distinguished initial and final configuration. Define a *coverability run* to be a run which begins in the initial state and ends in a configuration that is coordinate-wise greater or equal to the final configuration (and the state is the same). If we equip the system with a function f from transitions to some finite alphabet Σ, then we get a language

$$\{f(w) \in \Sigma^* \mid w \text{ is a coverability run}\}.$$

Such a language is called a *coverability language*. Show that

$$L \text{ is regular} \quad \Leftrightarrow \quad \text{both } L \text{ and its complement are coverability languages}$$

Open problems

The deadline for these problems is until the last week of the lectures.

Open problem 1. (We do not know the solution to this problem, could be easy, could be hard.) For a polynomial automaton, which computes a function $f : \Sigma^* \to \mathbb{Q}$, consider the function

$$n \in \{0, 1, \ldots\} \quad \mapsto \quad \sum_{w \in \Sigma^n} f(w).$$

Prove or disprove: The new function above is also computed by a polynomial automaton, assuming that numbers are represented as words over a one-letter alphabet.