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Languages, Automata and Computation II
Assignment 1

Problem 1

We define an ω-word u as *universal* if, and only if, for any $i \in \mathbb{N}$, there exists a $j > i$ such that $u[i] \neq u[j]$. Intuitively, this means that u contains an infinite subsequence of the form $(ab)^\infty$. Note that if $u \in \Sigma^\omega$ is universal, then for any $v \in \Sigma^\omega$, it holds that $v \sqsubseteq u$. This is because we can remove finitely or infinitely many letters from u to obtain $(ab)^\infty$, and for each i-th pair of consecutive ab occurrences, we can remove either a or b, depending on the value of $v[i]$.

If an ω-word u is not universal, then either $\#_a(u)$ or $\#_b(u)$ is finite. In the first case, u is of the form vb^∞, and we will refer to such an ω-word as *a-long*. In the second case, u is of the form va^∞, and we will call it *b-long*. In both cases, v is a word of finite length.

Let $u_1, u_2, \ldots$ be any infinite sequence of ω-words. We will prove that there exist indices $i < j$ such that $u_i \sqsubseteq u_j$. If there are at least two universal ω-words in this sequence, the proof is straightforward since u_1 can embed into the second universal word in the sequence. Otherwise, either there are infinitely many a-long words in the sequence, or there are infinitely many b-long words. Without loss of generality, assume there are infinitely many a-long words in the sequence. Let $k_1, k_2, \ldots$ be the indices of the a-long words. Denote $u_{k_l} = u'_{k_l} a^\infty$, where u'_{k_l} is finite. By Higman's lemma, there exist indices $i < j$ such that u'_{k_i} is a substring of u'_{k_j}, concluding the proof as $u_{k_i} \sqsubseteq u_{k_j}$.

Now, we will show that in the variant where only finitely many letters can be removed, the resulting relation is not a well-quasi-order. Consider the infinite sequence of ω-words $u_1, u_2, \ldots$ such that

$$u_i \;=\; a^i b a^{i+1} b a^{i+2} b \ldots$$

Clearly, for any $i \in \mathbb{N}^+$, we have $u_i \sqsupseteq u_{i+1}$ because the first $i+1$ letters of u_i can be removed to obtain u_{i+1}.

Suppose, for the sake of contradiction, that there exists $i \in \mathbb{N}^+$ such that $u_i \sqsubseteq u_{i+1}$. Let k be the smallest number such that a non expandable block of the form $a^k b$ remains intact, after removing finitely many letters from u_{i+1} to obtain u_i. The number of letters b in u_{i+1} before this block equals $k - i - 1$. However, the number of letters b in u_i before this block equals $k - i > k - i - 1$, which is a contradiction, as letters can only be removed.

Suppose, for the sake of contradiction, that there exists $i \in \mathbb{N}^+$ such that $u_i \sqsubseteq u_{i+1}$. Let k be the smallest number such that a non-expandable block of the form $a^k b$ remains intact after removing finitely many letters from u_{i+1} to obtain u_i. The number of b letters in u_{i+1} before this block is $k - i - 1$. However, the number of b letters in u_i before this block is $k - i$, which is greater than $k - i - 1$, contradicting the fact that letters can only be removed.

From the above, we conclude that $u_i \sqsupset u_{i+1}$ for any $i \in \mathbb{N}^+$. Therefore, $u_1, u_2, \ldots$ forms an infinite sequence of strictly decreasing elements, which shows that the relation is not a well-quasi-order.

Problem 2

Consider the structure $\big((\mathbb{N} \cup \{\omega\})^d, \leqslant\big)$, where $(x_1, \ldots, x_d) \leqslant (y_1, \ldots, y_d)$ if and only if, for each $i \in \{1, \ldots, d\}$, one of the following holds:

$$x_i = y_i = \omega \qquad \text{or} \qquad x_i \in \mathbb{N} \;\wedge\; y_i = \omega \qquad \text{or} \qquad x_i, y_i \in \mathbb{N} \;\wedge\; x_i \leqslant y_i.$$

By Dickson's lemma, this structure is a well-quasi-order because $(\mathbb{N} \cup \{\omega\}, \leqslant)$ is itself a well-quasi-order.

For any tuple $(x_1, \ldots, x_d) \in (\mathbb{N} \cup \{\omega\})^d$, we say $(x_1, \ldots, x_d) \in X$ if and only if, for every $n \in \mathbb{N}$, the following condition holds:

$$([x_1 = \omega] \cdot n + [x_1 \neq \omega] \cdot x_1, \ldots, [x_d = \omega] \cdot n + [x_d \neq \omega] \cdot x_1) \ \in \ X,$$

where $[P]$ denotes the Iverson bracket, i.e., $[P] = 1$ if P is true, and $[P] = 0$ otherwise.

A subset $A \subseteq \{1, \ldots, d\}$ is called *good* if and only if $([1 \in A] \cdot \omega, \ldots, [d \in A] \cdot \omega) \in X$. For any good set A, define $g(A)$ as the set of tuples $(x_1, \ldots, x_d) \in (\mathbb{N} \cup \{\omega\})^d$ satisfying:

1. $x_i = \omega$, for all $i \in A$;
2. $(x_1, \ldots, x_d) \in X$;
3. $(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_d) \notin X$, for all $i \in \{1, \ldots, d\} \setminus A$.

Suppose, for the sake of contradiction, that $g(A)$ is infinite. Let

$$g(A) \ = \ \{(x_1^1, \ldots, x_d^1), (x_1^2, \ldots, x_d^2), \ldots\}.$$

Since $((\mathbb{N} \cup \{\omega\})^d, \leqslant)$ is a well-quasi-order, there exist indices $i, j \in \mathbb{Z}^+$ such that $i < j$ and $(x_1^i, \ldots, x_d^i) \leqslant (x_1^j, \ldots, x_d^j)$, contradicting the third condition for tuples in $g(A)$.

Define $\mathrm{down}(g(A))$ as the set of all tuples $(y_1, \ldots, y_d) \in \mathbb{N}^d$ for which there exists a tuple $(x_1, \ldots, x_d) \in g(A)$ such that $(x_1, \ldots, x_d) \geqslant (y_1, \ldots, y_d)$ and $y_i = 0$ for all $i \in A$. If $g(A)$ is empty, let $\mathrm{down}(g(A))$ be the singleton containing the zero tuple. The set $\mathrm{down}(g(A))$ is finite, with an upper bound on its size:

$$|\mathrm{down}(g(A))| \ \leqslant \ \sum_{(x_1, \ldots, x_d) \in g(A)} \ \prod_{i \in \{1, \ldots, d\} \setminus A} (x_i + 1).$$

Using this, define a semilinear set

$$Y \ = \ \bigcup_{A \subseteq \{1, \ldots, d\} \ \wedge \ A \text{ is good}} \left(\mathrm{down}(g(A)) + \left(\bigcup_{i \in A} ([1 = i], \ldots, [d = i]) \right)^* \right).$$

By definition of $g(A)$, we have $Y \subseteq X$. Our goal is to prove that $Y = X$.

Suppose, for the sake of contradiction, that there exists a tuple $(x_1, \ldots, x_d) \in X \setminus Y$. We will construct a tuple $(y_1, \ldots, y_d) \in \mathbb{N}^d$ such that $(x_1, \ldots, x_d) \leqslant (y_1, \ldots, y_d)$ and $(y_1, \ldots, y_d) \in X$. Since Y is downward closed, this will imply $(y_1, \ldots, y_d) \in X \setminus Y$.

Proceed through indices $i = 1, \ldots, d$, choosing y_i iteratively while maintaining a set A, initially empty. Assume $y_1, \ldots, y_{i-1}$ have been chosen. For all $j \in \{1, \ldots, i-1\}$, define $y'_j = \omega$ if $j \in A$, and $y'_j = y_j$ otherwise. If there exists a number $z_i \in \mathbb{N}$ such that

$$(y'_1, \ldots, y'_{i-1}, z_i, x_{i+1}, \ldots, x_d) \ \in \ X \qquad \text{and} \qquad (y'_1, \ldots, y'_{i-1}, z_i + 1, x_{i+1}, \ldots, x_d) \ \notin \ X,$$

set $y_i = z_i$. Otherwise, let $y_i = x_i$ and add i to A. Note that $y_i \geqslant x_i$.

By construction, $(y_1, \ldots, y_d) \in X$, and A is good. Furthermore,

$$\forall_{i \in \{1, \ldots, d\} \setminus A} \ (y'_1, \ldots, y'_{i-1}, y_i + 1, y'_{i+1}, \ldots, y'_d) \notin X,$$

where y'_j is as defined earlier. Thus,

$$(y_1, \ldots, y_d) \ \in \ \text{down}(g(A)) + \left(\bigcup_{i \in A} ([1 = i], \ldots, [d = i]) \right)^* .$$

This contradiction completes the proof that X is a semilinear set.

Now we proceed to the second part of the problem. Let $T \subseteq \mathbb{Z}^d$ be the set of available transitions. Define X as the downward closure of configurations reachable from x.

Consider a directed graph G where $V(G) = (\mathbb{N} \cup \{\omega\})^d$ and $E(G)$ contains pairs $((y_1, \ldots, y_d), (z_1, \ldots, z_d))$ such that there exists a transition $(t_1, \ldots, t_d) \in T$ satisfying:

$$y_i = z_i = \omega \qquad \text{or} \qquad y_i, z_i \in \mathbb{N} \ \wedge \ y_i + t_i = z_i,$$

for each $i \in \{1, \ldots, d\}$. Run a breadth-first search (BFS) on this graph, starting at vertex x.

Whenever a vertex $(y_1, \ldots, y_d)$ is encountered such that there exists an ancestor $(z_1, \ldots, z_d)$ in the BFS-tree with $(z_1, \ldots, z_d) \leqslant (y_1, \ldots, y_d)$, perform the following:

1. For each $i \in \{1, \ldots, d\}$, set $y'_i = \omega$ if $y_i > z_i$, or $y'_i = y_i$ otherwise;
2. If $(y'_1, \ldots, y'_d)$ has not been visited, add it to the BFS queue and mark it as visited;

Note that $(y'_1, \ldots, y'_d) \geqslant (y_1, \ldots, y_d)$, so we do not need to process the neighbors of $(y_1, \ldots, y_d)$. We do this because if $(y_1, \ldots, y_d)$ and $(z_1, \ldots, z_d)$ belong to X, then $(y'_1, \ldots, y'_d)$ also belongs to X.

Suppose, for the sake of contradiction, that the BFS does not terminate. Then there exists an infinite sequence of vertices:

$$Y_0 \ = \ \left((y^1_1, \ldots, y^1_d), (y^2_1, \ldots, y^2_d), \ldots, \right)$$

where each vertex $(y^i_1, \ldots, y^i_d)$ is the parent of the next. Since $\left((\mathbb{N} \cup \{\omega\})^d, \leqslant \right)$ is a well-quasi-order, there exist indices $i, j \in \mathbb{Z}^+$ with $i < j$ such that $\left(y^i_1, \ldots, y^i_d \right) \leqslant \left(y^j_1, \ldots, y^j_d \right)$. Now consider the sequence:

$$Y_1 \ = \ \left((y^j_1, \ldots, y^j_d), (y^{j+1}_1, \ldots, y^{j+1}_d), \ldots, \right).$$

In this sequence, each tuple has at least one element equal to ω. Repeating this argument produces sequences $Y_2, Y_3, \ldots$, where each tuple in Y_k contains at least k elements equal to ω. The existence of Y_{d+1} leads to a contradiction. Therefore, the BFS terminates

Let S be the set of visited vertices. The answer is:

$$X \ = \ \bigcup_{y \in S} \left(\text{down}(\{y\}) + \left(\bigcup_{i \in \{1, \ldots, d\} \ \wedge \ y_i = \omega} ([1 = i], \ldots, [d = i]) \right)^* \right).$$

The algorithm is correct because for every y coverable by x, there exists $y' \in S$ such that $y' \geqslant y$.