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Languages, Automata and Computation II
Assignment 3

Problem 1

This problem was solved in collaboration with Kacper Bal and Mateusz Mroczka.

We say that a language $L \in \mathbb{A}^*$ is a *good*, if it satisfies the condition from the statement, i.e.

$$\underset{w \in \mathbb{A}^*}{\forall} \; \underset{\sigma : \mathbb{A} \to \mathbb{A}}{\forall} \; (w \in L \iff \sigma(w) \in L).$$

Let $A \in \mathbb{A}$ be an arbitrary element. Define σ_A as the function constantly equal to A. For any good language $L \in \mathbb{A}^*$, it must hold that

$$\underset{w \in \mathbb{A}^*}{\forall} \; (w \in L \iff \sigma_A(w) \in L),$$

since the order of quantifiers does not matter. This is equivalent to

$$\underset{w \in \mathbb{A}^*}{\forall} \; \big(w \in L \iff A^{|w|} \in L\big),$$

which implies that for every $n \in \mathbb{N}$, L either contains all words of length n or none of them.

The problem of determining whether a language $L \in \mathbb{A}^*$ is good is semi-decidable, since we can iterate through register automata that use zero registers, comparing each one with the automaton from the input.

The complement of this problem is also semi-decidable, as we can iterate over words of length n, for $n \in \mathbb{N}$, considering only a subset of n symbols from $\mathbb{A}$. For each such word, we can check all the possible functions σ limited to this subset and verify whether the condition holds.

Since both the problem and its complement are semi-decidable, the problem is decidable, which completes the proof.