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Languages, Automata and Computation II
Assignment 3

Problem 1

This problem was solved in collaboration with Kacper Bal and Mateusz Mroczka.

We say that a language $L \in \mathbb{A}^*$ is a *good*, if it satisfies the condition from the statement, i.e.

$$\underset{w \in \mathbb{A}^*}{\forall} \ \underset{\sigma : \mathbb{A} \to \mathbb{A}}{\forall} \ (w \in L \iff \sigma(w) \in L).$$

Let $A \in \mathbb{A}$ be an arbitrary element. Define σ_A as the function constantly equal to A. For any good language $L \in \mathbb{A}^*$, it must hold that

$$\underset{w \in \mathbb{A}^*}{\forall} \ (w \in L \iff \sigma_A(w) \in L),$$

since the order of quantifiers does not matter. This is equivalent to

$$\underset{w \in \mathbb{A}^*}{\forall} \ \big(w \in L \iff A^{|w|} \in L\big),$$

which implies that for every $n \in \mathbb{N}$, L either contains all words of length n or none of them.

The problem of determining whether a language $L \in \mathbb{A}^*$ is good is semi-decidable, since we can iterate through register automata that use zero registers, comparing each one with the automaton from the input.

The complement of this problem is also semi-decidable, as we can iterate over words of length n, for $n \in \mathbb{N}$, considering only a subset of n symbols from $\mathbb{A}$. For each such word, we can check all the possible functions σ limited to this subset and verify whether the condition holds.

Since both the problem and its complement are semi-decidable, the problem is decidable, which completes the proof.

Problem 2

This problem was solved in collaboration with Kacper Bal and Mateusz Mroczka.

1. Let $L = \{b^*a\}^\omega$, and $w = b^\omega$. Clearly, $w \notin L$, but any finite prefix of w can be extended to a word in L by concatenating it with a^ω. Therefore, L is ω-regular, but not closed, which concludes the proof.

2. Let $w = a^1b^1a^2b^2a^3b^3\ldots$, and $L = \{w\}$. As shown in the tutorial, L is not ω-regular. To prove that it is closed, consider any $v \in \Sigma^\omega$ different from w. Let $n \in \mathbb{N}$ be the smallest index at which w and v differ. Then, the prefix of v of length $n+1$ (assuming 0-indexing) cannot be extended to an ω-word in L. Additionally, any finite prefix of w can be extended to w. Hence, L is closed, which completes the proof.

3. We can construct a parity automaton that is equivalent to the Büchi automaton from the input, as discussed in the lecture. Let $\mathcal{A}$ be this parity automaton, and let A be the set of states from which there exists an accepting run. For each state q, we can verify whether $q \in A$ by setting the initial state to q and checking for nonemptiness.

   Let $\mathcal{A}'$ be the automaton obtained from $\mathcal{A}$ by keeping only the states in A, and increasing every weight by one. By checking the nonemptiness of $\mathcal{A}'$, we can determine whether there exists a run in $\mathcal{A}$ that visits only the states from A but is not accepting. If such a run exists, the language $L(\mathcal{A})$ is not closed, as there exists an ω-word corresponding to this run, whose every finite prefix can be extended to an ω-word in $L(\mathcal{A})$, but which is not in the language itself. If no such run exists, the language is closed, which concludes the proof.