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Languages, Automata and Computation II
Assignment 1

Problem 1

We define an ω-word u as *universal* if, and only if, for any $i \in \mathbb{N}$, there exists a $j > i$ such that $u[i] \neq u[j]$. Intuitively, this means that u contains an infinite subsequence of the form $(ab)^\infty$. Note that if $u \in \Sigma^\omega$ is universal, then for any $v \in \Sigma^\omega$, it holds that $v \sqsubseteq u$. This is because we can remove finitely or infinitely many letters from u to obtain $(ab)^\infty$, and for each i-th pair of consecutive ab occurrences, we can remove either a or b, depending on the value of $v[i]$.

If an ω-word u is not universal, then either $\#_a(u)$ or $\#_b(u)$ is finite. In the first case, u is of the form vb^∞, and we will refer to such an ω-word as *a-long*. In the second case, u is of the form va^∞, and we will call it *b-long*. In both cases, v is a word of finite length.

Let $u_1, u_2, \ldots$ be any infinite sequence of ω-words. We will prove that there exist indices $i < j$ such that $u_i \sqsubseteq u_j$. If there are at least two universal ω-words in this sequence, the proof is straightforward since u_1 can embed into the second universal word in the sequence. Otherwise, either there are infinitely many a-long words in the sequence, or there are infinitely many b-long words. Without loss of generality, assume there are infinitely many a-long words in the sequence. Let $k_1, k_2, \ldots$ be the indices of the a-long words. Denote $u_{k_l} = u'_{k_l} a^\infty$, where u'_{k_l} is finite. By Higman's lemma, there exist indices $i < j$ such that u'_{k_i} is a substring of u'_{k_j}, concluding the proof as $u_{k_i} \sqsubseteq u_{k_j}$.

Now, we will show that in the variant where only finitely many letters can be removed, the resulting relation is not a well-quasi order. Consider the infinite sequence of ω-words $u_1, u_2, \ldots$ such that

$$u_i \; = \; a^i b a^{i+1} b a^{i+2} b \ldots$$

Clearly, for any $i \in \mathbb{N}^+$, we have $u_i \sqsupseteq u_{i+1}$ because the first $i+1$ letters of u_i can be removed to obtain u_{i+1}.

Suppose, for the sake of contradiction, that there exists $i \in \mathbb{N}^+$ such that $u_i \sqsubseteq u_{i+1}$. Let k be the smallest number such that a non expandable block of the form $a^k b$ remains intact, after removing finitely many letters from u_{i+1} to obtain u_i. The number of letters b in u_{i+1} before this block equals $k - i - 1$. However, the number of letters b in u_i before this block equals $k - i > k - i - 1$, which is a contradiction, as letters can only be removed.
Suppose, for the sake of contradiction, that there exists $i \in \mathbb{N}^+$ such that $u_i \sqsubseteq u_{i+1}$. Let k be the smallest number such that a non-expandable block of the form $a^k b$ remains intact after removing finitely many letters from u_{i+1} to obtain u_i. The number of b letters in u_{i+1} before this block is $k - i - 1$. However, the number of b letters in u_i before this block is $k - i$, which is greater than $k - i - 1$, contradicting the fact that letters can only be removed.

From the above, we conclude that $u_i \sqsupset u_{i+1}$ for any $i \in \mathbb{N}^+$. Therefore, $u_1, u_2, \ldots$ forms an infinite sequence of strictly decreasing elements, which shows that the relation is not a well-quasi order.