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Languages, Automata and Computation II
Assignment 2

Problem 1

For any $w \in \Sigma^*$, let $P_w(x)$ denote the polynomial obtained by feeding w into the automaton. By definition, this function is a polynomial in one variable x. Let X be the set defined in the problem statement.

We consider two cases:

1. There exists a word $w \in \Sigma^*$ such that $P_w(x)$ is not the zero polynomial.

   In this case, let $R(P_w)$ represent the set of roots of $P_w(x)$. Since $P_w(x)$ is not identically zero, $R(P_w)$ is finite. Furthermore, X must be a subset of $R(P_w)$; otherwise, there would exist some $x \in X$ for which $P_w(x) \neq 0$, contradicting the definition of X. Thus, X is finite.

2. For every word $w \in \Sigma^*$, $P_w(x)$ is the zero polynomial.

   In this scenario, $P_w(x) = 0$ holds for all $x \in \mathbb{Q}$ and every $w \in \Sigma^*$. Consequently, $X = \mathbb{Q}$.

From these cases, we conclude that X is either finite or equal to $\mathbb{Q}$, completing the proof.