Dominik Wawszczak
Student ID Number: 440014
Group Number: 2

Selected Topics in Graph Theory
Assignment 1

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## Problem 1

For every $k \in \mathbb{N}$, define $G_k$ as a full ternary tree of depth $k-1$ with an additional vertex $s$, which is connected to all the vertices of the tree. The treewidth of this graph is 2 because we can take the classic decomposition of a ternary tree and insert $s$ into every bag. It remains to show that the spaghetti treewidth of $G_k$ is at least $k$.

Suppose, for the sake of contradiction, that there exists a spaghetti tree decomposition $(T, \{B_x : x \in V(T)\})$ of width at most $k-1$. Without loss of generality, assume that this decomposition is minimal in terms of $|V(T)|$.

We claim that $s$ belongs to every bag in the decomposition. Suppose instead that there exists a vertex whose bag does not contain $s$. Then, there is also a leaf $l \in V(T)$ with the same property. Let $p$ be the only neighbor of $l$ in $T$. Since $|V(T)|$ is minimal, there must be a vertex $u \in B_l$ such that $u \notin B_p$, as otherwise, $B_l$ would be redundant. Consequently, there is no bag that contains both $s$ and $u$ because the nodes whose bags contain $u$ must induce a path in $T$. This proves the claim.

Since $s$ is included in every bag, $T$ is a path. Thus, removing $s$ from each bag yields a path decomposition of a ternary tree of depth $k-1$. The width of this path decomposition is one less than that of the original decomposition, which is at most $k-2$. However, we showed in the tutorials that the pathwidth of a full ternary tree of depth $k-1$ is $k-1$. This contradiction completes the proof.

## Problem 2

Consider a bipartite graph $G = (X, C, E)$, where $X = \{x_1, x_2, \ldots, x_n\}$, $C = \{C_1, C_2, \ldots, C_m\}$, and $E = \{(x_i, C_j) \;:\; C_j \text{ contains either } x_i \text{ or } \neg x_i\}$. This graph is 3-regular, as each variable appears in exactly three clauses, and every clause contains exactly three literals corresponding to pairwise distinct variables. A direct consequence of this observation is that $n = m$, since $3n = |E| = 3m$.

We now prove that Hall's condition holds in $G$. Let $S$ be any subset of $X$. Define $E_S = \{(x_i, C_j) \;:\; (x_i, C_j) \in E \land x_i \in S\}$ as the set of edges between $S$ and $N_G(S)$. Since $|E_S| = 3|S|$, we obtain $3|S| \leqslant 3|N_G(S)|$, confirming that $G$ satisfies Hall's condition. Therefore, $G$ has a perfect matching.

We can satisfy each clause $C_j$ by assigning the appropriate truth value to its matched variable. This ensures that the entire formula is satisfied, completing the proof.

## Problem 3

Suppose, for the sake of contradiction, that $G$ does not have a perfect matching. This implies that Tutte's condition does not hold in $G$. Therefore, there exists a set $U \subseteq V(G)$ such that $|\mathcal{C}| > |U|$, where $\mathcal{C}$ is the set of odd connected components in $G \setminus U$. The size of $U$ must, of course, be at least $t$.

We say that a vertex $u \in U$ and a component $C \in \mathcal{C}$ are *adjacent* if and only if there exists a vertex $v \in C$ such that $\{u, v\} \in E(G)$. Define:

$$X \;=\; \{(u, C) \;:\; u \in U \;\land\; C \in \mathcal{C} \;\land\; u \text{ and } C \text{ are adjacent}\}.$$

For any $u \in U$, there can be at most $t$ components in $\mathcal{C}$ adjacent to it, as otherwise $K_{1,t+1}$ would be an induced subgraph of $G$. This gives us the following upper bound for the size of $X$: $|X| \leqslant t|U|$.

We now prove that for any $C \in \mathcal{C}$, there are at least $t$ vertices in $U$ adjacent to it. Suppose, for the sake of contradiction, that the size of the set

$$S = \{u : u \in U \wedge u \text{ and } C \text{ are adjacent}\}$$

is smaller than $t$. Since $S \subsetneq U$, there exists a vertex in $U \setminus S$ that is not adjacent to $C$. This leads to a contradiction, as $G \setminus S$ is not connected and $|S| < t$. Thus, we obtain a lower bound for the size of $X$: $t|\mathcal{C}| \leqslant |X|$.

From the above, we have $t|\mathcal{C}| \leqslant |X| \leqslant t|U|$, and therefore $|\mathcal{C}| \leqslant |U|$. This contradiction completes the proof.