Selected topics in graph theory 2025 — homework 3

Graph coloring, spectral graph theory

Deadline: June 22nd, 2025, 11:59am CET

Problem 1. Determine the largest integer k with the following property: every simple connected plane graph with at most k triangle faces contains a vertex of degree at most 4.

Problem 2. Let $\alpha \in (0, 1)$ and $d \in \mathbb{Z}_+$ be fixed. Design a data structure with the following specification.

Upon initialization, the data structure receives two integers n and k, a graph G with $V(G) = \{1, 2, \ldots, nk\}$ that is promised to consist of k connected components, each consisting of n vertices and being a d-regular expander with $\lambda \leqslant (1 - \alpha)d$. Furthermore, we are given k vertices $v_1, v_2, \ldots, v_k \in \{1, 2, \ldots, nk\}$ that are promised to lie in distinct connected components of G.

Upon query, the data structure receives a vertex $x \in \{1, 2, \ldots, nk\}$ and is supposed to answer with an index $i \in \{1, 2, \ldots, k\}$ such that x and v_i is within the same connected component of G.

Both initialization and query should work in time $\sqrt{n} \cdot \mathrm{poly}(k, \log n)$. The data structure can be randomized, and the answer to a query should be correct with probability at least 0.5.
Hint: Birthday paradox.

Problem 3. Let G be a plane graph and let $\mathcal{P}$ be a family of paths in G with the following property: no edge of G belongs to more than one path of $\mathcal{P}$, an endpoint of a path in $\mathcal{P}$ does not belong to any other path in $\mathcal{P}$, and if two paths $P_1, P_2 \in \mathcal{P}$ intersect at a vertex $v \in V(G)$, they only touch and do not intersect transversally, that is, if one looks at the cyclic order of the edges of P_1 and P_2 around v, then the two edges belonging to P_1 are not separated by the two edges belonging to P_2 (the order is P_1, P_1, P_2, P_2; and not P_1, P_2, P_1, P_2). Let $H(G, \mathcal{P})$ be the intersection graph of the paths $\mathcal{P}$, that is, the graph with vertex set $\mathcal{P}$ where any $P_1, P_2 \in \mathcal{P}$ are adjacent if and only if $V(P_1) \cap V(P_2) \neq \emptyset$. Prove that there exists a universal constant c such that

$$\chi(H(G, \mathcal{P})) \leqslant c \cdot \omega(H(G, \mathcal{P})).$$

Note: you can get partial credit for proving a bound $\chi(H(G, \mathcal{P})) \leqslant f(\omega(H(G, \mathcal{P})))$ for any function f.

Problem 4. Let G be a graph. Let a be the maximum chromatic number over radius-2 balls in G, that is,

$$a := \max\{\chi(G[N[N[v]]]) \mid v \in V(G)\}.$$

Let b be the maximum length of an induced cycle in G (or 1 if G is a forest). Prove that $\chi(G) = \mathcal{O}(ab)$.
Note: you can get partial credit for proving a bound $\chi(G) \leqslant f(a, b)$ for any function f.