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Selected Topics in Graph Theory
Assignment 3

Problem 1

Let G be a simple planar graph with v vertices, e edges, and f faces. Suppose G has t triangular faces and that every vertex has degree at least 5. Since the minimum degree is 5, we have $2e \geqslant 5v$. By Euler's formula:

$$v - e + f \;=\; 2 \quad\implies\quad f - 2 \;=\; e - v \;\geqslant\; \frac{3}{5}e.$$

Let f_i denote the number of faces of size i. Then:

$$2e \;=\; \sum_{i\geqslant 3} i f_i \;=\; 3t + \sum_{i \geqslant 4} i f_i \;\geqslant\; 3t + 4 \cdot \sum_{i\geqslant 4} f_i \;=\; 3t + 4(f-t) \;=\; 4f - t.$$

Thus:

$$t \;\geqslant\; 4f - 2e \;\geqslant\; 4f - \frac{10}{3}\cdot(f-2) \;=\; \frac{2f+20}{3} \;\geqslant\; \frac{2t+20}{3},$$

which implies:

$$3t \;\geqslant\; 2t + 20 \quad\implies\quad t \;\geqslant\; 20.$$

Hence, $k \geqslant 19$. Since the icosahedral graph is a 5-regular simple planar graph with exactly 20 triangular faces, $k = 19$.

Problem 2

For any $i \in \{1, 2, \ldots, k\}$ and any u in the same connected component as v_i, we have

$$\left| p_t(v_i, u) - \frac{1}{n} \right| \;\leqslant\; (1-\alpha)^t,$$

where $p_t(a, b)$ is the probability that a random walk of length t starting at a ends at b. By setting $t = \frac{\ln(2n)}{-\ln(1-\alpha)} = \mathcal{O}(\log n)$, we obtain

$$\left| p_t(v_i, u) - \frac{1}{n} \right| \;\leqslant\; \frac{1}{2n}, \quad \text{and therefore} \quad p_t(v_i, u) \in \left[\frac{1}{2n}, \frac{3}{2n}\right].$$

If we perform m such walks, the probability that at least one ends at u is at least

$$1 - \left(1 - \frac{1}{2n}\right)^m.$$

We want this probability to be at least $\frac{1}{\sqrt{n}}$. Since

$$\left(1 - \frac{1}{2n}\right)^m \;\leqslant\; e^{-\frac{m}{2n}},$$

it suffices that m satisfies

$$e^{-\frac{m}{2n}} \;\leqslant\; 1 - \frac{1}{\sqrt{n}} \;=\; e^{\ln\left(1 - \frac{1}{\sqrt{n}}\right)} \quad\implies\quad -\frac{m}{2n} \;\leqslant\; \ln\left(1 - \frac{1}{\sqrt{n}}\right),$$

hence

$$m \geqslant -2n \ln\left(1 - \frac{1}{\sqrt{n}}\right) = 2n \cdot \sum_{i=1}^{\infty} \frac{\left(\frac{1}{\sqrt{n}}\right)^i}{i} = 2\sqrt{n} + 1 + 2 \cdot \sum_{i=3}^{\infty} \frac{\left(\frac{1}{\sqrt{n}}\right)^{i-2}}{i}.$$

It is enough to set

$$\begin{aligned} m &= \left\lceil 2\sqrt{n} + 1 + 2 \cdot \sum_{i=3}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^{i-2} \right\rceil = \left\lceil 2\sqrt{n} + 1 + \frac{2}{\sqrt{n}} \cdot \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^{i} \right\rceil = \\ &= \left\lceil 2\sqrt{n} + 1 + \frac{2}{\sqrt{n}} \cdot \frac{1}{1 - \frac{1}{\sqrt{n}}} \right\rceil = \left\lceil 2\sqrt{n} + 1 + \frac{2}{\sqrt{n} - 1} \right\rceil, \end{aligned}$$

so $m = \mathcal{O}(\sqrt{n})$.

At initialization, for each $i \in \{1, 2, \ldots, k\}$, we perform m random walks of length t from v_i, and construct a set R_i containing all reached vertices. This ensures that if u is in the same component as v_i, then $u \in R_i$ with probability at least $\frac{1}{\sqrt{n}}$. The total time complexity of initialization is $\mathcal{O}(k\sqrt{n}\log n)$.

To answer a query, we perform m random walks of length t from x, and create a set Q of all reached vertices. We then search for $i \in \{1, 2, \ldots, k\}$ such that $Q \cap R_i \neq \emptyset$. The time complexity of a single query is $\mathcal{O}(k\sqrt{n}\log n)$, as it can be checked whether $Q \cap R_i \neq \emptyset$ in time $\mathcal{O}((|Q| + |R_i|)\log(|Q| + |R_i|)) = \mathcal{O}(\sqrt{n}\log n)$.

It remains to estimate the probability that $Q \cap R_i \neq \emptyset$, where i is such that x and v_i lie in the same connected component. Since for any u in this component, the probabilities that $u \in R_i$ and $u \in Q$ are both at least $\frac{1}{\sqrt{n}}$, the probability that u belongs to both these sets is at least $\frac{1}{n}$. Thus, the probability that no such u exists is at most

$$\left(1 - \frac{1}{n}\right)^n \leqslant e^{-1},$$

and so the probability that $Q \cap R_i \neq \emptyset$ is at least $1 - e^{-1} \geqslant \frac{1}{2}$.