Selected topics in graph theory 2025 — homework 2

Graph minors, spectral graph theory

Deadline: May 23rd, 2025, 21:00 CET

Problem 1. An ant walks on a d-regular graph G, whose $r \geqslant 1$ edges are colored red. Initially, the ant is walking along a red edge chosen uniformly at random (i.e., each red edge with probability $1/r$) and going towards an endpoint chosen uniformly at random (i.e., each endpoint with probability $1/2$). In a single step, the ant finishes walking along the current edge, reaches the endpoint in front of her, chooses the next edge uniformly at random among the edges incident with this endpoint (including the edge she just finished walking), and starts walking along the chosen edge.

Prove that for every $t \geqslant 1$, the probability that in the t-th step the ant chooses a red edge as the next edge to traverse is at most

$$\frac{2r}{d|V(G)|} + \left(\frac{\lambda(G)}{d}\right)^t.$$

Problem 2. Let $k \geqslant 100$ be an integer and let Γ_k be the $k \times k$ grid with the bottom-left-to-top-right diagonal added in every cell. (Formally, the vertex set of Γ_k is $\{1, \ldots, k\} \times \{1, \ldots, k\}$, and two distinct vertices (a, b) and (c, d) are adjacent if and only if $|a - c| \leqslant 1$, $|b - d| \leqslant 1$, and $(a - c)(b - d) \geqslant 0$.) Further, let I be any independent set in Γ_k. Prove that $\mathrm{tw}(\Gamma_k - I) \geqslant \frac{k}{100}$.

Problem 3. For a graph G and a subset of vertices X, the operation of *flipping* at X results in the graph $G \oplus X$ obtained from G by complementing the adjacency relation on X: for every pair of vertices $u, v \in X$, u and v are adjacent in $G \oplus X$ if and only if u and v are *not* adjacent in G. (The adjacency between any pair u, v where u or v is not in X stays the same.) We define a graph parameter *flip-depth* recursively as follows:

- If G has one vertex, then the flip-depth of G is 0.
- Otherwise, G has flip-depth at most k if and only if there exists a subset of vertices X such that every connected component of $G \oplus k$ has flip-depth at most $k - 1$.

Prove that for every fixed $k \in \mathbb{N}$, graphs of flip-depth at most k are well-quasi-ordered by the induced subgraph relation.

Problem 4. Let G be an n-vertex d-regular graph and let λ_n be the smallest eigenvalue of the adjacency matrix of G. Prove that any independent set in G is of size at most

$$n \cdot \frac{-\lambda_n}{d - \lambda_n}.$$