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Problem 1

Let $n = |V(G)|$, and define vectors $x_t \in \mathbb{R}^n$ for each $t \in \mathbb{N}$, where $x_t[v]$ denotes the probability that vertex v is visited at step t. Let $y \in \mathbb{R}^n$ be a vector such that $y[v]$ is the number of red edges incident to v.

We have

$$x_0 \;=\; \frac{y}{2r} \qquad \text{and} \qquad x_t \;=\; \left(\frac{A}{d}\right)^t \cdot x_0,$$

for all $t \in \mathbb{N}$, where A is the adjacency matrix of G. The goal is to estimate the value

$$p_t \;=\; x_t^T \cdot \frac{y}{d}.$$

We decompose y as

$$y \;=\; \frac{2r}{n} \cdot \mathbf{1} + z,$$

where z is orthogonal to $\mathbf{1}$, since

$$z^T\mathbf{1} \;=\; \left(y - \frac{2r}{n} \cdot \mathbf{1}\right)^T \cdot \mathbf{1} \;=\; y^T\mathbf{1} - \frac{2r}{n} \cdot \mathbf{1}^T\mathbf{1} \;=\; 2r - \frac{2r}{n} \cdot n \;= 0.$$

Substituting into p_t, we obtain

$$p_t \;=\; x_t^T \cdot \frac{y}{d} \;=\; \left(\left(\frac{A}{d}\right)^t \cdot \frac{y}{2r}\right)^T \cdot \frac{y}{d} \;=\; \frac{1}{2rd} \cdot \left(\left(\frac{A}{d}\right)^t \cdot \left(\frac{2r}{n} \cdot \mathbf{1} + z\right)\right)^T \cdot \left(\frac{2r}{n} \cdot \mathbf{1} + z\right) \;=$$
$$=\; \frac{1}{2rd} \cdot \left(\frac{4r^2}{n^2} \cdot \mathbf{1}^T \cdot \left(\frac{A}{d}\right)^t \cdot \mathbf{1} + \frac{2r}{n} \cdot \mathbf{1}^T \cdot \left(\frac{A}{d}\right)^t \cdot z + \frac{2r}{n} \cdot z^T \cdot \left(\frac{A}{d}\right)^t \cdot \mathbf{1} + z^T \cdot \left(\frac{A}{d}\right)^t \cdot z\right).$$

Since $\frac{A}{d}$ is a doubly stochastic matrix, we have

$$\mathbf{1}^T \cdot \left(\frac{A}{d}\right)^t \;=\; \mathbf{1}^T \qquad \text{and} \qquad \left(\frac{A}{d}\right)^t \cdot \mathbf{1} \;=\; \mathbf{1},$$

and hence

$$p_t \;=\; \frac{1}{2rd} \cdot \left(\frac{4r^2}{n^2} \cdot \mathbf{1}^T\mathbf{1} + \frac{2r}{n} \cdot \mathbf{1}^T z + \frac{2r}{n} \cdot z^T\mathbf{1} + z^T \cdot \left(\frac{A}{d}\right)^t \cdot z\right) \;=$$
$$=\; \frac{1}{2rd} \cdot \left(\frac{4r^2}{n^2} \cdot n + z^T \cdot \left(\frac{A}{d}\right)^t \cdot z\right) \;=\; \frac{2r}{dn} + \frac{1}{2rd} \cdot z^T \cdot \left(\frac{A}{d}\right)^t \cdot z.$$

Thus, it remains to prove that

$$\frac{1}{2rd} \cdot z^T \cdot \left(\frac{A}{d}\right)^t \cdot z \;\leqslant\; \left(\frac{\lambda(G)}{d}\right)^t.$$

Since

$$z^T \cdot \left(\frac{A}{d}\right)^t \cdot z \;\leqslant\; \left(\frac{\lambda(G)}{d}\right)^t \cdot \|z\|^2,$$

it suffices to show that $\|z\|^2 \leqslant 2rd$.

We compute

$$\begin{aligned}\|z\|^2 \;&=\; \left\|y - \frac{2r}{n}\cdot \mathbf{1}\right\|^2 \;=\; y^T y - 2\cdot\frac{2r}{n}\cdot y^T\mathbf{1} + \frac{4r^2}{n^2}\cdot \mathbf{1}^T\mathbf{1} \;=\; y^T y - 2\cdot \frac{2r}{n}\cdot 2r + \frac{4r^2}{n^2}\cdot n \;=\\ &=\; y^T y - \frac{4r^2}{n}.\end{aligned}$$

Let $s = 2r \bmod d$. The value of $y^T y$ is largest when there are $\left\lfloor \frac{2r}{d}\right\rfloor = \frac{2r-s}{d}$ vertices with d red edges, one vertex with s red edges, and the rest with none. Otherwise, one could increase $y^T y$ by transferring a red edge from a vertex with the fewest nonzero red edges to another with fewer than d red edges.

This yields the following upper bound:

$$y^T y \;\leqslant\; \left(\frac{2r-s}{d}\right)\cdot d^2 + s^2 \;=\; 2rd - sd + s^2 \;<\; 2rd,$$

as $s < d$. Therefore,

$$\|z\|^2 \;=\; y^T y - \frac{4r^2}{n} \;<\; 2rd - \frac{4r^2}{n} \;\leqslant\; 2rd,$$

which completes the proof.

Problem 2

We will show that a $\frac{k}{100} \times \frac{k}{100}$ grid is a minor of $\Gamma_k \setminus I$. To this end, we partition the vertices of Γ_k into $\left\lfloor \frac{k}{100}\right\rfloor^2$ squares of size 100×100. Formally, define

$$S_{i,j} \;=\; \Gamma_k\big[\{(a,b) \;:\; 100(i-1) < a \leqslant 100i \;\wedge\; 100(j-1) < b \leqslant 100j\}\big],$$

for all $i, j \in \left\{1, 2, \ldots, \left\lfloor \frac{k}{100}\right\rfloor\right\}$.

We now prove that for every such i, j, the subgraph $S_{i,j} \setminus I$ is connected. Take any distinct $(a,b), (a',b') \in S_{i,j} \setminus I$. Without loss of generality, assume $(a,b) < (a',b')$, i.e., $a < a'$ or $a = a'$ and $b < b'$. We will construct a path from (a,b) to (a',b') by induction on $|a-a'| + |b-b'|$.

<u>Base case:</u> If $|a-a'| + |b-b'| = 1$, then (a,b) and (a',b') are adjacent in $S_{i,j}$, and thus directly connected.

<u>Induction step:</u> We consider two cases:

1. $a = a'$

   If $(a', b'-1) \notin I$, we can move to that vertex and proceed. Otherwise, we go to $(a', b'-2)$ and then proceed via:

   1.1 $(a'+1, b'-1), (a'+1, b'), (a', b')$ if $a \bmod 100 \neq 0$;

   1.2 $(a'-1, b'-2), (a'-1, b'-1), (a', b')$ if $a \bmod 100 = 0$.

2. $a < a'$

   If $(a'-1, b') \notin I$, we extend the path through that vertex. Otherwise, we distinguish three subcases:

2.1 $b = b'$: The argument is analogous to the case $a = a'$.

2.2 $b < b'$: If $(a', b) \notin I$, we can proceed via (a', b) to (a', b'). Otherwise, we first go to $(a' - 1, b)$, then to $(a', b + 1)$, and finally to (a', b').

2.3 $b > b'$: As before, if $(a', b) \notin I$, we proceed through (a', b) to (a', b'). Otherwise, we go to $(a' - 1, b)$, then to $(a' - 1, b - 1)$, followed by $(a', b - 1)$, and continue to (a', b').

We have established that each $S_{i,j} \setminus I$ is connected. It remains to show that for any $i, j, i', j' \in \{1, 2, \ldots, \lfloor \frac{k}{100} \rfloor\}$ with $|i - i'| + |j - j'| = 1$, there exists an edge in $\Gamma_k \setminus I$ connecting $S_{i,j} \setminus I$ and $S_{i',j'} \setminus I$.

We consider only the case $i' = i + 1$ and $j' = j$, as other case is symmetric. Choose any $(a, b) \in V(S_{i,j} \setminus I)$ with $a = 100i$ and $b < 100j$. If $(a+1, b) \notin I$, then the edge $\{(a, b), (a+1, b)\}$ connects the two components. Otherwise, we can use the edge $\{(a, b), (a + 1, b + 1)\}$.

Since a $\frac{k}{100} \times \frac{k}{100}$ grid is a minor of $\Gamma_k \setminus I$, we conclude that $\operatorname{tw}(\Gamma_k \setminus I) \geqslant \frac{k}{100}$, completing the proof.

Problem 4

Let $S \subseteq V(G)$ be an independent set of G, and define a vector $x \in \mathbb{R}^n$ such that $x_i = 1$ if $i \in S$, and $x_i = 0$ otherwise. Decompose x as

$$x \;=\; \frac{|S|}{n} \cdot \mathbf{1} + y.$$

Then y is orthogonal to $\mathbf{1}$, since

$$y^T \mathbf{1} \;=\; \left(x - \frac{|S|}{n} \cdot \mathbf{1}\right)^T \cdot \mathbf{1} \;=\; x^T \mathbf{1} - \frac{|S|}{n} \cdot \mathbf{1}^T \mathbf{1} \;=\; |S| - \frac{|S|}{n} \cdot n \;=\; 0.$$

Let A be the adjacency matrix of G. Since S is an independent set, it follows that

$$x^T A x \;=\; \sum_{i,j \in \{1,2,\ldots,n\}} A_{i,j} x_i x_j \;=\; 0,$$

Expanding x using the decomposition, we get

$$0 \;=\; x^T A x \;=\; \left(\frac{|S|}{n} \cdot \mathbf{1} + y\right)^T \cdot A \cdot \left(\frac{|S|}{n} \cdot \mathbf{1} + y\right) \;=\; \frac{|S|^2}{n^2} \cdot \mathbf{1}^T A \mathbf{1} + 2 \cdot \frac{|S|}{n} \cdot \mathbf{1}^T A y + y^T A y.$$

Since $\mathbf{1}$ is an eigenvector of A with eigenvalue d, we have $A\mathbf{1} = d\mathbf{1}$, and thus

$$\frac{|S|^2}{n^2} \cdot \mathbf{1}^T A \mathbf{1} \;=\; \frac{|S|^2}{n^2} \cdot \mathbf{1}^T d \mathbf{1} \;=\; \frac{|S|^2}{n^2} \cdot dn \;=\; \frac{|S|^2 d}{n}.$$

Also,

$$2 \cdot \frac{|S|}{n} \cdot \mathbf{1}^T A y \;=\; 2 \cdot \frac{|S|}{n} \cdot \mathbf{1}^T d y \;=\; 0,$$

since $\mathbf{1}^T y = 0$ by orthogonality. Therefore,

$$0 \;=\; \frac{|S|^2 d}{n} + y^T A y.$$

Rearranging gives

$$-\frac{|S|^2 d}{n} \;=\; y^T A y \;\geqslant\; \lambda_n \cdot \|y\|^2.$$

We compute

$$\|y\|^2 \;=\; \left\|x - \frac{|S|}{n}\cdot \mathbf{1}\right\|^2 \;=\; |S| - 2\cdot\frac{|S|^2}{n} + \frac{|S|^2}{n} \;=\; |S| - \frac{|S|^2}{n},$$

so

$$-\frac{|S|^2 d}{n} \;\geqslant\; \lambda_n \cdot \left(|S| - \frac{|S|^2}{n}\right).$$

Dividing both sides by $|S|$ yields

$$-\frac{|S|d}{n} \;\geqslant\; \lambda_n \cdot \left(1 - \frac{|S|}{n}\right) \quad\Longrightarrow\quad |S|d \;\leqslant\; \lambda_n|S| - \lambda_n n \quad\Longrightarrow\quad |S| \;\leqslant\; n\cdot\frac{-\lambda_n}{d-\lambda_n},$$

which completes the proof.