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Selected Topics in Graph Theory
Assignment 2

Problem 2

We will show that a $\frac{k}{100} \times \frac{k}{100}$ grid is a minor of $\Gamma_k \setminus I$. To this end, we partition the vertices of Γ_k into $\left\lfloor \frac{k}{100} \right\rfloor^2$ squares of size 100×100. Formally, define

$$S_{i,j} \;=\; \Gamma_k\big[\{(a,b) \;:\; 100(i-1) < a \leqslant 100i \;\wedge\; 100(j-1) < b \leqslant 100j\}\big],$$

for all $i, j \in \left\{1, 2, \ldots, \left\lfloor \frac{k}{100} \right\rfloor\right\}$.

We now prove that for every such i, j, the subgraph $S_{i,j} \setminus I$ is connected. Take any distinct $(a,b), (a',b') \in S_{i,j} \setminus I$. Without loss of generality, assume $(a,b) < (a',b')$, i.e., $a < a'$ or $a = a'$ and $b < b'$. We will construct a path from (a,b) to (a',b') by induction on $|a - a'| + |b - b'|$.

Base case: If $|a - a'| + |b - b'| = 1$, then (a,b) and (a',b') are adjacent in $S_{i,j}$, and thus directly connected.

Induction step: We consider two cases:

1. $a = a'$

   If $(a', b'-1) \notin I$, we can move to that vertex and proceed. Otherwise, we go to $(a', b'-2)$ and then proceed via:

   1.1 $(a'+1, b'-1), (a'+1, b'), (a', b')$ if $a \bmod 100 \neq 0$;

   1.2 $(a'-1, b'-2), (a'-1, b'-1), (a', b')$ if $a \bmod 100 = 0$.

2. $a < a'$

   If $(a'-1, b') \notin I$, we extend the path through that vertex. Otherwise, we distinguish three subcases:

   2.1 $b = b'$: The argument is analogous to the case $a = a'$.

   2.2 $b < b'$: If $(a', b) \notin I$, we can proceed via (a', b) to (a', b'). Otherwise, we first go to $(a'-1, b)$, then to $(a', b+1)$, and finally to (a', b').

   2.3 $b > b'$: As before, if $(a', b) \notin I$, we proceed through (a', b) to (a', b'). Otherwise, we go to $(a'-1, b)$, then to $(a'-1, b-1)$, followed by $(a', b-1)$, and continue to (a', b').

We have established that each $S_{i,j} \setminus I$ is connected. It remains to show that for any $i, j, i', j' \in \left\{1, 2, \ldots, \left\lfloor \frac{k}{100} \right\rfloor\right\}$ with $|i - i'| + |j - j'| = 1$, there exists an edge in $\Gamma_k \setminus I$ connecting $S_{i,j} \setminus I$ and $S_{i',j'} \setminus I$.

We consider only the case $i' = i + 1$ and $j' = j$, as other case is symmetric. Choose any $(a,b) \in V(S_{i,j} \setminus I)$ with $a = 100i$ and $b < 100j$. If $(a+1, b) \notin I$, then the edge $\{(a,b), (a+1,b)\}$ connects the two components. Otherwise, we can use the edge $\{(a,b), (a+1, b+1)\}$.

Since a $\frac{k}{100} \times \frac{k}{100}$ grid is a minor of $\Gamma_k \setminus I$, we conclude that $\mathrm{tw}(\Gamma_k \setminus I) \geqslant \frac{k}{100}$, completing the proof.

Problem 4

Let $S \subseteq V(G)$ be an independent set of G, and define a vector $x \in \mathbb{R}^n$ such that $x_i = 1$ if $i \in S$, and $x_i = 0$ otherwise. Decompose x as

$$x \;=\; \frac{|S|}{n} \cdot \mathbf{1} + y.$$

Then y is orthogonal to $\mathbf{1}$, since

$$y^T\mathbf{1} \;=\; \left(x - \frac{|S|}{n} \cdot \mathbf{1}\right)^T \cdot \mathbf{1} \;=\; x^T\mathbf{1} - \frac{|S|}{n} \cdot \mathbf{1}^T\mathbf{1} \;=\; |S| - \frac{|S|}{n} \cdot n \;=\; 0.$$

Let A be the adjacency matrix of G. Since S is an independent set, it follows that

$$x^T Ax \;=\; \sum_{i,j \in \{1,2,\ldots,n\}} A_{i,j}x_ix_j \;=\; 0,$$

Expanding x using the decomposition, we get

$$0 \;=\; x^T Ax \;=\; \left(\frac{|S|}{n} \cdot \mathbf{1} + y\right)^T \cdot A \cdot \left(\frac{|S|}{n} \cdot \mathbf{1} + y\right) \;=\; \frac{|S|^2}{n^2} \cdot \mathbf{1}^T A\mathbf{1} + 2 \cdot \frac{|S|}{n} \cdot \mathbf{1}^T Ay + y^T Ay.$$

Since $\mathbf{1}$ is an eigenvector of A with eigenvalue d, we have $A\mathbf{1} = d\mathbf{1}$, and thus

$$\frac{|S|^2}{n^2} \cdot \mathbf{1}^T A\mathbf{1} \;=\; \frac{|S|^2}{n^2} \cdot \mathbf{1}^T d\mathbf{1} \;=\; \frac{|S|^2}{n^2} \cdot dn \;=\; \frac{|S|^2 d}{n}.$$

Also,

$$2 \cdot \frac{|S|}{n} \cdot \mathbf{1}^T Ay \;=\; 2 \cdot \frac{|S|}{n} \cdot \mathbf{1}^T dy \;=\; 0,$$

since $\mathbf{1}^T y = 0$ by orthogonality. Therefore,

$$0 \;=\; \frac{|S|^2 d}{n} + y^T Ay.$$

Rearranging gives

$$-\frac{|S|^2 d}{n} \;=\; y^T Ay \;\geqslant\; \lambda_n \cdot \|y\|^2.$$

We compute

$$\|y\|^2 \;=\; \left\|x - \frac{|S|}{n} \cdot \mathbf{1}\right\|^2 \;=\; |S| - 2 \cdot \frac{|S|^2}{n} + \frac{|S|^2}{n} \;=\; |S| - \frac{|S|^2}{n},$$

so

$$-\frac{|S|^2 d}{n} \;\geqslant\; \lambda_n \cdot \left(|S| - \frac{|S|^2}{n}\right).$$

Dividing both sides by $|S|$ yields

$$-\frac{|S|d}{n} \;\geqslant\; \lambda_n \cdot \left(1 - \frac{|S|}{n}\right) \quad\Longrightarrow\quad |S|d \;\leqslant\; \lambda_n|S| - \lambda_n n \quad\Longrightarrow\quad |S| \;\leqslant\; n \cdot \frac{-\lambda_n}{d - \lambda_n},$$

which completes the proof.