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Problem 4

Let $S \subseteq V(G)$ be an independent set of G, and define a vector $x \in \mathbb{R}^n$ such that $x_i = 1$ if $i \in S$, and $x_i = 0$ otherwise. Decompose x as

$$x = \frac{|S|}{n} \cdot \mathbf{1} + y.$$

Then y is orthogonal to $\mathbf{1}$, since

$$y^T \mathbf{1} = \left(x - \frac{|S|}{n}\mathbf{1}\right)^T \mathbf{1} = x^T \mathbf{1} - \frac{|S|}{n}\mathbf{1}^T\mathbf{1} = |S| - \frac{|S|}{n} n = 0.$$

Let A be the adjacency matrix of G. Since S is an independent set, it follows that

$$x^T A x = \sum_{i \in S} \sum_{j \in S} A_{ij} = 0,$$

Expanding x using the decomposition, we get

$$0 = x^T A x = \left(\frac{|S|}{n}\mathbf{1} + y\right)^T A \left(\frac{|S|}{n}\mathbf{1} + y\right) = \frac{|S|^2}{n^2}\mathbf{1}^T A \mathbf{1} + 2\frac{|S|}{n}\mathbf{1}^T A y + y^T A y.$$

Since $\mathbf{1}$ is an eigenvector of A with eigenvalue d, we have $A\mathbf{1} = d\mathbf{1}$, and thus

$$\frac{|S|^2}{n^2}\mathbf{1}^T A \mathbf{1} = \frac{|S|^2}{n^2}\mathbf{1}^T d \mathbf{1} = \frac{|S|^2}{n^2} dn = \frac{|S|^2 d}{n}.$$

Also,

$$2\frac{|S|}{n}\mathbf{1}^T A y = 2\frac{|S|}{n}\mathbf{1}^T d y = 0,$$

since $\mathbf{1}^T y = 0$ by orthogonality. Therefore,

$$0 = \frac{|S|^2 d}{n} + y^T A y.$$

Rearranging gives

$$-\frac{|S|^2 d}{n} = y^T A y \geqslant \lambda_n \cdot \|y\|^2.$$

We compute

$$\|y\|^2 = \left\|x - \frac{|S|}{n}\mathbf{1}\right\|^2 = |S| - 2\frac{|S|^2}{n} + \frac{|S|^2}{n} = |S| - \frac{|S|^2}{n},$$

so

$$-\frac{|S|^2 d}{n} \geqslant \lambda_n \cdot \left(|S| - \frac{|S|^2}{n}\right).$$

Dividing both sides by $|S|$ yields

$$-\frac{|S|d}{n} \geqslant \lambda_n \cdot \left(1 - \frac{|S|}{n}\right) \implies |S| \cdot d \leqslant \lambda_n |S| - \lambda_n \cdot n \implies |S| \leqslant n \cdot \frac{-\lambda_n}{d - \lambda_n},$$

which completes the proof.