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Selected Topics in Graph Theory
Assignment 1

Problem 2

Consider a bipartite graph $G = (X, C, E)$, where $X = \{x_1, x_2, \ldots, x_n\}$, $C = \{C_1, C_2, \ldots, C_m\}$, and $E = \{(x_i, C_j) \ : \ C_j \text{ contains either } x_i \text{ or } \neg x_i\}$. This graph is 3-regular, as each variable appears in exactly three clauses, and every clause contains exactly three literals corresponding to pairwise distinct variables. A direct consequence of this observation is that $n = m$, since $3n = |E| = 3m$.

We will now prove that Hall's condition holds in G. Let S be any subset of X. Define $E_S = \{(x_i, C_j) \ : \ (x_i, C_j) \in E \ \wedge \ x_i \in S\}$ as the set of edges between S and $N_G(S)$. Since $|E_S| = 3|S|$, we obtain $3|S| \leqslant 3|N_G(S)|$, confirming that G satisfies Hall's condition. Therefore, G has a perfect matching.

We can satisfy each clause C_j by assigning the appropriate truth value to its matched variable. This ensures that the entire formula is satisfied, completing the proof.

Problem 3

Suppose, for the sake of contradiction, that G does not have a perfect matching. This implies that Tutte's condition does not hold in G. Therefore, there exists a set $U \subseteq V(G)$ such that $|\mathcal{C}| > |U|$, where $\mathcal{C}$ is the set of odd connected components in $G \setminus U$. The size of U must, of course, be at least t.

We say that a vertex $u \in U$ and a component $C \in \mathcal{C}$ are *adjacent* if and only if there exists a vertex $v \in C$ such that $\{u, v\} \in E(G)$. Define:

$$X \ = \ \{(u, C) \ : \ u \in U \ \wedge \ C \in \mathcal{C} \ \wedge \ u \text{ and } C \text{ are adjacent}\}.$$

For any $u \in U$, there can be at most t components in $\mathcal{C}$ adjacent to it, as otherwise $K_{1,t+1}$ would be an induced subgraph of G. This gives us the following upper bound for the size of X: $|X| \leqslant t|U|$.

We now prove that for any $C \in \mathcal{C}$, there are at least t vertices in U adjacent to it. Suppose, for the sake of contradiction, that the size of the set

$$S \ = \ \{u \ : \ u \in U \ \wedge \ u \text{ and } C \text{ are adjacent}\}$$

is smaller than t. Since $S \subsetneq U$, there exists a vertex in $U \setminus S$ that is not adjacent to C. This leads to a contradiction, as $G \setminus S$ is not connected and $|S| < t$. Thus, we obtain a lower bound for the size of X: $t|\mathcal{C}| \leqslant |X|$.

From the above, we have $t|\mathcal{C}| \leqslant |X| \leqslant t|U|$, and therefore $\mathcal{C} \leqslant |U|$. This contradiction completes the proof.