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Selected Topics in Graph Theory
Assignment 1

Problem 2

Consider a bipartite graph $G = (X, C, E)$, where $X = \{x_1, x_2, \ldots, x_n\}$, $C = \{C_1, C_2, \ldots, C_m\}$, and $E = \{(x_i, C_j) \ : \ C_j \text{ contains either } x_i \text{ or } \neg x_i\}$. This graph is 3-regular, as each variable appears in exactly three clauses, and every clause contains exactly three literals corresponding to pairwise distinct variables. A direct consequence of this observation is that $n = m$, since $3n = |E| = 3m$.

We will now prove that Hall's condition holds in G. Let S be any subset of X. Define $E_S = \{(x_i, C_j) \ : \ (x_i, C_j) \in E \ \wedge \ x_i \in S\}$ as the set of edges between S and $N_G(S)$. Since $|E_S| = 3|S|$, we obtain $3|S| \leqslant 3|N_G(S)|$, confirming that G satisfies Hall's condition. Therefore, G has a perfect matching.

We can satisfy each clause C_j by assigning the appropriate truth value to its matched variable. This ensures that the entire formula is satisfied, completing the proof.