Selected topics in graph theory 2025 — homework 1

Matchings and treewidth

Deadline: April 18th, 2025, 21:00 CET

Problem 1. A *spaghetti tree decomposition* of a graph G is a tree T and a collection of bags $\{B_x : x \in V(T)\}$ satisfying the following properties:

- for every edge uv of G, there is a node x of T with $u, v \in B_x$; and
- for every vertex u of G, the set of nodes $\{x \in V(T) \mid u \in B_x\}$ induces a (nonempty) path in T.

That is, compared to standard tree decompositions, we require that for every vertex u, the nodes whose bags contain u induce a path in T, instead of a tree. The width of $(T, \{B_x : x \in V(T)\}$ is $\max_{x \in V(T)} |B_x| - 1$, and the *spaghetti treewidth* of G is the minimum possible width of a spaghetti tree decomposition of G.

Prove that for every $k \in \mathbb{N}$, there exists a graph G_k whose treewidth is 2 and whose spaghetti treewidth is at least k.

Problem 2. A boolean formula over a variable set $\{x_1, \ldots, x_n\}$ is in $(E3, 3)$-*CNF* if it is of the form $C_1 \wedge C_2 \wedge \ldots \wedge C_m$, where each *clause* C_i is a disjunction of three literals of pairwise distinct variables (a *literal* is a variable or its negation), and each variable appears in exactly three clauses. For example, here is a boolean formula in $(E3, 3)$-CNF:

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee x_3 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_3 \vee \neg x_4).$$

Prove that every boolean formula in $(E3, 3)$-CNF is satisfiable, that is, there is an assignment of values true/false to the variables that makes all the clauses true.

Problem 3. Let G be a graph with $|V(G)|$ even and let t be a positive integer. Suppose that G satisfies the following properties:

- for each $S \subseteq V(G)$ with $|S| < t$, the graph $G - S$ is connected, and
- G does not contain $K_{1,t+1}$ as an induced subgraph (that is, there is no vertex u with $t + 1$ pairwise non-adjacent neighbors).

Prove that G has a perfect matching.

Problem 4. Prove that there is a function $f : \mathbb{N} \to \mathbb{N}$ with the following property. Suppose G is a graph of treewidth at most k. Then one can partition $V(G)$ into two subsets, A and B, so that whenever two vertices u, v of G belong to the same connected component of $G[A]$ or to the same connected component of $G[B]$, we have $\mathrm{dist}_G(u, v) \leqslant f(k)$.