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Problem 1

Let G be a simple planar graph with v vertices, e edges, and f faces. Suppose G has t triangular faces and that every vertex has degree at least 5. Since the minimum degree is 5, we have $2e \geqslant 5v$. By Euler's formula:

$$v - e + f \;=\; 2 \quad \Longrightarrow \quad f - 2 \;=\; e - v \;\geqslant\; \frac{3}{5}e.$$

Let f_i denote the number of faces of size i. Then:

$$2e \;=\; \sum_{i \geqslant 3} i f_i \;=\; 3t + \sum_{i \geqslant 4} i f_i \;\geqslant\; 3t + 4 \cdot \sum_{i \geqslant 4} f_i \;=\; 3t + 4(f - t) \;=\; 4f - t.$$

Thus:

$$t \;\geqslant\; 4f - 2e \;\geqslant\; 4f - \frac{10}{3} \cdot (f - 2) \;=\; \frac{2f + 20}{3} \;\geqslant\; \frac{2t + 20}{3},$$

which implies:

$$3t \;\geqslant\; 2t + 20 \quad \Longrightarrow \quad t \;\geqslant\; 20.$$

Hence, $k \geqslant 19$. Since the icosahedral graph is a 5-regular simple planar graph with exactly 20 triangular faces, $k = 19$.