Dominik Wawszczak

Student ID Number: 440014

Group Number: 2

## Problem 1

Let $G$ be a simple planar graph with $v$ vertices, $e$ edges, and $f$ faces. Suppose $G$ has $t$ triangular faces and that every vertex has degree at least 5. Since the minimum degree is 5, we have $2e \geqslant 5v$. By Euler's formula:

$$v - e + f = 2 \quad \implies \quad f - 2 = e - v \geqslant \frac{3}{5}e.$$

Let $f_i$ denote the number of faces of size $i$. Then:

$$2e = \sum_{i \geqslant 3} i f_i = 3t + \sum_{i \geqslant 4} i f_i \geqslant 3t + 4 \cdot \sum_{i \geqslant 4} f_i = 3t + 4(f - t) = 4f - t.$$

Thus:

$$t \geqslant 4f - 2e \geqslant 4f - \frac{10}{3} \cdot (f - 2) = \frac{2f + 20}{3} \geqslant \frac{2t + 20}{3},$$

which implies:

$$3t \geqslant 2t + 20 \quad \implies \quad t \geqslant 20.$$

Hence, $k \geqslant 19$. Since the icosahedral graph is a 5-regular simple planar graph with exactly 20 triangular faces, $k = 19$.

## Problem 2

For any $i \in \{1, 2, \ldots, k\}$ and any $u$ in the same connected component as $v_i$, we have

$$\left| p_t(v_i, u) - \frac{1}{n} \right| \leqslant (1 - \alpha)^t,$$

where $p_t(a, b)$ is the probability that a random walk of length $t$ starting at $a$ ends at $b$. By setting $t = \frac{\ln(2n)}{-\ln(1-\alpha)} = \mathcal{O}(\log n)$, we obtain

$$\left| p_t(v_i, u) - \frac{1}{n} \right| \leqslant \frac{1}{2n}, \quad \text{and therefore} \quad p_t(v_i, u) \in \left[ \frac{1}{2n}, \frac{3}{2n} \right].$$

If we perform $m$ such walks, the probability that at least one ends at $u$ is at least

$$1 - \left( 1 - \frac{1}{2n} \right)^m.$$

We want this probability to be at least $\frac{1}{\sqrt{n}}$. Since

$$\left( 1 - \frac{1}{2n} \right)^m \leqslant e^{-\frac{m}{2n}},$$

it suffices that $m$ satisfies

$$e^{-\frac{m}{2n}} \leqslant 1 - \frac{1}{\sqrt{n}} = e^{\ln\left(1 - \frac{1}{\sqrt{n}}\right)} \quad \implies \quad -\frac{m}{2n} \leqslant \ln\left( 1 - \frac{1}{\sqrt{n}} \right),$$

hence

$$m \;\geqslant\; -2n \ln\left(1 - \frac{1}{\sqrt{n}}\right) \;=\; 2n \cdot \sum_{i=1}^{\infty} \frac{\left(\frac{1}{\sqrt{n}}\right)^{i}}{i} \;=\; 2\sqrt{n} + 1 + \sum_{i=3}^{\infty} \frac{\left(\frac{1}{\sqrt{n}}\right)^{i-2}}{i} \;\geqslant$$

$$\geqslant\; 2\sqrt{n} + 1 + \sum_{i=3}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^{i-2} \;=\; 2\sqrt{n} + 1 + \frac{1}{\sqrt{n}} \cdot \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^{i} \;=$$

$$=\; 2\sqrt{n} + 1 + \frac{1}{\sqrt{n}} \cdot \frac{1}{1 - \frac{1}{\sqrt{n}}} \;=\; 2\sqrt{n} + 1 + \frac{1}{\sqrt{n} - 1},$$

so $m = \mathcal{O}\left(\sqrt{n}\right)$.

At initialization, for each $i \in \{1, 2, \ldots, k\}$, we perform $m$ random walks of length $t$ from $v_i$, and construct a set $R_i$ containing all reached vertices. This ensures that if $u$ is in the same component as $v_i$, then $u \in R_i$ with probability at least $\frac{1}{\sqrt{n}}$. The total time complexity of initialization is $\mathcal{O}\left(k\sqrt{n}\log n\right)$.

To answer a query, we perform $m$ random walks of length $t$ from $x$, and create a set $Q$ of all reached vertices. We then search for $i \in \{1, 2, \ldots, k\}$ such that $Q \cap R_i \neq \emptyset$. The time complexity of a single query is $\mathcal{O}\left(k\sqrt{n}\log n\right)$, as it can be checked whether $Q \cap R_i \neq \emptyset$ in time $\mathcal{O}((|Q| + |R_i|)\log(|Q| + |R_i|)) = \mathcal{O}\left(\sqrt{n}\log n\right)$.

It remains to estimate the probability that $Q \cap R_i \neq \emptyset$, where $i$ is such that $x$ and $v_i$ lie in the same connected component. Since for any $u$ in this component, the probabilities that $u \in R_i$ and $u \in Q$ are both at least $\frac{1}{\sqrt{n}}$, the probability that $u$ belongs to both these sets is at least $\frac{1}{n}$. Thus, the probability that no such $u$ exists is at most

$$\left(1 - \frac{1}{n}\right)^{n} \;\leqslant\; e^{-1},$$

and so the probability that $Q \cap R_i \neq \emptyset$ is at least $1 - e^{-1} \geqslant \frac{1}{2}$.