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Computational Complexity
Assignment 4

Consider an instance G^* of the 3-Coloring Problem, with $V(G^*) = N$. Set $k = \lfloor \log_3 \log_2 N \rfloor - 1$[1]. We partition the vertices of G^* into groups of size k, with the final group potentially containing fewer than k vertices.

We construct a graph G – an instance of the problem from the statement. Let n denote the number of groups. We define $V(G)$ as $\{1, \ldots, n\} \times \{1, \ldots, n\}$. It holds that

$$n \leqslant \frac{N}{k} + 1 = \frac{N}{\lfloor \log_3 \log_2 N \rfloor - 1} + 1 \leqslant \frac{\alpha N}{\log_3 \log_2 N},$$

for some constant α, because

$$\frac{N}{\lfloor \log_3 \log_2 N \rfloor - 1} + 1 \leqslant \frac{\alpha N}{\log_3 \log_2 N} \iff \frac{\log_3 \log_2 N}{\lfloor \log_3 \log_2 N \rfloor - 1} + \frac{\log_3 \log_2 N}{N} \leqslant \alpha,$$

which is true since $\frac{x}{\lfloor x \rfloor - 1} < 3$ for $x \geqslant 2$, and $\frac{\log_3 \log_2 N}{N}$ is bounded by a constant, as it converges to 0. Thus, $n = o(N)$.

Let l_i denote the number of possible 3-colorings of the i-th group ($1 \leqslant i \leqslant n$). It follows that

$$l_i \leqslant 3^k = 3^{\lfloor \log_3 \log_2 N \rfloor - 1} \leqslant 3^{\log_3 \log_2 N - 1} = \frac{\log_2 N}{3} = \log_2 \sqrt[3]{N} < \log_2 n,$$

because

$$n \geqslant \frac{N}{k} = \frac{N}{\lfloor \log_3 \log_2 N \rfloor - 1} > \sqrt[3]{N}.$$

Consider an arbitrary ordering of all the colorings in each group, such as lexicographic order. For every $1 \leqslant i_1, i_2 \leqslant n$ with $i_1 \neq i_2$, and for every $1 \leqslant j_1 \leqslant l_{i_1}$, $1 \leqslant j_2 \leqslant l_{i_2}$, we create an edge between vertices (i_1, j_1) and (i_2, j_2) if and only if the colorings j_1 and j_2 are compatible. This construction satisfies both conditions stated in the problem.

We will prove the following equivalence:

$$G^* \text{ is 3-colorable} \iff G \text{ contains a clique of size } n. \tag{1}$$

Assume G^* is 3-colorable. Then, the vertices representing the partial colorings of each group form a clique. Conversely, if G contains a clique of size n, a valid 3-coloring of G^* can be constructed by selecting the corresponding colorings for every group.

Suppose there exists a constant c such that the problem from the statement can be solved in time $\mathcal{O}(c^n) = \mathcal{O}\big(2^{\lambda n}\big)$, where $\lambda = \log_2 c$. This implies that the 3-Coloring Problem could be solved in time $\mathcal{O}\big(2^{\lambda n}\big) = \mathcal{O}\big(2^{o(N)}\big)$, due to the equivalence (1). However, by the Sparsification Lemma and the linear reduction from 3-SAT to 3-Coloring, this contradicts the Exponential Time Hypothesis, completing the proof.

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[1] We assume that $N \geqslant 512$, as smaller values would make k either non-positive or undefined.