Dominik Wawszczak
Student ID Number: 440014
Group Number: 3

Task 1

To begin, we will show that verifying whether an input contains a tree can be checked using $\mathcal{O}(\log N)$ space, where N is the size of the input. A graph is a tree if and only if it satisfies the following two conditions:

1. The number of edges is one less than the number of vertices.

2. The graph is connected.

Checking the first condition is straightforward. We can iterate over the edges while maintaining a binary counter and then verify if it equals $n-1$. This counter will indeed require no more than $\mathcal{O}(\log N)$ space.

The second condition is more complex. We can iterate over all pairs $(s,t) \in \{1,\ldots,n\}^2$ using two binary counters. For each pair, we need to determine if a simple path exists from s to t. This problem belongs to the L class, as discussed in the lecture.

A *root* of a tree T with radius at most 2 is a vertex $r \in V(T)$ such that every other vertex in $V(T)$ is within distance 2 of r.

We will use a slightly different definition of an *induced subgraph* than the one in the assignment statement. A graph G_1 is an induced subgraph of G_2 if and only if there exists an injection $f : V(G_1) \to V(G_2)$ such that

$$\underset{u \in V(G_1)}{\forall} \; \underset{v \in V(G_1)}{\forall} \; (u,v) \in E(G_1) \iff (f(u), f(v)) \in E(G_2),$$

which we will refer to as the *map function*.

Our approach will follow these steps:

1. Find a root r of T_1.

   To do this, iterate over all vertices of T_1, treating each vertex as a candidate for r. To verify a candidate, check for each $v \in V(T_1) \setminus \{r\}$ whether either $(v,r) \in E(T_1)$ or there exists a vertex $u \in V(T_1) \setminus \{v,r\}$ with $(v,u) \in E(T_1)$ and $(u,r) \in E(T_1)$.

   This requires three binary counters (for r, v, and u). If no root is found, we can reject the input.

2. For each $s \in V(T_2)$, check if there exists a valid map function f such that $f(r) = s$.

   The remaining part of the solution will focus on this check.

<u>Lemma 1</u> Let T_1 be a tree of radius 2 with root r, and let T_2 be any tree. Let $v_1, \ldots, v_k$ denote the children of r, ordered so that

$$\underset{i \in \{2,\ldots,k\}}{\forall} \; (\deg(v_{i-1}), v_{i-1}) \;\geqslant\; (\deg(v_i), v_i),$$

where

$$(a_1, b_1) \;\geqslant\; (a_2, b_2) \quad\iff\quad a_1 > a_2 \;\vee\; (a_1 = a_2 \;\wedge\; b_1 \geqslant b_2).$$

Let s be any vertex of T_2, and denote $u_1, \ldots, u_l$ as the children of s when T_2 is rooted at s, ordered similarly. Then a map function f with the property that $f(r) = s$ exists if and only if

$$k \leqslant l \quad \wedge \quad \forall_{i \in \{1,\ldots,k\}} \deg(v_i) \leqslant \deg(u_i). \tag{1}$$

Proof of Lemma 1 If condition (1) holds, we can define $f(r) = s$ and set $f(v_i) = u_i$ for each $i \in \{1, \ldots, k\}$, ensuring there are enough vertices in T_2 to accommodate each grandson of r.

To derive a contradiction, assume there is a map function f, but condition (1) does not hold. Since $k \leqslant l$ must be true, there exists an $i \in \{1, \ldots, k\}$ such that $\deg(v_i) > \deg(u_i)$. Take the smallest such i. There is a unique j such that $f(v_i) = u_j$, which must be less than i, or there would not be enough vertices to match the children of v_i; thus $i > 1$. For each $i' < i$, let j' be the index with $f(v_{i'}) = u_{j'}$; similarly, j' must be less than i because $\deg(v_{i'}) \geqslant \deg(v_i)$. This creates a contradiction by the Pigeonhole principle, as f was assumed to be injective, completing the proof.

Returning to the second step, once we have the root r, finding the number k of its children requires only one binary counter. For a given $s \in V(T_2)$, we can similarly find the number of its children l. By Lemma 1, if $k > l$, we can discard s as a candidate. Otherwise, we need to verify for each $i \in \{1, \ldots, k\}$ whether $\deg(v_i) \leqslant \deg(u_i)$. To calculate the degree of a vertex, we can iterate over edges while using a single binary counter. However, finding v_i and u_i in logarithmic space is more challenging.

We can determine $(\deg(v_1), v_1)$ by iterating over the sons of r, comparing each $(\deg(v_{\mathrm{cur}}), v_{\mathrm{cur}})$ with the best pair found so far, and updating as needed. We can calculate $(\deg(u_1), u_1)$ in a similar way. After this, we compare $\deg(v_1) \leqslant \deg(u_1)$ and reject if the condition does not hold; otherwise, we move on to the next pair. Once $(\deg(v_i), v_i)$ is calculated, obtaining $(\deg(v_{i+1}), v_{i+1})$ is straightforward – simply ignore any $(\deg(v_{\mathrm{cur}}), v_{\mathrm{cur}})$ strictly greater than $(\deg(v_i), v_i)$. The same process applies to $(\deg(u_{i+1}), u_{i+1})$.

The entire algorithm requires only a constant number of binary counters, concluding the solution.

Task 2

To start, we will prove that the language belongs to the NP class. For any induced subgraph H of G, we need to perform the following steps:

1. Verify that H is a tree.
2. Identify the root of H, confirming that H is a tree of radius 2.
3. Compare the multisets of degrees of the children of the roots in both T and H.

Each of these steps can be completed in polynomial time.

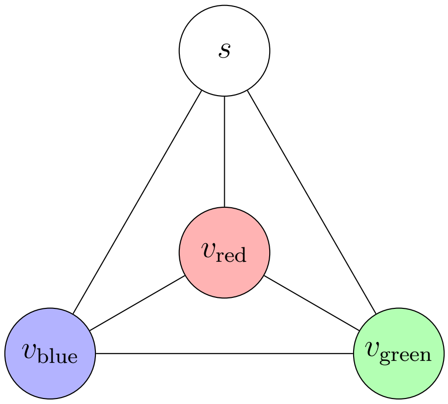
Now, we proceed to show that the problem is NP-hard. To do so, we will construct a Karp reduction from the 3-coloring problem to the problem specified in the assignment.

Let G^* be an arbitrary graph. Our goal is to construct a tree T of radius 2 and a graph G such that

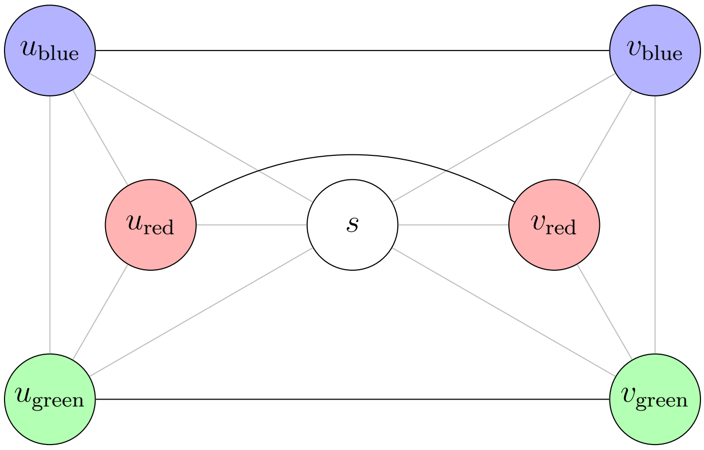
$$G^* \text{ is 3-colorable} \quad \Longleftrightarrow \quad T \text{ is isomorphic to an induced subgraph of } G. \tag{2}$$

Let $n = |V(G^*)|$ and $m = |E(G^*)|$. The tree T will consist of a root r and $n + 3$ children: a vertex v' for each $v \in V(G^*)$ and three auxiliary vertices: a_1, a_2, and a_3.

The graph G will include a special vertex s. For each $v \in V(G^*)$, the following structure will be added to G:



Additionally, for each edge $(u, v) \in E(G^*)$, we will add the following three edges to G:



Finally, s will have three auxiliary neighbors: b_1, b_2 and b_3.

It should be noted that the size of the new instance is polynomial in terms of the size of G^*, as T has $n + 4$ vertices and $n + 3$ edges, while G has $3n + 4$ vertices and $6n + 3m + 3$ edges. Therefore, the only remaining step is to prove the equivalence (2).

Assume that G^* is 3-colorable. For each $v \in V(G^*)$, let $\text{color}_v \in \{\text{blue}, \text{green}, \text{red}\}$ be the color assigned to v. We can then define a map function f as follows:

$$\begin{aligned} f(r) &= s; & \\ f(v') &= v_{\text{color}_v}, & \text{for any } v \in V(G^*); \\ f(a_i) &= b_i, & \text{for } i \in \{1, 2, 3\}. \end{aligned}$$

For every child t of r, we have $(f(r), f(t)) \in E(G)$, so we only need to show that for any two children t_1 and t_2 of r, $(f(t_1), f(t_2)) \notin E(G)$. If either t_1 or t_2 is one of the three auxiliary children, this is straightforward. Otherwise, assume $t_1 = u'$ and $t_2 = v'$ for some $u, v \in V(G^*)$. If $(u, v) \notin E(G^*)$, then $f(t_1) = u_{\text{color}_u}$ and $f(t_2) = v_{\text{color}_v}$ are not connected. On the other hand, if $(u, v) \in E(G^*)$, then $\text{color}_u \neq \text{color}_v$, meaning they are also not connected.

Conversely, assume there exists a map function f. Then we must have $f(r) = s$, as $\deg(r) = n + 3$ and s is the only vertex in G with degree at least $n + 3$. We can verify this since

$$\deg(v_{\text{color}}) \;=\; 3 + \deg(v) \;\leqslant\; n + 2, \quad \text{for any } v \in V(G^*) \text{ and color} \in \{\text{blue}, \text{green}, \text{red}\},$$

and $\deg(b_1) = \deg(b_2) = \deg(b_3) = 1$.

Suppose there exists a vertex $v \in V(G^*)$ and two distinct colors $\text{color}_1, \text{color}_2 \in \{\text{blue}, \text{green}, \text{red}\}$, such that both v_{color_1} and v_{color_2} belong to the image of f. Then the induced subgraph would contain a triangle $(s, v_{\text{color}_1}, v_{\text{color}_2})$, creating a contradiction. Therefore, at most one of v_{blue}, v_{green}, or v_{red} can belong to the image of f.

If none of v_{blue}, v_{green}, or v_{red} belongs to the image of f, then the size of the image of f is at most $n + 3$: consisting of s, b_1, b_2, b_3, and one vertex for each of the remaining $n - 1$ vertices. However, the size of the domain of is $n+4$, and because f is an injection, it follows that exactly one of v_{blue}, v_{green}, or v_{red} must appear in the image of f for each $v \in V(G^*)$.

For any $v \in V(G^*)$, let color_v be the unique color in $\{\text{blue}, \text{green}, \text{red}\}$ such that $v_{\text{color}_v} \in f^{-1}(V(T))$. If, for some $(u, v) \in E(G^*)$, $\text{color}_u = \text{color}_v$, then the induced subgraph would contain a triangle $(s, u_{\text{color}_u}, v_{\text{color}_v})$, which again results in a contradiction. Thus, we have a valid 3-coloring, completing the proof.