

Let $X, Y \subseteq \{0, 1\}^k$, where $|X| = |Y| = n$, be an instance of the Orthogonal Vectors Problem. We construct two new sets of binary vectors with dimension $d = 3k$:

$$A = \{xx0_k : x \in X\} \quad \text{and} \quad B = \{\bar{y}0_k y : y \in Y\},$$

where vu denotes the concatenation of vectors v and u , 0_l is a vector consisting of l zeros, and \bar{v} is the vector v with every coordinate negated. Note that $|A| = |B| = n$.

We will prove the following equivalence:

$$\exists_{x \in X} \exists_{y \in Y} x \perp y \iff \exists_{a \in A} \exists_{b \in B} \text{dist}_{\text{Hamming}}(a, b) \leq k. \quad (1)$$

Assume there exist $x \in X$ and $y \in Y$ such that $x \perp y$. Take $a = xx0_k \in A$ and $b = \bar{y}0_k y \in B$. Define

$$\text{cnt}_{p,q} = |\{i : x[i] = p \wedge y[i] = q\}|,$$

for any $p, q \in \{0, 1\}$. We have:

$$\begin{aligned} \text{dist}_{\text{Hamming}}(a, b) &= \text{dist}_{\text{Hamming}}(xx0_k, \bar{y}0_k y) = \\ &= \text{dist}_{\text{Hamming}}(x, \bar{y}) + \text{dist}_{\text{Hamming}}(x, 0_k) + \text{dist}_{\text{Hamming}}(0_k, y) = \\ &= (\text{cnt}_{0,0} + \text{cnt}_{1,1}) + (\text{cnt}_{1,0} + \text{cnt}_{1,1}) + (\text{cnt}_{0,1} + \text{cnt}_{1,1}) = \\ &= (k - \text{cnt}_{0,1} - \text{cnt}_{1,0}) + (\text{cnt}_{0,1} + \text{cnt}_{1,0} + 2 \cdot \text{cnt}_{1,1}) = k + 2 \cdot \text{cnt}_{1,1} = k, \end{aligned}$$

because $\text{cnt}_{1,1} = 0$, as $x \perp y$. We also used the fact that $\text{cnt}_{0,0} + \text{cnt}_{0,1} + \text{cnt}_{1,0} + \text{cnt}_{1,1} = k$.

Conversely, assume there exist $a \in A$ and $b \in B$ such that $\text{dist}_{\text{Hamming}}(a, b) \leq k$. Take $x \in X$ and $y \in Y$ where $a = xx0_k$ and $b = \bar{y}0_k y$. As before, we can show that $\text{dist}_{\text{Hamming}}(a, b) = k + 2 \cdot \text{cnt}_{1,1}$ with $\text{cnt}_{p,q}$ defined as earlier. Therefore, it must hold that $\text{cnt}_{1,1} = 0$, which means $x \perp y$, completing the proof of the equivalence (1).

Suppose there exists a constant $\delta > 0$ such the problem from the statement can be solved in time $\mathcal{O}(d^{100} \cdot n^{2-\delta})$. Then, due to the equivalence (1), the Orthogonal Vectors Problem could be solved in time

$$\mathcal{O}(d^{100} \cdot n^{2-\delta}) = \mathcal{O}((3k)^{100} \cdot n^{2-\delta}) = k^{\mathcal{O}(1)} \cdot n^{2-\delta},$$

which contradicts the Orthogonal Vectors Conjecture, concluding the proof, since

$$\text{Strong Exponential Time Hypothesis} \implies \text{Orthogonal Vectors Conjecture}.$$