The Basic, Two-Dimensional Heat Equation

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We are considering the two dimensional heat equation on a square with side lengths L that is centered at the origin and has an initial point-source of heat. Solving this particular problem will give us a template for which we can figure out what happens with different initial and boundary conditions. Firstly, as stated above, we have

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x,y) &= a^2\nabla^2 u(t,x,y) \\ &= a^2\frac{\partial^2}{\partial x^2}u(t,x,y) + a^2\frac{\partial^2}{\partial y^2}u(t,x,y) \end{aligned} \tag{1}$$

where a^2 is the *thermal diffusivity* of the material of which the plate is made. We also have our initial and boundary conditions:

$$\begin{aligned} u(0,x,y) &= \delta(x,y) \\ \nabla u\left(t, \pm\frac{L}{2}, \pm\frac{L}{2}\right) &= 0. \end{aligned}$$

To solve this, we begin by taking the two dimensional Fourier transform of our solution; if it exists, we can write

$$\mathcal{F}\left(u(t,x,y)\right) = \widehat{u}(t,s_1,s_2)$$

which we can plug into (1).

$$\begin{aligned} \frac{\partial}{\partial t}\widehat{u}(t,s_1,s_2) &= a^2\frac{\partial^2}{\partial x^2}\widehat{u}(x,s_1,s_2) + a^2\frac{\partial^2}{\partial y^2}\widehat{u}(x,s_1,s_2) \\ &= a^2(2\pi i s_1)^2\widehat{u}(t,s_1,s_2) + a^2(2\pi i s_2)^2\widehat{u}(t,s_1,s_2) \\ &= -\left[4\pi^2 a^2(s_1+s_2)\right]\widehat{u}(t,x,y) \end{aligned} \tag{2}$$

This however, is merely an ODE of the form

$$x'(t) + \alpha x(t) = 0$$

with the general solution

$$x(t) = x_0 e^{-\alpha t}.$$

By applying this to (2), we get

$$\widehat{u}(t,s_1,s_2) = \widehat{f}(s_1,s_2)e^{-4\pi^2 a^2(s_1+s_2)t} \quad \text{for } t \geq 0. \tag{3}$$

Taking the inverse Fourier transform will give the general solution to our two dimensional heat equation. To do so, we have to use the translation and modulations rules associated with the process.

$$
\begin{aligned}
u(t,x,y) &= \mathcal{F}^{-1}\left(\widehat{u}(t,s_1,s_2)\right) \\
&= \mathcal{F}^{-1}\left(e^{-4\pi^2 a^2 (s_1+s_2)t}\right) * f(x,y) \\
&= k(t,x,y) * f(x,y)
\end{aligned}
\tag{4}
$$

where

$$
k(t,x,y) \equiv \begin{cases} \delta(x,y) & \text{if } t=0 \\ \dfrac{e^{-(x^2+y^2)/4a^2 t}}{4\pi a^2 t} & \text{if } t>0. \end{cases}
\tag{5}
$$

Now that we have (4) and (5), we can use those equations to solve our problem. Because our boundary conditions require that no heat (zero flux conditions) leaks from the sides, we have to reflect our function over the entire x and y plane and keep it even. Now, the heat source is at the origin and at points $\{mL, nL\}$ for $m, n \in \mathbb{Z}$. Our initial condition now becomes

$$
\sum_m \sum_n \delta(x-mL, y-nL) = \frac{1}{L^2} \text{III}\left(\frac{x}{L}, \frac{y}{L}\right).
\tag{6}
$$

Using this to compute (4), we get our solution

$$
\begin{aligned}
u(t,x,y) &= k(t,x,y) * \frac{1}{L^2}\text{III}\left(\frac{x}{L}, \frac{y}{L}\right) \\
&= \sum_m \sum_n k(t, x-mL, y-nL) \\
&= \frac{1}{4\pi a^2 t} \sum_m \sum_n e^{-(x-mL)^2/4a^2 t} e^{-(y-nL)^2/4a^2 t}.
\end{aligned}
\tag{7}
$$

To simply the equation we acquired, we use Poisson's relation,

$$
\sum_m f(x-mp) = \frac{1}{p} \sum_m \widehat{f}\left(\frac{m}{p}\right) e^{2\pi i m x/p},
\tag{8}
$$

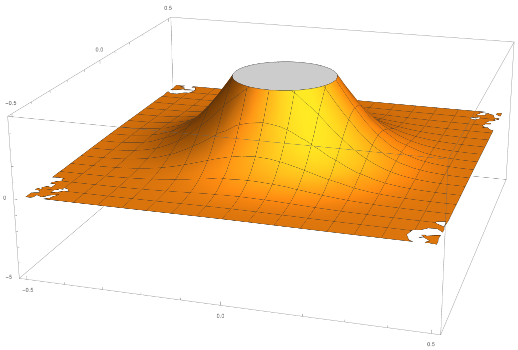
and write

$$
\begin{aligned}
u(t,x,y) &= \frac{1}{L^2} \sum_m \sum_n e^{-(4\pi^2 a^2 m^2/L^2)t} e^{2\pi i m x/L} e^{-(4\pi^2 a^2 n^2/L^2)t} e^{2\pi i n y/L} \\
&= \frac{1}{L^2} \sum_m \sum_n e^{(mx+ny)2\pi i/L} e^{-(m^2+n^2)(2\pi a/L)^2 t}
\end{aligned}
\tag{9}
$$

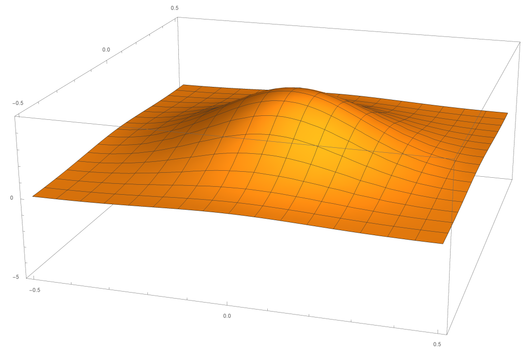
which is our simplified solution. (See 1a - 1d). Now, we have to check if it satisfies our boundary conditions.

$$
\begin{aligned}
\frac{\partial}{\partial x} u(t,x,y)\bigg|_{x=\pm L/2} &= \frac{1}{L^2} \sum_m \sum_n \frac{\partial}{\partial x} e^{2\pi i m x/L} e^{2\pi i n y/L} e^{-(m^2+n^2)(2\pi a/L)^2 t}\bigg|_{x=\pm L/2} \\
&= \frac{1}{L^2} \sum_m \sum_n \frac{2\pi i m}{L} e^{\pi i m} e^{2\pi i n y/L} e^{-(m^2+n^2)(2\pi a/L)^2 t} \\
&= \frac{1}{L^2} \sum_{m=1}^{\infty} \sum_n -\frac{4\pi m}{L} \sin(\pi m) e^{2\pi i n y/L} e^{-(m^2+n^2)(2\pi a/L)^2 t} \\
&= 0 \qquad\qquad \text{since } \sin(\pi m) = 0 \text{ for all } m \in \mathbb{N}.
\end{aligned}
\tag{10}
$$

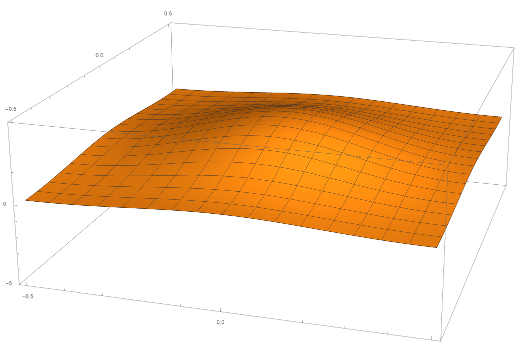
By the same method above, our solution (9) also satisfies the boundary condition $u_x(t, x, y = \pm L/2) = 0$.



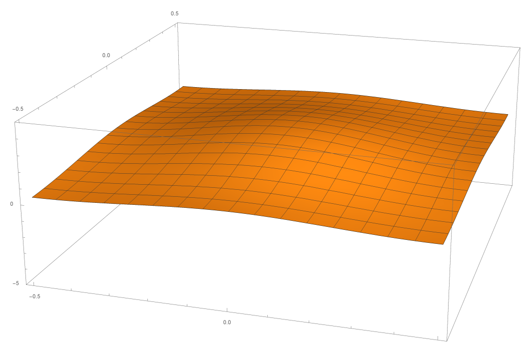
(a) $t = 0.01$



(b) $t = 0.02$.



(c) $t = 0.03$.



(d) $t = 0.04$.

Figure 1: graph of equation (9)

```
Manipulate[
 Plot3D[
  Sum[
   Exp[2 Pi i(m x + n y] Exp[-(m^2 + n^2) 4 Pi^2 t],
  {m, -10, 10}, {n, -10, 10}],
 {x, -.5, .5}, {y, -0.5, 0.5}, PlotRange -> 5 ],
{t, 0.00001, 0.1}]
```

Figure 2: Mathematica code for Figure 1

As seen in Figure 1, we were able to plot this particular solution using Mathematica (Figure 2), the code being given above. In this case, $L = 1$ and $a = 1$.

A more specific example of using the two-dimensional heat equation is starting with the initial condition, or temperature in this case, $u(x, y, 0) = \Pi(x, y)$. In order to better understand our steps it is best to show what happens to the one-dimensional heat equation using the initial temperature $u(x, 0) = \Pi(x)$. To start, we use the convolution product in (4) for the one-dimensional heat equation with the diffusion kernel

$$k(t,x,) \equiv \begin{cases} \delta(x) & \text{if } t=0 \\ \dfrac{e^{-x^2/4a^2t}}{\sqrt{4\pi a^2 t}} & \text{if } t>0 \end{cases} \tag{11}$$

to show us that the product is defined for every choice f and for every $t \geq 0$. When f is a suitably regular ordinary function and $t > 0$, we can also write

$$u(x,t) = \frac{1}{\sqrt{4\pi\alpha^2 t}} \int_{\xi=-\infty}^{\infty} f(\xi) e^{-(x-\xi)/4\alpha^2 t}\, d\xi. \tag{12}$$

From here, knowing our initial temperature $u(x, 0) = \Pi(x)$, we have

$$u(x,t) = \frac{1}{\sqrt{4\pi\alpha^2 t}} \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} e^{-(x-\xi)/4\alpha^2 t}\, d\xi. \tag{13}$$

which is an integral expressed as the error function, where

$$erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2}\, d\xi.$$

Also, using (3) in one dimension gives us

$$U(t,s) = \operatorname{sinc}(s) e^{-4\pi^2 a^2 st} \tag{14}$$

and therefore

$$\begin{aligned} u(x,t) &= \int_{s=-\infty}^{\infty} U(s,t) e^{2\pi i s x}\, ds \\ &= \int_{-\infty}^{\infty} \operatorname{sinc}(s) e^{-4\pi^2 a^2 st} \cos(2\pi s x)\, dx. \end{aligned} \tag{15}$$

The plot on the left in Figure 3 shows us how our initial value approaches an approximate Gaussian rapidly as time increases, while the plot on the right shows us that the discontinuities in our initial data are smoothed instantly.

Going back to our original example with initial condition of $u(x, y, 0) = \Pi(x, y)$, and following our previous steps, we can look at (4) and (5) and write

$$u(x,y,t) = \frac{1}{4\pi\alpha^2 t} \int_{\xi=-\infty}^{\infty} \int_{\eta=-\infty}^{\infty} f(\xi,\eta) e^{-(x-\xi)/4\alpha^2 t} e^{-(y-\eta)/4\alpha^2}\, d\eta\, d\xi. \tag{16}$$

Solving the heat equation $u_t = u_{xx} + u_{yy}$ with the initial temperature $u(x, y, 0) = \Pi(x, y)$, using (11), we write

$$u(x,y,t) = \frac{1}{4\pi\alpha^2 t} \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \int_{\eta=-\frac{1}{2}}^{\frac{1}{2}} e^{-(x-\xi)/4\alpha^2 t} e^{-(y-\eta)/4\alpha^2}\, d\eta\, d\xi. \tag{17}$$

We can also use (3) to write

$$U(t,s_1,s_2) = \operatorname{sinc}(s_1) e^{-4\pi^2 a^2 s_1 t} + \operatorname{sinc}(s_2) e^{-4\pi^2 a^2 s_2 t} \tag{18}$$

and synthesize

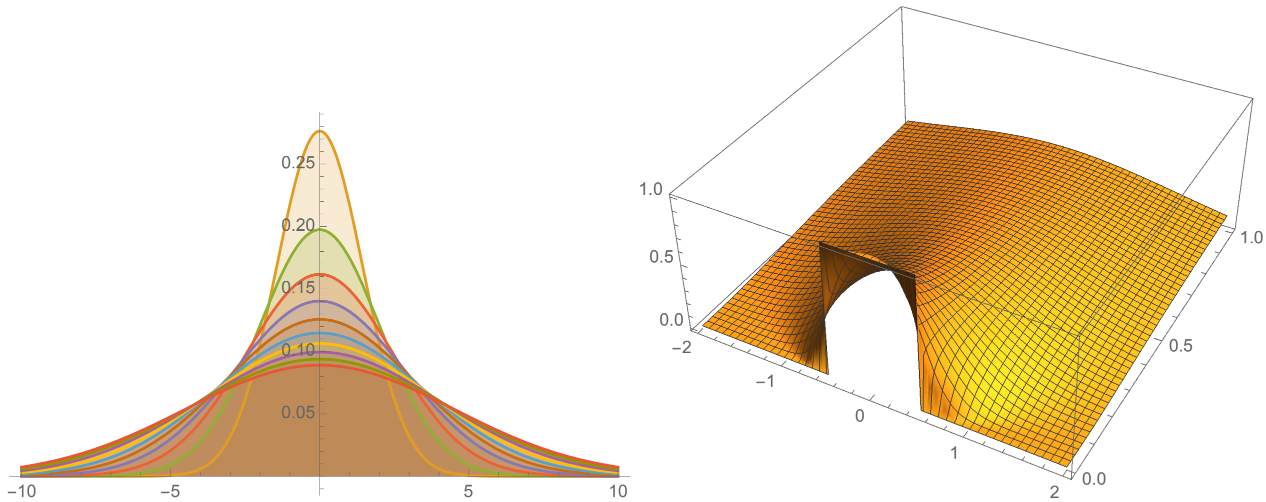


Figure 3: graph of equation (15)

```
heqn = D[u[x, t], t] == D[u[x, t], {x, 2}];
ic = u[x, 0] == UnitBox[x];
sol = DSolveValue[{heqn, ic}, u[x, t], {x, t}]
Plot[Evaluate[Table[sol, {t, 0, 10}]], {x, -10, 10}, PlotRange -> All, Filling -> Axis]
Plot3D[sol, {x, -2, 2}, {t, 0, 1}, PlotRange -> All, PlotPoints -> 50, Mesh -> Full]
```

Figure 4: Mathematica code for Figure 3

$$
\begin{aligned}
u(x,y,t) &= \int_{s_2=-\infty}^{\infty}\int_{s_1=-\infty}^{\infty} U(s_1,s_2,t)e^{2\pi i s_1 x}e^{2\pi i s_2 y}\,ds_1\,ds_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{sinc}(s_1)e^{-4\pi^2a^2s_1t}\cos(2\pi s_1 x) + \operatorname{sinc}(s_2)e^{-4\pi^2a^2s_2t}\cos(2\pi s_2 y)\,dx\,dy.
\end{aligned} \tag{19}
$$

The following graphs in Figure 6 show the process of the heat equation over time with our given initial condition. Say for example we have a plate of a certain metal that has been heated in its center. Overtime we will see the temperature of the heated area decrease, starting at its edges, and then moving inwards. Meanwhile, the plate heats up the outside of this region.

We see that our initial temperature becomes a super smooth temperature profile an instant $t \geq 0$ later. Between (7a) and (7b), a time lapse of only one second, the temperature flattens and spreads across the specified domain. Over time the heat will spread until there is a uniform intermediate temperature, i.e., affecting the whole plate as spoken previously.

Lastly, an interesting topic to point out is the behavior of these solutions when t is really close to 0. As can be seen in Figure 7, the graph behaves rather chaotically due to the fact that we are adding up a finite number of terms, but in order to get an exact representation, an infinite amount of terms have to be added. This brings to question the compromises one has to make in order to gain the most efficiency and accuracy, though most of the time, adding up the a couple dozen terms suffices. However, with more complex boundary and initial conditions, it is most often the case that supercomputers have to be used to carry out those calculations.

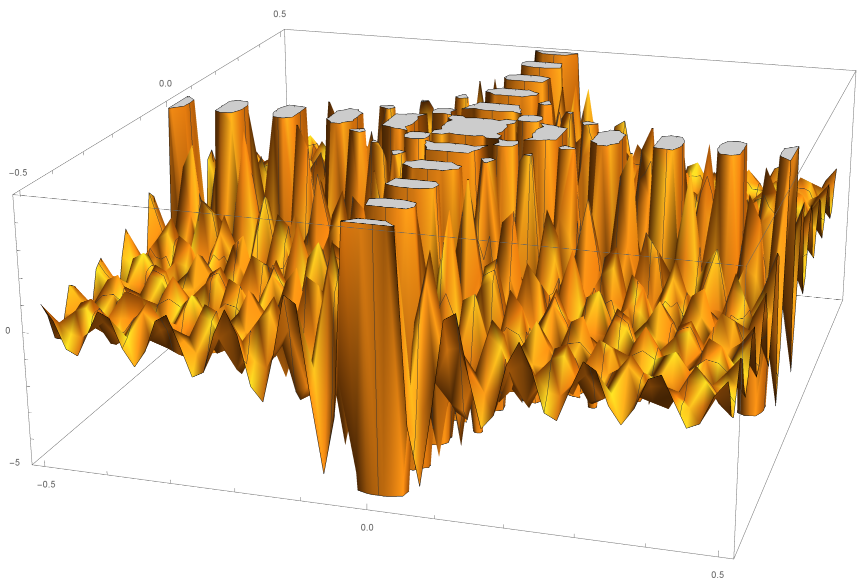
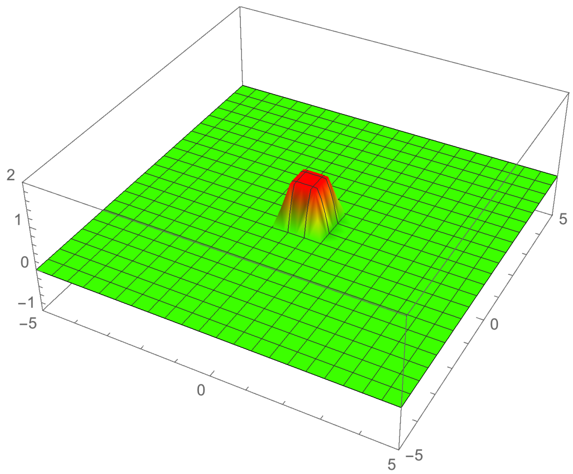


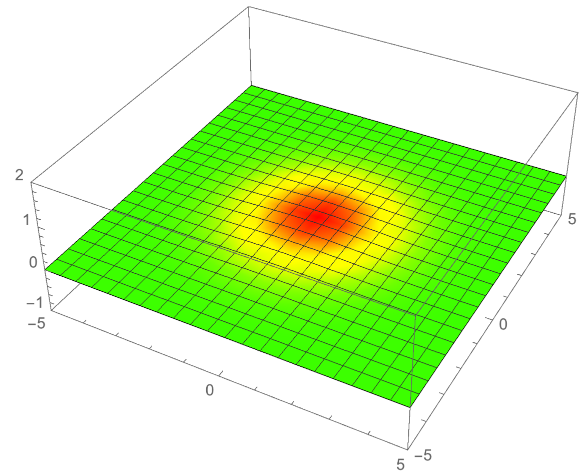
Figure 5: graph of equation (9) at t
close to 0

```
FUNC = NDSolve[{D[T[x, y, t], t] == RT*(D[T[x, y, t], x, x] + D[T[x, y, t], y, y]),
T[x, y, 0] == UnitBox[x, y], T[-5, y, t] == 0, T[5, y, t] == 0,
T[x, -5, t] == 0, T[x, 5, t] == 0},
T, {x, -5, 5}, {y, -5, 5}, {t, 0, 5}];
a = Table[
 Plot3D[
  T[x, y, t] /. FUNC, {x, -5, 5}, {y, -5, 5},
  Mesh -> 20, PlotRange -> {{-5, 5}, {-5, 5}, {-1, 2}},
  ColorFunction -> Function[{x, y, z}, Hue[.3 (1 - z)]]
 ],
{t, 0, 5}]
```

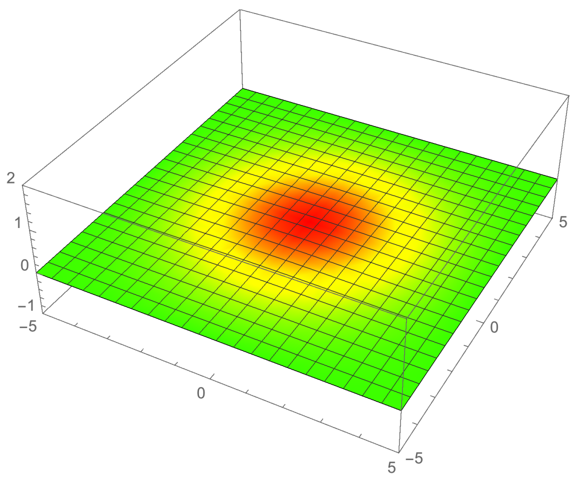
Figure 6: Mathematica code for Figure 7



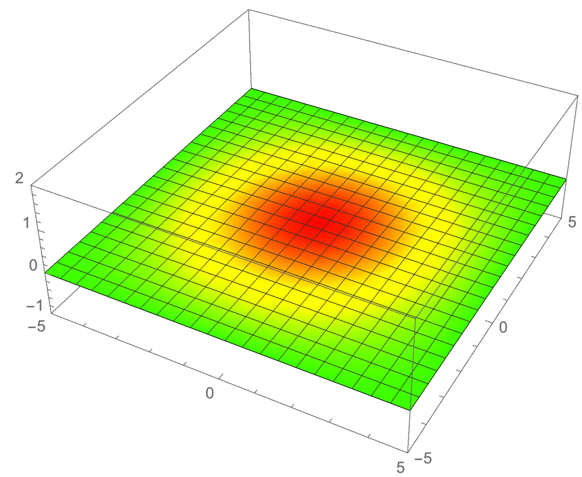
(a) $t = 0$



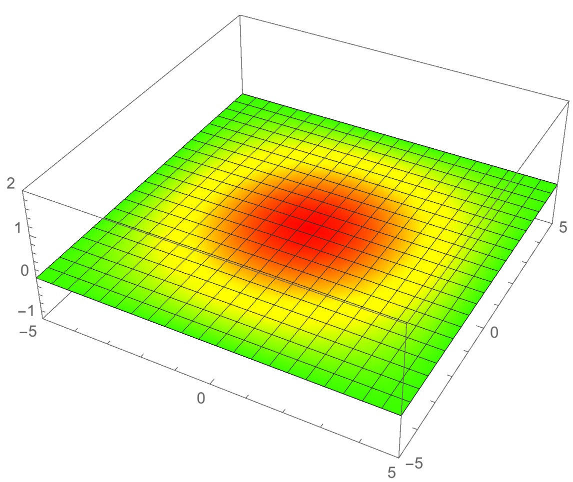
(b) $t = 1.$



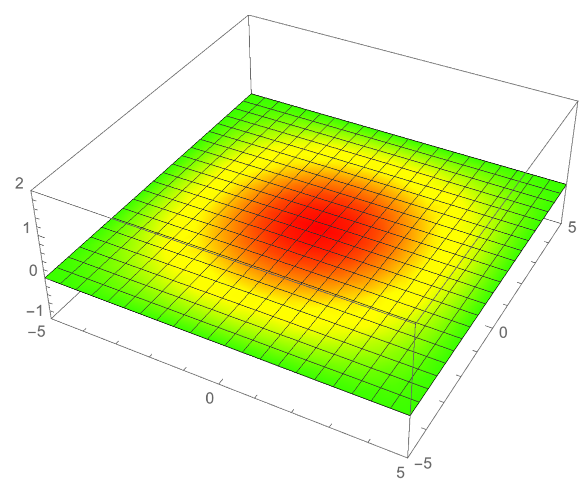
(c) $t = 2.$



(d) $t = 3.$



(e) $t = 4$



(f) $t = 5.$

Figure 7: graph of equation 19