Introduction to Cosmology

ASTR 434

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Lecture 3 / The Friedmann Equation / October 1st 2024

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This lecture: the Friedmann Equation

- The Friedmann equation (Newtonian derivation)
- Critical density
- The Fluid and Acceleration equations
- Expansion history for a flat, matter-dominated universe
- Radiation

Recap

The <u>general</u> FLRW metric for a <u>Universe with spatial curvature</u>

$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + S_\kappa^2(r)\, d\Omega^2]$$

$$S_\kappa(r) = \begin{cases} R_0 \sin(r/R_0), & \text{positive curvature} \\ r, & \text{zero curvature} \\ R_0 \sinh(r/R_0) & \text{negative curvature} \end{cases}$$

(note: $r \ll R_0 \implies S_\kappa \sim r$ regardless of κ)

$$R_0 \equiv R(t = t_0)$$

$$0 < a(t) < 1$$

The GR Field Equations

Einstein's **field equations** describe the relationship between energy density and spacetime curvature:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$G_{\mu\nu}$ is a symmetric 4×4 **tensor** that describes the curvature of (4-dimensional) spacetime. It is obtained via manipulation of the **metric** tensor $g_{\mu\nu}$.

$T_{\mu\nu}$ is another symmetric 4×4 tensor that describes the energy density as a function of spacetime coordinates.

$T_{\mu\nu}$ depends on what kind of "stuff" you think is in the Universe.

This tensor equation is a compact way to write 10 simultaneous equations, connecting each (independent) combination of two coordinates. The art of GR is specifying these tensors and then solving the resulting equations, which in general is extremely difficult.

The GR Field Equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Spacetime curvature $\propto$ *density*

This is analogous to Poisson's equation, $\nabla^2 \phi = 4\pi G \rho$

This is a two-way relationship. If the density changes, the curvature changes, which changes the density… Hence this relationship is *dynamic*.

If the problem is simple (symmetrical) enough, the set of simultaneous equations can be reduced to a much smaller number of equations that are easier to work with.

The Friedmann Equation

Aleksandr A. Friedmann

Александр Фридман

From Wikipedia:

Friedmann **fought in World War I** on behalf of Imperial Russia, as an army aviator, an instructor, and eventually, under the revolutionary regime, as the **head of an airplane factory**.

Friedmann in 1922 **introduced the idea of an expanding universe** that contained moving matter. **Correspondence with Einstein** suggests that Einstein was unwilling to accept the idea of an evolving Universe and worked instead to revise his equations to support the static, eternal Universe of Newton's time.

In June 1925 Friedmann was given the job of the director of the Main Geophysical Observatory in Leningrad. In July 1925 he **participated in a record-setting balloon flight**, reaching the elevation of 7,400 m.

Friedmann died on September 16, 1925 [**age 37**], from misdiagnosed typhoid fever. He had allegedly contracted the bacteria on return from his honeymoon in Crimea, **when he ate an unwashed pear bought at a railway station**.

The Friedmann Equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Under the cosmological principle, and hence with the FLRW metric, this reduces to a **pair** of coupled equations.

The most important of these is "**the" Friedmann equation**, which has the form:

$$\textit{Expansion rate} \equiv H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 \propto \textit{energy density of stuff}$$

The others are the **fluid equation** (expressing conservation of energy / first law of thermodynamics) and the **acceleration equation** (sometimes 'second Friedmann equation') for $\ddot{a}(t)$.

Any one of these three equations can be derived from the other two.

The Friedmann Equation

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\bar{\varepsilon}(t) - \frac{\kappa c^2}{a^2}$$

"Expansion Rate $\propto$ Energy Density"

$\bar{\varepsilon}$: mean density of <u>all types</u> of "stuff"

$k = 1/R_0^2$: curvature constant

Newtonian Derivation

The full GR derivation yields both the Friedmann equation and the acceleration equation, and naturally includes the curvature term.

However, *once we know the answer from GR*, we can obtain the Friedmann equation more 'intuitively' via Newtonian arguments (Milne & McCrea, 1934).

The Newtonian treatment is not rigorous or complete on its own, but in the end, it is equivalent to that from the full GR derivation (up to a constant term, which we will meet later).

It has to be that way, because the cosmological principle (homogeneity) implies the Hubble expansion must happen in exactly the same way on all scales. This can only be true if the expansion can be described correctly in the Newtonian limit of GR.

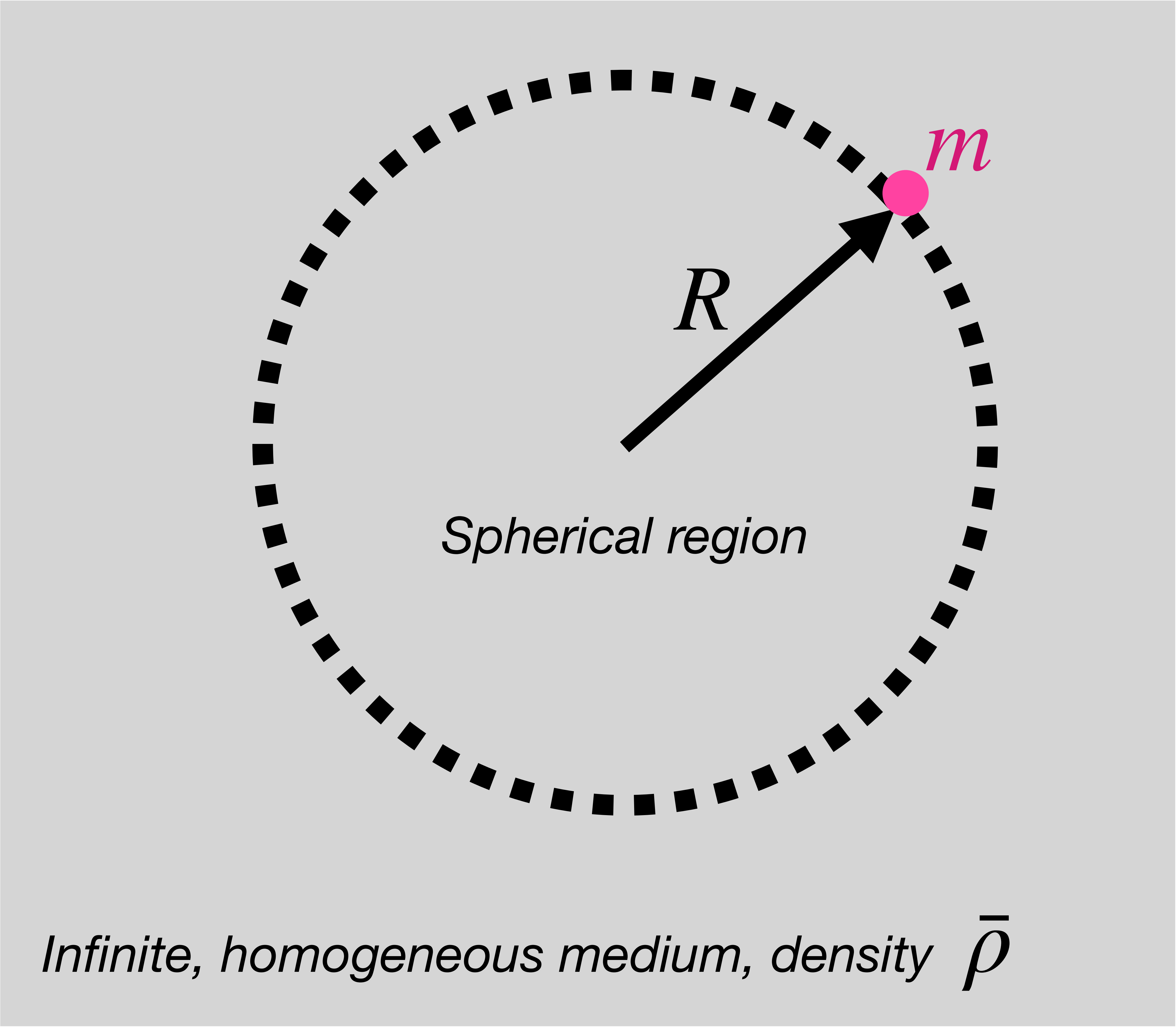
Newtonian Derivation

The idea is to consider a small mass m at a distance $R(t)$ from the center of a spherical region in an homogeneous medium enclosing a total mass M with density $\bar{\rho}$.

The mass enclosed at R is $M = \dfrac{4}{3}\pi R^3 \bar{\rho}$.

By Newton's shell theorem, the gravitational force on the small mass is due to M (the exterior forces cancel out).

This is also true in GR (Birkhoff's theorem).



Newtonian Derivation

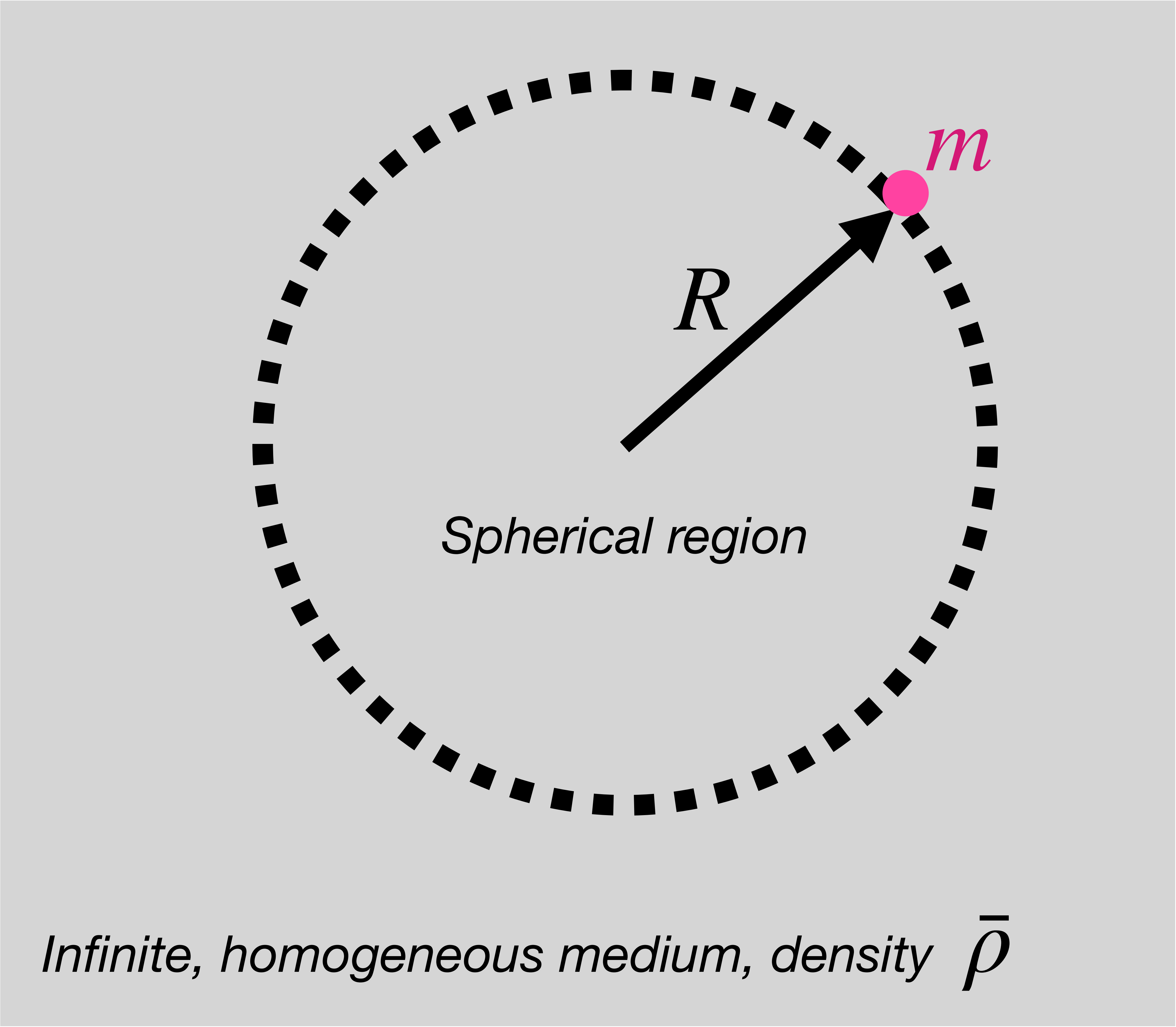
Imagine the small mass has some total energy E *(we will worry about where the energy comes from in a moment).*

In simple Newtonian mechanics terms:

$E = KE + PE$

Our task is to write down terms for the KE and PE.

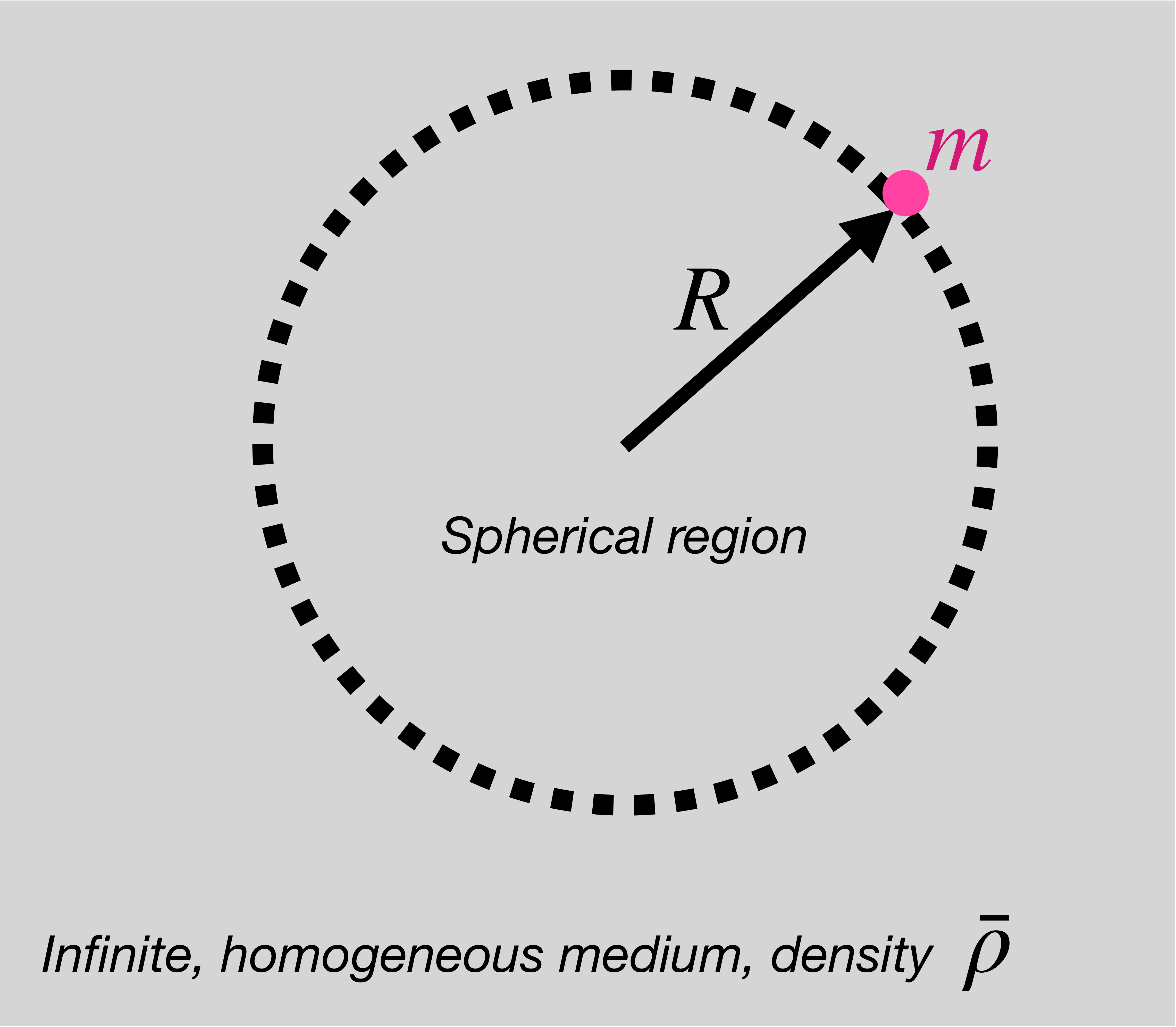
This is exactly the same problem as a ball thrown straight up from the surface of a planet.



Newtonian Derivation

As the test mass moves, the radius will change.

Take the total mass M within the sphere to be constant. In that case, the density is a function of time, $\bar{\rho}(t)$.



Newtonian Derivation

Starting with Newton's 2nd law for the test mass: $m\ddot{R} = -\dfrac{GMm}{R^2}$

If we integrate this with respect to time, we will have an expression for the rate of expansion of the spherical region, $\dot{R}$. To do this integral, note that:

$$\int \dot{R}\ddot{R}dt = \frac{1}{2}\dot{R}^2 \text{ and } \int -\frac{\dot{R}}{R^2}dt = \frac{1}{R}$$

Hence multiplying both sides of the 2nd law equation by $\dot{R}$ and integrating gives

$$\frac{1}{2}\dot{R}^2 = \frac{GM}{R} + \epsilon$$

Where ϵ appears as a constant of the integration. This is just conservation of mechanical energy, "$E = KE + PE$".

Newtonian Derivation

Expanding $M = \frac{4}{3}\pi R^3 \bar{\rho}$ and rearranging,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\bar{\rho} + \frac{2\epsilon}{R^2}$$

and writing the radius of the sphere as the product of a scale factor and comoving radius,

$$R = aR_0 \implies \dot{R} = \dot{a}R_0$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho} + \frac{2\epsilon}{a^2 R_0} = \frac{8\pi G}{3}\bar{\rho} + \frac{U}{a^2}\text{, with } U = \frac{2\epsilon}{R_0^2}$$

Newtonian Derivation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho}(t) + \frac{U}{a^2}$$

We can see that there are 3 possible scenarios for the expansion of the sphere (in the case where the right-hand side is positive at $t = 0$, i.e. the sphere starts off expanding):

$U > 0 \implies$ RHS always $> 0 \implies \dot{a} > 0$ always — endless expansion

$U < 0 \implies$ RHS negative at finite $a \implies$ turnaround and collapse in finite time

$U = 0 : \dot{a} \to 0$ when $\rho = 0$ at $t \to \infty \implies$ boundary between endless expansion and collapse

Newtonian Derivation

The Newtonian derivation is valid but obscures some important details. The full GR derivation accounts for the contribution of energy density, not just mass density, and also naturally introduces a contribution from curvature.

Comparing the GR result, $H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{a^2}$,

to the Newtonian result, $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho} + \frac{U}{a^2 R_0^2}$,

we see the implied substitutions are $\bar{\rho}(t) \rightarrow \varepsilon(t) = \bar{\rho}c^2$ and $U \rightarrow -kc^2 = -\frac{\kappa c^2}{R_0^2}$, with $\kappa = \{-1,0,1\}$.

Energy Density

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{a^2}$$

Recall $E^2 = m^2c^4 + p^2c^2$

For $v \ll c$, we have $p \approx mv$, hence in the non-relativistic case $E \approx mc^2\,(1 + v^2/c^2)^{1/2} \approx mc^2 + \frac{1}{2}mv^2$.

A universe of slowly-moving particles with mass will have $\bar{\varepsilon} \simeq \frac{\bar{\rho}}{c^2}$.

A universe of massless particles moving at the speed of light has energy density $\bar{\varepsilon} = pc = h\nu$.

A universe with **multiple components** will have $\varepsilon_{tot} = \varepsilon_1 + \varepsilon_2 + \dots$

Curvature

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{a^2}$$

In GR, the curvature term does the same job as the "total energy" in the Newtonian derivation. Remember that $k = -\kappa c^2 / R_0$ is fixed.

The balance between this **curvature term** and the density term determines the ultimate fate of the universe.

Critical Density

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{a^2}$$

If $\kappa = 0$ (flat geometry) then the density must be

$$\frac{\varepsilon_{\text{crit}}}{c^2} = \rho_{\text{crit}}(t) = \frac{3H^2(t)}{8\pi G}$$

This is called the **critical density for closure** (or just **critical density**). For the Universe to be geometrically flat, the density must have exactly this value.

The critical density depends on time. Since the sign of the curvature is an intrinsic property that can't change over time, a flat universe must **always** have the critical density.

Critical Density

The present-day value of the critical density is

$$\rho_{\mathrm{crit},0} = \frac{3H_0^2}{8\pi G} = 2.78\, h^2 \times 10^{11}\,\mathrm{M_\odot\,Mpc^{-3}}$$

$$\approx 1.36 \times 10^{11}\,\mathrm{M_\odot\,Mpc^{-3}} \text{ for } h = 0.7$$

This is approximately equivalent to one hydrogen atom per cubic meter, or a typical spacing of $\sim 1\,\mathrm{Mpc}$ between galaxies with the total mass of the Milky Way ($\sim 10^{12}\,\mathrm{M_\odot}$) — since we observe more or less this density of galaxies, we might guess the Universe is *pretty close* to the critical density.

Total Density Parameter

The critical density ρ_c provides a meaningful scale for cosmic density: the fate of the Universe depends on the ratio of the actual density to ρ_c.

It is therefore conventional to write the Friedman equation in terms of the **density parameter:**

$$\Omega(t) \equiv \frac{\bar{\rho}(t)}{\rho_{\mathrm{crit}}(t)}$$

This is the **total** density parameter — the density at a given time, relative to $\rho_{\mathrm{crit}}(t) = \dfrac{3H^2(t)}{8\pi G}$ at that time.

Total Density Parameter

With the definition of the density parameter, $\Omega(t) \equiv \dfrac{\rho(t)}{\rho_{\text{crit}}(t)}$, we can write the Friedmann equation as:

$$1 - \Omega(t) = -\frac{\kappa c^2}{R_0^2 a^2 H^2(t)}$$

Since the sign of κ cannot change, and the denominator on the righthand side is always positive, the sign of the righthand side — and hence the left hand side — cannot change with time.

If $\Omega(t) > 1$ at any time, the Universe has a closed geometry (positive curvature) and will always have that geometry. Likewise if $\Omega(t) < 1$, the Universe will always have an open geometry (negative curvature).

And, of course, a flat universe, $\Omega(t) = 1$, is always flat.

Note this means we can measure the curvature, in principle, if we can measure $\Omega(t = t_0)$: $\dfrac{\kappa}{R_0^2} = \left(\dfrac{H_0}{c}\right)^2 (\Omega_0 - 1)$

Total Density Parameter

However, to measure the cosmic density, we need to know **all** the relevant contributions!

For a long time (up to the late 1990s), the (reasonable) assumption was that the most important contribution was the visible, gravitating matter in galaxies. This led to the conclusion that $\Omega \sim \Omega_{m,0} \approx 0.3$, and hence that the Universe is **open** (negative curvature).

However, we now know we were missing a **very important contribution** to the total energy density — **dark energy**. We will talk about this later.

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{a^2} +$$ 

Current measurements suggest $\Omega \simeq 1$ to a precision of $\lesssim 1\,\%$, i.e. the Universe is extremely close to flat.

Solutions to the Friedmann Equation

Friedmann Equation: example solution

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{a^2}$$

To use the Friedmann equation to predict $a(t)$, we first need to know how $\rho(t)$ changes with time. We will look at this in more detail later.

As a quick example, **ordinary matter** is <u>defined</u> (in this context) as non-relativistic "stuff", the total mass is of which is fixed, and hence the density of which scales in direct proportion to the volume, i.e. $\rho_m(t) \propto a(t)^{-3}$.

This is the kind of "stuff" that we are made of, and the stars and gas in galaxies — most of this "stuff" is hydrogen atoms. This "stuff" also includes the "dark matter", which we will meat later.

We will further assume the Universe is flat. With these assumptions, we have the <u>specific</u> Friedmann equation for a universe that **<u>only</u>** contains ordinary matter:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^3}$$

Friedmann Equation: example solution

Since this Universe is flat, we can use the definition of the critical density, $\rho_{\mathrm{crit},0} = \dfrac{3H_0^2}{8\pi G}$, to write

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2}{a^3}.$$

It's pretty easy to solve this to find $a(t)$. We have $\dfrac{da}{dt} = H_0 a^{-1/2}$ and hence

$$\int a^{1/2} da = H_0 \int dt,$$

from which it follows that

$$a(t) = \left(\frac{3}{2} H_0 t\right)^{2/3}$$

Friedmann Equation: example solution

$$a(t) = \left(\frac{3}{2}H_0 t\right)^{2/3}$$

When $a = 1$, $t = t_0 = \dfrac{2}{3H_0}$, so we can finally write the scale factor (expansion history) in a **flat, matter-dominated Universe** (often called the **Einstein-de Sitter model**) as:

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

This is a very useful and simple result to remember: in a flat, matter-dominated Universe, the expansion factor scales with time as $t^{2/3}$, and the age of the universe is $\dfrac{2}{3}t_H$. For $h = 0.7$, $t_0 \sim 9.8\,\mathrm{Gyr}$ (a rather low value!).

The derivations of $a(t)$ for different components of the total density will get more complicated later, so it's worth <u>making sure you understand this very simple case!</u>

Friedmann Equation: example solution

We have shown that a universe in which the density of **ordinary matter** is equal to the **critical density** will **expand**.

Is this expansion speeding up or slowing down with time?

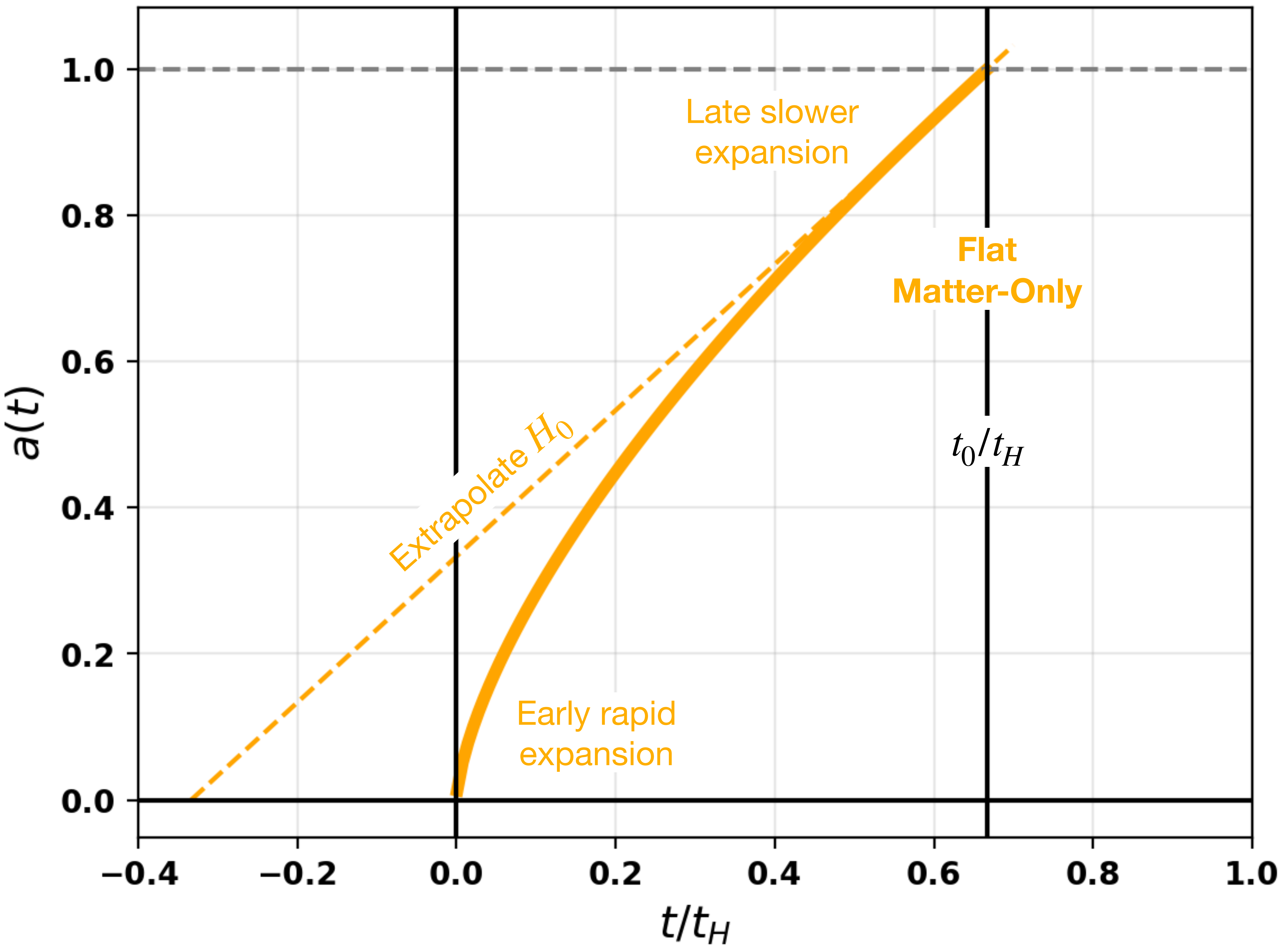
$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

$$\dot{a}(t) = \frac{2}{3t_0}\left(\frac{t}{t_0}\right)^{-1/3} = H_0\left(\frac{t}{t_0}\right)^{-1/3} = H_0\, a^{-1/2}$$

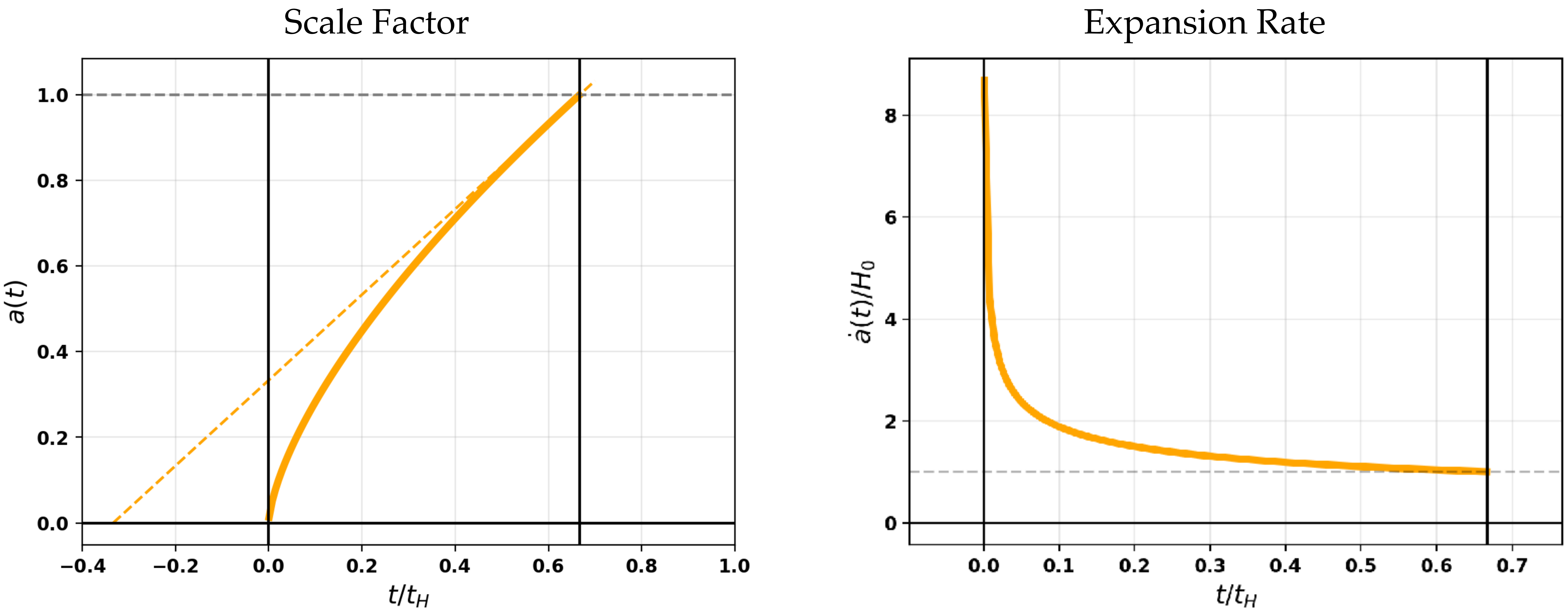
$$\ddot{a}(t) = \frac{d}{da}\frac{da}{dt}\dot{a} = \left(\frac{H_0}{2}a^{-3/2}\right)\left(H_0 a^{-1/2}\right) = -\frac{H_0^2}{2a^2}$$

So the expansion is always **decelerating**, but the rate of deceleration decreases with time.

Friedmann Equation: example solution



Friedmann Equation: example solution



The Fluid Equation

The Friedman equation contains two unknown function: to solve it, we need to write $\varepsilon(t)$ in terms of $a(t)$, i.e. specify how (energy) density changes as the volume of the universe increases (or decreases).

In the simple example, we stated the relationship between density and scale factor for non-relativistic ordinary matter: $\rho(t) \propto a^{-3}$. We know there are other contributions to the energy density $\varepsilon(t)$ — for example, radiation.

The Friedman equation expresses conservation of mechanical energy; we can also consider conservation of energy more generally though the first law of thermodynamics:

$$dE = dQ - PdV$$

dQ : heat transfer, PdV: mechanical work (pressure P, change in volume dV).

Adiabatic Expansion

$$dE = dQ - PdV$$

No heat can flow into the Universe, out of the Universe, or from place to place within the Universe (to preserve homogeneity). Hence $dQ = 0$, the expansion is **adiabatic**.

Therefore, as the Universe expands, we have:

$$\frac{dE}{dt} + P\frac{dV}{dt} = 0$$

where P is the total pressure associated with all the components of the energy density.

Adiabatic Expansion

Consider a spherical region of the Universe (as previously) of radius $R(t) = a(t)R_0$. The total energy in that region is related to the volume and energy density by:

$$E(t) = \varepsilon(t)V(t).$$

As the sphere expands, the rate of change of the total energy is therefore

$$\dot{E} = V\dot{\varepsilon} + \dot{V}\varepsilon.$$

Hence from the first law of thermodynamics:

$$V\dot{\varepsilon} + \dot{V}(\varepsilon + P) = 0$$

The Fluid (or Continuity) Equation

The volume is $V = \frac{4}{3} a^3 R_0^3$, so $\dot{V} = \frac{4}{3}\pi R_0^3 (3\dot{a}^2 a) = 3\left(\frac{\dot{a}}{a}\right) V = 3HV$.

Therefore $V\dot{\varepsilon} + \dot{V}(\varepsilon + P) = 0$ becomes

$$V(\dot{\varepsilon} + 3H\varepsilon + 3HP) = 0$$

And hence we have a differential equation expressing the **conservation of energy** (i.e. **continuity of energy density**):

$$\dot{\varepsilon} + 3H(\varepsilon + P) = 0$$

This equation is "relativistically correct", no further modifications needed for GR.

The Acceleration Equation

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{a^2} \qquad \textit{(Friedmann Equation)}$$

$$\dot{\varepsilon} + 3H(\varepsilon + P) = 0 \qquad \textit{(Fluid Equation)}$$

With a bit of algebra, we can combine these two equations into a third equation that describes the rate at which the expansion **accelerates** or decelerates with time:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) \qquad \textit{(Acceleration Equation)}$$

Positive ε causes the expansion to slow down over time (provided $P < -(1/3)\varepsilon$; but what kind of substance would have *negative* pressure…?)

Equations of state

$$\dot{\varepsilon} + 3H(\varepsilon + P) = 0$$

If we know $P(\varepsilon)$ then we can solve the fluid equation for $\varepsilon(t)$ and hence solve the Friedmann equation. So now we have shifted the problem to specifying $P(\varepsilon)$, called the **equation of state**.

When we talk about different "components" of the energy density, we really mean components with different equations of state.

Although equations of state can be very complicated in principle, all those relevant to cosmology have a very simple general form:

$$P = w\varepsilon$$

where w is a dimensionless constant,

$$w = \frac{P}{\rho c^2}$$

Different components have different w.

Equations of state

We can re-write the continuity equation in terms of w:

$$\dot{\varepsilon} = -3H\varepsilon(1+w)$$

This doesn't look too hard to solve. Notice that $H = \dot{a}/a \implies \dfrac{d}{dt} = H\dfrac{d}{d\ln a}$ which means we can write

$$\frac{1}{H}\frac{1}{\varepsilon}\frac{d\varepsilon}{dt} = \frac{1}{H}\frac{d\ln\varepsilon}{dt} = \frac{d\ln\varepsilon}{d\ln a} = -3(1+w)$$

and therefore

$$\varepsilon = \varepsilon_0\, a^{-3(1+w)}$$

equivalently, $\rho = \rho_0\, a^{-3(1+w)}$

Of course, we can only solve the integral like this if w is **constant**. For all the components we will consider, w is constant — there might be more exotic components that break this rule, we just don't know about them yet!

Equation of state for non-relativistic matter

For a "gas" of non-relativistic matter, $w \simeq 0$. We can determine this as follows.

When we say "non-relativistic", we mean the velocities of the particles in the gas are $v \ll c$. The ideal gas equation is $P = \dfrac{\rho}{\mu} kT$, for mean molecular eight μ and temperature T. For a non-relativistic gas, $\varepsilon \simeq \rho c^2$ and therefore:

$$P \simeq \frac{kT}{\mu c^2} \epsilon$$

The temperature is related to the RMS speed of the gas particles by $3kT = \mu \langle v^2 \rangle$, so

$$P \simeq \frac{\langle v^2 \rangle}{3c^2} \varepsilon = w\varepsilon$$

with $w \ll 1 \sim 0$ by definition. So for non-relativistic matter, $\rho = \rho_0\, a^{-3}$ as we used earlier. Notice there is no way the pressure can be negative in this case.

Check: acceleration in the EdS model

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) = -\frac{4\pi G}{3c^2}\varepsilon(1 + 3w)$$

Therefore, for non-relativistic matter ($w \simeq 0$):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\varepsilon$$

$$\implies \ddot{a} = -\frac{4\pi G}{3c^2}\rho_0 a^{-3} a = -\frac{H_0^2}{2a^2}$$

as we had earlier. Once again, the key fact here is that the number (hence mass) of non-relativistic matter is conserved, so its density just goes down (or up) in the simplest possible way as the volume it occupies increases or decreases.

Note: the "dark matter" we will introduce later is also thought to be non-relativistic matter with $w \simeq 0$.

Radiation

"Radiation" means massless, relativistic "particles" that propagate at the speed of light.

This includes photons, but also things like massless neutrinos (in principle, although it's now known that neutrinos have mass).

For radiation, $w = 1/3$, so $\varepsilon_\gamma = \varepsilon_{\gamma,0}\, a^{-3(1+1/3)} = \varepsilon_{\gamma,0}\, a^{-4}$. What's going on here?

Radiation

The energy of a photon is $E_\gamma = h\nu = \dfrac{hc}{\lambda}$. In our discussion of redshift we said that photon wavelengths were stretched with the expansion as $\lambda \propto a$, so their energy must go down as $E_\gamma \propto a^{-1}$. Thus the energy density of photons must scale as

$$\varepsilon_\gamma = \frac{N_\gamma}{V} E_\gamma \propto a^{-4}$$

where N_γ is the number density of photons. The extra "redshifting" effect means the photon energy density **goes down faster with the expansion compared to the matter density**.

Note that $w = 1/3$ implies there is a pressure (or "momentum density") $P_\gamma = \dfrac{1}{3}\rho c^2$ associated with radiation — you might encounter this "radiation pressure" elsewhere, for example in discussions of stellar structure. Particles that have mass but move with $v \sim c$ have $0 < w < 1/3$.

Radiation and matter

Since we know that our Universe contains non-relativistic matter and radiation, the simplest "realistic" Friedmann equation we might have is the flat case with these two components:

$$H^2(t) = \frac{8\pi G}{3c^2}(\varepsilon_m + \varepsilon_\gamma) = \frac{8\pi G}{3c^2}(\rho_{m,0}a^{-3} + \rho_{\gamma,0}a^{-4})$$

We can use the definition of the density parameter to write this in a more compact way:

$$H^2 = H_0^2\left[\Omega_{m,0}a^{-3} + \Omega_{\gamma,0}a^{-4}\right]$$

Remember that for a flat universe, $\Omega = \Omega_m + \Omega_\gamma + \ldots = 1$ always.

Since $\frac{\kappa}{R_0^2} = \left(\frac{H_0}{c}\right)^2(\Omega_0 - 1)$ we can easily account for the curvature in this simple form as well:

$$H^2 = H_0^2\left[\Omega_{m,0}a^{-3} + \Omega_{\gamma,0}a^{-4} + (1 - \Omega_0)a^{-2}\right]$$

From radiation to matter

The fact that radiation and matter densities scale at different rates has very important implication for how the Universe evolved at early times.

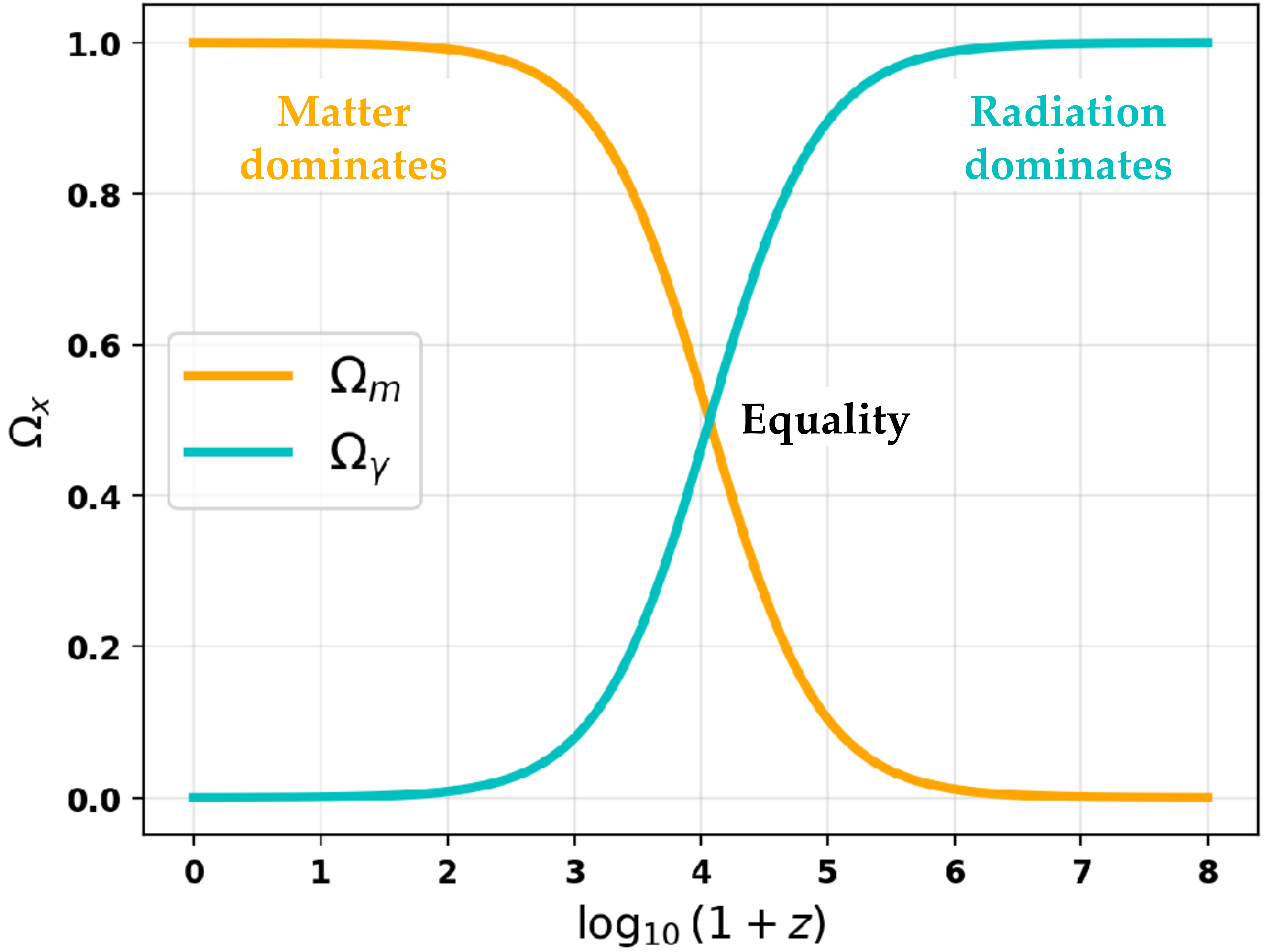
We can measure $\Omega_{m,0}$ and $\Omega_{\gamma,0}$ today; we therefore know that $\Omega_{\gamma,0} \ll \Omega_{m,0}$. We will say more about this later (see e.g. 5.1 in Ryden).

However, as we go **back** in time towards the big bang, **radiation** becomes more and more important, i.e. $\rho_\gamma(t)$ goes up faster than $\rho_m(t)$ as $t \to 0$. At early times $\Omega_\gamma \gg \Omega_m$.

There must therefore be a time when their contributions to the expansion "cross over".

We call this the time of **matter-radiation equality**. The time before this is called the **radiation-dominated era** and the time afterwards is called the **matter-dominated era**.

From radiation to matter



Proper Distances

Now that we can solve the Friedmann equation for $a(t)$, we can go back to the FLRW metric and compute expressions for physical distances in specific expanding universe models.

We will do this in more detail next time — this is just a quick preview.

First recall $adr = cdt \Longrightarrow \int_r^0 dr = D_p(t) = c\int_t^{t_0} \frac{dt}{a}$ **[Always take a good look at the limits on these integrals]**

(In the last lecture we used s_{proper} as the symbol for the proper distance; we now use $D_p(t)$ — the proper distance to the source of a light signal emitted at time t and received at $t = t_0$.)

Now we know how to obtain $a(t)$. For the EdS model, we have $a(t) = \left(\frac{t}{t_0}\right)^{2/3}$ with $t_0 = \frac{2}{3H_0}$.

Distances

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \implies \frac{da}{dt} = \frac{2}{3t_0}\left(\frac{t}{t_0}\right)^{-1/3} \implies dt = \frac{\sqrt{a}}{H_0}da, \text{ hence } D_p(a) = \frac{c}{H_0}\int_a^1 a^{-1/2} = 2\frac{c}{H_0}\left[1 - \sqrt{a}\right]$$

The integral could also be done in terms of t, although it's usually the case that we want the answer in terms of expansion factor or redshift, and doing the integral directly in a can often be easiest.

So for a flat Universe, the instantaneous physical distance **now** between us and a galaxy with redshift z is:

$$D_p(z=0) = \frac{2c}{H_0}\left[1 - (1+z)^{-1/2}\right]$$

When the light was **emitted**, the physical distance between us and the galaxy was $D_p(z) = aD_p(z=0) = \dfrac{D_p(z=0)}{1+z}$.

Particle Horizon Distance

$$D_p(z=0) = \frac{2c}{H_0}\left[1 - (1+z)^{-1/2}\right]$$

Notice that there is a maximum physical distance from which we can receive light at the present day. The light reaching us now from that maximum distance has infinite redshift:

$$D_{\text{hor}}(z=0) = \frac{2c}{H_0}.$$

This is equivalent to saying $D_{\text{hor}} = \int_0^{t_0} \frac{dt}{a(t)}$. Nothing beyond this distance can have an *causal* effect on us.

Notice that $D_{\text{hor}} \sim D_H$, the Hubble distance, but they are not identical. They are often mixed up.

Summary

The cosmic expansion predicted by GR is described by the Friedmann equation:

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\bar{\varepsilon}(t) - \frac{\kappa c^2}{a^2}$$

This implies a critical cosmic density associated with flat geometry: $\rho_{\rm crit}(t) = \dfrac{3H^2(t)}{8\pi G}$

To solve, we need to specify $\varepsilon = \varepsilon_0\, a^{-3(1+w)}$ for one or more components of the energy density; for example, **non-relativistic matter has** $w = 0$, hence its density scales as a^{-3}.

For a flat, **matter-dominated** universe (EdS model) $a(t) = \left(\dfrac{t}{t_0}\right)^{2/3}$ with $t_0 = \dfrac{2}{3H_0}$.

We can proceed from this, with the FLRW metric, to derive an equation for the **proper distance as a function of redshift**.

The energy density of **radiation** scales as a^{-4}, such that it becomes relatively more important in driving the expansion at earlier times.

Next time

Friedman models with multiple components.

How do distances based on size and luminosity change with redshift in different models?

The Cosmological Constant.

For next time: Ch. 5 in Ryden (+6.1, 6.2, 6.3); 3.3, 3.4 in Huterer