Introduction to Cosmology

ASTR 434

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Lecture 4 / Friedmann Models / October 8th 2024

Recap

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\bar{\varepsilon}(t) - \frac{kc^2}{a^2} = \frac{8\pi G}{3}\bar{\rho}(t) - \frac{kc^2}{a^2}$$

$$\rho_{\text{crit}}(t) = \frac{3H^2(t)}{8\pi G}$$

($\rho = \rho_{\text{crit}} \implies$ flat geometry)

$$\frac{\rho_i(t)}{\rho_{\text{crit}}(t)} \equiv \Omega_i(t)$$

where "i" labels a contribution to the mass density with a particular equation of state.

For simple EOS, "$P = w\varepsilon$" with constant w, $\rho_i(t) = \rho_{0,i}\, a^{-3(1+w_i)}$ *(e.g. $w \approx 0$ for non-relativistic matter)*

Recap

The Friedmann equation says $H^2(t) = H_0^2 \Omega(t) - \dfrac{kc^2}{a^2} = H_0^2 \left[\Omega(t) - \dfrac{kc^2}{H_0^2 a^2} \right]$, where $\Omega = \Omega_m + \Omega_r + \ldots$ for all the energy density contributions.

We can define a *curvature density parameter* $\Omega_\kappa = 1 - \Omega = 1 - \Omega_m + \Omega_r + \ldots$

So a easy-to-remember and general version of the Friedmann equation is:

$$H^2 = H_0^2 \left[\Omega + \Omega_k \right] = H_0^2 \left[\Omega_{m,0}\, a^{-3} + \Omega_{r,0}\, a^{-4} + \ldots + \Omega_{k,0}\, a^{-2} \right]$$

(Later we will fill in the … with another term, $\Omega_{\Lambda,0}$)

We can see that $\Omega_{k,0} = 1 - \Omega_0 = -\dfrac{kc^2}{H_0^2} = -\dfrac{\kappa c^2}{H_0^2 R_0^2}$, so **positive curvature** $\implies \kappa > 0 \implies \Omega_k < 0$.

A quick summary of distances (so far)

Comoving distance of an object seen at redshift z (= physical distance at $t = t_0$):

$$r(z) = c\int_{t_0}^{t} \frac{1}{a(t)}dt$$

Proper distance to object at the time the light was emitted: $D'_p(z) = \dfrac{r(z)}{1+z}$

Hubble length: $d_H = c/H(t)$

Particle Horizon: $r_{\rm hor}(z) = c\displaystyle\int_{t_0}^{\infty} \frac{1}{a(t)}dt \;\; = \frac{2c}{H_0}$ for matter-only "EdS" universe.

A small clarification

Last time, I talked about the limits on integrals like $r(z) = c \int_{t_0}^{t} \frac{1}{a(t)} dt$.

Here t is the "cosmic clock", so $t = 0$ is the Big Bang and $t = t_0 \sim 13.6\,\mathrm{Gyr}$ is the present day age of the Universe.

As written above, this is the integral **<u>from</u>** t_0 **<u>to</u>** t. Of course, the light signal really propagates the other way, **from** the source galaxy at t **to** the observer (us) at $t = t_0$.

The integral as written is "backwards" in that sense, although it is not incorrect. If we wrote the limits the other way around, we would end up with a **negative** comoving distance.

That's OK, because the distance is really the length r of the comoving radius vector $\mathbf{r} \equiv r\hat{\mathbf{r}}$, so a "negative" distance is just a distance measured **towards the observer** rather than **away from the observer**.

A quick summary of times (so far)

Age of the universe: $t_0 = \dfrac{2}{3(1+w)H_0}$ (for a single-component universe with constant w)

The **lookback time**, $t_{\rm lbk}$, is the difference between t_0 and some cosmic time t in the past, i.e.

$$t_{\rm lbk} \equiv t_0 - t$$

Lookback time answers questions like "how long ago did X happen" (as opposed to the cosmic clock time t, which answers questions like "how old was the universe when X happened").

Small note: Can we really have a universal cosmic clock in GR?

In the case of an FLRW metric, the answer is yes — for example, because the cosmological principle means that every observer has to measure the same mean density $\varepsilon(t)$, so (in principle) everyone can agree on what cosmic time it is just by measuring their local density, without having to talk to anyone else.

This lecture: Friedmann Models

- Angular size distance
- Curved matter-only universes
- Luminosity distance
- Radiation-only universes
- Matter+radiation universes
- An empty universe
- Introducing the cosmological constant

Important point of today's lecture: *a realistic expansion history $a(t)$ has multiple "stages" corresponding to different energy density components that dominate the Friedmann equation at different times.*

Pictures of Galaxies

Pictures of Galaxies

ESA/Webb, NASA & CSA, A. Martel

Pictures of Galaxies

ESA/Webb, NASA

Distances

Apparent Distance

The photons we receive from distant galaxies have travelled across expanding (and possibly curved) space.

This produces some interesting and observationally significant “distortions” into our view of those galaxies.

Specifically, it changes the relationships between their **apparent size**, **apparent brightness** and **distance.**

Apparent Distance

We already saw that, in an expanding universe, far-away galaxies look closer than they really are today. Specifically, in the case of an **EdS universe:**

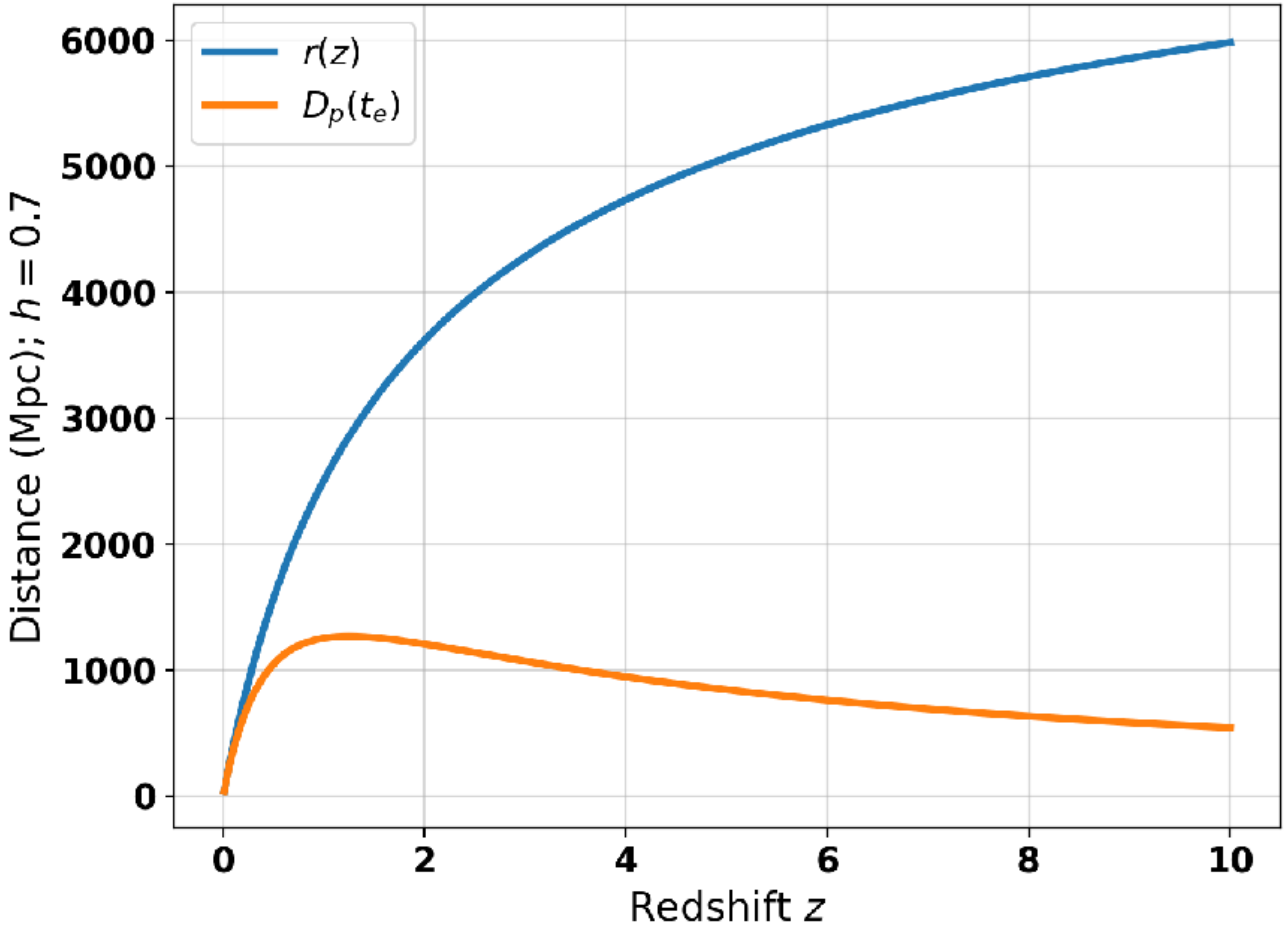
$$D_p = \frac{2c}{H_0}\left[1 - (1+z)^{-1/2}\right]$$

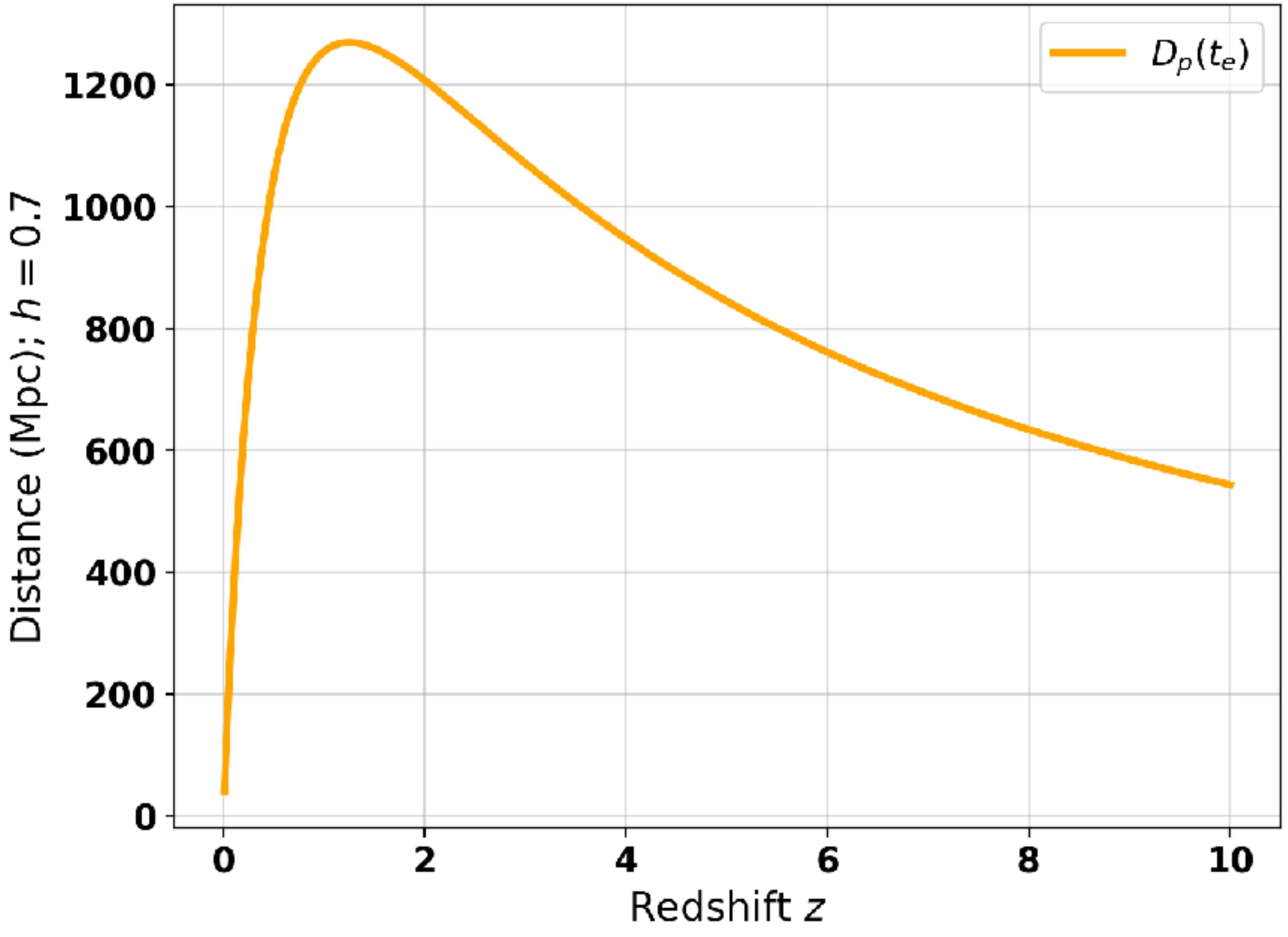
A galaxy observed with a redshift $z = 3$ is at a **present-day** physical distance $D_p = c/H_0$ (this is also its comoving distance, by definition).

When the light left the galaxy, the distance between us the galaxy was $D_p' = \dfrac{D_p}{1+z} = \dfrac{c}{4H_0}$.

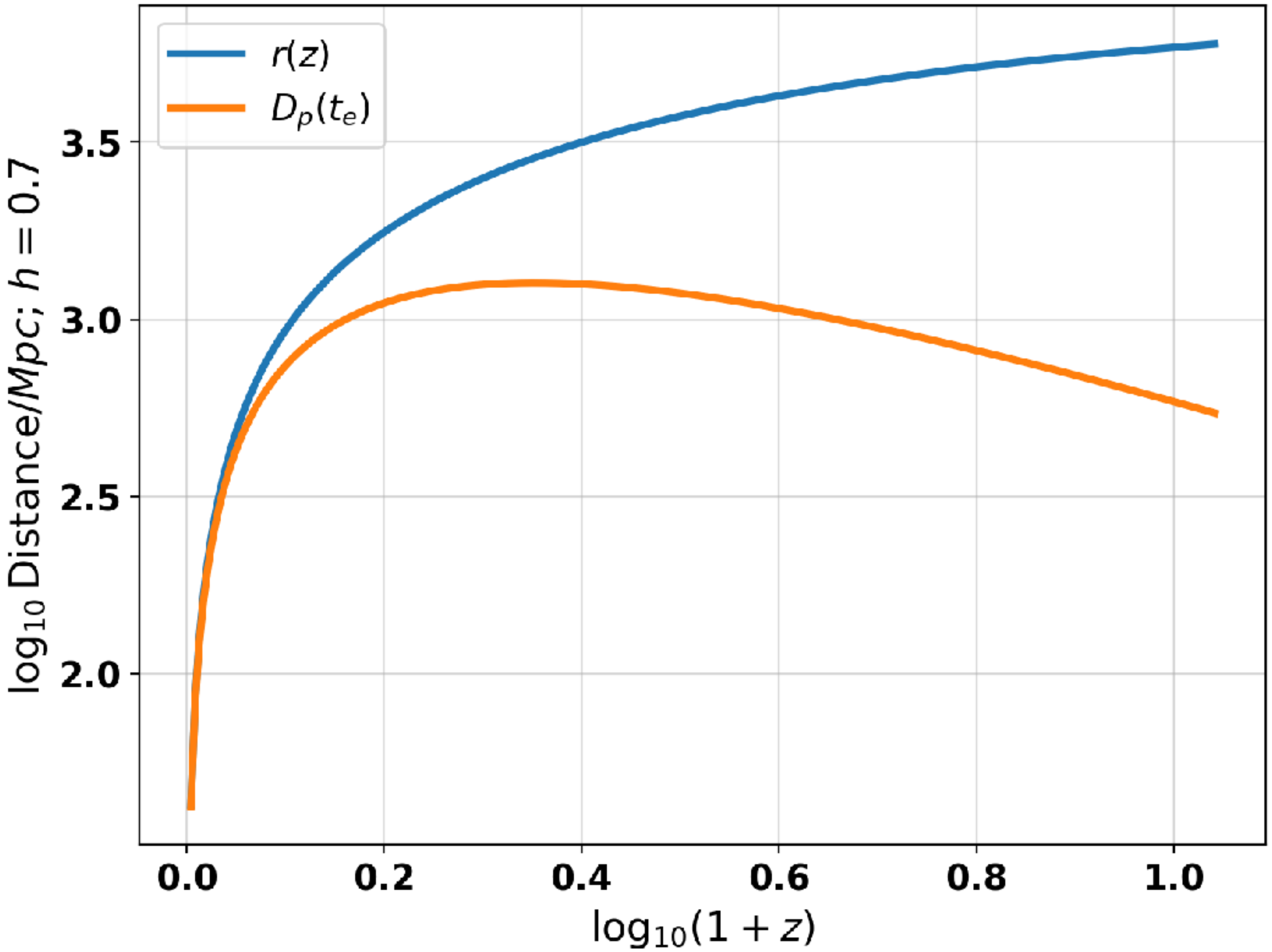
We learn about the galaxy from the light we receive. So the galaxy “looks like” it is at a distance 1/4 of its actual distance today. Only when $z \ll 1$ do galaxies “look like” they are as far away as they actually are ($D_p' \sim D_p$).

Apparent Distance vs. redshift, EdS





Apparent Distance (log scales)



Standard rulers and candles

Light signals emitted in the early universe take longer to reach us because of the expansion.

If the signal is sent out at a time when the expansion is very rapid, it will take *even longer* to arrive, so the **apparent** distance will be even less.

The lengths of rulers and the brightness of lightbulbs depend on this **apparent** distance.

So, in an FLRW metric, the relationship between **redshift** and the apparent sizes of rulers or apparent brightness of lightbulbs **depends on the expansion history.**

Standard rulers and candles

We might guess that we can get the correct expressions for size or brightness by replacing the “true” (we should say “comoving”) distance in the Euclidean equations by the **apparent** distance D_p' .

However, that guess would be correct **only** in the case of **angular sizes** in a **flat** universe.

Things are more complicated in all other cases (including **brightness** in a flat but expanding universe, and angular size in a **curved** universe).

To work out the correct expressions for size and brightness as a function of **comoving distance** or **redshift**, we need to use the FLRW metric.

Angular Sizes

$$ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + S_\kappa^2(r)\,d\Omega^2\right]$$

(Note: this $d\Omega$ is a differential solid angle, nothing to do with density parameters!)

$$S_\kappa(r) = \begin{cases} R_0 \sin(r/R_0), & \text{positive curvature} \\ r, & \text{zero curvature} \\ R_0 \sinh(r/R_0) & \text{negative curvature} \end{cases}$$

The length of a "ruler" transverse to our line of sight at a comoving distance is related to its angular size $d\theta$ by

$$ds = a(t)\,S_\kappa(r)\,d\theta$$

(Where t is the time re observe the ruler; it might help to think of light signals sent from both ends of the ruler, separated by a distance $d\theta$)

Angular Sizes

In Euclidean space, we have the angular diameter distance $D_A \equiv \dfrac{\Delta s}{\Delta \theta}$

In FLRW space we have $\Delta s = a(t)\, S_\kappa(r)\, \Delta\theta$

For $\kappa = 0$ this gives the result we expected: $D_A \equiv \dfrac{\Delta s}{\Delta \theta} = \dfrac{r}{1+z}(= D_p')$

i.e. the distance we would infer from observing the angular size of a standard ruler (light signals from two points with a known transverse separation) **is <u>not</u> the current proper distance**.

Instead, it is the proper distance *at time the light signal was emitted*.

(i.e. the FLRW metric supports our earlier guess that this was the case…)

Angular Sizes

However, for **positively curved** space, $\kappa = 1$, we have

$$D_A = \frac{R_0 \sin(r/R_0)}{1+z}$$

When $r \ll R_0$, this is approximately the flat-space formula. But for $r/R_0 \to \pi/2$, $D_A \to \dfrac{R_0}{1+z}$.

From earlier, $\dfrac{1}{R_0^2} = \left(\dfrac{-1}{\kappa}\right)\left(\dfrac{H_0}{c}\right)^2(1-\Omega_0) = \left(\dfrac{-1}{\kappa}\right)\dfrac{\Omega_{k,0}}{d_{H,0}^2}$; we can use this to write $R_0 = \dfrac{d_{H,0}}{\sqrt{|\Omega_k|}}$

e.g. if $\Omega_{k,0} = 1 - \Omega_0 = 0.7$, $R_0 = 4283\,\mathrm{Mpc}\,h^{-1}/\sqrt{0.7} = 4283/\left(0.7\sqrt{0.7}\right) \sim$ 5119 Mpc for $h = 0.7$.

Models with matter and curvature

$$D_A = \frac{R_0 \sin(r/R_0)}{1+z}\ (\kappa = 1) \text{ or } D_A = \frac{R_0 \sinh(r/R_0)}{1+z}\ \ (\kappa = -1),\ \ R_0 = \frac{d_{H,0}}{\sqrt{|\Omega_k|}}$$

To find D_A we still need $r(z)$. Using the Friedmann equation, we found $r(z) = \dfrac{2c}{H_0}\left[1 - (1+z)^{-1/2}\right]$.

However, this was for a flat, matter-dominated model, $\Omega_0 = \Omega_{m,0} = 1$.

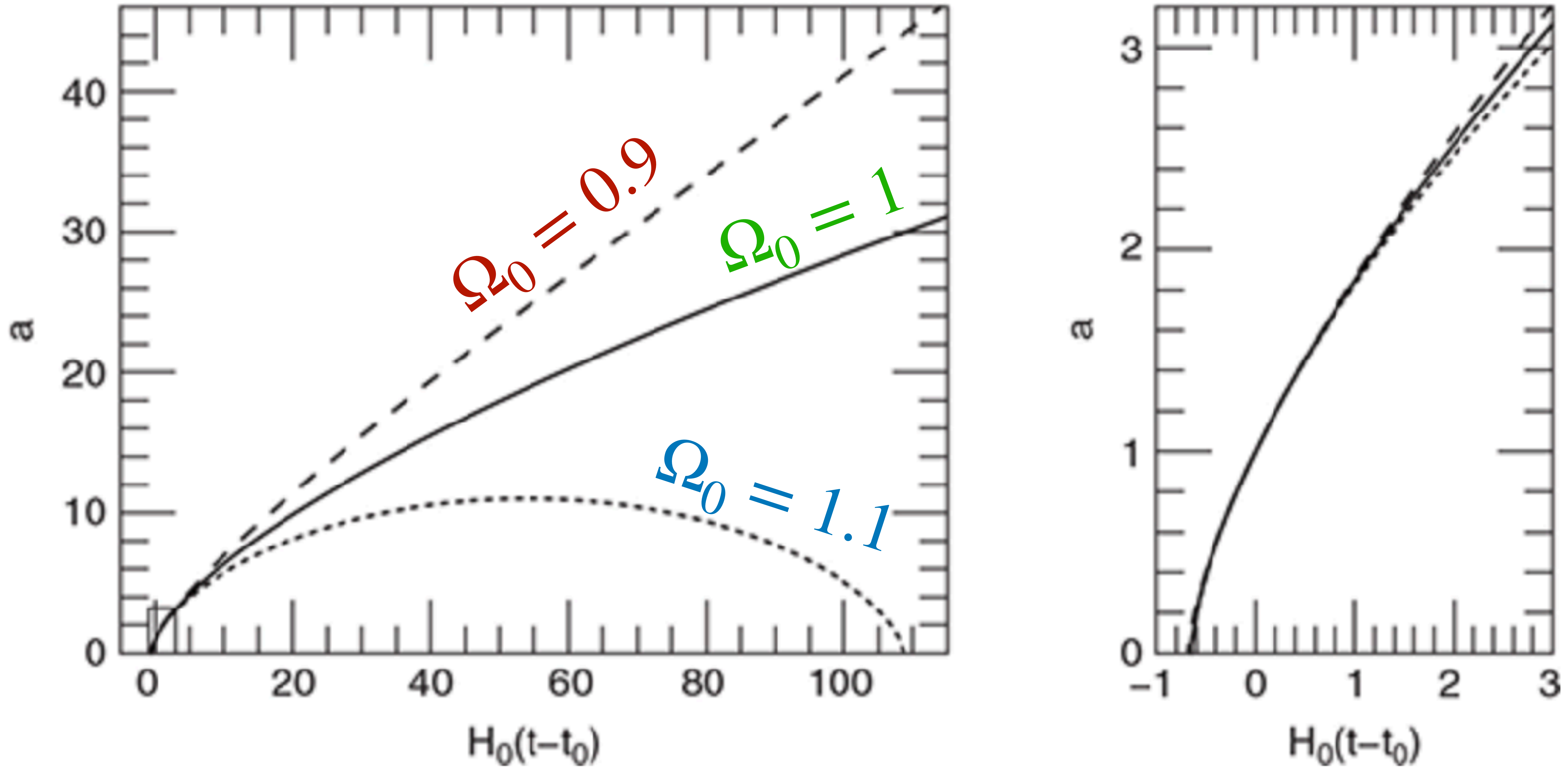
Consider a negatively curved matter-only universe with $\Omega_{m,0} < 1$ and hence $\Omega_k = 1 - \Omega > 0$.

The Friedmann equation becomes

$$H^2(t) = H_0^2\left[\Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2}\right]$$

Models with matter and curvature

$$H^2(t) = H_0^2\left[\Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2}\right]$$



Ryden

Figure 5.4 Scale factor versus time for universes containing only matter. Solid line: $a(t)$ for a universe with $\Omega_0 = 1$ (flat). Dashed line: $a(t)$ for a universe with $\Omega_0 = 0.9$ (negatively curved). Dotted line: $a(t)$ for a universe with $\Omega_0 = 1.1$ (positively curved). The right panel is a blow-up of the small rectangle near the lower left corner of the left panel.

Models with matter and curvature

The time of maximum expansion for positive curvature corresponds to $H(t_{\max}) = 0$:

$$0 = \frac{\Omega_0}{a_{\max}^3} + \frac{1 - \Omega_0}{a_{\max}^2} \implies a_{\max} = \frac{\Omega_0}{\Omega_0 - 1}$$

A maximum is only reached for $\Omega_0 > 1$. After reaching $a_{\max}$, this universe will recollapse to a “big crunch”.

The matter+curvature models is an example of the general fact that the formula for $a(t)$ is usually a bit harder to derive in multiple-component models…

Starting is easy:

$$\frac{\dot{a}}{H_0} = \left[\frac{\Omega_0}{a} + (1 - \Omega_0)\right]^{1/2} \implies H_0 t = \int_0^a \left[\frac{\Omega_0}{a} + (1 - \Omega_0)\right]^{-1/2} da$$

Models with matter and curvature

Textbooks then say (e.g. Ryden 5.4.1) “This equation has a parametric solution…”. For $\Omega_0 > 1$

$$a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1 - \cos\theta)$$

$$t(\theta) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta)$$

With $0 < \theta < 2\pi$. Clearly $\theta = 2\pi \implies a(\theta) = 0$ — This universe **recollapses** into a “big crunch”.

The age of the universe at the big crunch is:

$$t_{\text{crunch}} = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{-3/2}}$$

So for example, $\Omega_0 = 1.1$ gives $t_{\text{crunch}} \approx 1069\,h\,\text{Gyr} \sim 748\,\text{Gyr}$ for $h = 0.7$.

Models with matter and curvature

The parametric solution for $\Omega_0 < 1$ is similar (with $0 < \eta < \infty$):

$$a(\eta) = \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}(\cosh\eta - 1)$$

$$t(\eta) = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh\eta - \eta)$$

In the limit of low Ω_0 and long times, this tends to the solution for an empty universe, which we will look at next.

Some important points to take away from this part are:

In general, including multiple components in the Friedmann equation leads to non-trivial integrals for $a(t)$.

As often the case for differential equations, there are often multiple solutions with fundamentally different behaviours.

Model of an empty universe

In an empty universe, $\Omega = 0$ and hence $\Omega_k = 1$, so $\dot{a}^2 = -\dfrac{\kappa c^2}{R_0^2}$.

One solution is $\kappa = 0 \implies \dot{a} = 0$. This flat universe with nothing in it does not expand. This is Minkowski "special relativity" space (i.e. FLRW with no curvature and $a =$ constant).

The solution with curvature is $\kappa < 0$, $\Omega_k > 1$ (we can't have a positively-curved closed universe with no energy density — there's nothing to create the positive curvature). In this case,

$$\dot{a} = \pm \frac{c}{R_0} \implies a(t) = \frac{t}{t_0} \text{ with } t_0 = \frac{R_0}{c}.$$

This negatively curved empty space expands at a constant rate forever, because there is nothing to accelerate or decelerate the expansion (this checks out in the acceleration equation). As the empty universe expands, the energy density in curvature goes down.

In this case, $t_0 = H_0^{-1}$ is always true.

Model of an empty universe

The empty universe (sometimes called the **Milne model)** is an approximation for the curvature-dominated case when $\Omega \ll 1$. If it was not for dark energy (see later), this would be the future of our $\Omega_0 \sim 0.3$ universe.

In this model, the relationship between redshift and time is just

$$t = t_0 a(t) = \frac{t_0}{1+z} = \frac{1}{H_0(1+z)}$$

The comoving distance to a source is therefore

$$r(z) = c \int \frac{dt}{a} = \frac{c}{H_0} \int_t^{t_0} \frac{dt}{t} = \frac{c}{H_0} \ln\left(\frac{t_0}{t}\right) = \frac{c}{H_0} \ln(1+z)$$

Luminosity distance

This is the distance inferred from the brightness of a lightbulb of known luminosity (energy per second emitted into a spherical volume).

In an FLRW metric, there are **three reasons** why the luminosity distance is not the same as the apparent distance.

One reason is **geometric, i.e. related to the curvature** (like the angular diameter distance). The other two reasons are related to the effects of the expansion: **redshifting** and **arrival time delay.**

In the static Euclidean case, far-away lightbulbs look dimmer because the same light energy (total luminosity, L) is spread over a sphere of surface area $4\pi D_L^2$, decreasing the observed energy per unit area (**flux**, F)

$$D_L = \left(\frac{L}{4\pi F}\right)^{1/2}$$

Luminosity distance

We can write $F = \dfrac{L_0}{4\pi D_L^2}$ where the subscript refers to what an observer sees at $t = t_0$. Recalling the metric,

$$ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_\kappa^2(r)\, d\Omega^2\right]$$

$$S_\kappa(r) = \begin{cases} R_0 \sin(r/R_0), & \text{positive curvature} \\ r, & \text{zero curvature} \\ R_0 \sinh(r/R_0) & \text{negative curvature} \end{cases}$$

The **proper area** of a sphere of comoving radius r is $A = 4\pi S_\kappa^2(r)$, and hence:

$$F = \frac{L_0}{4\pi S_\kappa^2(r)}$$

Luminosity distance: redshifting

What about L_0?

Last time, when talking about the evolution of the photon energy density, we argued that the energy of individual photons goes down as $E_\gamma \propto (1+z) = a^{-1}$, because photon wavelengths are stretched by the expansion.

In other words,

$$E_0 = \frac{E}{1+z}$$

But this is just half the story!

Luminosity distance: arrival time delay

Our photon detector collects so many photons per second per area, and records their energies. We know the energies will be lower than when the photons were emitted.

However, the **rate of arrival of the photons** will also be less than the rate at which they were emitted! If the source emits one photon every interval δt_e, those photons will arrive at a rate

$$\delta t_0 = \delta t_e(1+z)$$

This is an *extra* effect, because our definition of luminosity is energy per unit area *per unit time*.

The energy density of the universe in the Friedman equation doesn't care about the "per unit time" part, so the energy density is only affected by red-shifting.

Luminosity distance

Putting these two effects together, $L_0 \equiv \dfrac{E_0}{\delta t_0} = \dfrac{E}{1+z}\dfrac{1}{\delta t(1+z)} = \dfrac{L}{(1+z)^2}$ and hence

$$F = \frac{L}{4\pi S_\kappa^2(r)(1+z)^2}$$

Thus the "equivalent" luminosity distance is

$$D_L(z) = S_\kappa(r)(1+z)$$

"Equivalent" just means we can write something that looks like the Euclidean formula: $F = \dfrac{L}{4\pi D_L^2(z)}$

Even in a flat FLRW universe, galaxies at higher redshift will look **fainter than they would in a static universe**.

Luminosity and angular size distance

$$D_L(z) = S_\kappa(r)(1+z) \text{ and } D_A(z) = \frac{S_\kappa(r)}{1+z}$$

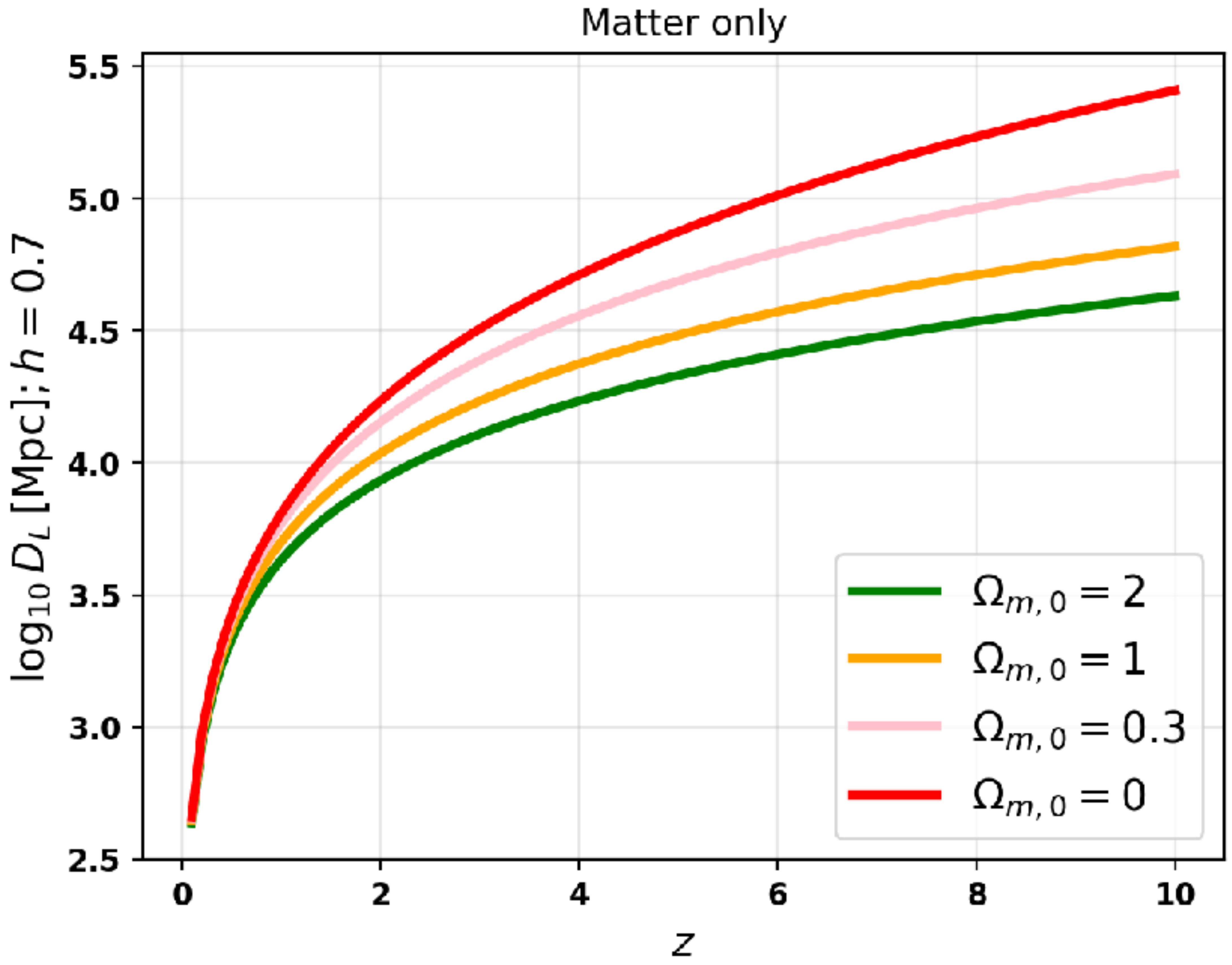
$$\implies D_A = \frac{D_L}{(1+z)^2} \text{ i.e. } \frac{D_L}{D_A} = (1+z)^2$$

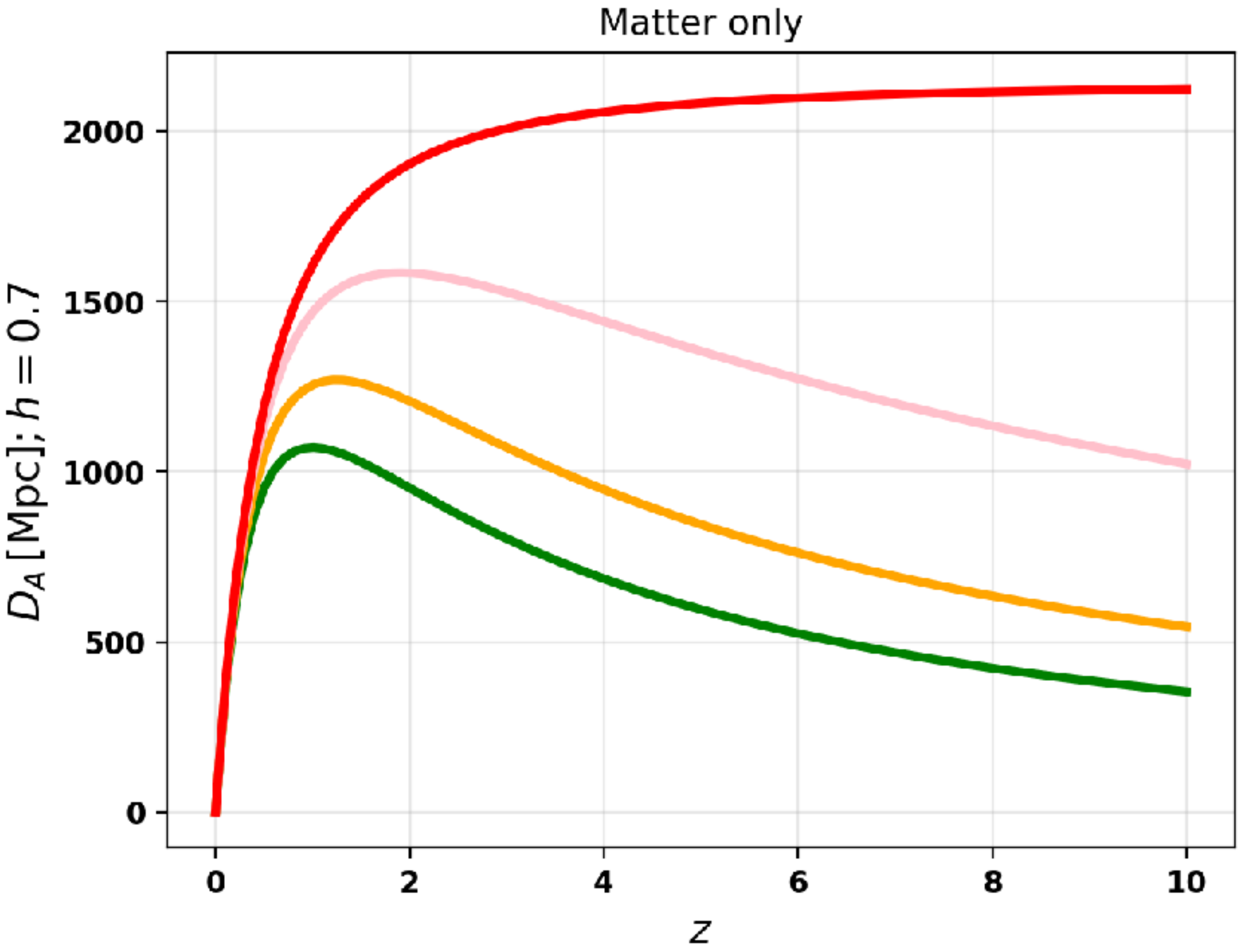
The angular size distance is smaller than the luminosity distance, by a factor that doesn't depend on the curvature.

So, even though the angular sizes of standard rulers might get larger at high redshift, those big rulers would be very faint! At high redshift this difference is huge.

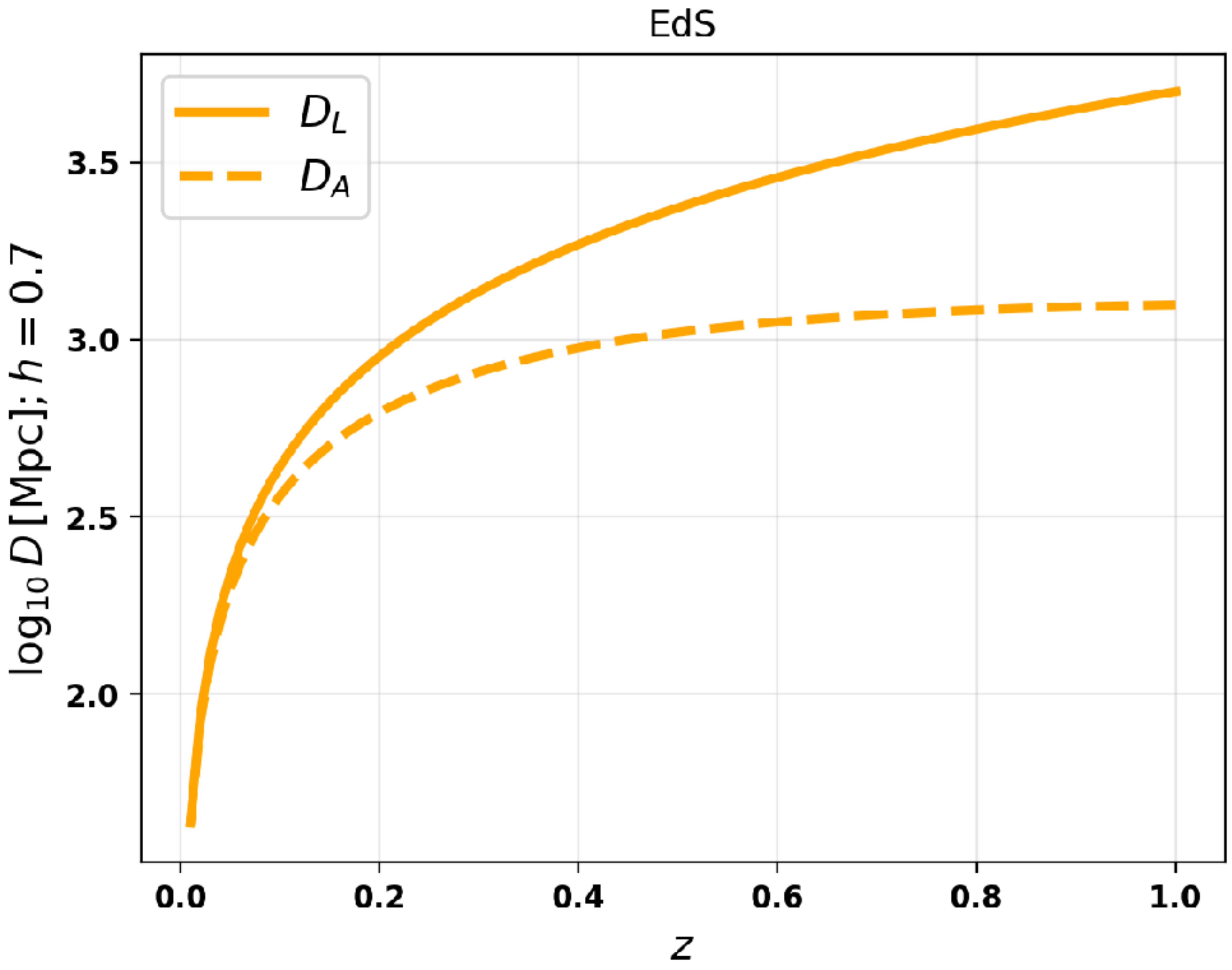
Remember: *when $z \ll 1$, all distances are approximately the same as the proper distance D_p, i.e. the same as they would be in a static, Euclidean universe.*

Luminosity and angular size distance





Luminosity and angular size distance



Cosmic distances

For practical work in cosmology, luminosity and angular diameter distances are ***very important***.

How big something looks $\Longrightarrow$ use the angular diameter distance.

How bright something looks $\Longrightarrow$ use the luminosity distance.

How the universe looks depends a lot on how those two distances change with redshift.

Cosmic distance measures

The procedure for finding the two distance measures is:

- *Start with the cosmological parameters and solve the Friedmann equation for* $a(t)$*;*
- *Use the FLRW metric to find* $r(z)$ *and hence* $S_\kappa(r)$ *[which is just the same as* $r(z)$ *in the flat case]*
- *Use the (simple) expressions on the previous slides to find* D_A *and* D_L*.*

Complicated mixes of energy density components can give Friedmann equations that are hard to solve, so the first two steps might need a numerical solution.

A "cosmological calculator" is a program that does this calculation for arbitrary combinations of parameters. Astronomers (not just cosmologists) have to use such calculators **all the time** if the things they are observing are further than $z \sim 0.1$.

Cosmological calculators

You can do quick calculators with Ned Wright's Cosmology Calculator:

https://www.astro.ucla.edu/~wright/CosmoCalc.html

For more detailed calculators in Python, I recommend the **cosmology module** of the **astropy** package (i.e. `astropy.cosmology`):

https://www.astropy.org/

By now, you should know enough to follow this tutorial:

https://learn.astropy.org/tutorials/redshift-plot.html

Cosmological calculators

https://www.astro.ucla.edu/~wright/CosmoCalc.html

Enter values, hit a button

69.6 H_o
0.286 Omega_M
3 z
Open Flat
0.714 Omega_{vac}
General

Open sets $\mathrm{Omega}_{vac} = 0$ giving an open Universe [if you entered $\mathrm{Omega}_M < 1$]
Flat sets $\mathrm{Omega}_{vac} = 1\text{-}\mathrm{Omega}_M$ giving a flat Universe.
General uses the Omega_{vac} that you entered.
Source for the default parameters.

For $H_o = 69.6$, $\mathrm{Omega}_M = 0.286$, $\mathrm{Omega}_{vac} = 0.714$, $z = 3.000$

- It is now 13.721 Gyr since the Big Bang.
- The age at redshift z was 2.171 Gyr.
- The light travel time was 11.549 Gyr.
- The comoving radial distance, which goes into Hubble's law, is 6481.3 Mpc or 21.139 Gly.
- The comoving volume within redshift z is 1140.389 Gpc^3.
- The angular size distance D_A is 1620.3 Mpc or 5.2846 Gly.
- This gives a scale of 7.855 kpc/".
- The luminosity distance D_L is 25924.3 Mpc or 84.554 Gly.

1 Gly = 1,000,000,000 light years or 9.461×10^{26} cm.
1 Gyr = 1,000,000,000 years.
1 Mpc = 1,000,000 parsecs = 3.08568×10^{24} cm, or 3,261,566 light years.

Tutorial: Part 1 | Part 2 | Part 3 | Part 4
FAQ | Age | Distances | Bibliography | Relativity

See the advanced and light travel time versions of the calculator.

James Schombert has written a Python version of this calculator.

Ned Wright's home page

Back to the Friedmann Equation

Radiation-only universe

The real universe was radiation-driven at early times, and became matter-driven later.

Following the same process as for a flat matter-only universe, we find that, for a **flat** universe with $w = 1/3$,

$$t_0 = \frac{2}{3(1+w)}\frac{1}{H_0} = \frac{1}{2H_0}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}$$

$$r(t) = 2ct_0\left[1 - \left(\frac{t}{t_0}\right)^{1/2}\right] \implies r(z) = \frac{c}{H_0}\frac{z}{1+z}$$

Radiation + Matter

In this case, $H(t)^2 = H_0^2 \left[\Omega_{m,0}\, a^{-3} + \Omega_{r,0}\, a^{-4}\right]$.

After a bit of rearrangement, $H_0 dt = \dfrac{a}{\sqrt{\Omega_{r,0}}} \left[1 + \dfrac{a}{a_{\text{eq}}}\right]^{-1/2}$

Integrating, $H_0 t = \dfrac{4a_{\text{eq}}^2}{3\sqrt{\Omega_{r,0}}} \left[1 - \left(1 - \dfrac{a}{2a_{\text{eq}}}\right)\left(1 + \dfrac{a}{a_{\text{eq}}}\right)^{1/2}\right]$

where a_{eq} is the scale factor at the time of matter-radiation equality, given by $\dfrac{\Omega_{m,0}}{a^3} = \dfrac{\Omega_{r,0}}{a^4} \implies a_{\text{eq}} = \dfrac{\Omega_{r,0}}{\Omega_{m,0}}$.

Radiation + Matter

$$H_0 t = \frac{4a_{\text{eq}}^2}{3\sqrt{\Omega_{r,0}}}\left[1 - \left(1 - \frac{a}{2a_{\text{eq}}}\right)\left(1 + \frac{a}{a_{\text{eq}}}\right)^{1/2}\right]$$

In the limit $a \ll a_{\text{eq}}$, the part in square brackets goes to $\frac{3}{8}\left(\frac{a}{a_{\text{eq}}}\right)^2$ to leading order; hence

$$a(t) \sim \left(2\sqrt{\Omega_{r,0}}H_0 t\right)^{1/2} \qquad (a \ll a_{\text{eq}})$$

Likewise, at times when this universe is matter-dominated

$$a(t) \sim \left(\frac{3}{2}\sqrt{\Omega_{m,0}}H_0 t\right)^{2/3} \qquad (a \gg a_{\text{eq}})$$

Matter-radiation equality

We can find the time of matter-radiation equality by setting $a = a_{\rm eq}$ in

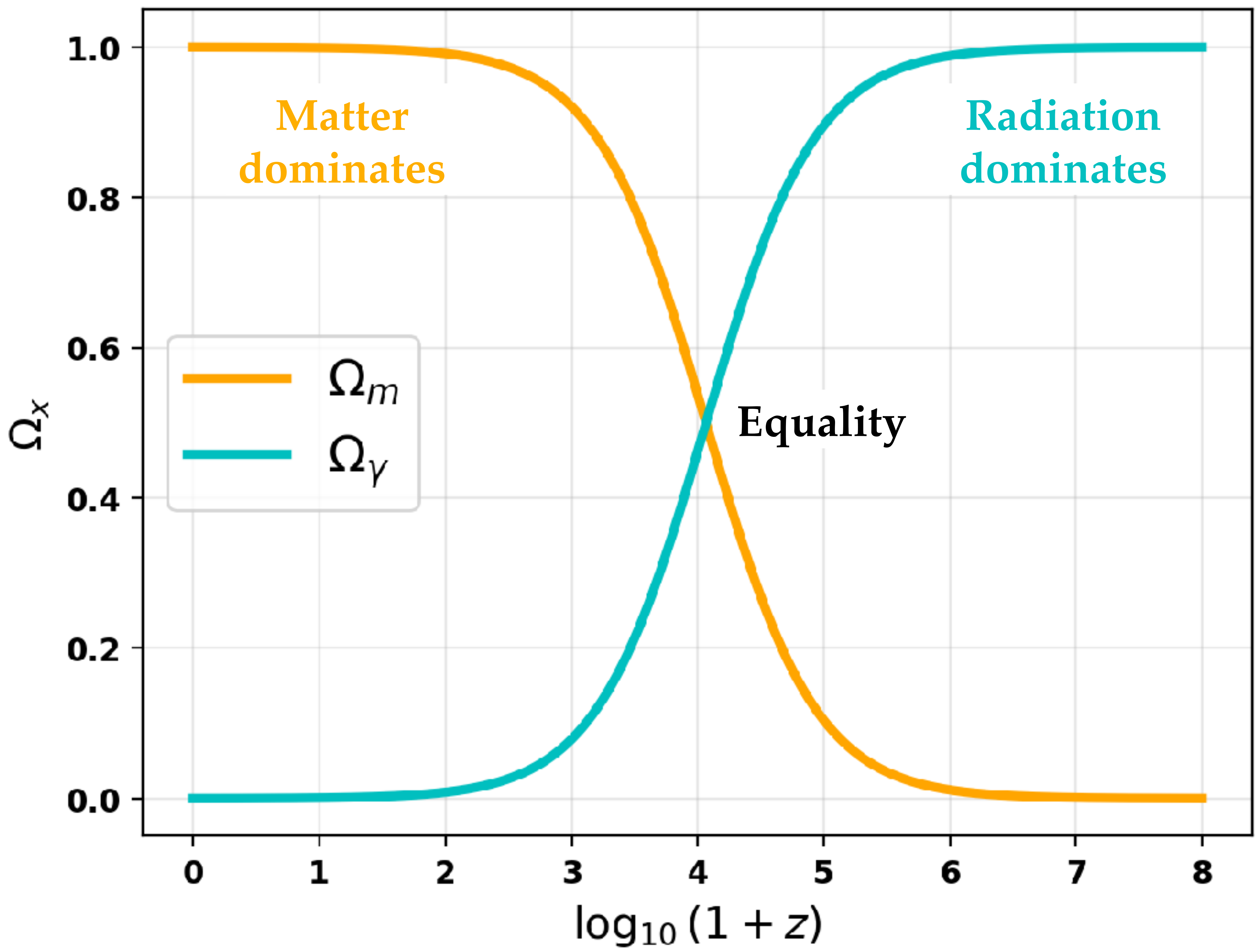
$$H_0 t = \frac{4a_{\rm eq}^2}{3\sqrt{\Omega_{r,0}}}\left[1 - \left(1 - \frac{a}{2a_{\rm eq}}\right)\left(1 + \frac{a}{a_{\rm eq}}\right)^{1/2}\right]$$

$$t_{\rm eq} = \frac{4}{3H_0}\left(1 - \frac{1}{\sqrt{2}}\right)\frac{a_{\rm eq}^2}{\sqrt{\Omega_{r,0}}} \approx \frac{0.391}{H_0}\frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2}$$

For $\Omega_{r,0} = 9 \times 10^{-5}$ (*), $\Omega_{m,0} = 0.3$, $h = 0.7$, we find $t_{\rm eq} \approx 52{,}000$ years.

(includes neutrinos)*

From radiation to matter



Radiation background

Most of the energy density of **photons** contributing to $\Omega_{r,0}$ is contributed by the **Cosmic Microwave Background**.

Sufficiently sensitive microwave detectors measure a uniform background of photons with characteristic energy

$$\varepsilon_{\gamma,CMB} = 4.175 \times 10^{-14}\,\mathrm{J\,m^{-3}} = 0.2606\,\mathrm{MeV\,m^{-3}}$$

arriving from all directions.

The spectrum of this radiation is that of a near-perfect blackbody →

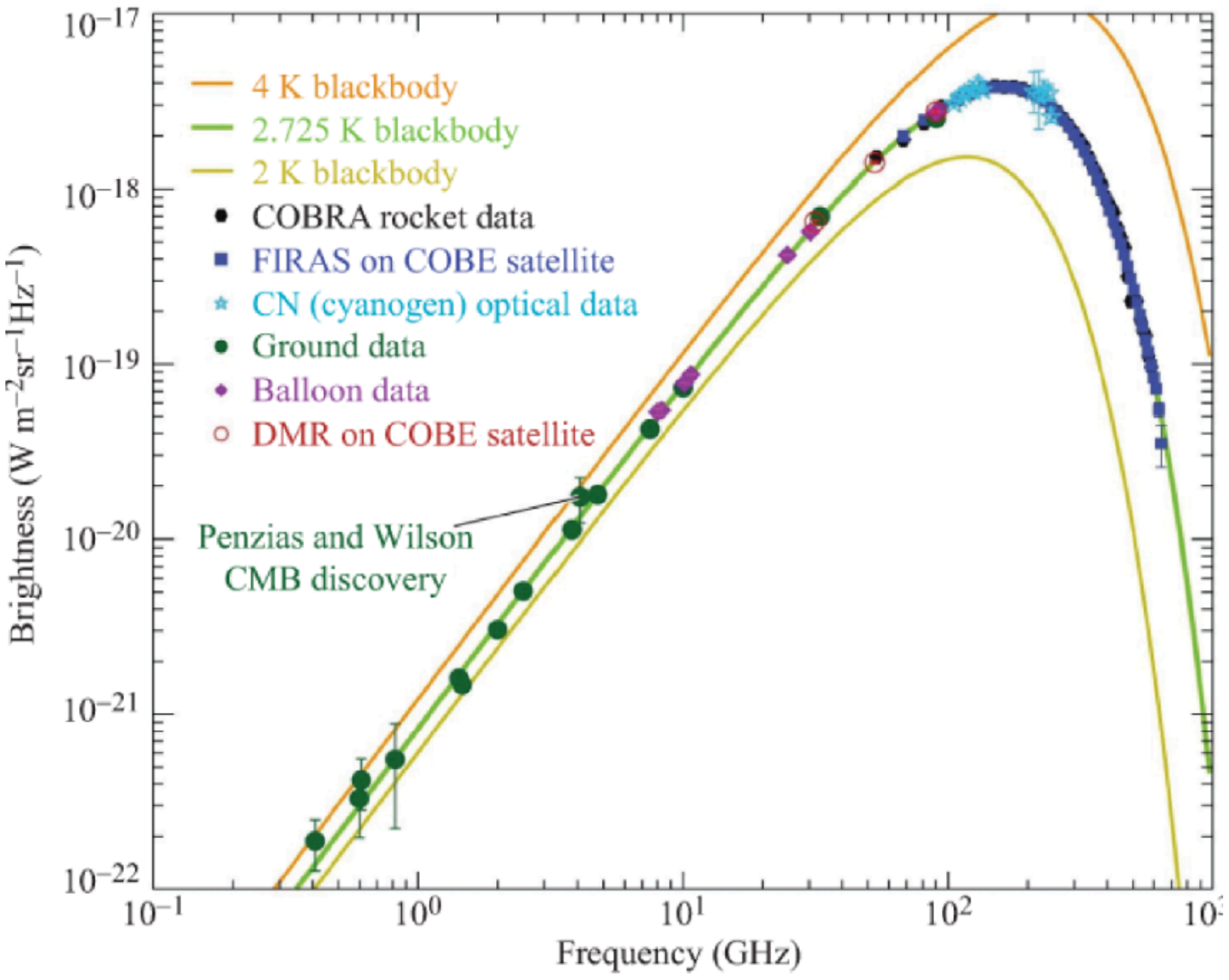


Fig 13.2 from Huterer, reproduced from Samtleben et al. (2007; *Annual Reviews of Astronomy and Astrophysics*)

Radiation background

From the Stephan-Boltzmann law, $\varepsilon_\gamma = \alpha T^4$, the equivalent temperature of the CMB is

$$T_0 = 2.7255 \pm 0.0006\,\mathrm{K}$$

This **CMB temperature** is another fundamental cosmological parameter. It is the most accurately measured of all the cosmological parameters.

From blackbody statistics (see Ryden Ch. 3), this temperature is equivalent to $n_\gamma \sim 4 \times 10^8$ photons per cubic meter, with mean energy $\bar{E}_\gamma \sim 6 \times 10^{-4}\,\mathrm{eV}$.

Since $\varepsilon_\gamma \propto a^{-4}$, we have that $T \propto a^{-1}$ for blackbody radiation. Hence at $a_{\mathrm{eq}} \sim 3333$, $T \sim 9000\,\mathrm{K}$ — a bit hotter than the surface of the Sun!

For $a \to 0$, the Universe was clearly *very* hot at early times…

Neutrinos

This is an obvious opportunity to talk about neutrinos, but I will mostly leave that for next time.

Short version:

Neutrinos are almost-but-not-quite massless particles that act either as a form of dark matter or as a form of radiation (i.e. relativistic matter), depending on the details of their actual mass and the age of the Universe.

When extrapolating backwards from the present day, we can correct the radiation density parameter to include the contribution from neutrinos when they were relativistic — this is where $\Omega_{r,0} \sim 9 \times 10^{-5}$ comes from. $\Omega_{\gamma,0} \sim 5 \times 10^5$, $\Omega_{\nu,0} \lesssim 4 \times 10^5$.

Cosmological observations can tell us something about the particle physics of neutrinos, and vice versa.

See Ryden 2.4, 5.1.

Accelerated Expansion

The deceleration parameter

The next slides have a lot of maths, so let's first get the basic point:

In general the relationship between redshift and comoving distance is not linear, because the expansion rate is changing over time.

How exactly it changes depends on things we don't know (i.e. all the Ω_i parameters).

We can measure the **current** expansion rate H_0 by fitting a **straight line** to the relationship between **proper distance** and the velocity ($\approx cz$). That only works if H_0 is roughly constant, i.e. for small lookback times.

We can also introduce a similar, observable parameter associated with **acceleration**, **called** q_0.

Distance-redshift relation

The **Taylor Series** expansion of the scale factor at the present time is:

$$a(t) = a(t_0) + \dot{a}\Big|_{t=t_0}(t - t_0) + \frac{1}{2}\ddot{a}\Big|_{t=t_0}(t - t_0)^2 + \dots$$

Then, dividing by $a(t_0)$, we have

$$\frac{a(t)}{a(t_0)} = 1 + \frac{\dot{a}}{a}\Big|_{t=t_0}(t - t_0) + \frac{1}{2}\frac{\ddot{a}}{a}\Big|_{t=t_0}(t - t_0)^2 + \dots,$$

in which we can recognise H_0 (and use $a(t_0) = 1$),

$$a(t) = 1 + H_0(t - t_0) + \frac{1}{2}\frac{\ddot{a}}{a}\Big|_{t=t_0}(t - t_0)^2 + \dots$$

Distance-redshift relation

$$a(t) \approx 1 + H_0(t - t_0) + \frac{1}{2}\frac{\ddot{a}}{a}\bigg|_{t=t_0}(t - t_0)^2$$

The first two terms tell us that, if we go back **a little bit** in time from $t = t_0$, we can estimate that the scale factor will be a little less that 1, by an amount we can deduce from the current expansion rate, H_0.

In this case, we don't need to know any physics behind $a(t)$; we just have to approximate the expansion rate as constant over the period $\Delta t = t - t_0$, which is OK if this interval is small.

Of course, the expansion rate might not be exactly constant. The third term tells us the **second order** correction to account for an acceleration (or deceleration) of the expansion.

Including higher order terms means we can extend our approximation to larger t (provided we know the coefficients). Do we know the coefficient on the second order term?

Distance-redshift relation

$$a(t) \approx 1 + H_0(t - t_0) + \frac{1}{2}\frac{\ddot{a}}{a}\Big|_{t=t_0}(t - t_0)^2$$

Recall the acceleration equation $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\sum_i \left(\varepsilon_i + 3P_i\right) = -\frac{4\pi G}{3c^2}\sum_i \varepsilon_i(1 + w_i)$

Recall the definition of the critical density, $\varepsilon_{\text{crit}} = \frac{3c^2H^2}{8\pi G}$, so we have

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\frac{H^2}{\varepsilon_c}\sum_i \varepsilon_i(1 + w_i) = -\frac{1}{2}H^2\sum_i \Omega_i(1 + w_i)$$

Which, if we wanted, we could rearrange to make both sides dimensionless numbers:

$$-\frac{1}{H^2}\frac{\ddot{a}}{a} = \frac{1}{2}\sum_i \Omega_i(1 + w_i)$$

Distance-redshift relation

We have $a(t) \approx 1 + H_0(t-t_0) + \frac{1}{2}\frac{\ddot{a}}{a}\Big|_{t=t_0}(t-t_0)^2$ and $-\frac{\ddot{a}}{aH^2} = \frac{1}{2}\sum_i \Omega_i(1+w_i)$.

Define a new **deceleration parameter** $q_0 = -\frac{\ddot{a}}{aH^2}\Big|_{t=t_0} = -\frac{1}{H_0^2}\frac{\ddot{a}}{a}\Big|_{t=t_0}$ so we can write

$$a(t) \approx 1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2 (t-t_0)^2$$

Notice that $q_0 > 0$ implies $\ddot{a} < 0$, i.e. the expansion is decelerating. There is still no "specific" physics in the Taylor series expression, all this comes from the definition of the scale factor and Hubble constant.

Distance-redshift relation

$$a(t) \approx 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2$$

However, we did use some physics to connect the q_0 parameter to the density parameters:

$$q_0 = \frac{1}{2} \sum_i \Omega_i \left(1 + w_i\right)$$

For example, for a flat universe with just matter and radiation, $q_0 = \frac{\Omega_{m,0}}{2} + \Omega_{r,0}$.

Clearly q_0 is also a parameter we can measure (although it's not as "easy" to measure as H_0).

Distance-redshift relation

$$a(t) \approx 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2$$

Significance of this: even if we don't know really know what the constituents of the energy density are, we can still estimate the relation between distance and redshift **close to** t_0 using this expansion.

$$r(z) = c\int_{t_0}^{t} \frac{1}{a(t)}dt \approx c\int_{t_0}^{t} 1 - H_0(t - t_0) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t - t_0)^2 dt$$

$$\implies r(z) \approx c(t_0 - t) + \frac{cH_0}{2}(t_0 - t)^2$$

(we discard terms of order higher that $(t - t_0)^2$, so q_0 does not appear here)

If there was no expansion ($H_0 = 0$), this would just be "distance = velocity × time" for a light signal.

Distance-redshift relation

$r(z) \approx c(t_0 - t) + \frac{cH_0}{2}(t_0 - t)^2$ isn't useful, because one side is in terms of redshift and the other is in terms of (lookback) time. Since only redshift can be measured, we can convert the RHS to redshift.

To do this, we start from $z(t) = \frac{1}{a(t)} - 1$ and use the expansion for $1/a$ again to get

$$z \approx H_0(t - t_0) + \left(1 + \frac{q_0}{2}\right) H_0^2(t - t_0)$$

and hence

$$t - t_0 \approx \frac{1}{H_0}\left[z - \left(1 + \frac{q_0}{2}\right) z^2\right]$$

Distance-redshift relation

After all this messing around:

$$r(z) \equiv D_p|_{t=t_0} \approx \frac{cz}{H_0}\left[1 - \frac{1+q_0}{2}z\right]$$

This is the second-order version of Hubble's law, which should "work" to a little higher redshift than the linear version, $D_p|_{t=t_0} = cz/H_0 \sim 4500z\,\mathrm{Mpc}$.

For $q_0 > -1$, relatively nearby galaxies at redshift z should be a little closer than we would estimate from the linear Hubble law. For an EdS model, $q_0 = 1/2$; for a low-density matter-only model $q_0 = \Omega_{m,0}/2$. etc.

In other words, in principle, **we can measure q_0 in the same way we measure H_0, using measurements of distance from slightly greater redshift**.

Accelerated expansion

The q_0 parameter is **"history"** — it was introduced in the 1950s when it was thought that the universe was $\Omega_m \lesssim 1$.

Observations in the 1990s found compelling evidence that the present-day universe is undergoing **accelerated expansion** with $q_0 < 0$.

This evidence comes from the relationship between redshift and luminosity distance for supernovae, which we look at in more detail next time.

We also have a strong constraint from the cosmic microwave background (see later) that $\Omega \approx 1$. Both the CMB and local measurements imply $\Omega_{m,0} < 1$, so something very weird must be going on.

We **missing an explanation for most of the energy density**, and whatever we're missing somehow has to **accelerate the expansion** *right now*, but not much earlier or later!

The cosmological constant

The current solution is to introduce a **cosmological constant, Λ,** into the Friedmann equation:

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\bar{\varepsilon}(t) + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}$$

You can read about the history of the cosmological constant in your textbooks.

This is justified within the framework of GR. It was used by Einstein as a means of obtaining a static universe.

After the Hubble expansion was discovered, there was no need to make the universe static: the cosmological constant was still justified, but there wasn't any obvious reason to include it.

When we needed to explain an *accelerating* universe, it came back into fashion…

The cosmological constant

The equivalent energy density of the cosmological constant is also (obviously) a constant, which we can write:

$$\varepsilon_\Lambda = \frac{\Lambda}{8\pi G} \text{ or } \rho_\Lambda = \frac{\Lambda}{8\pi G c^2}$$

The origin of this energy density is usually imagined to be the **vacuum energy** associated with empty space in quantum mechanics: particles are the excitations of space-filling fields, which fluctuate.

Large energy density fluctuations can occur on short time and length scales (the “uncertainty principle” $\Delta E \Delta t = \hbar$), corresponding to the near-instantaneous creation and annihilation of particle-antiparticle pairs.

There can, in principle, be a constant, nonzero “background” energy density associated with these fluctuations. However, it has so far not been possible to predict what this vacuum energy should be from first principles in a way we can connect to the cosmological measurement of ρ_Λ.

The cosmological constant

From a practical point of view, we **define** the cosmological constant as a form of energy density with $w = -1$ and hence $P = -\varepsilon_\Lambda$. The cosmological constant has **negative pressure**.

The "complete" Friedmann equation, as far as we know at the moment, is

$$H^2(t) = H_0^2 \left(\Omega_{m,0}\, a^{-3} + \Omega_{r,0}\, a^{-4} + \Omega_{\Lambda,0} + (1 - \Omega_0)\, a^{-2} \right)$$

The more space expands, the more Λ there is, so ρ_Λ is always the same!

Ω_Λ has mostly "replaced" q_0 as the parameter of interest.

Keep in mind that we don't know for sure that the Λ model is correct. We could be dealing with something that *looks a lot like* a cosmological constant, but it doesn't have to be exactly constant. This is the difference between the cosmological constant and **dark energy**: we may have $\Omega_{DE}(t)$ instead of Ω_Λ.

Cosmological constant only

For a **flat** universe containing **only** a cosmological constant:

$$\dot{a} = H_0\, a \implies a(t) = e^{H_0(t-t_0)}$$

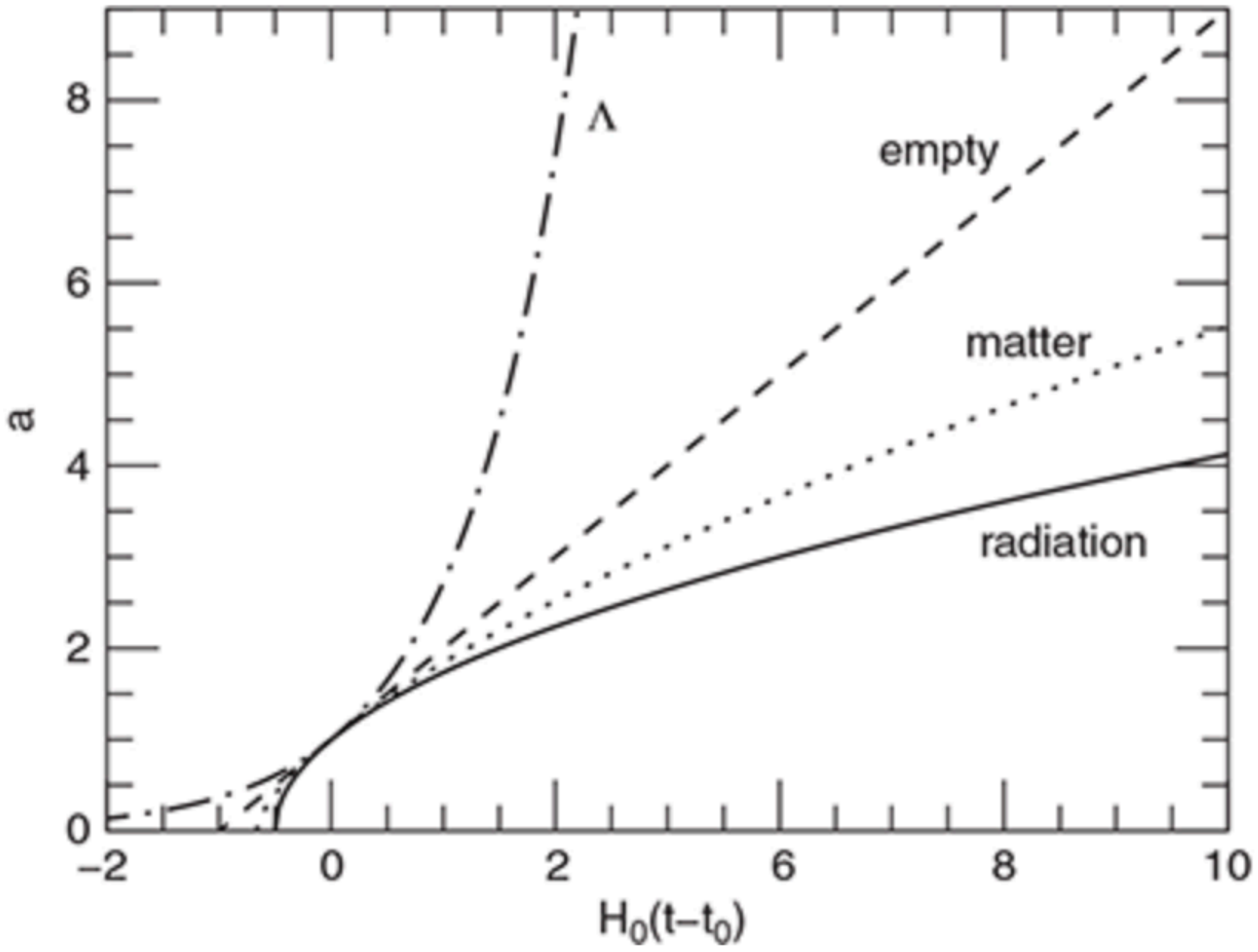
The universe undergoes **exponential expansion**.

As far as we know, this is what will happen to our universe in the far future when $\Omega_m(t) \sim 0$.

Later, we will also see that the universe probably had a phase of exponential expansion in the very beginning, called inflation, which mysteriously stopped (and thus was not strictly due to a cosmological constant, but something that behaved in the same way in the Friedmann equation).

Single-component expansion histories

Ryden



Cosmological constant only

$$a(t) = e^{H_0(t-t_0)}$$

The comoving distance to a given redshift in this universe is easy:

$$r(z) = c\int e^{H_0(t-t_0}dt = \frac{c}{H_0}\left[e^{H_0(t-t_0)} - 1\right] = \frac{c}{H_0}z$$

Hubble's law $cz = H_0 D_p(t = t_0)$ holds at all z in this universe, not just at $z \ll 1$. Galaxies at high redshift have an apparent distance

$$D_p(z) = \frac{c}{H_0}\frac{z}{1+z}$$

and hence the particle horizon is the same as the Hubble length.

The cosmological constant

In a universe dominated by a combination of Ω_m and Ω_Λ.

$$q_0 = \frac{\Omega_m}{2} - \Omega_\Lambda$$

We should be able to estimate Ω_Λ from the combination of galaxy distances and Ω_m (because Ω_Λ drives acceleration). We will see this next time. If we have other reasons to believe the universe is flat, then

$$\Omega_m + \Omega_\Lambda = 1$$

$$\implies q_0 = \frac{\Omega_m}{2} - 1 + \Omega_m = \frac{3\Omega_m}{2} - 1 \sim -0.55$$

The cosmological constant

At late times ($z \lesssim 1$) the cosmological constant "takes over" in the Friedmann equation and the expansion accelerates.

Ryden

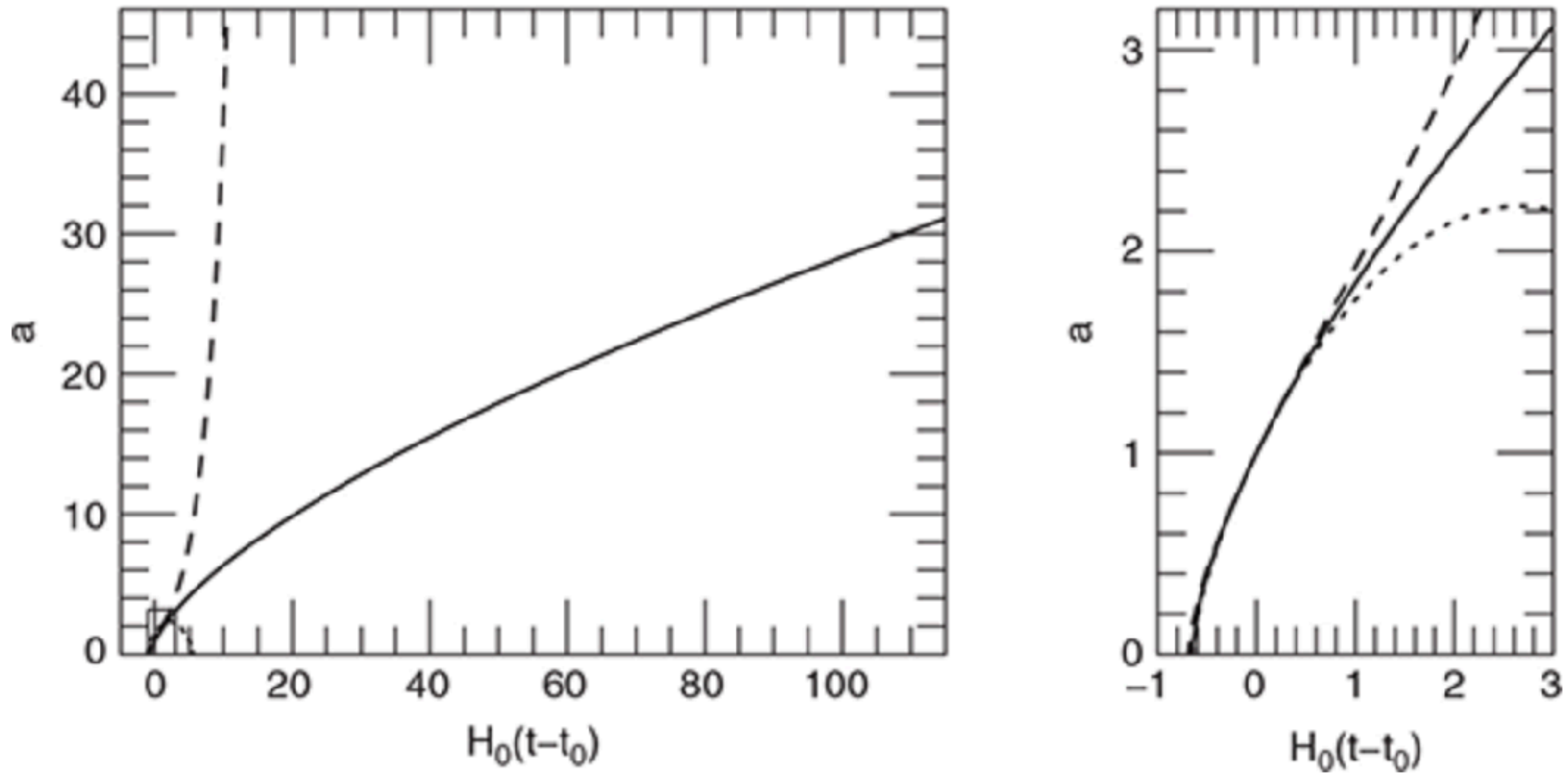
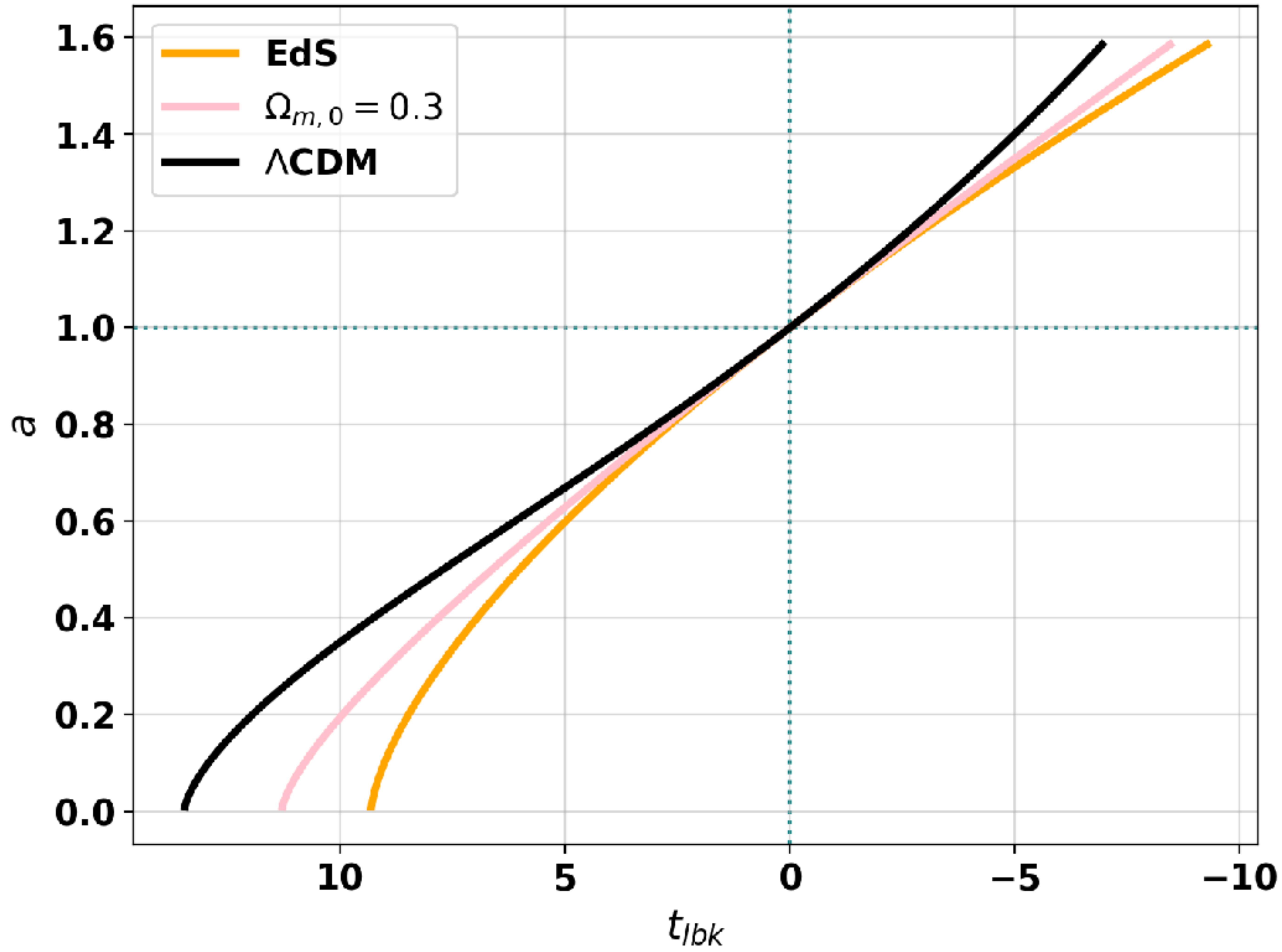


Figure 5.5 Scale factor versus time for flat universes containing both matter and a cosmological constant. Solid line: $a(t)$ for a universe with $\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$. Dashed line: $a(t)$ for a universe with $\Omega_{m,0} = 0.9$, $\Omega_{\Lambda,0} = 0.1$. Dotted line: $a(t)$ for a universe with $\Omega_{m,0} = 1.1$, $\Omega_{\Lambda,0} = -0.1$. The right panel is a blow-up of the small rectangle near the lower left corner of the left panel.

The cosmological constant



Summary 1

In general Friedmann models, the standard-ruler (angular diameter) and standard-candle (luminosity) distances are **different**.

Luminosity distance: $D_L(z) = S_\kappa(r)(1+z)$, angular diameter distance: $D_A(z) = \dfrac{S_\kappa(r)}{1+z}$

Hence: $\dfrac{D_L}{D_A} = (1+z)^2$

General Friedmann models with multiple components can have complex expansion histories and complicated expressions for $r(z)$, $t(z)$, and hence $D_L(z)$ and $D_A(z)$.

In some cases these solutions are parametric; in others only a numerical solution is possible.

In general, we use "cosmological calculators", but looking at single-component solutions gives us some intuition for the different types of behaviour.

Summary 2 (proper distances)

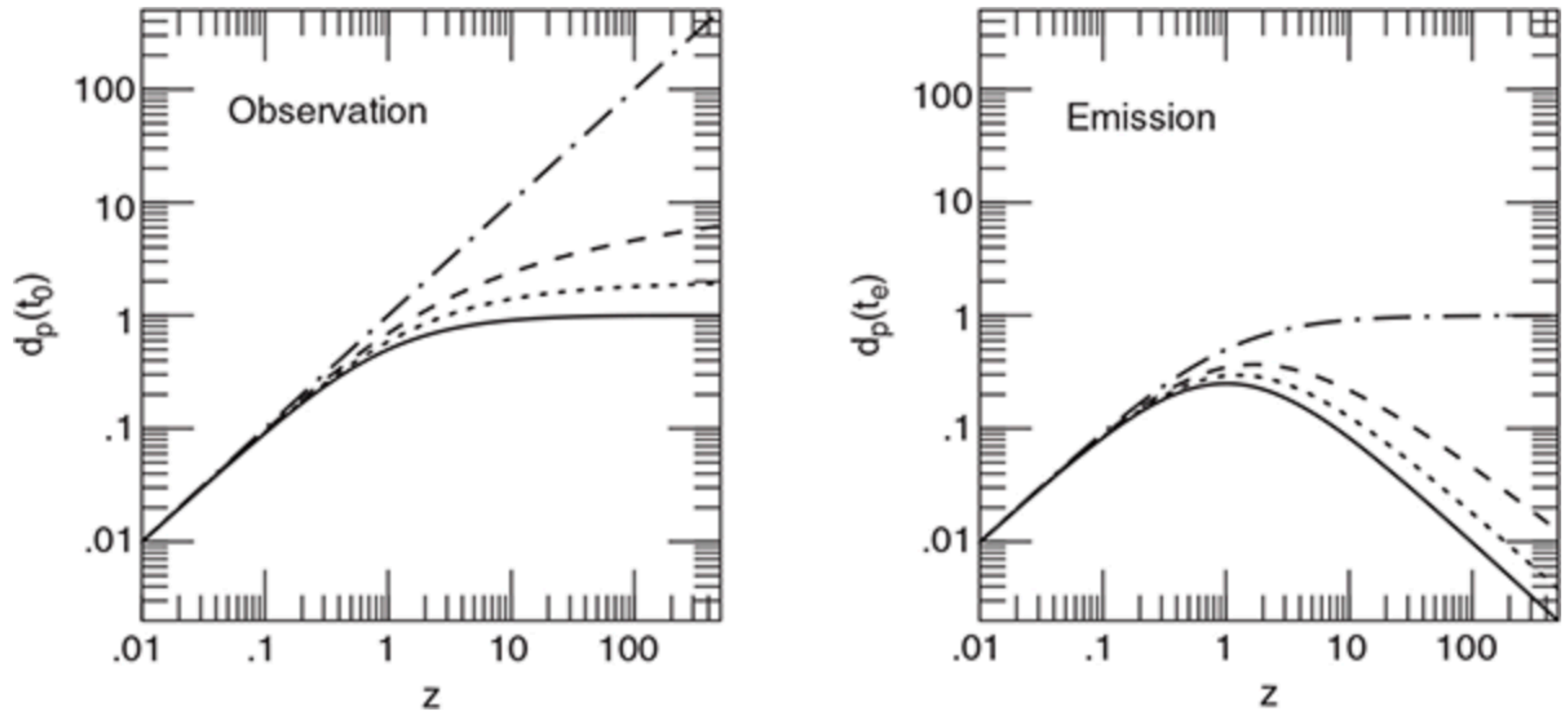


Figure 5.3 The proper distance to an object with observed redshift z, measured in units of the Hubble distance, c/H_0. Left panel: the proper distance at the time the light is observed. Right panel: proper distance at the time the light was emitted. Line types are the same as those of Figure 5.2.

Summary 3

The expansion history close to $z \sim 0$ can be approximated by a "physics free" Taylor expansion; the observable parameters H_0 and q_0 appear in the coefficients of the first- and second-order terms.

A cosmological constant Λ can be added to the GR field equations, and hence to the Friedmann equation.

Vacuum energy is one possible physical origin of a cosmological constant.

A model dominated by constant energy density expands **exponentially:** $a(t) = e^{H_0(t-t_0)}$.

The current best model based on observations (more details next time) is **flat,** $\Omega = 1$, with matter, radiation and a cosmological constant that dominates near the present day:

$$H^2(t) = H_0^2 \left(\Omega_{m,0}\, a^{-3} + \Omega_{r,0}\, a^{-4} + \Omega_{\Lambda,0} + (1 - \Omega_0)\, a^{-2}\right).$$

Next time

The ΛCDM model

Low-redshift evidence for dark matter and dark energy

For next time:

In Ryden: go back over **4.5, 5.3.3, 5.4.2, 5.4.3**, then **6.4, 6.5** and skim **Ch. 7** (we will skip the maths of the virial theorem and gravitational lensing).

In Huterer: 3.2.3, 3.3.2; skim Ch 11; 12.1, 12.3.1

This is a lot! Hopefully you're getting more familiar with the general idea now, so these sections will be easier to digest…