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BIFURCATION ANALYSIS OF A SIMPLE ANALYTIC MODEL OF SELF-PROPAGATING STAR FORMATION

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ABSTRACT

We investigate the structure and stability of rotationally symmetric nonhomogeneous time-independent solutions derived from a simple analytic model of self-propagating star formation. For this purpose we employ two methodologies: We use bifurcation theoretical methods to prove the existence of nonhomogeneous axisymmetric stationary solutions of an appropriate nonlinear evolution equation for the stellar density. We show that the nonhomogeneous solution branch bifurcates from the homogeneous one at a critical parameter value of the star formation rate. Further, the analytical theory allows us to show that the new solution set is stable in the weakly nonlinear regime near the bifurcation point. To follow the solution branch further, we use numerical methods. The numerical calculation shows the structure and stability of these solutions. We conclude that no periodic time-dependent solutions of this special model exist, and no further bifurcations can be found. The same results have been found in simulations of stochastic self-propagating star formation based on similar models. Therefore, our findings provide a natural explanation, why long-lived large-scale structure have not been found in those simulations.

Subject heading: stars: formation

1. INTRODUCTION

Self-propagating star formation (SPSF) has been discussed as a mechanism for the creation of galactic structures for more than a decade (e.g., Mueller & Arnett 1976; Gerola & Seiden 1978). Several authors proposed that self-propagating star formation could be understood as a dissipative process causing structure formation in the interstellar medium (ISM) (Shore 1983; Nozakura & Ikeuchi 1984, 1988; Feitzinger 1985). The line of thought behind this reasoning is as follows. The lack of regular two-armed spirals in the SPSF simulations (e.g., Seiden & Gerola 1982) is generally considered as a major defect of the general concept of self-propagating star formation as a mechanism for the formation of grand design spiral structure. On the other hand, it has been shown analytically (Cowie & Rybicki 1982; Balbus 1984) that large-scale detonation waves may exist under special circumstances (e.g., the neglect of feedback mechanisms). These detonation waves were also found in SPSF simulations (Freedman, Madore, & Mehta 1984).

Therefore, analytical models of self-propagating star formation may be able to advance the understanding further. If namely the conjecture regarding self-propagating star formation as a dissipative process in the ISM is true, an analytic model may show that self-organization leads to regular structures, e.g., two-armed spirals in analogy to other dissipative processes which tend to exhibit self-organized structures like spiral waves (e.g., Nicolis & Prigogine 1977).

The first attempts to obtain analytic descriptions of self-propagating star formation were based on simple reaction-diffusion equations (Shore 1983; Nozakura & Ikeuchi 1984, 1988; Feitzinger 1985), where the diffusion part of the equations is a Laplacian with a constant diffusion coefficient, quite similar to models in, e.g., theoretical chemistry. Recently, Neukirch & Feitzinger (1988) showed that such a description leads to difficulties and introduced an analytic theory of self-propagating star formation which includes the retardation effects of self-propagation and the nonlocal character of induced star formation. They could show that the spatial coupling induced by the propagation of star formation is generally more complicated than a simple diffusive coupling. Furthermore, this theory provides a way to couple the dynamics of the galactic matter including, e.g., self-gravitation in a definite way.

Neukirch & Feitzinger (1988) investigated the stability properties of homogeneous (i.e., spatially constant) stationary solutions of a simple model for triggered star formation and different rotation curves. They could show that in all cases considered a stability transition exists for a certain value of a parameter describing the relative strength of induced star formation. A problem encountered by Neukirch & Feitzinger is the rotational symmetry of their solutions. In their simple example the most unstable mode is axisymmetric. Thus the stability transition they investigated does not lead to an appropriate description of galaxies with sustained two-armed spiral structure. Nevertheless the nonlinearity allows multiple solutions to exist. Therefore, it is possible that another stability transition on another solution branch exists leading to more complex geometries.

To investigate this possibility, it is necessary to study the stationary solutions in more detail and search for indications of another stability transition. Thus in the following section we review the analytic theory, its homogeneous solution, and the stability transition found by Neukirch & Feitzinger (1988). In § 3 we use bifurcation theory to calculate neighboring axisymmetric equilibria near the transition to instability and to assess the stability of the new solution branch. In § 4 we then calculate solutions along the new branch numerically and search for further stability transitions.

The argumentation presented in this paper is purely theoretical. We try to point out the consequences of certain assumptions within the confines of the model. We would like to emphasize that this approach should be considered as complementary to comparing the model predictions with observations. From the observational point of view, there is evidence that other large scale dynamical phenomena like, e.g., density waves play a major role in the generation of spiral structure (e.g., Kaufman et al. 1989). Our argumentation will show, why the simplest SSPSF models are not capable of generating large-scale grand design spiral structure. For a recent compilation and discussion of the physical processes relevant for large scale star formation, see, e.g., Franco (1992).

2. THE MODEL EQUATION AND ITS CONSTANT STATIONARY SOLUTION

In this section we briefly summarize the approach of Neukirch & Feitzinger (1988); for details we refer to the original publication, in particular to Appendix B. For a general star formation rate, described by the function $\Psi(\rho_s, \rho_g)$ (where ρ_s and ρ_g denote densities of stars and interstellar gas, respectively; Ψ may contain terms depending only on the gas density), the time evolution of ρ_s is given by

$$\frac{\partial \rho_s}{\partial t} = \frac{1}{1 + T(\partial\Psi/\partial\rho_s)} \left[-\nabla \cdot (\boldsymbol{v}\rho_s) + \Psi - F_{\rm sd} + D \frac{\partial \Psi}{\partial \rho_s} \Delta\rho_s + D \frac{\partial^2 \Psi}{\partial \rho_s^2} (\nabla\rho_s)^2 \right] . \quad (1)$$

Here T denotes a typical propagation time scale $D^{1/2}$, a typical propagation length scale, $\boldsymbol{v}$ is the velocity field which in the case of galactic motions simply represents differential rotation, and $F_{\rm sd}$ describes the destruction of stars in the course of stellar evolution. The propagation time scale T is important only in the case of a dynamical treatment, but, as we will see below, T influences the damping time scale of the normal modes. Since our approach is based on stationary solutions of equation (1), the more important quantity is $D^{1/2}$. $D^{1/2}$ may be identified with the cell size of the SSPSF simulations.

Conservation of mass requires that

$$\rho_s(\boldsymbol{r}, t) + \rho_g(\boldsymbol{r}, t) = \rho(\boldsymbol{r}) \quad (2)$$

independent of time. Equation (2) implies a stationary distribution of total mass. In this paper, we restrict our analysis to a stationary total mass distribution. In principle, the theory allows for more general total mass distributions, e.g., a total density, which appears stationary only in a suitable rotating frame of reference, characterizing a rotating spiral pattern. In this way, one could model the interaction of a density wave with the self-propagating star formation process in a simple approximation, without having to incorporate the dynamics of the disk. Since in the present paper we are interested in the consequences of the pure SPSF mechanism, we do not investigate this possibility any further.

Similar to Neukirch & Feitzinger (1988) we now specialize in disk galaxies which may be described as two-dimensional systems. Hence we replace the densities ρ_s, ρ_g, and ρ by the surface densities σ_s, σ_g, and σ, where we take σ to be constant on the disk and normalized to unity. Furthermore let the radius of the galaxy considered be R. The last two assumptions are made for mathematical convenience in order to allow at least a partially analytic treatment. The method we use could in principle as well be applied to a σ depending on the radial coordinate without a sharp edge. This would not change the qualiative conclusion drawn by using a constant σ, but make the analysis much harder and less transparent.

The simplest reasonable form of Ψ, discarding terms depending only on the gas density for simplicity, but including the dependence of star formation on both stellar and gas density is given by

$$\Psi = a\sigma_s\sigma_g . \quad (3)$$

clearly, the stellar density in our simple two-phase model is identical to the "active density of stars" in multiphase models, i.e., to the stars participating in SPSF.

The destruction of stars to lowest order is taken to be proportional to the stellar density σ_s

$$F_{\rm sd} = b\sigma_s , \quad (4)$$

where a and b are constant. Substituting equations (3) and (4) into equation (1) and using equation (2) we get the evolution equation

$$\frac{\partial \sigma_s}{\partial t} = \frac{1}{1 + Ta(1 - \sigma_s)} \times [-\nabla \cdot (\boldsymbol{v}\sigma_s) + a\sigma_s(1 - \sigma_s) - b\sigma_s + Da(1 - \sigma_s)\Delta\sigma_s] . \quad (5)$$

Clearly the stellar density is to vanish at the boundary of the disk, i.e., introducing polar coordinates r, θ, we take $\sigma_s(r = R, \theta) = 0$.

Since we are searching for axisymmetric stationary solutions of equation (5), we need not specify the rotation curve $v_\theta(r)$. We consider purely rotational motion of the galactic matter; hence we take v_r to vanish identically. Furthermore the assumption of axial symmetry implies that all quantities depend on the radial coordinate r alone.

Introducing the dimensionless radial coordinate

$$\varpi = \frac{r}{R} , \quad (6a)$$

and the ratio of the propagation length scale to the galactic radius

$$\xi = \frac{D^{1/2}}{R} , \quad (6b)$$

the equation for the equilibria follows from equation (5)

$$G(\lambda, \xi, s) = s(1 - s) - \lambda s + \xi^2(1 - s) \frac{1}{\varpi} \frac{d}{d\varpi} \left(\varpi \frac{d}{d\varpi} \right) s = 0 , \quad (7)$$

where we have substituted s for σ_s in order to avoid double indices and introduced $\lambda = b/a$, corresponding to the ratio of a typical star formation time scale to the typical stellar destruction time scale. An obvious solution of equation (7) satisfying the boundary condition is given by

$$s_0(\varpi) = 0 , \quad (8)$$

i.e., a galaxy consisting of gas only.

To investigate the stability of this solution, the linearized operator $G_s(\lambda, \xi, s_0)$ has to be considered. It is given by

$$G_s(\lambda, \xi, s_0) = \xi^2 \frac{1}{\varpi} \frac{d}{d\varpi} \left(\varpi \frac{d}{d\varpi} \right) + (1 - \lambda) . \quad (9)$$

Determining the stability of s_0 requires knowledge of the eigenvalues of $G_s(\lambda, \xi, s_0)$. The eigenvalues are determined by the boundary value problem

$$G_s(\lambda, \xi, s_0)\psi_n = \eta_n \psi_n \tag{10}$$

with the boundary condition

$$\psi_n(\varpi = 1) = 0 . \tag{11}$$

The solutions of the equations (10) and (11) are given by

$$\psi_n(\varpi) = N_n J_0(j_{0,n}\varpi) \tag{12a}$$

$$\eta_n = 1 - \lambda - \xi^2 j_{0,n}^2 . \tag{12b}$$

Here $J_0(x)$ denotes the ordinary Bessel function of the first kind and zeroth order, $j_{0,n}$ its nth zero, and N_n a normalizing factor.

Stability depends on the sign of the highest eigenvalue

$$\eta_1 = 1 - \lambda - \xi^2 j_{0,1}^2 \tag{13}$$

of $G_s(\lambda, \xi, s_0)$. Clearly, when λ is decreased, η_1 changes sign from negative to positive at

$$\lambda_0 = 1 - \xi^2 j_{0,1}^2 \tag{14}$$

and s_0 becomes unstable. Decreasing λ corresonds to increasing star formation rate. Now bifurcation theory can be used to investigate the possible existence of neighbouring equilibria.

3. BIFURCATION THEORETICAL APPROACH

Bifurcation theoretical methods (e.g., Sattinger 1980) are widely used in the nonlinear stability analysis of physical systems, e.g, current carrying plasmas (Hesse & Schindler 1986; Hesse & Kiessling 1987). We do not state details of the mathematical background here; for this we refer to the above mentioned references.

To assure the existence of a bifurcation point in the situation encountered above, the derivative of η_1 with respect to λ must be less than zero within our sign convention. This property is confirmed by inspection of equation (13). Hence we find that bifurcation point exists at $\lambda = \lambda_0$ and $s = s_0$.

To determine the bifurcating solution branch, we expand s and λ in a power series with respect to some expansion parameter ϵ. This gives

$$s = \sum_n \epsilon^n s_n \tag{15a}$$

$$\lambda = \sum_j^n \epsilon^j \lambda_j , \tag{15b}$$

where $s_0 = 0$ is the equilibrium which changes stability and λ_0 the value of λ at the bifurcation point. (The existence of such an expansion is guaranteed by the mathematical theory [Sattinger 1980].)

The next step is to substitute the expansions (15) into the equilibrium equation (7) and expand the equilibrium equation (7) with respect to ϵ:

$$G(\lambda, \xi, s) = \sum_n \frac{1}{n!}\left[\frac{d^n}{d\epsilon^n} G(\lambda, \xi, s)\right]_{\epsilon=0} \epsilon^n . \tag{16}$$

The linear independence of the powers ϵ^n requires each term of the sum to vanish separately. Clearly, $O(\epsilon^0)$ is identical with the equilibrium equation at the bifurcation point and hence fulfilled. The first order in ϵ reads

$$(1 - \lambda_0)s_1 + \xi^2 \frac{1}{\varpi}\frac{d}{d\varpi}\left(\varpi \frac{d}{d\varpi}\right) s_1 = 0 \tag{17}$$

which just states that s_1 is the eigenfunction of the linearized operator G_s belonging to the eigenvalue zero. Obviously, the solution of equation (17) is given by

$$s_1 = \psi_1 . \tag{18}$$

The stability of the bifurcating solution branch in a neighborhood of the bifurcation point is determined by the first nonvanishing λ_j in the expansion (15) (Sattinger 1980). To determine the value of λ_1, the second order in ϵ has to be investigated. It is given by

$$(1 - \lambda_0)s_2 + \xi^2 \frac{1}{\varpi}\frac{d}{d\varpi}\left(\varpi \frac{d}{d\varpi}\right)s_2$$

$$= s_1^2 + \lambda_1 s_1 + s_1 \xi^2 \frac{1}{\varpi}\frac{d}{d\varpi}\left(\varpi \frac{d}{d\varpi}\right)s_1 \tag{19a}$$

$$= \lambda_0 s_1^2 + \lambda_1 s_1 , \tag{19b}$$

where the last identity follows from the first order equation. Equation (19) is solvable if and only if its right-hand side is perpendicular to s_1 with respect to a suitable scalar product. This implies

$$\int_0^1 \varpi\, d\varpi s_1(\lambda_0 s_1^2 + \lambda_1 s_1) = 0 . \tag{20}$$

We have to choose λ_1 such that equation (20) is satisfied. Hence λ_1 becomes

$$\lambda_1 = -\lambda_0 \int_0^1 \varpi\, d\varpi s_1^3 . \tag{21}$$

Since a and b are positive, λ is always positive and we therefore only need to consider the case of positive λ_0. Thus, λ_1 is always negative. This indicates that the bifurcating branch is stable for small ϵ (Schaeffer & Golubitzki 1985). At this point, if we emphasize again that we search for a bifurcation with decreasing λ and not with increasing λ as usual. Therefore, within our conventions we have encountered a forward bifurcation and therefore the bifurcating branch must be stable as discussed above.

Since further analytical exploration of the bifurcating branch appears to be rather tedious, we use numerical methods to follow the solution branch. This will be discussed in the following section.

4. CALCULATION AND STABILITY OF THE BIFURCATING BRANCH

After having calculated the bifurcating branch analytically near the bifurcation point, we will now extend this calculation numerically to the strongly nonlinear regime. To perform this task, we employ a method developed by Keller (1977) and utilized by, e.g., Zwingmann (1983, 1987) to study magnetostatic force equilibria applicable to Earth's magnetotail and solar coronal structures. This method is capable of automatically following the solution branch and monitoring the stability of the solutions. Furthermore, if a stability transition involves a bifurcation point, the code can either continue on the old solution branch or transit to the new one. For details, we refer to Zwingmann (1985).

In order to account for the possibility of neighboring non-axisymmetric solutions, we solve the equilibrium problem in two dimensions by including the ϕ-dependent part of the Laplacian operator. We do not need to include the part of the operator which depends on the rotation curve $v_\theta(\varpi)$ for the

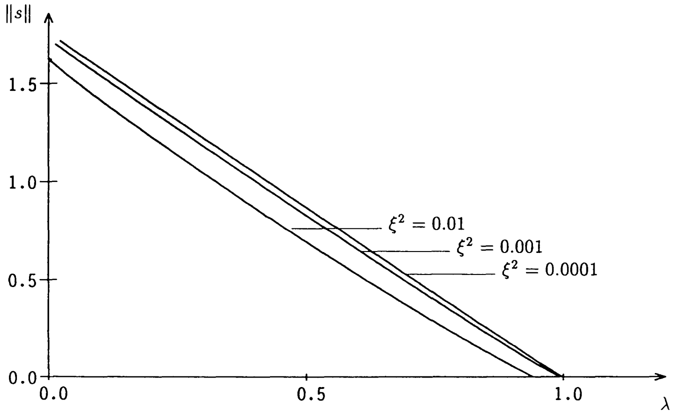

FIG. 1.—Solution diagram for different values of ξ. The figure shows the L_2-norm of s and its dependence on λ. The values of ξ from left to right are 0.1, 0.316, and 0.01.

following reason: The local stability of the axisymmetric equilibria is determined by the linearized evolution equation. In the Appendix we show that the linearized evolution equation may be decomposed into a Hermitian and an anti-Hermitian part. We derive then a sufficient stability criterion for the equilibria based solely on the Hermitian part of the linearized equilibrium operator. The Hermitian part does not involve the part of the operator including the rotation curve. The stability criterion derived in the Appendix may be stated as follows:

If there is no bifurcation of the new branch of axisymmetric solutions calculated with the equilibrium operator including only the ϕ-dependence of the Laplacian operator, then the same will be true for the full operator. The difference between the full operator and the reduced one is that the linearized full operator will have complex eigenvalues and the reduced operator real eigenvalues, if a bifurcation to nonaxisymmetric solutions occurs.

Therefore our analysis shows that the stability of the equilibrium is completely determined by the self-adjoint part of the operator alone. If we want to explicitly calculate the eigenmodes and eigenvalues of the problem, however, we would have to solve the complete problem including the rotation curve dependent part. The method we utilize here gives us information about the sign of the real part of the eigenvalues without having to solve the eigenvalue problem explicitly. The disadvantage of the method is that it does not give us the timescales involved. For mathematical details we refer to the Appendix.

The discretization of the equation is carried out with the method of finite elements. The element used is a triangle with quadratic shape functions (e.g., Wait & Mitchell 1985). We used 200 elements with a nonequidistant grid in the radial coordinate to resolve especially the strong gradients showing up at the outer boundary during the calculations. The results of the calculations are shown in the Figures 1–4. Figure 1 shows the solution diagram for different values of ξ. We see that the L_2-norm of the solution s, defined as

$$\| s \| := \left[\int_0^{2\pi} \int_0^1 s^2(\varpi, \phi) \varpi \, d\varpi \, d\phi \right]^{1/2} \tag{22}$$

approaches a value close to $\sqrt{\pi}$ for $\lambda \to 0$. This may be understood if one looks at the radial shape of the solutions when λ

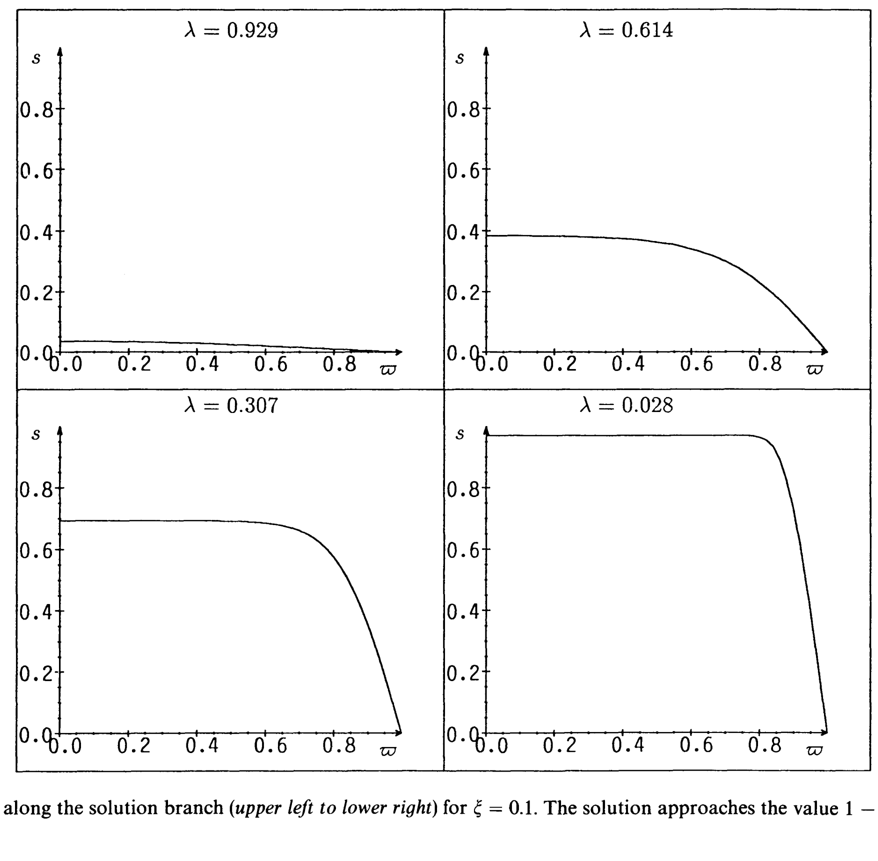

FIG. 2.—Radial shape of s along the solution branch (*upper left to lower right*) for $\xi = 0.1$. The solution approaches the value $1 - \lambda$ except for a thin sheet at the boundary.

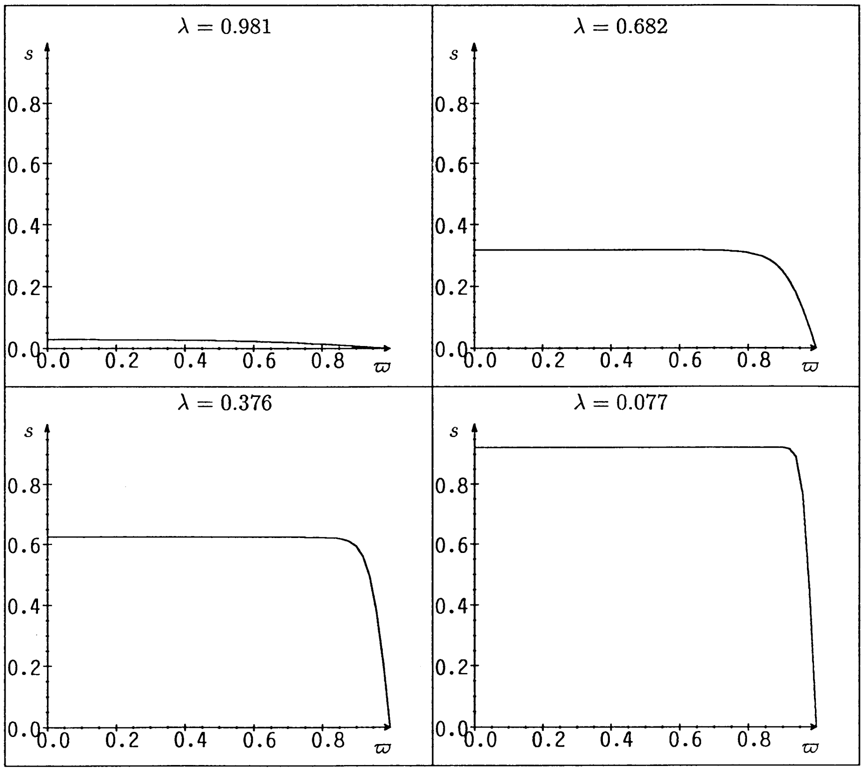

FIG. 3.—Same as Fig. 2 for $\xi = 0.0316$

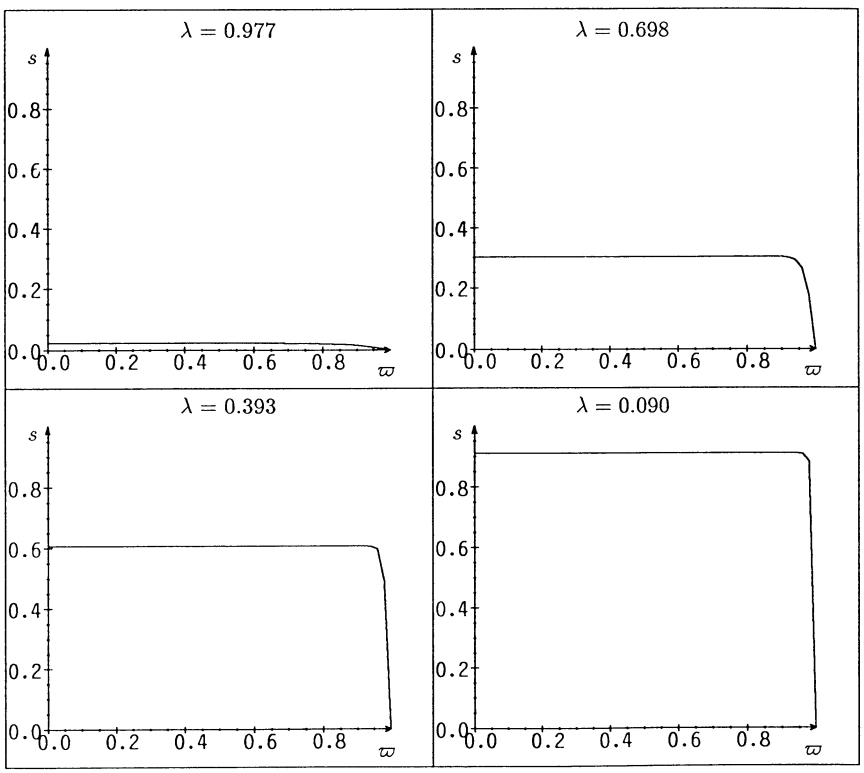

FIG. 4.—Same as Figs. 2 and 3 for $\xi = 0.01$. Notice that $1 - \lambda$ is approached at higher values of λ for smaller ξ and that the sheet at the boundary is thinner.

approaches 0, shown for different representative values of ξ in Figures 2–4. As λ decreases, the average stellar density grows and finally tends to $1 - \lambda$ everywhere, except in a thin sheet near the boundary, where it is fixed to 0. $1 - \lambda$ is the second homogeneous solution of the equilibrium equation (7), if we do not impose boundary conditions at $\varpi = 1$. Inserting $1 - \lambda$ into equation (22), we obtain the approximate value $\| s \| \simeq \sqrt{\pi}(1 - \lambda)$. This expression tends to $\sqrt{\pi}$ for $\lambda = 0$, corresponding to a state where all the matter is concentrated in stars. This is not surprising, because $\lambda = 0$ implies that either there is no destruction of stars or that the time scale for star formation tends to zero.

The radial shape of the solutions shown in Figures 2–4 is of course not very realistic for a real galactic disk. It should be emphasized that in the present investigation we do not aim to model a realistic disk galaxy, but that we focus our attention on some special properties of the theory of self-propagating star formation, which provide a deeper insight into the possibilities and the possible deficiencies of the underlying theoretical concept.

The calculations show no indication of a stability transition on the calculated branch, whatever value of ξ was chosen. This is exactly the same result as that of the SSPSF simulations (e.g., Seiden & Gerola 1982), where a percolation phase transition occurs at a critical propagation probability, but no other instability was found, when the propagation probability was increased further. Therefore, with the help of the analytic approach developed by Neukirch & Feitzinger (1988) our results provide an explanation for the nonexistence of long-lived large-scale structures in the simplest stochastic model calculations. The existence of a stable branch consisting of axisymmetric equilibria only implies that perturbation of an equilibrium located on this branch will be damped. This furthermore explains why stochasticity is a fundamental ingredient of the simple SSPSF models. Since in a deterministic model any perturbations will be damped and the stellar density will relax to the axisymmetric equilibrium, a mechanism is necessary which permanently excites new perturbations. This mechanism is provided by the stochastic element in the SSPSF simulations. Since the perturbations are nevertheless damped, they will vanish on certain time scale and thus do not lead to new structures.

The method used here does not allow us to calculate this time scale explicitly. The time scale will in general depend on the form of the rotation curve and on the propagation time scale T. For a rough estimate of the damping time scales, we replace s_0 by its approximate value $1 - \lambda$, which is a reasonable approximation for most values of λ. Calculating the eigenvalues for the approximated problem, and setting $m = 0$ (see eq. [A2]), which provides a lower bound for the damping time scale, we get

$$\omega_{0k} = \frac{a\lambda}{1 + Ta\lambda}\left(1 - \frac{1}{\lambda} - \xi^2 j_{0k}^2\right) \tag{23}$$

corresponding to a time scale

$$T_{\text{damp}} \simeq \left| \frac{\lambda}{b} \frac{1 + Tb}{2\pi(\lambda - \lambda\xi^2 j_{0k}^2 - 1)} \right| . \tag{24}$$

We see that T_{damp} is essentially determined by the destruction time scale $1/b$, but modulated by a factor depending on λ and the propagation time scale T. Therefore the damping time scale is roughly 10^7 yr for the stellar component active in the SPSF process. We may therefore conclude that the modes will certainly be damped after one galactic rotation period.

5. SUMMARY AND DISCUSSION

In this paper we investigated the structure and stability of the solution set of a nonlinear operator describing steady state equilibria of a simple model of self-propagating star formation (Neukirch & Feitzinger 1988). Starting from a homogeneous axissymetric solution to the equilibrium equations we found a critical value of the star formation probability at which the original solution loses stability. Using bifurcation theoretical models we could then analytically prove the existence of another equilibrium branch, exhibiting nonhomogeneous axisymmetric star densities, which intersected the original branch at the point of stability transition. Furthermore, the analytic theory provided a nonlinear stability analysis as well. We found the bifurcation solutions to be stable in the weakly nonlinear regime near the bifurcation point. This result prompted us to use numerical tools to investigate the behavior of the new solution branch for star formation rates far different from the critical rate at the bifurcation point.

Furthermore, we could derive a sufficient stability criterion for axisymmetric solutions, which enabled us to check the stability of the new solution branch. We found that the whole branch is stable against axisymmetric and nonaxisymmetric perturbations. Any nonaxisymmetric instability would have led necessarily to time-dependent behavior due to the complex eigenvalues associated with the corresponding linearized operator (see Appendix).

On the basis of the results presented here, we may for the first time give an explanation for the nonexistence of persistent large-scale structures in the simple SPSF simulations (e.g., Mueller & Arnett 1976; Gerola & Seiden 1978). For any possible value of the star formation rate, parameterized by λ in this work, there exists one stable axisymmetric equilibrium solution of the evolution equation (5). The bifurcation point found here corresponds to the percolation phase transition of the SPSF models (e.g., Schulman & Seiden 1982). At the bifurcation point, the homogeneous solution, which is the only equilibrium solution for low star formation rates (large λ), becomes unstable and a new solution branch of inhomogeneous axisymmetric solutions emerges for larger star formation rates (small λ). Exactly the same features appear in the corresponding SPSF model. Since the new branch is stable, any perturbation will be damped and vanish after a certain short time scale, which we roughly estimated to be of the order of the stellar destruction time scale. In the SPSF models the stochasticity permanently excites new perturbations, which are responsible for the flocculent structure seen in those models. A smooth, grand design structure would have only been possible if a nonaxisymmetric instability of the new branch existed. As we have shown, this is not the case and therefore no large-scale structure exists in simple models of self-propagating star formation.

A well-known property of the spiral features seen in SPSF simulations is the dependence of their pitch angle on the form of the rotation curve. At first sight, our result seem to contradict these findings, because they are completely independent of the rotation curve. But a closer look at our results reveals that there is no contradiction at all. As explained above, the spiral features in the SPSF simulations correspond to stochastically excited normal modes in our approach. The shape of the normal modes will of course not be independent of the shape of the rotation curve because in order to calculate the normal modes we would have to solve the full eigenvalue problem containing the part depending on $\Omega(\varpi)$. What we presented here is a way to circumvent the explicit calculation of the normal modes and nevertheless get information about the stability properties of the equilibria. This information does not depend on the particular form of the rotation curve. The price we have to pay for using this method is that we get no information about the time scale on which the normal modes will be damped. To get an estimate of this time scale we have to solve an approximate form of the eigenvalue problem.

The complete match between the results of the SPSF simulations and of the analytic approach by Neukirch & Feitzinger (1988) shows that this approach is a useful tool, which supplements the numerical simulations and allows a reliable and efficient interpretation of their results. This will be even more important for more complicated and realistic models, which might be developed in the future.

APPENDIX

In this appendix, we want to perform the local stability analysis of a general axisymmetric equilibrium. Assume that the solution $s_0(\varpi)$ of equation (7) for given λ and ξ is known. Linearizing equation (5) about s_0 we get

$$\frac{\partial s_1}{\partial t} = \frac{1}{1 + Ta(1 - s_0)}\left[\!\left[-\frac{v_\phi(\varpi)}{\varpi}\frac{\partial s_1}{\partial \phi} + a\left\{\left[(1 - s_0) - \frac{\lambda}{1 - s_0}\right]s_1 + \xi^2(1 - s_0)\Delta s_1\right\}\right]\!\right] . \tag{A1}$$

Setting $s_1 = \exp(\omega_{mj} t)\exp(im\phi)s_{mj}(\varpi)$ we obtain the eigenvalue problem:

$$\omega_{mj} s_{mj} = \frac{1}{1 + Ta(1 - s_0)}\left\{ -im\frac{v_\phi(\varpi)}{\varpi} s_{mj} + a\left(1 - s_0 - \frac{\lambda}{1 - s_0}\right)s_{mj} + a\xi^2(1 - s_0)\left[-\frac{m^2}{\varpi^2} s_{mj} + \frac{1}{\varpi}\frac{d}{d\varpi}\left(\varpi\frac{ds_{mj}}{d\varpi}\right)\right]\right\} . \tag{A2}$$

Multiplying equation (A2) by the complex conjugate of s_{mj} denoted by $\bar{s}_{mj}$ and integrating, we get for ω_{mj} the expression

$$\omega_{mj} = \frac{\int_0^1 \bar{s}_{mj}\hat{L}_m s_{mj}\varpi\, d\varpi}{\int_0^1 [1 + Ta(1 - s_0)/a(1 - s_0)]\,|s_{mj}|^2\varpi\, d\varpi} , \tag{A3}$$

where

$$\hat{L}_m \cdot = \left[-\frac{imv_\phi(\varpi)}{\varpi a(1 - s_0)}\cdot + 1 - \frac{\lambda}{(1 - s_0)^2} - \xi^2\frac{m^2}{\varpi^2}\right]\cdot + \xi^2\frac{1}{\varpi}\frac{d}{d\varpi}\left(\varpi\frac{d}{d\varpi}\cdot\right) \tag{A4}$$

Introducing the scalar product

$$\langle f, g\rangle = \int_0^1 \bar{f}\cdot g\varpi\, d\varpi \tag{A5}$$

we may define the Hermitian part of $\hat{L}_m$ as

$$\hat{L}_m^H \cdot = \left[1 - \frac{\lambda}{(1 - s_0)^2} - \xi^2\frac{m^2}{\varpi^2}\right]\cdot + \xi^2\frac{1}{\varpi}\frac{d}{d\varpi}\left(\varpi\frac{d}{d\varpi}\cdot\right) . \tag{A6}$$

The eigenvalue problem

$$\hat{L}_m^H v_{m,k}^H = \lambda_{m,k}^H v_{m,k}^H \tag{A7}$$

is of the Sturm-Liouville type; therefore the functions $v_{m,k}^H$ form complete and orthogonal system of functions on the interval [0, 1] for fixed m. Since $\hat{L}_m^H$ is real the $v_{m,k}^H$ may be chosen real, too. We now expand the eigenfunctions of the operator $\hat{L}_m$ into a series of $v_{m,k}^H$:

$$s_{mj} = \sum_k c_{m,k} v_{m,k}^H , \tag{A8}$$

where $c_{m,k}$ are complex coefficients. Inserting equation (A8) into equation (A3) gives

$$\omega_{mj} = \frac{\sum_k |c_{kj}|^2\lambda_{m,k}^H + I}{\int_0^1 [1 + Ta(1 - s_0)/a(1 - s_0)]\,|s_{mj}|^2\varpi\, d\varpi} \tag{A9}$$

with

$$I = \sum_{j,n} \bar{c}_{kj} c_{kn}\int_0^1 v_{mj}^H \hat{L}_m^A v_{mn}^H \varpi\, d\varpi = -\bar{I} , \tag{A10}$$

where $\hat{L}_m^A$ is the anti-Hermitian part of $\hat{L}_m$. Therefore, I determines the imaginary part of ω_{mj}. Since the denominator in equation (A9) is always positive, the real part of ω_{mj} is in any case negative if all eigenvalues of $\hat{L}_m^H$ are negative. This is a sufficient stability criterion for $s_0(\varpi)$. Since we know that the new branch is stable near the bifurcation point, it suffices to investigate the Hermitian part of equation (A2) to assess the stability of $s_0(\varpi)$. This is done during the numerical procedure of calculating the equilibrium s_0. In all cases which were investigated, we found that the stability criterion was fulfilled.

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