Operations Research, Spring 2022 (110-2)
Suggested Solution for Homework 3

Tim Kuo[*], Yuan Ting Lin[†], Yu-Chieh Kuo[‡]

May 21, 2022

1. (a) First, observe six connected points of boards within the valley, including two point A and B labeling the two sides of the valley. Therefore, let $I = \{0, ..., 5\}$ be the set of connected points of boards, where point 0 and point 5 represent the two sides of the valley, i.e., the point A and B, respectively. Next, let (x_i, y_i) be the position of the point $i \in I$, where $(x_0, y_0) = (0, 100)$ and $(x_5, y_5) = (100, 150)$.
The formulation is

$$\begin{aligned} \min \quad & \sum_{i=1}^{4} y_i \\ \text{s.t.} \quad & (x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = L^2 \quad \forall i \in I \setminus \{5\} \\ & x_0 = 0,\, x_5 = 100,\, y_0 = 100,\, y_5 = 150 \\ & x_i \geq 0 \quad \forall i \in I \\ & y_i \geq 0 \quad \forall i \in I. \end{aligned}$$

The formulation is **not** a convex program since the constraint contains an equation.

(b) We depict the positions as Fig. 1.

(a) $L = 25$ (b) $L = 35$

Figure 1: The position of the boards.

*Department of Information Management, National Taiwan University. E-mail: r10725025@ntu.edu.tw.
†Department of Information Management, National Taiwan University. E-mail: b07705036@ntu.edu.tw.
‡Department of Information Management, National Taiwan University. E-mail: b07611039@ntu.edu.tw.

2. We propose two formulations to solve this problem so that you can compare a similar one with your answer.

Method 1: (a) Let x_i be 1 if we obtain patent $i \in I$ or 0 otherwise, and y_j be 1 if we can produce the product $j \in J$ or 0 otherwise. The linear integer programming to maximize the total profits is

$$\begin{aligned} \max \quad & \sum_{j \in J} P_j y_j \\ \text{s.t.} \quad & \sum_{i \in I} F_i x_i \le B \\ & y_j - x_i \le 1 - A_{ij} \quad \forall i \in I,\, j \in j \\ & x_i \in \{0,1\} \quad \forall i \in I \\ & y_j \in \{0,1\} \quad \forall j \in J. \end{aligned}$$

(b) The linear relaxation is

$$\begin{aligned} \max \quad & \sum_{j \in J} P_j y_j \\ \text{s.t.} \quad & \sum_{i \in I} F_i x_i \le B \\ & y_j - x_i \le 1 - A_{ij} \quad \forall i \in I,\, j \in j \\ & x_i \ge 0 \quad \forall i \in I \\ & x_i \le 1 \quad \forall i \in I \\ & y_j \ge 0 \quad \forall j \in J \\ & y_j \le 1 \quad \forall j \in J. \end{aligned}$$

and the dual linear relaxation is

$$\begin{aligned} \min \quad & \sum_{i \in I} \sum_{j \in J} (1 - A_{ij}) z_{ij} + Bw + \sum_{i \in I} u_i + \sum_{j \in J} v_j \\ \text{s.t.} \quad & \sum_{i \in I} z_{ij} + v_j - P_j \ge 0 \quad \forall j \in J \\ & -\sum_{j \in J} z_{ij} + F_i w + u_i \ge 0 \quad \forall i \in I \\ & w \ge 0 \\ & z_{ij} \ge 0 \quad \forall i \in I,\, j \in J \\ & u_i \ge 0 \quad \forall i \in I \\ & v_j \ge 0 \quad \forall j \in J. \end{aligned}$$

(c) The optimal solution for the linear relaxation is $x^* = (1, 1, 0.08, 1)$ and $y^* = (1, 1, 0.08, 1, 0.08, 1)$ with the optimal objective value of 5416. That is, obtaining the complete license of patent 1,2,4 and partial license (precisely, 0.08) of patent 3 is the optimal strategy and will result in the highest profit at \$5416. Note that the solution to the linear relaxation differs from the solution to the original formulation; therefore, it might give the unsatisfactory answer.

(d) The shadow price of our budget constraint is $w^* = 1.08$. Namely, the profit will increase \$1.08 if we have a \$1 more budget.

Method 2: (a) Let x_i be 1 if we obtain patent $i \in I$ or 0 otherwise, and y_j be 1 if we can produce the product $j \in J$ or 0 otherwise. The linear integer programming to maximize the total profits is

$$\begin{aligned}
\max \quad & \sum_{j\in J} P_j y_j \\
\text{s.t.} \quad & \sum_{i\in I} F_i x_i \le B \\
& |S_j| y_j \le \sum_{i\in I} A_{ij} x_i \quad \forall j \in J \\
& x_i \in \{0,1\} \quad \forall i \in I \\
& y_j \in \{0,1\} \quad \forall j \in J.
\end{aligned}$$

(b) The linear relaxation of the program is

$$\begin{aligned}
\max \quad & \sum_{j\in J} P_j y_j \\
\text{s.t.} \quad & \sum_{i\in I} F_i x_i \le B \\
& |S_j| y_j \le \sum_{i\in I} A_{ij} x_i \quad \forall j \in J \\
& x_i \ge 0 \quad \forall i \in I \\
& x_i \le 1 \quad \forall i \in I \\
& y_j \ge 0 \quad \forall j \in J \\
& y_j \le 1 \quad \forall j \in J.
\end{aligned}$$

The dual linear relaxation can be represented as

$$\begin{aligned}
\min \quad & Bw + \sum_{i\in I} u_i + \sum_{j\in J} v_j \\
\text{s.t.} \quad & F_i w + u_i - \sum_{j\in J} A_{ij} z_j \ge 0 \quad \forall i \in I \\
& |S_j| z_j + v_j - P_j \ge 0 \quad \forall j \in J \\
& w \ge 0 \\
& z_j \ge 0 \quad \forall j \in J \\
& u_i \ge 0 \quad \forall i \in I \\
& v_j \ge 0 \quad \forall j \in J.
\end{aligned}$$

(c) The optimal solution for the linear relaxation is $x^* = (1, 1, 0.08, 1)$ and $y^* = (1, 1, 0.54, 1, 0.54, 1)$ with the optimal objective value of 6658. That is, obtaining the complete license of patent 1,2,4 and partial license (precisely, 0.08) of patent 3 is the optimal strategy and will result in the highest profit at \$6658. Note that the solution to the linear relaxation differs from the solution to the original formulation; therefore, it might give the unsatisfactory answer.

(d) The shadow price of our budget constraint is $w^* = 0.54$. Namely, the profit will increase \$0.54 if we have a \$1 more budget.

3. (a) Following the definition of variables and extending the formulation in problem 2, the maximization problem alters to

$$\begin{aligned}
\max \quad & \sum_{j\in J} P_j y_j - \sum_{i\in I} F_i x_i \\
\text{s.t.} \quad & y_j - x_i \le 1 - A_{ij} \quad \forall i \in I,\ j \in J \\
& x_i\, y_j \in \{0,1\} \quad \forall i \in I,\ j \in J.
\end{aligned}$$

Note that the first constraint is equivalent to

$$y_j - x_i \le 0 \quad \forall j \in J,\ i \in S_j.$$

(b) Observe the coefficient matrix C derived from the formulation in problem 3a:

$$
C^{mn\times(m+n)} = \begin{array}{c}
\begin{array}{cccccccc} y_1 & y_2 & \cdots & y_m & x_1 & x_2 & \cdots & x_n \end{array} \\
\left[\begin{array}{cccc:cccc}
1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\
0 & 1 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1
\end{array}\right]
\end{array}.
$$

We want to prove the sufficient condition for total unimodularity by considering its transpose matrix C^T:

- All elements in the matrix C^T are either 1, 0, or -1: **True**.
- Each column in C^T contains at most two nonzero elements: **True**.
- Rows in C^T can be divided into two groups so that for each column, two nonzero elements are in the same group if and only if they are different: **True**. (Flip the sign of all x_i, and divide them into a group of x_i and a group of y_i)

Consequently, C^T is totally unimodular, which concludes that C **is totally unimodular** as given a totally unimodular matrix A, the transpose matrix A^T satisfies total unimodularity as well.

4. Recall Definition 2: *For a convex domain $F \subseteq \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex over F if*

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2).$$

for all $\lambda \in [0,1]$ and $x_1, x_2 \in F$.

Moreover, the lecture materials and the supplement for convex analysis guide convexity proof. We follow such steps to examine these functions.

(a) Since $f(x)$ is twice differentiable, its first and second derivatives are

$$f'(x) = 8x^3 - 6x + e^x \quad \text{and} \quad f''(x) = 24x^2 - 6 + e^x.$$

As the method to find the closed-form solution to such a function is beyond the scope of this course, the function $f(x)$ is convex over the region such that $f''(x) \ge 0$, i.e., $\{x \in \mathbb{R} \,|\, 24x^2 - 6 + e^x \ge 0\}$.

(b) The gradient and the Hessian of $f(x_1, x_2)$ is

$$\nabla f(x_1,x_2) = \begin{bmatrix} 3(x_1^2 - 10x_1 + 24) \\ -2(x_2 - 5) \end{bmatrix} \quad \text{and} \quad \nabla^2 f(x_1,x_2) = \begin{bmatrix} 6x_1 - 30 & 0 \\ 0 & -2 \end{bmatrix}.$$

The Hessian is not positive semi-definite since $-2 < 0$, therefore, the function $f(x_1, x_2)$ is convex nowhere.

(c) The gradient and the Hessian of $f(x_1, x_2, x_3)$ is

$$\nabla f(x_1,x_2,x_3) = \begin{bmatrix} 3x_1^2 - 2x_2 \\ -2x_1 + x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \nabla^2 f(x_1,x_2,x_3) = \begin{bmatrix} 6x_1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

After checking all the principle minors, we found that the Hessian is positive semi-definite if and only if $6x_1 - 4 \ge 0 \iff x_1 \ge \frac{2}{3}$, that is, the function $f(x_1, x_2, x_3)$ is convex over the region $\{x_1, x_2, x_3 \in \mathbb{R} \,|\, x_1 \ge \frac{2}{3}\}$.

5. (a) The relaxed program is

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} v_i x_i + \lambda \left(B - \sum_{i=1}^{n} w_i x_i \right) \\ \text{s.t.} \quad & x_i \in \{0, 1\} \quad \forall i \in \{1, ..., n\}. \end{aligned}$$

Note that the Lagrange multiplier λ should be nonnegative to reward feasibility.

(b) The objective function in problem 5a can be rewritten as

$$\sum_{i=1}^{n} v_i x_i + \lambda \left(B - \sum_{i=1}^{n} w_i x_i \right) = \lambda B + \sum_{i=1}^{n} (v_i - \lambda w_i) x_i.$$

Given λ , λB is a constant, and we only need to maximize $\sum_{i=1}^{n}(v_i - \lambda w_i)x_i$. Thus, the method to find an optimal solution can be simply described as letting x_i be 1 if $v_i - \lambda w_i \geq 0$ or 0 otherwise.

(c) With $\lambda = 0.8$, the optimal solution to the relaxed program is $x^* = (1, 1, 1, 1, 1, 0, 1)$. It is not feasible to the original knapsack problem.

(d) With $\lambda = 1.1$, the optimal solution to the relaxed program is $x^* = (1, 1, 1, 1, 1, 0, 0)$. It is feasible to the original knapsack problem.

(e) After trying all possible values of λ listed in Table 1 (for each interval, pick one valid value of λ arbitrarily), we can select the best solution among all feasible solutions. The solution reported by the heuristic algorithm is $x^{\text{ALG}} = (1, 1, 1, 1, 1, 0, 0)$ with the value of $\lambda \in (1, \frac{8}{7}]$ and the correspondingly objective value 41. We can solve this IP directly and find that the optimal solution is also $x^* = (1, 1, 1, 1, 1, 0, 0)$. Therefore, the resulting optimality gap is 0%, and we conclude this heuristic algorithm is good for this instance.

Value of λ	solution	objective value	feasible solution?
$\lambda \in [0, \frac{4}{7}]$	$(1, 1, 1, 1, 1, 1, 1)$	48	No
$\lambda \in (\frac{4}{7}, 1]$	$(1, 1, 1, 1, 1, 0, 1)$	44	No
$\lambda \in (1, \frac{8}{7}]$	$(1, 1, 1, 1, 1, 0, 0)$	41	Yes
$\lambda \in (\frac{8}{7}, \frac{5}{4}]$	$(1, 1, 0, 1, 1, 0, 0)$	33	Yes
$\lambda \in (\frac{5}{4}, \frac{7}{5}]$	$(1, 1, 0, 1, 0, 0, 0)$	28	Yes
$\lambda \in (\frac{7}{5}, \frac{3}{2}]$	$(1, 1, 0, 0, 0, 0, 0)$	21	Yes
$\lambda \in (\frac{3}{2}, \infty)$	$(0, 0, 0, 0, 0, 0, 0)$	0	Yes

Table 1: All possible values of λ and the corresponding solutions