Operations Research, Spring 2022 (110-2)
Suggested Solution for Homework 1

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1. (a) Let $I = \{1, 2, \dots, 10\}$ be the set of players and $J = \{\mathrm{S}, \mathrm{L}, \mathrm{H}, \mathrm{M}, \mathrm{O}\}$ be the set of positions. Let S_i, P_i be player i's spiking and passing ability, respectively, $i \in I$. Let x_{ij} be 1 if player i plays the position j or 0 otherwise; A_{ij} be 1 if player i is allowed to play the position j or 0 otherwise, $i \in I, j \in J$.

   The IP formulation is

$$
\begin{aligned}
\max \quad & \sum_{i \in I} \sum_{j \in \{\mathrm{H},\mathrm{M},\mathrm{O}\}} S_i x_{ij} + 3 \sum_{i \in I} P_i x_{i,\mathrm{S}} \\
\text{s.t.} \quad & \sum_{j \in J} x_{ij} \leq 1 \quad \forall i \in I \\
& \sum_{i \in I} A_{ij} x_{ij} = 1 \quad \forall j \in \{\mathrm{S}, \mathrm{L}, \mathrm{O}\} \\
& \sum_{i \in I} A_{ij} x_{ij} = 2 \quad \forall j \in \{\mathrm{H}, \mathrm{M}\} \\
& \sum_{i \in I} \sum_{j \in J} A_{i,\mathrm{S}} x_{ij} \geq 2 \\
& \sum_{i \in I} \sum_{j \in J} A_{i,\mathrm{H}} x_{ij} \geq 3 \\
& \sum_{j \in J} (x_{1,j} + x_{3,j}) \leq 1 \\
& \sum_{j \in J} (x_{1,j} + x_{3,j} + x_{4,j} + x_{6,j} + x_{8,j}) \geq 2 \\
& x_{ij} \in \{0, 1\} \quad \forall i \in I,\ j \in J.
\end{aligned}
$$

   (b) Let B_i, R_i be player i's blocking and receiving ability, respectively, $i \in I$. The additional constraints are

$$
\begin{aligned}
& \sum_{i \in I} \sum_{j \in \{\mathrm{L},\mathrm{H},\mathrm{O}\}} x_{ij} R_i \geq 14 \\
& \sum_{i \in I} \sum_{j \in \{\mathrm{M},\mathrm{O}\}} x_{ij} B_i \geq 10.5.
\end{aligned}
$$

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2. (a) Let $I = \{1, \dots, 8\}$ be the set of slots, $J = \{1, \dots, 5\}$ be the set of days, and $K = \{1, \dots, 10\}$ be the set of groups. Let D_{ij} be the set of demand at slot i in the day j, $i \in I$, $j \in J$. Let A_{mnk} be 1 if students in the group k should work at day m and day n or 0 otherwise, $m = 1, \dots, 4$, $n = m+1, \dots, 5$. Let x_k be the number of students belonging to group k, $k \in K$.
The LP formulation is

$$\begin{aligned} \min \quad & \sum_{k \in K} x_k \\ \text{s.t.} \quad & \sum_{k \in K} \sum_{m=1}^{4} \sum_{n=m+1}^{5} A_{mnk} x_k \geq D_{ij} \quad \forall i \in I,\ j \in J \\ & x_k \geq 0 \quad \forall k \in K. \end{aligned}$$

(b) Let $I = \{1, \dots, 8\}$ be the set of slots, $J = \{1, \dots, 5\}$ be the set of days, $K = \{1, \dots, 80\}$ be the set of groups, and $S = \{1, 2\}$ be the set of two segments of slots, i.e., slots 1 to 4 and slots 5 to 8. Let D_{ij} be the set of demand at slot i in the day j, $i \in I$, $j \in J$. Let $A_{d_1 d_2 d_3 d_4, s_1 s_2 s_3 s_4, k}$ be 1 if students in the group k should work at day d_1, d_2, d_3, d_4 in the slot segment s_1, s_2, s_3, s_4, respectively, or 0 otherwise, $d_1, d_2, d_3, d_4 \in J$, $d_1 < d_2 < d_3 < d_4$, $s_1, s_2, s_3, s_4 \in S$. Let x_k be the number of students belonging to group k, $k \in K$.
The LP formulation is

$$\begin{aligned} \min \quad & \sum_{k \in K} x_k \\ \text{s.t.} \quad & \sum_{k \in K} \sum_{d_1=1}^{2} \sum_{d_2=d_1+1}^{3} \sum_{d_3=d_2+1}^{4} \sum_{d_4=d_3+1}^{5} \sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{s_3 \in S} \sum_{s_4 \in S} A_{d_1 d_2 d_3 d_4, s_1 s_2 s_3 s_4, k} x_k \geq D_{ij} \quad \forall i \in I,\ j \in J \\ & x_k \geq 0 \quad \forall k \in K. \end{aligned}$$

3. Let $I = \{1, \dots, 40\}$ be the set of CSRs and K be the predetermined number of days off per month. Let x_{ijd} be 1 if the i-th CSR works in the j shift in day d and 0 otherwise, $i \in I$, $j \in J$, and $d \in D$. Let e_{td} be the number of lacks in time period t in day d, $t \in T$ and $d \in D$.
The IP formulation is

$$\begin{aligned} \min \quad & \sum_{t \in T} \sum_{d \in D} e_{td} \\ \text{s.t.} \quad & \sum_{j \in J} x_{ijd} = 1 \quad \forall i \in I,\ d \in D \\ & \sum_{j \in J} \sum_{d \in D} B_{j,\text{leave}} x_{ijd} = K \quad \forall i \in I \\ & \sum_{j \in J} \sum_{d=n}^{n+6} B_{j,\text{night}} x_{ijd} \leq 1 \quad \forall i \in I,\ n \in \{1, \dots, |D| - 6\} \\ & \sum_{j \in J} \sum_{d=n}^{n+6} B_{j,\text{afternoon}} x_{ijd} \leq 2 \quad \forall i \in I,\ n \in \{1, \dots, |D| - 6\} \\ & \sum_{j \in J} \sum_{d=n}^{n+6} B_{j,\text{leave}} x_{ijd} \geq 1 \quad \forall i \in I,\ n \in \{1, \dots, |D| - 6\} \\ & e_{td} \geq C_{td} - \sum_{i \in I} \sum_{j \in J} A_{jt} x_{ijd} \quad \forall t \in T,\ d \in D \\ & e_{td} \geq 0 \quad \forall t \in T,\ d \in D \\ & x_{ijd} \in \{0, 1\} \quad \forall i \in I,\ j \in J,\ d \in D. \end{aligned}$$

4. (a) The optimal solution is $x_{12} = 0, x_{13} = 0, x_{14} = 0, x_{15} = 8, x_{23} = 2, x_{24} = 5, x_{25} = 0, x_{34} = 4, x_{35} = 0$, and $x_{45} = 0$ (there may be multiple optimal solutions). The optimal value obtained is 19.

The leader of STIM should hire 8 students working on Monday and Friday, 2 students working on Tuesday and Wednesday, 5 students working on Tuesday and Thursday, and 4 students working on Wednesday and Thursday to meet all the requirements. The total amount of students hired is 19.

(b) The optimal solution is $x_{12} = 0, x_{13} = 0, x_{14} = 0, x_{15} = 5, x_{23} = 2, x_{24} = 2, x_{25} = 0, x_{34} = 2, x_{35} = 0$, and $x_{45} = 0$ (there may be multiple optimal solutions). The optimal value is 11.

The leader of STIM should hire 5 students working on Monday and Friday, 2 students working on Tuesday and Wednesday, 2 students working on Tuesday and Thursday, and 2 students working on Wednesday and Thursday to meet all the requirements. The total amount of students hired is 11.

(c) The solutions are different. The students needed are more evenly distributed in the second part, and the requirements can be met by hiring fewer students with the given set of shifts.