Operations Research, Spring 2022 (110-2)
Homework 1

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1 Rules

- This homework is due at ***23:59, March 19***. Those who submit their works late but are late by less than one hour gets 10 points off. Works that are late more than one hour get no point.
- For this homework, students should work individually. While discussions are encouraged, copying is prohibited.
- Please submit a ***PDF file*** through NTU COOL and make sure that the submitted work contains the student ID and name. Those who fail to do these will get 10 points off.
- You are ***required*** to ***type*** your work with LaTeX (***strongly suggested***) or a text processor with a formula editor. Hand-written works are not accepted. You are responsible to make your work professional in mathematical writing by following at least the following rules:[1]
  1. When there is a symbol denoted by an English letter, make it italic. For example, write $a + b = 3$ rather than a + b = 3.
  2. An operator (e.g., +) should not be italic. A function with a well-known name (e.g., log, max and sin) is considered as an operator.
  3. A number should not be italic. For example, it should be $a + b = 3$ rather than $a + b = 3$.
  4. Superscripts or subscripts should be put in the right positions. For example, a_1 and $a1$ are completely different: The former is a variable called a_1 while the latter is actually $a \times 1$.
  5. When there is a subtraction, write $-$ rather than -. For example, write $a - b = 3$ rather than a - $b = 3$. The same thing applies to the negation operator. For example, write $a = -3$ rather than a = -3.
  6. If you want to write down the multiplication operator, write $\times$ rather than *.
  7. For an exponent, write it as a superscript rather than using ^. For example, write 10^2 rather than 10^2.
  8. There should be proper space beside a binary operator. For example, it should be $a + b = 3$ rather than a+b=3.

  Those who fail to follow these rules may get at most 10 points off.
- As we may see, there are many students, many problems, but only a few TAs. Therefore, when the TAs grade this homework, it is possible for only some problems to be randomly selected and graded. For all problems, detailed suggested solutions will be provided.

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[1] A more complete list of formatting rules is on NTU COOL.

2 Problems

1. (25 points) Mikasa is trying to choose seven out of ten volleyball players for a contest. Ten players have been rated (on a scale of 1 for poor to 5 for excellent) according to their spiking, blocking, receiving, and passing abilities. The positions that each player is allowed to play and the player's abilities are listed in Table 1, where S, L, H, M, and O stand for setters, liberals, outside hitters, middle hitters, and opposite.

Player	Position	Spiking	Blocking	Receiving	Passing
1	S	2	1	3	5
2	S or L	2	3	4	3
3	S or H or O	4	4	4	3
4	H or L or O	3	3	5	3
5	H or M	5	4	3	3
6	H or M or O	3	3	3	2
7	M	4	5	1	1
8	H or L	4	2	5	2
9	L	1	1	5	3
10	M or H	2	3	3	2

Table 1: Player information

The seven-player team must satisfy the following restrictions:

- There must be exactly one player playing setter, one playing liberal, two playing outside hitter, two playing middle hitter, and one playing opposite. For example, you may not select players 3, 4, 5, 6, 7, 8, and 10, because then only player 3 may play setter and opposite, but she may only play one position.
- To make the team flexible enough, there must be at least two players who may play setter and at least three players who may play outside hitter.
- If player 1 is chosen, then player 3 cannot be chosen.
- At least two among players 1, 3, 4, 6, 8 must be chosen.

Given these constraints, Ikuta wants to maximize

$$\text{total spiking level of the four hitters and opposite} + 3 \times \text{passing level of the setter}$$

of the starting team.

(a) (15 points) Formulate an integer program that solves Ikuta's problem.

(b) (10 points) Add constraints to the IP to account for the following restrictions:

- The average receiving level of the two outside hitters, opposite, and liberal must be at least 3.5.
- The average blocking level of the two middle hitters and the opposite must be at least 3.5.

2. (30 points; 15 points each) For the coming IM Week, the student association of the IM department (STIM) needs to assign students to work in five days. There are eight time slots in each day. The time slots 9–10, 10–11, ..., and 4–5 are labeled as slots 1, 2, ..., and 8. Monday, Tuesday, ..., and Friday are labeled as days 1, 2, ..., and 5. The number of students that are needed for slot i in day j is D_{ij}. An example of demand is in Table 2. For example, $D_{11} = 6$, $D_{23} = 1$, etc.

   Each hired student will work for exactly sixteen slots. By considering the constraints specified in the following parts, please formulate a linear program that may help the leader of STIM to find a hiring and assignment plan that minimizes the number of students needed to fulfill all the demands.

Slot	Number of students needed				
	Day 1	Day 2	Day 3	Day 4	Day 5
1	6	4	4	5	4
2	3	2	1	3	4
3	3	2	1	4	3
4	8	7	4	9	5
5	8	6	6	8	3
6	4	2	2	2	3
7	3	2	2	2	5
8	2	1	1	2	8

Table 2: An example of demand

Note 1. Please answer Parts (a) and (b) separately.

Note 2. Your program must be linear. It cannot contain nonlinear constraints or integer variables. However, the solution of your program needs not to contain only integer values. A program is helpful as long as it provides a suggestion to the decision maker, even if the suggestion needs to be manually adjusted into a feasible plan.

(a) When one works in a day, she/he must work for the all eight time slots. In other words, if she/he is going to work for sixteen hours, she/he must work in exactly two days.
Hint. Each student must be assigned to one of $\binom{5}{2} = 10$ groups, where students in a group work in the same way.

(b) When one works in a day, she/he must work in either slots 1 to 4 or slots 5 to 8. In other words, she/he must work for exactly four hours in a day if she/he works in that day, and she/he must start to work at slot 1 or 5 for four consecutive slots.
Hint. Each student must be assigned to one of $\binom{5}{4} \times 2^4 = 80$ groups, where students in a group work in the same way.

3. (25 points) Recall your Case Assignment 0 (the in-class challenge on the first lecture day). We now ask you to formulate an integer program (with no nonlinear formulation, of course) that may solve the scheduling problem. What we want is not just a schedule for a specific month; we want a *solution*, which is a mathematical model, that may generate different schedules given different instance parameters. Therefore, we will consider the following abstraction.

In Figure 1 of Case Assignment 0, there is a time table for shifts. We now define the following notations which should provide information regarding that table. Let T be the set of time periods (9:00–9:30, 9:30–10:00, etc.) and J be the set of shifts (0 for on leave and others for on duty). Let $A_{jt} = 1$ if a CSR in shift $j \in J$ works in time period $t \in T$ or 0 otherwise. Moreover, we define S as the set of shift types, where $S = \{\text{morning, afternoon, night, leave}\}$. We then set the notation $B_{js} = 1$ if shift $j \in J$ belongs to type $s \in S$ or 0 otherwise.[2] Your integer program should take A and B as given and generate a schedule accordingly. In other words, when formulating your integer program, please ignore the concrete numbers contained in Figures 1 and 2 of Case Assignment 0.

Another dimension of abstraction is regarding Figure 2 of Case Assignment 0. Let D be the set of days in the coming month, your integer program should take C as given, where C_{td} is the demand quantity of CSRs in time period $t \in t$ in day $d \in D$.[3]

Now, please use the above notations (define more if you need) to formulate your integer program.

[2] As an example, in Case Assignment 0 there are 24 time periods (from 9:00–9:30 to 20:30–21:00) and fourteen shifts. We have $A_{11} = 1$ but $A_{17} = 0$ if we label 9:00–9:30 as time period 1 and 12:00–12:30 as time period 7. We also know that $B_{1,\text{afternoon}} = 0$ and $B_{11,\text{night}} = 1$. Nevertheless, please note that this footnote is provided to you simply to help you understand the notations.

[3] As an example, in Case Assignment 0 there are 31 days for the coming March (from day 1 to day 31). We have $C_{11} = 29$, $C_{1,10} = 21$, etc.

4. (20 points) Recall Problem 2a.

   (a) (8 points) Solve Problem 2a with the data given in Table 2, which are also contained in the sheet "Demand1" in the accompanying MS Excel file. First write down an optimal solution you obtain (or conclude that the instance is infeasible or unbounded) by solving your linear program. Then make some manual adjustment to general an executable plan and present the plan with business language.

   (b) (8 points) Repeat Part (a) with the data given in the sheet "Demand2" in the accompanying MS Excel file.

   (c) (4 points) Note that the average numbers of students needed per time slot are identical in the above two parts. Compare the two optimal solutions, is there any difference? If so, where does the difference come from? Intuitively explain.

   Note 1. I understand that there are always some students who do not own MS Excel. In this case, please use Google Spreadsheet or a similar product to open the file. You are also not required to solve the instances with MS Excel solver (though for Homework 1 this is suggested). You may write Python to invoke Gurobi Optimizer, as we will introduce in the second module of this course, or solve the instance in any way you like.

   Note 2. When we say "in a business language," we mean your boss (who know nothing about Operations Research and your mathematical programs) should be able to understand your plan. If you find it helpful, you may of course use some tables or figures to demonstrate your plan.