Operations Research, Spring 2022 (110-2)
Homework 3

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1 Rules

- This homework is due at ***23:59, May 21***. Those who submit their works late but are late by less than one hour gets 10 points off. Works that are late more than one hour get no point.
- For this homework, students should work individually. While discussions are encouraged, copying is prohibited.
- Please submit a ***PDF file*** through NTU COOL and make sure that the submitted work contains the student ID and name. Those who fail to do these will get 10 points off.
- You are required to ***type*** your work with LaTeX (strongly suggested) or a text processor with a formula editor. Hand-written works are not accepted. You are responsible to make your work professional in mathematical writing by following at least the following rules:[1]
  1. When there is a symbol denoted by an English letter, make it italic. For example, write $a + b = 3$ rather than a + b = 3.
  2. An operator (e.g., +) should not be italic. A function with a well-known name (e.g., log, max and sin) is considered as an operator.
  3. A number should not be italic. For example, it should be $a + b = 3$ rather than $a + b = \mathit{3}$.
  4. Superscripts or subscripts should be put in the right positions. For example, a_1 and $a1$ are completely different: The former is a variable called a_1 while the latter is actually $a \times 1$.
  5. When there is a subtraction, write $-$ rather than -. For example, write $a - b = 3$ rather than a - $b = 3$. The same thing applies to the negation operator. For example, write $a = -3$ rather than $a =$ -3.
  6. If you want to write down the multiplication operator, write $\times$ rather than *.
  7. For an exponent, write it as a superscript rather than using ^. For example, write 10^2 rather than 10^2.
  8. There should be proper space beside a binary operator. For example, it should be $a + b = 3$ rather than $a{+}b{=}3$.

  Those who fail to follow these rules may get at most 10 points off.
- As we may see, there are many students, many problems, but only a few TAs. Therefore, when the TAs grade this homework, it is possible for only some problems to be randomly selected and graded. For all problems, detailed suggested solutions will be provided.

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[1] A more complete list of formatting rules is on NTU COOL.

2 Problems

1. (20 points; 10 points each) There is a river at the bottom of a valley. The width of the valley is 100 meters, and the heights of the two sides of the valley are 100 and 150 meters, respectively. Figure 1 depicts the valley with the two points A and B labeling the two sides of the valley.

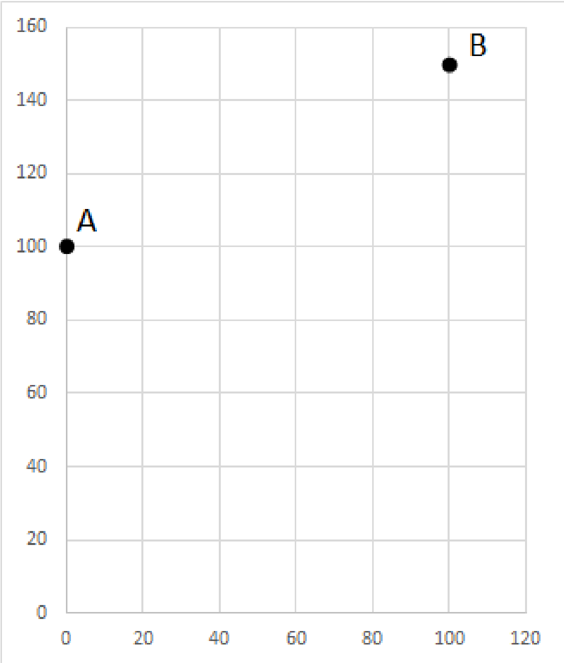

Figure 1: The valley

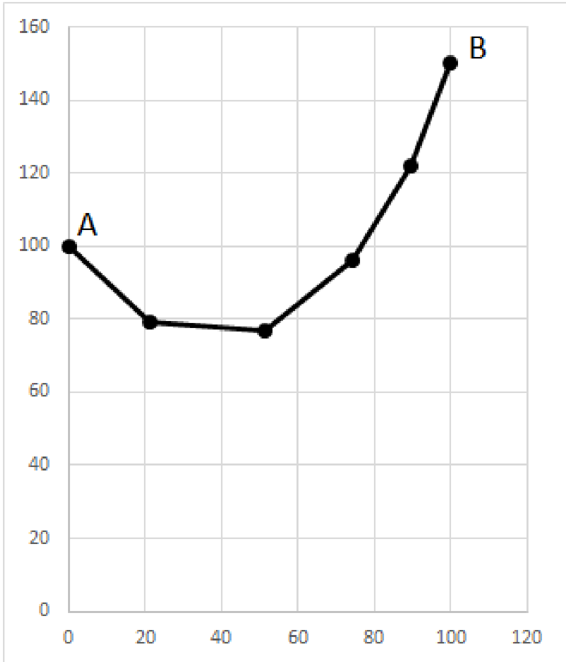

Figure 2: The position of the board when $L = 30$

You are given a set of five flat boards, each of length L. The right end point of board i is connected with the left end point of board $i+1$, $i = 1, 2, 3, 4$. Moreover, The left end point of board 1 and the right end point of board 5 are fixed at points A and B, respectively. Let's assume that the weights of a board and a connection point (please just imagine that there is something to connect two boards) are 0 and 1, respectively. As long as $5L > \sqrt{(100-0)^2 + (150-100)^2} \approx 111.8$, the nature will set the position of the five boards (or the four connection points) in a way that minimizes the total potential energy (energy of position), i.e., to minimize the sum of the heights of the four connection points. As an example, suppose that $L = 30$, Figure 2 depicts the final positions of the boards.

(a) Formulate a nonlinear program that finds the position (i.e., their x and y coordinates) of the four connection points. Determine whether your formulation is a convex program. Briefly explain why.

Hint. There is no need to have square roots in your formulation.

(b) Write a computer program that solves two instances, one with $L = 25$ and the other one with $L = 35$. Depict the positions in a way similar to that in Figure 2. Copy and paste your computer program onto your report.

Hint 1. There are certainly some ways to write down nonlinear formulations in Gurobi Optimizer. For this problem, being able to write down quadratic formulations is enough.

Hint 2. If your Gruobi Optimizer spends quite a long time (say, 20 minutes) with no answer, try the following: (1) Update to the latest version (which should be 9.5.1 on 2022/5/11), (2) add constraints saying that the x-coordinates of the four connection points are from left to right (say, $x_1 \leq x_2 \leq x_3 \leq x_4$), or (3) use appropriate inequality constraints rather than equality ones to set the distances among connection points.

2. (20 points; 5 points each) Among a set of patents $I = \{1, 2, ..., 4\}$, our company is considering whether to pay to obtain their licenses. Obtaining the license of patent i requires our company to pay a licensing fee F_i. Once a collection of patents S_j are obtained, product $j \in J = \{1, 2, ..., 6\}$ can be produced, and a profit P_j can be earned. More precisely, let $A_{ij} = 1$ if patent $i \in I$ is required for product $j \in J$ or 0 otherwise, we have $S_j = \{i \in I | A_{ij} = 1\}$. Once our company obtains the license of a patent after paying the licensing fee, it may be used in making multiple products with no additional cost. Whether to obtain the license of a patent is a binary decision.

   For the following subproblems, let

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}, F = \begin{bmatrix} 1200 \\ 1800 \\ 2500 \\ 1300 \end{bmatrix}, \text{ and } P = \begin{bmatrix} 1600 \\ 1000 \\ 1600 \\ 1200 \\ 1100 \\ 1400 \end{bmatrix}.$$

   (a) Formulate a linear integer program to maximize our company's total profit earned by making and selling products subject to a constraint that the total licensing fees cannot exceed the budget amount $B = 4500$. The licensing fees should exist in constraints only and should not appear in the objective function.

   (b) Find the linear relaxation of the program you formulate in Part (a). Then find the dual of the linear relaxation.

   (c) Use whatever way you like to find an optimal solution for the linear relaxation formulated in Part (b). Write down the optimal solution in the business language. If you use an analytical approach, write down your process. If you write a computer program, copy and paste your program onto your report.

   (d) Use the argument of complementary slackness to find the shadow price of your budget constraint in your optimal solution obtained in Part (c). Provide economic intuition for your answer.

3. (15 points) Among a set of patents $I = \{1, 2, ..., n\}$, our company is considering whether to pay to obtain their licenses. Obtaining the license of patent i requires our company to pay a licensing fee F_i. Once a collection of patents S_j are obtained, product $j \in J = \{1, 2, ..., m\}$ can be produced, and a profit P_j can be earned. More precisely, let $A_{ij} = 1$ if patent $i \in I$ is required for product $j \in J$ or 0 otherwise, we have $S_j = \{i \in I | A_{ij} = 1\}$. Once our company obtains the license of a patent after paying the licensing fee, it may be used in making multiple products with no additional cost. Whether to obtain the license of a patent is a binary decision.

   (a) (5 points) Formulate a linear integer program to maximize our company's net profit, which is the total profit earned by making and selling products minus the total licensing fees. There is no budget constraint.

   (b) (10 points) Prove or disprove that the coefficient matrix of your program formulated in Part (a) is totally unimodular. If it is not, find a formulation whose coefficient matrix is totally unimodular with a proof.

4. (15 points; 5 points each) For each of the following function, find the region over which the given function is convex (or conclude that the function is nowhere convex). You will get full points if and only if the function is convex over the reported region and nonconvex outside the region.

   (a) $f(x) = 2x^4 - 3x^2 + e^x - 2$.

   (b) $f(x_1, x_2) = (x_1 - 3)(6 - x_1)^2 + (x_2 - 2)(8 - x_2)$.

   (c) $f(x_1, x_2, x_3) = x_1^3 - 2x_1x_2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2$.

   Please note that to write down a region, you may simply list all constraints that form the region. For example, it is good to write $\{x \in \mathbb{R}^2 | x_1 + x_2 \leq 5, x_1 \geq 0, x_2 \leq 0\}$. You do not need to say it is "a triangle whose end points are $(0, 0)$, $(0, 5)$, and $(5, 0)$".

5. (30 points) While Lagrange relaxation may be used for nonlinear programs, it may also be used to design heuristic algorithms for linear integer programs. In this problem, let's demonstrate the idea with the knapsack problem

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} v_i x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} w_i x_i \leq B \\
& x_i \in \{0,1\} \quad \forall i = 1, ..., n.
\end{aligned}$$

(a) (5 points) Let the Lagrange multiplier of the knapsack constraint $\sum_{i=1}^{n} w_i x_i \leq B$ be λ. Use λ to move the knapsack constraint to the objective function to find the relaxed program. Do not change those binary constraints. Should your λ be nonnegative, nonpositive, or unrestricted in sign? Briefly explain why.

(b) (5 points) Suppose that the value of λ has been determined by somebody. Design a method to find an optimal solution to the relaxed program formulated in Part (a).

Note. Note that after we move the knapsack constraint to the objective function, the problem becomes *separable*, i.e., the problem may be decomposed into several subproblems, one with a subset (or just one) decision variables.

(c) (5 points) Let $n = 7$, $v = (12, 9, 8, 7, 5, 4, 3)$, $w = (8, 6, 7, 5, 4, 7, 3)$, and $B = 30$. Suppose that the Lagrange multiplier has been chosen to be $\lambda = 0.8$. Using your method developed in Part (b) to solve the given instance of the relaxed program. Is your optimal solution to the relaxed program feasible to the original knapsack problem?

(d) (5 points) Continue from Part (c). Suppose that now the Lagrange multiplier has been chosen to be $\lambda = 1.1$. Do Part (c) again.

(e) (10 points) A simple heuristic framework based on Lagrange relaxation may now be described:

- Step 1. Use Lagrange relaxation to move some constraints to the objective function so that the relaxed program may be decomposed into several subproblems that may be easily solved.
- Step 2. For each reasonable sets of values of the Lagrange multipliers:
  - Step 2.1. Solve all subproblems.
  - Step 2.2. Combine the optimal solutions to all subproblems to form a solution to the original problem. Note that this solution may be infeasible to the original problem. In this case, either ignore this solution or adjust it using some other heuristics to make it feasible.
- Step 3. Among all feasible solutions obtained above, report the one that is the best.

In general, there may be a lot of Lagrange multipliers, so researchers have developed many ways (e.g., subgradient methods) to choose (optimize) the values of Lagrange multipliers. This is beyond the scope of this course.

In a knapsack problem, however, there is only one Lagrange multiplier, and looking for a good value for the Lagrange multiplier is easy. In this part, let's consider the instance specified in Part (c) again. Please simply try all possible values of λ. Moreover, if a value of λ results in an infeasible solution to the original knapsack problem, please simply ignore it. Use this method to report a feasible solution to the original knapsack problem. Evaluate how good it is using some methods taught in this course.