Operations Research, Spring 2022 (110-2)
Suggested Solution for Homework 2

Tim Kuo*, Yuan Ting Lin†, Yu-Chieh Kuo‡

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1. (a) The standard form is

$$\begin{aligned}
\min \quad & x_1 + 2x_2 + 3x_3 \\
\text{s.t.} \quad & x_1 + x_2 - x_4 = 4 \\
& x_1 + x_2 + x_3 + x_5 = 9 \\
& x_3 - x_6 = 3 \\
& x_i \geq 0 \quad \forall i = 1, ..., 6.
\end{aligned}$$

   (b) Since we have six variables and three constraints in the standard form, there should be three basic variables and three nonbasic variables in a basic solution. The twenty possible ways to choose three (nonbasic) variables to be 0 are listed in the table below.

x_1	x_2	x_3	x_4	x_5	x_6	basis	Is basic solution?	Is basic feasible solution?
-	-	-	0	0	0	(x_1, x_2, x_3)	No	No
-	-	0	-	0	0	(x_1, x_2, x_4)	No	No
-	-	0	0	-	0	(x_1, x_2, x_5)	No	No
-	-	0	0	0	-	(x_1, x_2, x_6)	No	No
6	0	3	2	0	0	(x_1, x_3, x_4)	Yes	Yes
4	0	3	0	2	0	(x_1, x_3, x_5)	Yes	Yes
4	0	5	0	0	2	(x_1, x_3, x_6)	Yes	Yes
-	0	0	-	-	0	(x_1, x_4, x_5)	No	No
9	0	0	5	0	-3	(x_1, x_4, x_6)	Yes	No
4	0	0	0	5	-3	(x_1, x_5, x_6)	Yes	No
0	6	3	2	0	0	(x_2, x_3, x_4)	Yes	Yes
0	4	3	0	2	0	(x_2, x_3, x_5)	Yes	Yes
0	4	5	0	0	2	(x_2, x_3, x_6)	Yes	Yes
0	-	0	-	-	0	(x_2, x_4, x_5)	No	No
0	9	0	5	0	-3	(x_2, x_4, x_6)	Yes	No
0	4	0	0	5	-3	(x_2, x_5, x_6)	Yes	No
0	0	3	-4	6	0	(x_3, x_4, x_5)	Yes	No
0	0	9	-4	0	6	(x_3, x_4, x_6)	Yes	No
0	0	-	0	-	-	(x_3, x_5, x_6)	No	No
0	0	0	-4	9	-3	(x_4, x_5, x_6)	Yes	No

*Department of Information Management, National Taiwan University. E-mail: r10725025@ntu.edu.tw.

†Department of Information Management, National Taiwan University. E-mail: b07705036@ntu.edu.tw.

‡Department of Information Management, National Taiwan University. E-mail: b07611039@ntu.edu.tw.

(c) The exterme points are listed as follows.

x_1	x_2	x_3
(6 ,	0 ,	3)
(4 ,	0 ,	3)
(4 ,	0 ,	5)
(0 ,	6 ,	3)
(0 ,	4 ,	3)
(0 ,	4 ,	5)

(d) The standard form of LP is

$$\begin{aligned} \min \quad & x_1 + 2x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + x_2 - x_4 = 4 \\ & x_1 + x_2 + x_3 + x_5 = 9 \\ & x_3 - x_6 = 3 \\ & x_i \geq 0 \quad \forall i = 1, ..., 6. \end{aligned}$$

We use the two-phase implementation.

i. The Phase-I standard form LP is

$$\begin{aligned} \min \quad & x_7 + x_8 \\ \text{s.t.} \quad & x_1 + x_2 - x_4 + x_7 = 4 \\ & x_1 + x_2 + x_3 + x_5 = 9 \\ & x_3 - x_6 + x_8 = 3 \\ & x_i \geq 0 \quad \forall i = 1, ..., 8. \end{aligned}$$

First, we solve the Phase-I LP, which tries to minimize $x_7 + x_8$.

$$\begin{array}{cccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & x_7 = 4 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & x_5 = 9 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & x_8 = 3 \end{array} \rightarrow \begin{array}{cccccccc|c} 1 & 1 & 1 & -1 & 0 & -1 & 0 & 0 & 7 \\ \hline \boxed{1} & 1 & 0 & -1 & 0 & 0 & 1 & 0 & x_7 = 4 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & x_5 = 9 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & x_8 = 3 \end{array}$$

$$\rightarrow \begin{array}{cccccccc|c} 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 3 \\ \hline 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & x_1 = 4 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & x_5 = 5 \\ 0 & 0 & \boxed{1} & 0 & 0 & -1 & 0 & 1 & x_8 = 3 \end{array} \rightarrow \begin{array}{cccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & x_1 = 4 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & x_5 = 2 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & x_3 = 3 \end{array}$$

ii. Then, we solve the Phase-II LP.

$$\begin{array}{cccccc|c} -1 & -2 & -3 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & -1 & 0 & 0 & x_1 = 4 \\ 0 & 0 & 0 & 1 & 1 & 1 & x_5 = 2 \\ 0 & 0 & 1 & 0 & 0 & -1 & x_3 = 3 \end{array} \rightarrow \begin{array}{cccccc|c} 0 & -1 & 0 & -1 & 0 & -3 & 13 \\ \hline 1 & 1 & 0 & -1 & 0 & 0 & x_1 = 4 \\ 0 & 0 & 0 & 1 & 1 & 1 & x_5 = 2 \\ 0 & 0 & 1 & 0 & 0 & -1 & x_3 = 3 \end{array}$$

Finally, we use the simplex method with the smallest index rule. The optimal solution to the LP is $(x_1^*, x_2^*, x_3^*) = (4, 0, 3)$ with an objective value $z^* = 13$. There is no iteration that has no improvement.

2. (a) For the greedy algorithm, first, we calculate value-to-weight ratios and show them in Table 1. We then sort all items by their value-to-weight ratios from large to small. Finally, we select items one by one according to the order. Considering the linear relaxation of the integer program, we replace $x_i \in \mathbb{Z}_+$ as $x_i \geq 0$ for $i = 1, ..., 5$. With the greedy algorithm, the optimal solution of the linear relaxation of this integer program is $\left(0, 0, 0, 0, \frac{20}{3}\right)$, and the objective value is $66\frac{2}{3}$.

	Item				
	1	2	3	4	5
Value	2	2	5	11	10
Weight	1	4	3	5	3
Value-to-weight ratio	2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{11}{5}$	$\frac{10}{3}$

Table 1: Item table

(b) The full branch-and-bound tree is shown in Figure 1. The optimal solution is $x^* = (2, 0, 0, 0, 6)$ with an objective value 64.

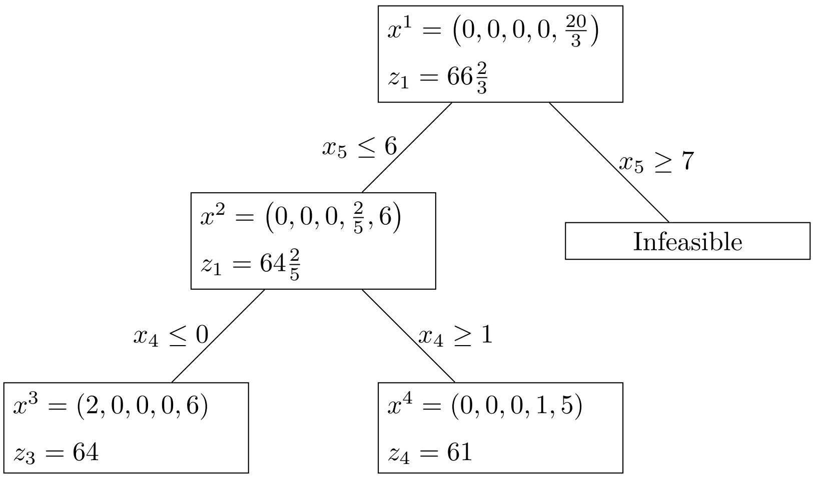

Figure 1: Branch-and-bound tree for Problem 2b

3. (a) Let x_{ij} be 1 if job j is assigned to machine i or 0 otherwise, and w be the benefit earned by the machine earning the least benefit among all machines.
The IP formulation is

$$\begin{aligned}
\min \quad & w \\
\text{s.t.} \quad & w \leq \sum_{j=1}^{n} b_j x_{ij} \quad \forall i = 1, ..., m \\
& \sum_{i=1}^{m} x_{ij} \leq 1 \quad \forall j = 1, ..., n \\
& \sum_{j=1}^{n} p_j x_{ij} \leq K \quad \forall i = 1, ..., m \\
& x_{ij} \in \{0, 1\} \quad \forall i = 1, ..., m \quad \forall j = 1, ..., n \\
& w \geq 0.
\end{aligned}$$

(b) The maximum of the benefit earned by the machine earning the least benefit among all machines is 17.417. It is an upper bound of the objective value of an IP-optimal solution.

(c) The schedule is to assign jobs 2 and 10 to machine 1, jobs 11 and 20 to machine 2, jobs 3, 15 and 25 to machine 3, and jobs 6, 14 and 19 to machine 4. Each machine earns 23, 23, 23, and 28, respectively. The least benefit among all machines is 23 (earned by machines 1, 2, and 3). The upper bound of the benefit earned by the machine earning the least benefit among all machines obtained by linear relaxation is 30.39, and the optimality gap is 24.324%.

(d) The schedule is to assign jobs 3 and 20 and 24 to machine 1, jobs 6, 10 and 13 to machine 2, jobs 5, 12, 18 and 22 to machine 3, and jobs 1, 8, 15 and 19 to machine 4. Each machine earns 34, 29, 26 and 25, respectively. The least benefit among all machines is 25 (earned by machine 4). The upper bound of the benefit earned by the machine earning the least benefit among all machines obtained by linear relaxation is 30.39 and the optimality gap is 17.744%.

4. (a) The objective values of the two methods and their optimality gaps is shown in Table 2.

Instance No.	Objective value			Optimality gap	
	Gurobi optimizer	HBF algorithm	HRF algorithm	HBF gap	HRF gap
1	146.7	144	144	1.84%	1.84%
2	95.279	89	92	6.591%	3.442%
3	54.72	50	51	8.626%	6.798%
4	144.368	136	140	5.797%	3.026%
5	59.82	49	55	18.088%	8.058%
6	115.571	108	111	6.551%	3.956%
7	83.964	76	81	9.485%	3.53%
8	76.575	70	71	8.586%	7.28%
9	85.75	80	80	6.706%	6.706%
10	138.4	137	136	1.012%	1.734%
11	76.067	63	72	17.178%	5.346%
12	154.75	146	150	5.654%	3.069%
13	59.5	55	53	7.563%	10.924%
14	87.333	78	82	10.687%	6.107%
15	115.95	108	109	6.856%	5.994%
16	62.653	59	58	5.83%	7.426%
17	54.632	54	51	1.157%	6.649%
18	87.68	76	82	13.321%	6.478%
19	56.176	54	50	3.874%	10.995%
20	61.25	55	57	10.204%	6.939%

Table 2: Comparison of the two methods

The average gap of HBF is 7.780%, and the average gap of HRF is 5.815%. In conclusion, HRF is better in this experiment.

(b) We invent a heuristic algorithm by modifying HRF. In HRF, we break tie by choosing the smallest index. In the modified HRF, we break tie by choosing the smallest processing time, and if it's still tie, we choose the smallest index. The objective values of the modified HRF and its optimality gaps is shown in Table 3.

Instance No.	Objective value		Optimality gap
	Gurobi optimizer	Modified HRF algorithm	Modified HRF gap
1	146.7	144	1.84%
2	95.279	92	3.442%
3	54.72	51	6.798%
4	144.368	143	0.948%
5	59.82	55	8.058%
6	115.571	114	1.36%
7	83.964	78	7.103%
8	76.575	72	5.975%
9	85.75	80	6.706%
10	138.4	137	1.012%
11	76.067	69	9.29%
12	154.75	150	3.069%
13	59.5	53	10.924%
14	87.333	82	6.107%
15	115.95	109	5.994%
16	62.653	58	7.426%
17	54.632	51	6.649%
18	87.68	85	3.057%
19	56.176	50	10.995%
20	61.25	57	6.939%

Table 3: Modified HRF

The average gap of modified HRF is 5.685%, which is better then the results of HBF and HRF. In conclusion, modified HRF is better in this experiment.

5. (a) Let $f(x_1, x_2) = 3x_1^2 + 2x_2^2 + 4x_1x_2 + 6e^{x_1} + x_2$. The gradient in general is

$$\nabla f(x_1, x_2) = (6x_1 + 4x_2 + 6e^{x_1}, 4x_2 + 4x_1 + 1).$$

In iteration 1, we have

$$\nabla f(x^0) = \nabla f(0,0) = (6,1).$$

$$a_0 = \operatorname*{argmin}_{a \geq 0} f(x^0 - a\nabla f(x^0)) = \operatorname*{argmin}_{a \geq 0} f(-6a, -a) = \operatorname*{argmin}_{a \geq 0} 134a^2 + 6e^{-6a} - a = 0.08459.$$

Thus, we derive our next solution as

$$x^1 = x^0 - a_0 \nabla f(x^0) = (0,0) - 0.08459(6,1) = (-0.50754, -0.08459).$$

(b) At any x^k, the second order derivative is

$$\begin{bmatrix} 6 + 6e^{x_1} & 4 \\ 4 & 4 \end{bmatrix}.$$

Therefore, in iteration 1, we have

$$x^1 = x^0 - \nabla^2 f(x^0)^{-1} \nabla f(x^0) = \left(-\frac{5}{8}, \frac{3}{8}\right).$$