Operations Research, Spring 2022 (110-2)
Suggested Solution for Case Assignment 1

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1 Formulation

We provide two formulations, and you can compare them with your solution.

Method 1

1. Before diving into the formulation, we parametrize the problem at first. Let I be the set of job, M be the set of machines, P_i be the processing time of the job i, and D_i be the deadline of the job i, $i \in I$.

   Moreover, we define the decision variables as follows. Let x_i be the end time of job i, y_{im} be 1 if the job i is processed at the machine m, z_{ij} be 1 if the job i is processed before the job j and both jobs are processed at the same machine and 0 otherwise, e_i be 1 if the job i is late and 0 otherwise, and w be the makespan at each day, where $i \in I$, $j \in I$ and $m \in M$. Besides, let N_1 and N_2 be two sufficiently large numbers for further use, and we specify the lower bound of them later.

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Our minimization LP formulation for the first stage is

$$
\begin{aligned}
\min \quad & \sum_{i \in I} e_i \\
\text{s.t.} \quad & x_i + P_j - x_j \le N_1 \left[(1 - z_{ij}) + (2 - y_{im} - y_{jm}) \right] \quad \forall i, j \in I,\, m \in M,\, i < j \\
& x_j + P_i - x_i \le N_1 \left[z_{ij} + (2 - y_{im} - y_{jm}) \right] \quad \forall i, j \in I,\, m \in M,\, i < j \\
& x_i \le D_i + N_2 e_i \quad \forall i \in I \\
& \sum_{m \in M} y_{im} = 1 \quad \forall i \in I \\
& x_i \ge P_i \quad \forall i \in I \\
& x_i \ge 0 \quad \forall i \in I \\
& y_{im} \in \{0, 1\} \quad \forall i \in I,\, m \in M \\
& z_{ij} \in \{0, 1\} \quad \forall i, j \in I \\
& e_i \in \{0, 1\} \quad \forall i \in I.
\end{aligned}
$$

Here we specify the lower bound of N_1 and N_2:

$$
\begin{aligned}
N_1 &\ge \sum_{i \in I} P_i + P_j \quad \forall j \in I \\
N_2 &\ge \sum_{i \in I} P_i.
\end{aligned}
$$

After deriving the minimum number of tardy jobs, says T_1, we next try to minimize the makespan w but also restrict the number of tardy jobs at T_1 in the second stage. Our objective function alters to

$$\min \quad w,$$

and the constraints are

$$
\begin{aligned}
& w \ge x_i \quad \forall i \in I \\
& \sum_{i \in I} e_i = T_1 \\
& x_i + P_j - x_j \le N_1 \left[(1 - z_{ij}) + (2 - y_{im} - y_{jm}) \right] \quad \forall i, j \in I,\, m \in M,\, i < j \\
& x_j + P_i - x_i \le N_1 \left[z_{ij} + (2 - y_{im} - y_{jm}) \right] \quad \forall i, j \in I,\, m \in M,\, i < j \\
& x_i \le D_i + N_2 e_i \quad \forall i \in I \\
& \sum_{m \in M} y_{im} = 1 \quad \forall i \in I \\
& x_i \ge P_i \quad \forall i \in I \\
& w \ge 0 \\
& x_i \ge 0 \quad \forall i \in I \\
& y_{im} \in \{0, 1\} \quad \forall i \in I,\, m \in M \\
& z_{ij} \in \{0, 1\} \quad \forall i, j \in I \\
& e_i \in \{0, 1\} \quad \forall i \in I.
\end{aligned}
$$

Note that the constraints in the second stage also contain those in the first stage.

2. In problem 2, jobs are available to split now, and we need to modify the parameters and variables to fit this change as a result.

   We use $S = \{1, 2\}$ to demonstrate the phase where the job is. To be more precise, $s = 1$ indicates the first piece of splitted job, and $s = 2$ indicates the second piece. Let P_{is} be the processing time of the job i at phase s, x_{is} be the end time of job i at phase s, y_{ism} be 1 if the job i at phase s is processed in the machine m, z_{isjt} be 1 if the job i at phase s is processed before the job j at phase t, **and** both jobs are processed at the same machine or 0 otherwise, where $i, j \in I$, $s, t \in S$, and $m \in M$. Likewise, we define N_3 and N_4 as two sufficiently large numbers for the following use and specify the lower bound later. Other parameters remain the same as the problem 1.

   The minimization LP formulation for the first stage is remodeled as

$$
\begin{aligned}
\min \quad & \sum_{i \in I} e_i \\
\text{s.t.} \quad & x_{is} + P_{jt} - x_{jt} \le N_3 \left[(1 - z_{isjt}) + (2 - y_{ism} - y_{jtm})\right] \quad \forall i, j \in I,\, m \in M,\, s, t \in S,\, i < j \\
& x_{jt} + P_{is} - x_{is} \le N_3 \left[z_{isjt} + (2 - y_{ism} - y_{jtm})\right] \quad \forall i, j \in I,\, m \in M,\, s, t \in S,\, i < j \\
& x_{i2} \le D_i + N_4 e_i \quad \forall i \in I \\
& \sum_{m \in M} y_{ism} = 1 \quad \forall i \in I,\, s \in S \\
& x_{is} \ge P_{is} \quad \forall i \in I,\, s \in S \\
& x_{i2} \ge x_{i1} + P_{i2} \quad \forall i \in I \\
& x_{is} \ge 0 \quad \forall i \in I,\, s \in S \\
& y_{ism} \in \{0, 1\} \quad \forall i \in I,\, s \in S,\, m \in M \\
& z_{isjt} \in \{0, 1\} \quad \forall i, j \in I,\, s, t \in S \\
& e_i \in \{0, 1\} \quad \forall i \in I.
\end{aligned}
$$

   Here we specify the lower bound of N_3 and N_4 as well:

$$
\begin{aligned}
N_3 &\ge \sum_{i \in I} \sum_{s \in S} P_{is} + P_{jt} \quad \forall j \in I,\, t \in S \\
N_4 &\ge \sum_{i \in I} \sum_{s \in S} P_{is}.
\end{aligned}
$$

   After deriving the minimum number of tardy jobs, says T_2, we then try to minimize the makespan w but also restrict the number of tardy jobs at T_2 in the second stage. Our objective function is

$$
\min \quad w
$$

as well, and the constraints are

$$w \geq x_{i2} \quad \forall i \in I$$

$$\sum_{i \in I} e_i = T_2$$

$$x_{is} + P_{jt} - x_{jt} \leq N_3 \left[(1 - z_{isjt}) + (2 - y_{ism} - y_{jtm})\right] \quad \forall i, j \in I,\, m \in M,\, s, t \in S,\, i < j$$

$$x_{jt} + P_{is} - x_{is} \leq N_3 \left[z_{isjt} + (2 - y_{ism} - y_{jtm})\right] \quad \forall i, j \in I,\, m \in M,\, s, t \in S,\, i < j$$

$$x_{i2} \leq D_i + N_4 e_i \quad \forall i \in I$$

$$\sum_{m \in M} y_{ism} = 1 \quad \forall i \in I,\, s \in S$$

$$x_{is} \geq P_{is} \quad \forall i \in I,\, s \in S$$

$$x_{i2} \geq x_{i1} + P_{i2} \quad \forall i \in I$$

$$w \geq 0$$

$$x_{is} \geq 0 \quad \forall i \in I,\, s \in S$$

$$y_{ism} \in \{0, 1\} \quad \forall i \in I,\, s \in S,\, m \in M$$

$$z_{isjt} \in \{0, 1\} \quad \forall i, j \in I,\, s, t \in S$$

$$e_i \in \{0, 1\} \quad \forall i \in I.$$

3. To utilize machines with incomplete functions, we need to specify these machines and what jobs can be done by them. In this problem, machine 1 has the incomplete function that it can only do the boiling process. Hence, we define three additional parameters: let C_{is} be 1 if the job i at phase s requires the boiling process only and 0 otherwise, $M^{\mathrm{I}} = \{1\}$ be a specified set of machines with incomplete function, here is machine 1, and $M^{\mathrm{C}} = \{2, \ldots, 5\}$ be the set of other machines, where $i \in I$ and $s \in S$. Other parameters and variables remain the same as the problem 1 and problem 2.

   The minimization LP formulation for the first stage is identical with the one in the problem 2, but includes an extra constraint:

$$y_{ism} \leq C_{is} \quad \forall i \in I,\, s \in S,\, m \in M^{\mathrm{C}}.$$

   The formulation for the second stage is modeled as similar as which in problem 2. Let the minimum number of tardy jobs derived here be T_3. The objective function is identical to minimizing the makespan w, and other constraints remain the same; the only modification is to add an extra constraint mentioned above, and substitute T_3 for T_2, that is,

$$\sum_{i \in I} e_i = T_3.$$

Method 2

1. Let I be the set of jobs, and M be the set of machines, D_i be the deadline of job i, and P_i be the processing time of the job i, $i \in I$.

   Next, we define the decision variables. Let x_i be the completion time of the job i, y_{im} be 1 if the job i is assigned to machine m and 0 otherwise, z_{ijm} be 1 if the job i is processed before the job j **and** both job i and j are assigned to machine m or 0 otherwise, e_i be 1 if the job i is tardy, where $i, j \in I, i \neq j$, and $m \in M$. In addition, we define N_1 and N_2 as sufficiently large numbers for the further use.

The minimization LP formulation for the first stage is

$$
\begin{aligned}
\min \quad & \sum_{i \in I} e_i \\
\text{s.t.} \quad & z_{ijm} + z_{jim} \leq 1 \quad \forall i, j \in I,\, i \neq j,\, m \in M \\
& z_{ijm} + z_{jim} \geq y_{im} + y_{jm} - 1 \quad \forall i, j \in I,\, i \neq j,\, m \in M \\
& x_i + P_j - x_j \leq N_1(1 - z_{ijm}) \quad \forall i, j \in I,\, i \neq j,\, m \in M \\
& x_i \leq D_i + N_2 e_i \quad \forall i \in I \\
& \sum_{m \in M} y_{im} = 1 \quad \forall i \in I \\
& x_i \geq P_i \quad \forall i \in I \\
& x_i \geq 0 \quad \forall i \in I \\
& y_{im} \in \{0, 1\} \quad \forall i \in I,\, m \in M \\
& z_{ijm} \in \{0, 1\} \quad \forall i, j \in I,\, i \neq j,\, m \in M \\
& e_i \in \{0, 1\} \quad \forall i \in I.
\end{aligned}
$$

The lower bound of N_1 and N_2 is

$$
\begin{aligned}
N_1 &\geq \sum_{i \in I} P_i + P_j \quad \forall j \in I \\
N_2 &\geq \sum_{i \in I} P_i.
\end{aligned}
$$

After deriving the minimum number of tardy jobs, says T_1, we next try to minimize the makespan w but also restrict the number of tardy jobs at T_1 in the second stage. Our formulation alters to

$$
\begin{aligned}
\min \quad & w \\
\text{s.t.} \quad & \sum_{i \in I} e_i = T_1 \\
& w \geq x_i \quad \forall i \in I \\
& \textbf{All constraints in the first stage.}
\end{aligned}
$$

2. Now jobs are available to split, and we define $S = \{1, 2\}$ as the set of phase to demonstrate the phase where the job is. The parameters remain the same as the problem 1, but we have to modify the decision variables to conquer the problem.

   Let x_{is} be the completion time of the job i at phase s, y_{ism} be 1 if the job i at phase s is assigned to the machine m and 0 otherwise, z_{ijstm} be 1 if the job i at phase s is processed before the job j at phase t **and** both job i and j are processed in the machine m, where $i, j \in I$, $s, t \in S$, $m \in M$, and for any two pair (i, s) and (j, t), $i \neq j$ or $s \neq t$.

The minimization LP formulation for the first stage in problem 2 is remodeled as

$$
\begin{aligned}
\min \quad & \sum_{i \in I} e_i \\
\text{s.t.} \quad & z_{ijstm} + z_{jitsm} \leq 1 \quad \forall i,j \in I,\ s,t \in S,\ m \in M,\ s \neq t \text{ or } i \neq j \\
& z_{ijstm} + z_{jitsm} \geq y_{ism} + y_{jtm} - 1 \quad \forall i,j \in I,\ s,t \in S,\ m \in M,\ s \neq t \text{ or } i \neq j \\
& x_{is} + P_{jt} - x_{jt} \leq N_1(1 - z_{ijstm}) \quad \forall i,j \in I,\ s,t \in S,\ m \in M,\ s \neq t \text{ or } i \neq j \\
& x_{i2} \leq D_i + N_2 e_i \quad \forall i \in I \\
& \sum_{m \in M} y_{ism} = 1 \quad \forall i \in I,\ s \in S \\
& x_{i1} \geq P_{i1} \quad \forall i \in I \\
& x_{i2} \geq x_{i1} + P_{i2} \quad \forall i \in I \\
& x_{is} \geq 0 \quad \forall i \in I,\ s \in S \\
& y_{ism} \in \{0,1\} \quad \forall i \in I,\ s \in S,\ m \in M \\
& z_{ijstm} \in \{0,1\} \quad \forall i,j \in I,\ s,t \in S,\ m \in M,\ s \neq t \text{ or } i \neq j \\
& e_i \in \{0,1\} \quad \forall i \in I.
\end{aligned}
$$

The lower bound of N_1 and N_2 turns into

$$
\begin{aligned}
N_1 &\geq \sum_{i \in I} \sum_{s \in S} P_{is} + P_{jt} \quad \forall j \in I,\ t \in S \\
N_2 &\geq \sum_{i \in I} \sum_{s \in S} P_{is}.
\end{aligned}
$$

After deriving the minimum number of tardy jobs, says T_2, we next try to minimize the makespan w but also restrict the number of tardy jobs at T_2 in the second stage. The formulation alters to

$$
\begin{aligned}
\min \quad & w \\
\text{s.t.} \quad & \sum_{i \in I} e_i = T_2 \\
& w \geq x_{i2} \quad \forall i \in I \\
& \textbf{All constraints in the first stage.}
\end{aligned}
$$

3. We can add one additional parameter and modify the formulation in problem 2 to solve problem 3. Let A_{ism} be 1 if the job i at phase s can be processed in the machine m and 0 otherwise. The formulation is identical to what in problem 2 with one additional constraint, that is,

$$
\begin{aligned}
\min \quad & \sum_{i \in I} e_i \\
\text{s.t.} \quad & y_{ism} \leq A_{ism} \quad \forall i \in I,\ s \in S,\ m \in M \\
& \textbf{All constraints in problem 2.}
\end{aligned}
$$

The second stage's formulation is similar to the one in problem 2. What we need to do is to add the extra constraint mentioned above and update the optimal number of tardy jobs in this scenario (says T_3), that is, substitute $\sum_{i \in I} e_i = T_3$ for $\sum_{i \in I} e_i = T_2$.

2 Optimal Schedules

We invoke the `gurobi` package in Python to implement our formulations among each problem in Section 1 and test our algorithm by three instances provided by the professor. The optimal makespan and tardy jobs for all instances in each problem are shown in the following Table 1.

Table 1: Computational results

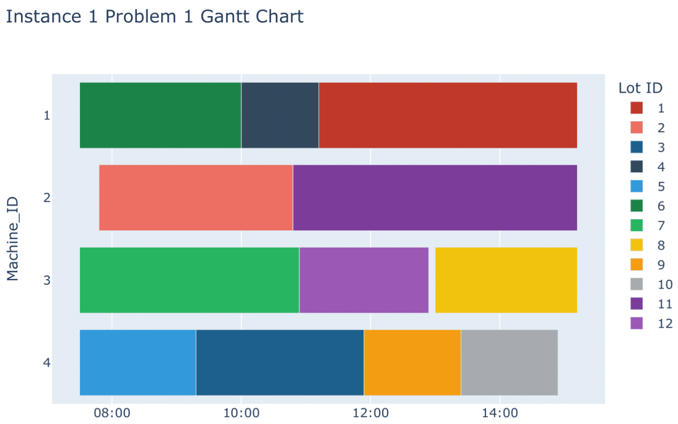
(a) Number of tardy jobs

	Problem 1	Problem 2	Problem 3
Instance 1	1	0	0
Instance 2	2	1	0
Instance 3	2	2	1

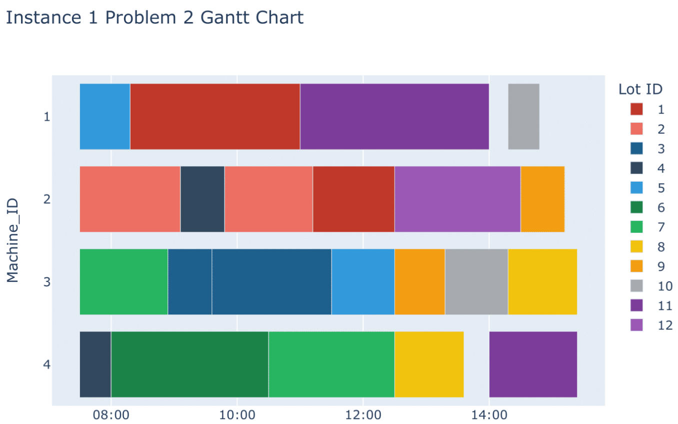
(b) Makespan in hours

	Problem 1	Problem 2	Problem 3
Instance 1	7.7	7.9	6.1
Instance 2	9.6	9.3	7.9
Instance 3	10.1	9.8	9.4

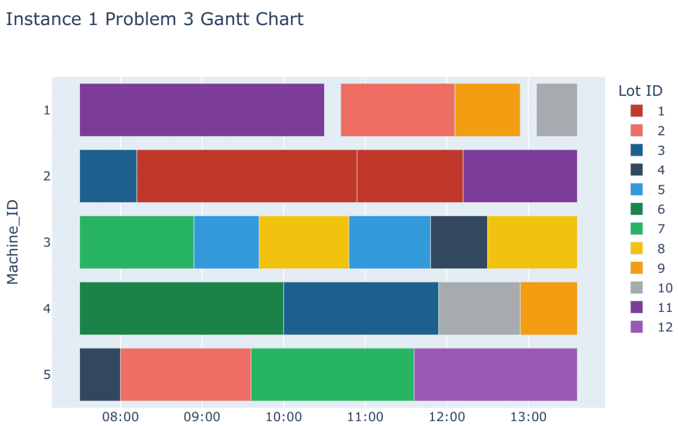
Subsequently, we visualize the computational results with the Gantt charts in Fig. 1, 2, and 3. The visualization can help demonstrate your idea intuitively and check whether your solution is feasible. By comparing the optimal schedule proposed from three problems for the different instances, we can observe that the schedule becomes more and more efficient as the assumptions are released, and the number of tardy jobs and the makespan decrease as a consequence. The computational results can easily find the latter; however, we need these charts to indicate the job distribution if we want to propose a scheduling plan to the company. Now, we can provide the IEDO company with an optimal schedule to advance its food manufacturing efficiency.



(a) Problem 1

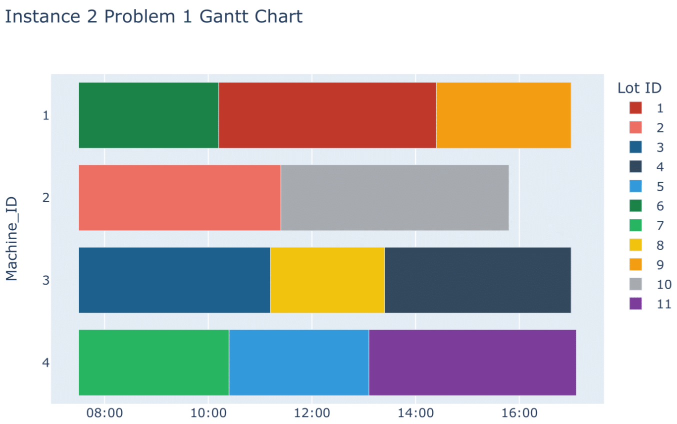


(b) Problem 2

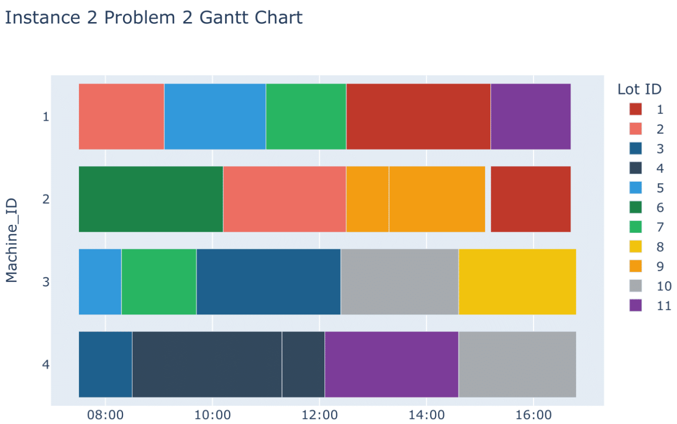


(c) Problem 3

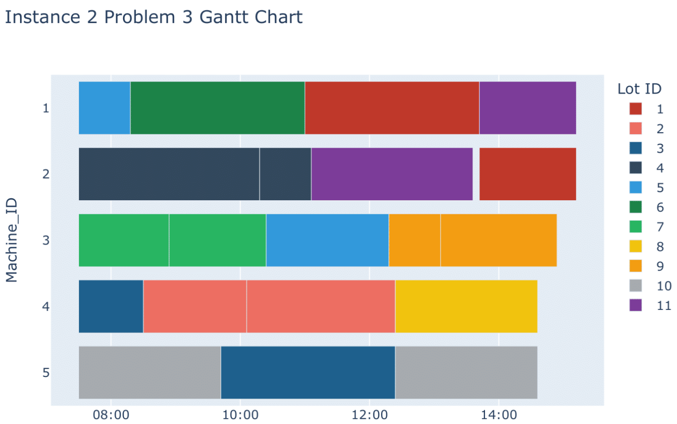
Figure 1: Optimal schedule for the instance 1



(a) Problem 1

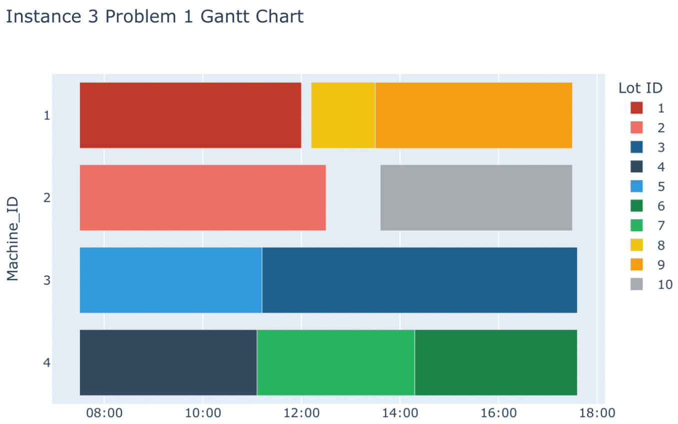


(b) Problem 2

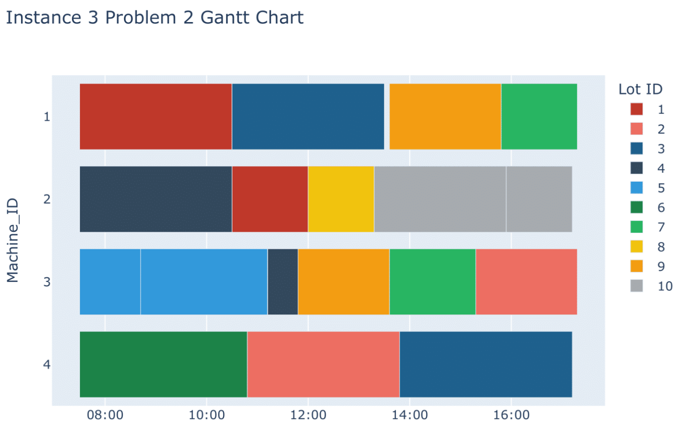


(c) Problem 3

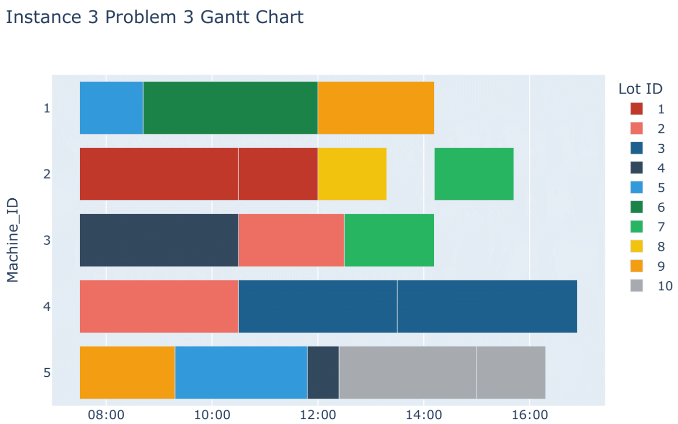
Figure 2: Optimal schedule for the instance 2



(a) Problem 1



(b) Problem 2



(c) Problem 3

Figure 3: Optimal schedule for the instance 3