Introduction to Iterative Methods for solving linear systems

The Jacobian Method

Outline

- Introduction to iterative method
- The general form of iterative method
- Convergence analysis
- Vector and matrix norms
- Jacobian method
- Diagonal dominant matrices

Iterative Methods

- General schema:
  - Original linear system: $Ax = b$.
  - Let $A = N – P$ and rewrite $Ax = b$ into

$(N\text{-}P)x = b \rightarrow Nx = b + Px.$

  - Where N is an invertible matrix and is *similar* to A.
- Iterative updating x using N and P:
  - Select an initial guess $x^{(0)}$;
  - Modify the solution by solving

$$Nx^{(k+1)} = b + Px^{(k)}$$

Residual Correction (1/3)

- How do we achieve such schema? By residual correction!
- The *residual* system:
  - Initial residual $r^{(0)} = b - Ax^{(0)}$.
  - Since $b = Ax$, we have

$$r^{(0)} = Ax - Ax^{(0)} = A\left(x - x^{(0)}\right) = Ae^{(0)}.$$

  - If $e^{(0)}$ had been solved, then $x = x^{(0)} + e^{(0)}$.

- However, this residual system is as hard as the original system.

  $Ax = b$ vs. $Ae^{(0)} = r^{(0)}$,
  - Alternative approaches are needed.

Residual Correction (2/3)

- Chose ***N*** which is similar to ***A*** but much simpler.
  - Instead of solving $Ae^{(0)} = r^{(0)}$, we solve
    $N\tilde{e}^{(0)} = r^{(0)}$
    with less efforts. (since ***N*** is simpler.)
  - Then, update the solution $x^{(1)} = x^{(0)} + \tilde{e}^{(0)}$.

- Since we didn't correct $x^{(0)}$ by using the true error, the resultant $x^{(1)}$ is just another approximation of x.
  - We have to correct it further.
  - Thus, we fall in a repeatedly improving process.

Residual Correction (3/3)

- By repeating the correction process, we have
  1. $r^{(k)} = b - Ax^{(k)}$,
  2. Solve $N\tilde{e}^{(k)} = r^{(k)}$,
  3. Update $x^{(k+1)} = x^{(k)} + \tilde{e}^{(k)} = x^{(k)} + N^{-1}r^{(k)}$.
  4. Expand the residual, $x^{(k+1)} = x^{(k)} + N^{-1}(b - Ax^{(k)})$.

- Multiplying both sides with ***N***,

  $Nx^{(k+1)} = Nx^{(k)} + b - Ax^{(k)} = b + (N - A)x^{(k)}$.
- Since ***A*** = ***N***-***P*** → ***P*** = ***N***-***A***, we have the general form

  $\boldsymbol{Nx}^{(\boldsymbol{k}+\mathbf{1})} = \boldsymbol{b} + \boldsymbol{Px}^{(\boldsymbol{k})}$.

  [note] We solve this equation to obtain the new solution $x^{(k+1)}$.

Convergence Criteria (1/2)

- Theorem: The residual correction process converges if $||\boldsymbol{N}^{-1}\boldsymbol{P}||<1$, where $||.||$ is a matrix norm operator.
- Proof:
  - Define error $e^{(k)} = x - x^{(k)}$.
  - Subtract $Nx^{(k+1)} = b + Px^{(k)}$ from $Nx = b + Px$,

    We have

    $N\left(x - x^{(k+1)}\right) = P(x - x^{(k)})$.
  - Rewrite the equation as $Ne^{(k+1)} = Pe^{(k)}$ or

    $e^{(k+1)} = (N^{-1}P)e^{(k)}$.

Convergence Criteria (2/2)

- Theorem:

  The residual correction process converges if $\|N^{-1}P\|<1$.

- Proof (continued)
  - Let $M = N^{-1}P$, then $e^{(k+1)} = Me^{(k)}$.
  - For k=0, $e^{(1)} = Me^{(0)}$.
  - For k=2, $e^{(2)} = Me^{(1)} = M\left(Me^{(0)}\right) = M^2e^{(0)}$.
  - In general, $e^{(k)} = M^ke^{(0)}$.
  - To be converged, we must have $\lim_{k\to\infty}\left\|e^{(k)}\right\| = 0$.

$$\left\|e^{(k)}\right\| = \left\|M^ke^{(0)}\right\| \le \left\|M^k\right\| \cdot \left\|e^{(0)}\right\| \le \|M\|^k\left\|e^{(0)}\right\|.$$

  - If $\|\boldsymbol{M}\|<1$, the above convergence condition will be met.

Review: Vector Norms

- 1-norm

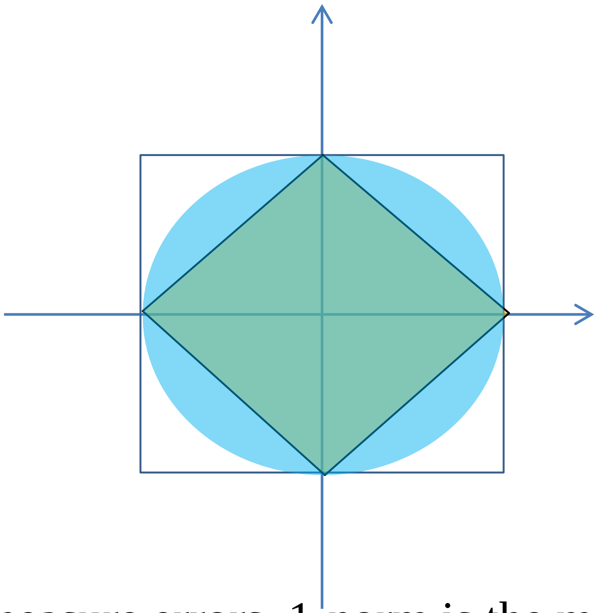
  $$\|x\|_1 = \sum_{i=0}^{n-1} |x_i|$$

- 2-norm

  $$\|x\|_2 = \sqrt{\sum x_i{}^2}$$

- ∞-norm

  $$\|x\|_\infty = \max_i |x_i|$$



To measure errors, 1-norm is the most restrict norm while ∞-norm is the most loose one.

Review: Matrix Norms

- Vector-induced matrix norm

  $\|A\| = \sup\{\|Ax\|, x \in R^n \; with \; \|x\| = 1\}.$
- 2-norm of matrix

  $\|A\|_2 = |\lambda|_{max} \le \left(\sum\sum {a_{ij}}^2\right)^{1/2}$
- 1-norm

  $\|A\|_1 = \max\limits_{j} \sum_i |a_{ij}|$, max column sum of absolute values.
- ∞-norm

  $\|A\|_\infty = \max\limits_{i} \sum_j |a_{ij}|$, max row sum.

Review: Properties of Norm

- Norms are used to measure the magnitude of an entity in R^n space.
- Norms must satisfy the following conditions

$\|x\| \geq 0,$

$\|x+y\| \leq \|x\| + \|y\|,$

$\|x * y\| \leq \|x\| * \|y\|,$

$\|ax\| = |a| \cdot \|x\|$, a is a scalar.

- For matrix norms

$\|A^{-1}\| \geq \|A\|^{-1}$

$\|A^{-1}\|_2 = |\frac{1}{\lambda_{min}}|.$

The Jacobian Method

- General iterative form:
  $Nx^{(k+1)} = b + Px^{(k)}$
  $x^{(k+1)} = N^{-1}(b + Px^{(k)})$.
- In Jacobian method
  $\boldsymbol{N}$ = diag(A), $N_{ii} = A_{ii}$, $N_{ij} = 0$, if $i \neq j$.
  $\boldsymbol{P} = \boldsymbol{N} - \boldsymbol{A}$, $P_{ii} = 0$, $P_{ij} = -A_{ij}$, if $i \neq j$.

$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i -\right.$

The matrices ***N*** and ***P*** in the Jacobi method::

$$N = \begin{bmatrix} a_{00} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & -a_{01} & -a_{02} \\ -a_{10} & 0 & -a_{12} \\ -a_{20} & -a_{21} & 0 \end{bmatrix}$$

Jacobian Method

```
//Initialize the solution.
 xold[] = {0.0};
 err = ∞;
//Iterate the correction until being converged
 while(err>ε){
     //The correction process
     for(i=0;i≤n-1;i++){
        sum = b[i];
        for(j=0;j≤n-1;j++)
           if(j!=i) sum = sum – A[i][j]*oldx[j];
        newx[i] = sum/A[i][i];
     }
     //Compute the delta vector
     e[] = newx[] – oldx[];
     //Copy the new results for the next iteration.
     oldx[] = newx[];
     err = norm_inf(e, n); //Compute the norm
  }
  return (newx);
```

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j \neq i} a_{ij} x_j^{(k)}\right]$$

Convergence Analysis

- In Jacobian method, $\boldsymbol{M} = \boldsymbol{N}^{-1}\boldsymbol{P}$.
  - Assume we use the **∞-norm** to measure the error.

$$\|M\| = \|N^{-1}P\| = \max_i \frac{1}{a_{ii}} \left(\sum_{i \neq j}^{n-1} |a_{ij}|\right).$$

- A sufficient condition for convergence is

$$\max_i \frac{1}{a_{ii}} \left(\sum_{i \neq j}^{n-1} |a_{ij}|\right) < 1.$$

- If so, the coefficient matrix is a **diagonal dominant** matrix.

$|a_{ii}| > \sum_{i \neq j} |a_{ij}|$ for all i. (all rows)

The matrices N and P:

$$N = \begin{bmatrix} a_{00} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & -a_{01} & -a_{02} \\ -a_{10} & 0 & -a_{12} \\ -a_{20} & -a_{21} & 0 \end{bmatrix}$$

Time Complexity Analysis

- Assume A is diagonal dominant and Jacobian method converges.
- Based on the previous result,
  $\left\|e^{(k)}\right\| = \left\|M^k e^{(0)}\right\| \leq \left\|M^k\right\| \cdot \left\|e^{(k)}\right\| \leq \|M\|^k \left\|e^{(0)}\right\|$.
- The program will stop if the norm is less than ε
  $\|M\|^k \left\|e^{(0)}\right\| < \varepsilon$, or $\frac{1}{\|M\|^k \left\|e^{(0)}\right\|} > \frac{1}{\varepsilon}$,
- Taking logarithmic values on both sides
  $\log \frac{1}{\|M\|^k} > \log \frac{1}{\varepsilon} - \log \frac{1}{\left\|e^{(0)}\right\|}$,
  $-k \log\|M\| > \log \frac{1}{\varepsilon} + \log\left\|e^{(0)}\right\|$.
- Let $\|M\| = a$ and $\left\|e^{(0)}\right\| = b$.
  $-k > \frac{\log \frac{1}{\varepsilon}}{\log a} + \frac{\log b}{\log a}$, or $k < \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{a}} - \frac{\log b}{\log a} = O(-\log\|M\| \cdot \log \frac{1}{\varepsilon})$.
  $k = O(-\log \frac{1}{\varepsilon} \cdot \log\|M\|)$. Or $k = O(\frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{\|M\|}})$.

Time Complexity Analysis

- Each iteration takes O(n^2) steps.
- If k iterations is required:

$$k = O(\log\frac{1}{\varepsilon}/\log\frac{1}{\|M\|}).$$

$$\|M\| = \|N^{-1}P\| = \max_i \frac{1}{a_{ii}}(\sum_{i\neq j}^{n-1}|a_{ij}|).$$

- If $\boldsymbol{A}$ is more diagonal dominant, then k is smaller.
- Proof:
  - If $\boldsymbol{M}$ is more diagonal dominant $\|\boldsymbol{M}\|$ is smaller.
  - Thus, $\frac{1}{\|M\|}$ and $\log\frac{1}{\|M\|}$ are larger.
  - And, $k = O\left(\frac{\log\frac{1}{\epsilon}}{\log\frac{1}{\|M\|}}\right)$ is smaller.

Example

- Matrix $\boldsymbol{A} = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 8 & 1 \\ 3 & 1 & 5 \end{bmatrix}$, how many iterations are required to make the system converged? Assume $\left\|e^{(0)}\right\| = b$ and $\varepsilon = 10^{-7}$.

$$M = N^{-1}P = \begin{bmatrix} 0 & -\frac{2}{6} & -\frac{3}{6} \\ -\frac{2}{8} & 0 & -\frac{1}{8} \\ -\frac{3}{5} & -\frac{1}{5} & 0 \end{bmatrix},$$

- Let $a = \|M\|_\infty = \frac{5}{6} < 1$. $\log\frac{1}{a} = \log\frac{6}{5} = 0.07918$

$$k < \frac{\log\frac{1}{\varepsilon}}{\log\frac{1}{a}} - \frac{\log b}{\log a} = \frac{7}{0.07918} + \frac{\log b}{0.07918} \approx 88 + \mathrm{C}.$$

Example

- Matrix $A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$. //Hilbert matrix

- $M = N^{-1}P = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{3} \\ -\frac{3}{2} & 0 & -\frac{4}{3} \\ -\frac{5}{3} & -\frac{5}{4} & 0 \end{bmatrix}$.

- $||M||_\infty \approx 2.9167 > 1$.

- This system cannot be solved by using Jacobi method.

Conclusion

- General iterative form for $\boldsymbol{A}x = b$:
  $Nx^{(k+1)} = b + Px^{(k)}$, where $\boldsymbol{A} = \boldsymbol{N}\text{-}\boldsymbol{P}$.
- Convergence theorem:
  - The correction process converges if $\|N^{-1}P\|<1$.
- Jacobi method:
  $\boldsymbol{N}$ = diag($\boldsymbol{A}$) and $\boldsymbol{P} = \boldsymbol{N} - \boldsymbol{A}$.
- Sufficient convergence condition of Jacobi method
  Matrix $\boldsymbol{A}$ is **diagonal-dominant**.
  $\|M\| = \|N^{-1}P\| < 1$.
- The time complexity of the Jacobi method is:

  $$k = O(\log\frac{1}{\varepsilon} \cdot (\frac{1}{\log\|\mathrm{M}\|})^{-1}).$$

  //k = number of iterations
  //In each iteration $O(n^2)$ operations are required.