Gauss-Seidel & SOR Methods

Solvers for Symmetric Positive Definite Systems

Outline

- Review
- Gaussian Seidel iteration
- The algorithm
- Convergence conditions
- The Successive Over Relaxation (SOR) method
- Optimal SOR

Review

- Linear system: $Ax = b$,
- General form of iterative method:

  $A = N - P$, $N \approx A$, $P = N - A$.

  $Ax = b \rightarrow Nx = b + Px$.

  $Nx^{(k+1)} = b + Px^{(k)}$.

- The convergence condition

  $e^{(k)} = x - x^{(k)}$.

  $e^{(k+1)} = (N^{-1}P)e^{(k)}$, $M = N^{-1}P$.

  $e^{(k)} = M^k e^{(0)}$.

  $\|e^{(k)}\| = \|M^k e^{(0)}\| \leq \|M^k\| \cdot \|e^{(0)}\| \leq \|M\|^k \|e^{(0)}\|$.

  $\|M\| < 1$.

Gauss-Seidel Iteration

- ***N*** is the lower triangular matrix and ***P*** is the strictly upper triangular matrix of ***A***.

$$n_{ij} = \begin{cases} a_{ij}, i \geq j. \\ 0, \text{otherwise.} \end{cases}$$

$$p_{ij} = \begin{cases} -a_{ij}, i < j, \\ 0, \text{otherwise.} \end{cases}$$

The shape of ***A***, ***N***, and ***P***

A* = *N* - *P

- $A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}$
- $N = \begin{bmatrix} a_{00} & 0 & 0 & 0 \\ a_{10} & a_{11} & 0 & 0 \\ a_{20} & a_{21} & a_{22} & 0 \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}$
- $P = \begin{bmatrix} 0 & -a_{01} & -a_{02} & -a_{03} \\ 0 & 0 & -a_{12} & -a_{13} \\ 0 & 0 & 0 & -a_{23} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Gauss-Seidel Iteration

- $\boldsymbol{N}$ is the lower triangular matrix and P is the strictly upper triangular matrix of A.
- The iteration:

$$Nx^{(k)} = b + Px^{(k-1)}.$$

//consider the i^th equation:

$$\sum_{j=0}^{i} a_{ij}x_j^{(k)} = b_i - \sum_{j=i+1}^{n-1} a_{ij}x_j^{(k-1)},$$

$$a_{i0}x_0^{(k)} + \cdots + a_{ii-1}x_{i-1}^{(k)} + a_{ii}x_i^{(k)} = b_i - \sum_{j=i+1}^{n-1} a_{ij}x_j^{(k-1)},$$

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=0}^{i-1} a_{ij}x_j^{(k)} - \right.$$

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$N = \begin{bmatrix} a_{00} & 0 & 0 & 0 \\ a_{10} & a_{11} & 0 & 0 \\ a_{20} & a_{21} & a_{22} & 0 \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -a_{01} & -a_{02} & -a_{03} \\ 0 & 0 & -a_{12} & -a_{13} \\ 0 & 0 & 0 & -a_{23} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The Pseudo-Codes

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=0}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n-1} a_{ij}x_j^{(k-1)}\right]$$

```
//Initialize the solution.
 x[] = {0.0};
 err = ∞;
//Iterate until being converged
 while(err>ε){
     for(i=0;i≤n-1;i++){
       sum = b[i];
       for(j=0;j≤n-1;j++) if(j!=i) sum = sum – A[i][j]*x[j];
       x[i] = sum/A[i][i];
     }
     //Compute the residual
     r[] = b[] – A[][]*x[];
     err = norm_inf(r, n); //Compute the norm
  }
  return (x[]);
```

Convergence of Gauss-Seidel

- Theorem1: Gauss-Seidel converges, if A is diagonal dominant.
  - See the proof of convergence of Jacobi method.

  $M = N^{-1}P, ||M|| < 1$. //using ∞ norm.

- Theorem2: Gauss-Seidel converges, if A is **Symmetric Positive Definite** (SPD).

- Definition of SPD: if A is SPD, then A is symmetric and $x^T Ax > 0, \forall x \in R^n, x \neq 0$.

Properties

- P1, If A is SPD, then $a_{ii} > 0$.

Proof by contradiction:

  - Assume $a_{ii} \leq 0$ for some i.
  - Select $e = [0, \ldots 0,1,0 \ldots 0]^T$, $e[i] = 1$, and $e[j]=0$, $i \neq j$.
  - Then $e^T A e = a_{ii} \leq 0$.
  - Thus A is not SPD, a contradiction!

- P2, if A is SPD, then its eigenvalues $\lambda > 0$.

Proof:

  - Let w be an eigenvector of A and λ the corresponding eigenvalue.

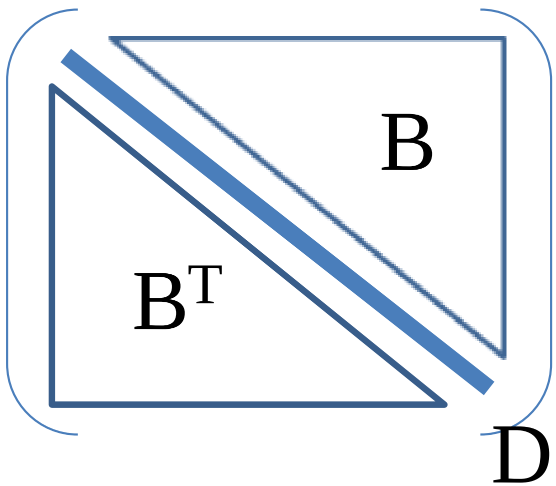
  $Aw = \lambda w$.

  $w^T A w = \lambda w^T w = \lambda \|w\|^2 > 0$, //by definition of SPD
  - It implies $\lambda > 0$.

Properties

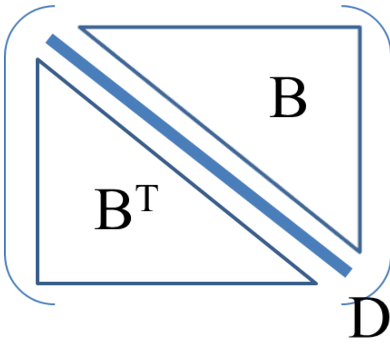
- P3, Let $D = diag(A)$, the diagonal matrix of A, then D is SPD.
  - Using property P1.
- P4, If A is SPD, then $N = D + B^T$, where B is the <u>strictly upper triangle matrix</u> of A.
  - See the right figure. $B^T \sim B$.
- P5, $D = A - B - B^T$
- P6, $w^T B w = w^T B^T w$.
  - Proof by expanding the equation and using $B_{ij} = B_{ji}$.

- A is SPD and it is divided into 3 parts in Gaussian-Seidel method.



Spectral Radius of the Correction Matrix

- Lemma: In Gauss-Seidel method, eigenvalues of $M=N^{-1}P$ are in the range of (-1,1), if A is SPD.
- Proof:
  - In Gauss-Seidel iteration $(A-B)x^{(k+1)} = b - Bx^{(k)}$.
  $x^{(k+1)} = (A-B)^{-1}b - (A-B)^{-1}Bx^{(k)}$.



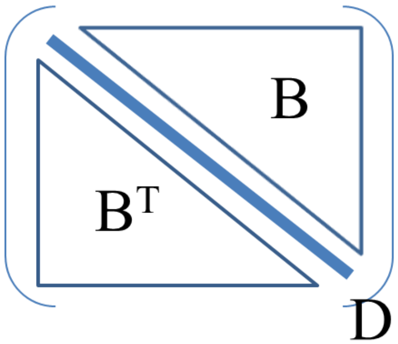
  - Thus $M = N^{-1}P = -(A-B)^{-1}B$.
  - Let w and λ be an eigenvector and an eigenvalue of M.
  - $Mw = \lambda w, -(A-B)^{-1}Bw = \lambda w$.
  - Multiply both sides by (A-B), we have $-Bw = \lambda(A-B)w$.
  - Inner-product both sides with w: $-w^T Bw = \lambda w^T(A-B)w = \lambda w^T Aw - \lambda w^T Bw$.
  $w^T Bw = \lambda w^T Bw - \lambda w^T Aw. (1-\lambda)w^T Bw = -\lambda w^T Aw$.
  - Thus, $w^T Bw = \frac{\lambda}{\lambda - 1} w^T Aw$. ---------------------------- (1).

Spectral Radius of the Correction Matrix

- Lemma: In Gauss-Seidel method, eigenvalues of $M=N^{-1}P$ are in the range of (-1,1), if A is SPD.

- Proof: (continued)
  - Since D is SPD,
  $0 < w^T D w = w^T(A - B - B^T)w.$
  - Using (1) and P6,
  $0 < w^T A w - w^T B w - w^T B^T w = (1 - 2\frac{\lambda}{\lambda-1}) w^T A w.$
  - Since $\boldsymbol{A}$ is SPD, $w^T A w > 0$ and
  $0 < 1 - 2\frac{\lambda}{\lambda-1} = 1 - \frac{2\lambda(\lambda-1)}{(\lambda-1)^2} = \frac{1-\lambda^2}{(\lambda-1)^2}.$
  - Thus,
  $1 - \lambda^2 > 0, -1 < \lambda < 1.$

Shape of B^T, D, and B.



Convergence of SPD Systems

- Theorem: If A is SPD, Gauss-Seidel converges.
- Proof:
  - Define $e^{(k)} = x - x^{(k)}$.
  $e^{(k+1)} = (N^{-1}P)e^{(k)}$, $M = N^{-1}P$.

  - From a previous proof, $e^{(k)} = M^k e^{(0)}$
  $\left\|e^{(k)}\right\| = \left\|M^k e^{(0)}\right\| \leq \left\|M^k\right\| \cdot \left\|e^{(0)}\right\| \leq \|M\|^k \left\|e^{(0)}\right\|$.

  - By the previous lemma, if A is SPD then $\|M\| < 1$,
  - Thus, $\lim_{k \to \infty} \|M\|^K = 0$.
  - The Gauss-Seidel iteration converges.

Relaxation

- In Gauss-Seidel method

  $$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=0}^{i-1} a_{ij}x_j^{(k+1)} + \sum_{j=i+1}^{n-1} a_{ij}x_j^{(k)}\right].$$

- Adding and subtracting $a_{ii}x_i^{(k)}$ in the right side

$$x_i^{(k+1)} = x_i^{(k)} + \frac{1}{a_{ii}}\left[b_i - \sum_{j=0}^{i-1} a_{ij}x_j^{(k+1)} + \sum_{j=i}^{n-1} a_{ij}x_j^{(k)}\right].$$

- The 2nd term of the right side contains an approximation of the residual $r = b-Ax$.
  - We modify the value of x by using the approximation.
- It is called a *relaxation*.

Successive Over Relaxation

- The relaxation of Gauss-Seidel

$$x_i^{(k+1)} = x_i^{(k)} + \frac{1}{a_{ii}}\left[b_i - \sum_{j=0}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i}^{n-1} a_{ij} x_j^{(k)}\right].$$

- By parameterizing the **relaxation** part

$$x_i^{(k+1)} = x_i^{(k)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=0}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i}^{n-1} a_{ij} x_j^{(k)}\right].$$

- If $\omega = 1.0$, we have Gauss-Seidel Iteration.
- If $1 < \omega < 2$, we have **Successive Over-Relaxation** (SOR) method.
- If $0 < \omega < 1$, we have under-relaxation.

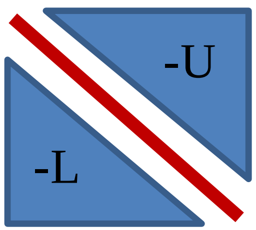
SOR Algorithm

$$x_i^{(k+1)} = x_i^{(k)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=0}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i}^{n-1} a_{ij}x_j^{(k)}\right]$$

```
//Initialize the solution.
 x[] = {0.0};
 err = ∞;
//Iterate until being converged
 while(err>ε){
     for(i=0;i≤n-1;i++){
       sum = b[i];
       for(j=0;j≤n-1;j++)  sum = sum – A[i][j]*x[j];
       x[i] = x[i] + ω*sum/A[i][i];
     }
     //Compute the residual
     r[] = b[] – A[][]*x[];
     err = norm_inf(r, n); //Compute the norm of the residual
  }
  return (x[]);
```

Relaxation Parameter

- The best ω is hard to compute.
  - But some results are known.
- P1, if A is SPD and $0 < \omega < 2$, SOR converges.
- P2, SOR can converge only if $0 < \omega < 2$.
- P3, Optimal ω:
  - Split the coefficient matrix $A = D - L - U$.
  - Let $T = D^{-1}(L+U)$



$$\omega = \frac{2}{1+\sqrt{1-\rho^2}},$$

  - where ρ is the spectral radius of T,

$$\rho = |\lambda_{max}|.$$

ADI Method

- In Jacobi, Gauss-Seidel, and SOR methods, we update x[] from x[0] to x[n-1] in each iteration.
- In Altered Direction Iteration (ADI) method,
  - We update x[] from x[0] to x[n-1] in most cases.
  - But, in some iterations, we update x[] from x[n-1] to x[0].
- SSOR (Symmetric Successive Over-Relaxation) method:
  - Using SOR to solve the system.
  - Update x[] from x[0] to x[n-1] in odd-numbered iterations.
  - Update x[] from x[n-1] to x[0] in even-numbered iterations.
- SSOR may be slower than SOR.
- SSOR is used as a preconditioner for other iterative methods.
  - What is a preconditioner?

Preconditioners

- The original system $Ax = b$ has a large condition number.
- Modify the system as follows
  $(PA)x = Pb$. Then solve the system.
- Or with more efforts
  - Solve $(AP)y = b$ at first.
  - Then solve $P^{-1}x = y$ later to obtain the solution.
- Method: left precondition and right precondition.
- Criteria:
  - **P** should be simple
  - Don’t form **AP** or **PA** explicitly. Why? Think about the time complexity of matrix-multiplication.

Reminder

- Hilbert matrix is symmetric positive definite, $a_{ij} = \frac{1}{i+j+1}, 0 \le i,j \le n-1.$
- However, linear systems with Hilbert matrices as coefficient matrices are almost intractable by using Gauss-Seidel method.
- Why?
- Answer:
  - The condition number $cond(A) = O\left(\frac{(1+\sqrt{2})^{4n}}{\sqrt{n}}\right)$ is exponentially grows with the dimension n.
  - Numerical errors (perturbations) prevent the computation from converge.

References

- Broyden, C. G. (1964). On convergence criteria for the method of successive over-relaxation. Mathematics of Computation, 18(85), 136-141.
  - Symmetric matrix, $0 < \omega < 2$ converges
  - Positive definite & low-triangular dominant → convergence
- James, K. R., & Riha, W. (1975). Convergence criteria for successive overrelaxation. SIAM Journal on Numerical Analysis, 12(2), 137-143.
  - Convergence criteria of SOR method for general and special matrices
  - Most referenced paper about SOR.
- Yang, S., & Gobbert, M. K. (2009). The optimal relaxation parameter for the SOR method applied to the Poisson equation in any space dimensions. Applied Mathematics Letters, 22(3), 325-331.
  - Optimal ω for Poisson equations (PDE, elliptic PDE)