Eigen Systems

An introduction to basic properties of eigenvalue and eigenvector

Outline

- Introduction
- Characteristic polynomial
- Basic properties of eigen-systems
- Diagonalization process
- Similarity transformation
- Singular values

Introduction

- Definition: $\boldsymbol{A}$ is an n by n matrix and $\boldsymbol{x}$ is a non-zero vector in the R^n space. Then $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ and λ is the associate eigenvalue if and only if

  $\boldsymbol{Ax} = \lambda\boldsymbol{x}$.

- Applications of eigensystem
  - Image analysis, computer vision, computer graphics and visualization
  - Dimension reduction and data classification
  - Vibration analysis
  - Flow field analysis
  - …

2- and 3-D Space Eigenvectors

- If $\boldsymbol{x}$ is an eigenvector of matrix $\boldsymbol{A}$, then we have
  $Ax = \lambda x,$
  $(\lambda I - A)x = 0$ where I is the identity matrix.
- Characteristic polynomial
  - The previous system has non-zero solution if the determinant $|\lambda I - A| = 0.$ //利用反證法證明，假設橫列式值≠0…..
  - By expanding the determinant, we have $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$
  - $p(\lambda)$ is the characteristic polynomial of A.
- In 2- and 3-D spaces, we can solve the eigenvalues analytically.
- But, in higher dimensional spaces, we have to compute eigenvalues by using numerical algorithms.

Properties of Eigen System (1/2)

1. If ***A*** is symmetric, all its eigenvalues are real and the eigenvectors are mutually orthogonal.
   – The eigenvectors form a basis for the n-D space.
2. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.
3. If λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1}.
4. The eigenvectors of A and A^{-1} are the same.

Proofs

- If ***A*** is symmetric, then its eigenvalues are real and its eigenvectors are mutually orthogonal.
- Proof:

  Let (λ_i, x_i) and (λ_j, x_j) be two different eigen-pairs and the eigenvectors be unit vectors.

  $\langle Ax_i, Ax_i \rangle = x_i^T A^T A x_i = x_i^T A^2 x_i = \lambda_i^2 > 0.$

  Thus λ_i is real.

  $\lambda_i < x_i, x_j > =< Ax_i, x_j > x_i^T A^T x_j = x_i^T A x_j$

  $=< x_i, \lambda_j x_j > \lambda_j x_i^T x_j = \lambda_j < x_i, x_j >.$

  However, $\lambda_i \neq \lambda_j$. The previous equation holds if and only if $< x_i, x_j > = 0$ and $x_i \perp x_j$.

Proofs

- If $A\vec{x} = \lambda\vec{x}$, then $A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$.
- Proof:

$$A\vec{x} = \lambda\vec{x} \rightarrow A^{-1}A\vec{x} = A^{-1}\lambda\vec{x},$$

$$\vec{x} = \lambda A^{-1}\vec{x} \rightarrow \frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}.$$

1. If λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1}.
2. The eigenvectors of A and A^{-1} are the same.

Properties of Eigen System (2/2)

5. If $\boldsymbol{A}$ is SPD, all its eigenvalues > 0.
6. If $\boldsymbol{L}$ is a lower triangular matrix, $\lambda_i = l_{ii}$.
   - Similar property holds for upper triangular matrices.
   - Eigenvalues of a diagonal matrix = the diagonal entries.
   - 利用 characteristic polynomial證明
7. The trace of $\boldsymbol{A}$, $trace(A) = \sum a_{ii} = \sum \lambda_i$.
   - 利用 characteristic polynomial證明
8. Determinant of A, $|A| = \prod \lambda_i$.
   - 利用 characteristic polynomial證明

Diagonalization

- Let $\Lambda = \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$ and $P = [v_0, v_1, v_2]$, where λ_i and v_i are eigenvalues and eigenvectors of matrix $\boldsymbol{A}$.

- Then $AP = P\Lambda$ (column operation) and $P^{-1}AP = P^{-1}P\Lambda = \Lambda$.
  $A = P\Lambda P^{-1}, \Lambda = P^{-1}AP.$

- Conclusion 1: we can decompose $\boldsymbol{A}$ into the product of matrices $\boldsymbol{P}$, $\boldsymbol{\Lambda}$, and $\boldsymbol{P}^{-1}$.

- Conclusion 2: we can diagonalize $\boldsymbol{A}$ by using the column matrix of the eigenvectors and the diagonal matrix of the eigenvalues.

Similarity Transformation

- Theorem: If matrix $\boldsymbol{P}$ is invertible, the following transformation preserves the eigenvalues of $\boldsymbol{A}$.

  $P^{-1}AP.$

Proof:

- The characteristic polynomial of matrix $\boldsymbol{A}$ is:

  $f(\lambda) = \det(\lambda I - A).$

- Consider the characteristic polynomial of the transformed matrix

  $\det(\lambda I - P^{-1}AP) = \det(\lambda P^{-1}P - P^{-1}AP)$

  $= \det(P^{-1}\lambda IP - P^{-1}AP) = \det(P^{-1}(\lambda I - A)P)$

  $= \det(P^{-1}) \det(\lambda I - A) \det(P)$

  $= \det(\lambda I - A) = f(\lambda).$

- Both matrices have the same characteristic polynomial, and their eigenvalues are the same.

Similarity Transformation

- Theorem: Let $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$. Then $\boldsymbol{y} = \boldsymbol{P}^{-1}\boldsymbol{x}$ is an eigenvector of $\boldsymbol{B}$.
- Proof:
  - Since y is an eigenvector of $\boldsymbol{B}$, we have

    $B * y = \lambda y.$
  - Expanding $\boldsymbol{B}*y$, we have

    $B * y = P^{-1}AP(P^{-1}x)$ // $y = P^{-1}x$

    $= P^{-1}Ax = P^{-1}\lambda x = \lambda(P^{-1}x) = \lambda y.$
  - Thus $y = P^{-1}x$ is an eigenvector of $\boldsymbol{B}$.

Singular Values

- Definition: $\boldsymbol{A}$ is a M x N matrix and $\boldsymbol{B} = \boldsymbol{A}^T\boldsymbol{A}$. The singular values of $\boldsymbol{A}$ are the square roots of the eigenvalues of $\boldsymbol{B}$.

- Question: Are singular values and eigenvalues of a square matrix the same?
- Answer: No !
  - If $\boldsymbol{A}$ is symmetric, then yes.