The Entropy

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1 Introduction

Since the time of R. Clausius, entropy has been an important idea in the domain of physics due to its characterization of irreversibility of the dynamics of a system. Clausius defined the entropy in the manner of thermodynamical process, which is the integral of reciprocal temperature on the path of heat change. After the work of Clausius, Ludwig Boltzmann suggested that the entropy can be calculated from a microscopic point of view, which is the famous formula S = $k_B ln(\Omega)$ that is engraved on his tomb. Ludwig Boltzmann even derived the primordial form of H-theorem via his Boltzmann transport equation in the phase space. H-theorem indicates that we can construct an H function which monotonically grows in free semi-classical theory or some limited dynamical processes. The following works by Gibbs and Maxwell also deepened the realization of entropy in the statistical point of view.

Partly based on the work of Boltzmann's, Shannon introduced the idea of Shannon Entropy for the variety of mathematics, statistics, and information theory, with this name suggested by von Neumann. Shannon characterized entropy as the information of a randomly distributed discrete system by his famous coding theorems, and realized by the following work by Huffman and his Huffman code. Thus, humans scientific community got a deeper perspective about the entropy. After the primitive birth of quantum mechanics, von Neumann rigorized it in the language of functional analysis in his 1932 book "Mathematical Foundations of Quantum Mechanics", where the idea "Hilbert space" introduced to physics for the first time. In this work, he also introduced the idea density matrix, which is useful in our era, independently as Landau suggested the same idea for the study of subsystem of a quantum system. Based on the Shannon entropy, von Neumann introduce a quantum version of it, which is called the von Neumann entropy nowadays. Since it turns out that this entropy qualifies the entanglement between two systems, it got another famous name "entanglement entropy".

In 1957, Edwin T. Jaynes suggested that the arrival of equilibrium was exactly the maximization of entropy, or the Principle of Entropy Maximization. Thus, the idea entropy reclaimed its thermodynamics meaning with deeper realization of it under the interpretation of information and statistics. Nowadays, the statistics approach to thermal statistical physics becomes the main one, and even in the field of non-equilibrium physics, how entropy grows in the evolution of an open quantum system is still an interesting problem. In the history of modern physics, the entropy got its importance without a doubt, and it seems that it will still play a role in the future on the understanding and applications of the quantum information theory and the non-equilibrium statistical mechanics. Therefore, this report introduced some works of entropy done, most of which is based on the work of Calabrese's on conformal field theory.

In the first part of this report, we will mathematically define entanglement entropy in quantum systems and related concept. Also, we will briefly introduce some rigorous theorems about entropy such as "Strong Subadditivity",or SSA, and the monotonicity of entropy proved by Lindblad. However, since the degree of freedom of a quantum field theory is uncountably infinite because of the inseparability of the Fock space, one must introduce some regularization to make it finite or at least describe how it diverges along some scale. Hence, in the third chapter, we will define the entanglement entropy in quantum field theory via replica's trick.

In the forth chapter, we'll talk about some important ideas in conformal field theory in order to understand the fifth and central part of this report : entanglement entropy in conformal field theory, which is based on the work of Calabrese [1]. Calabrese's result indicates a general areal behaviour of entropy in conformal field theory, thus, the entanglement reduces to geometric properties of systems.

2 Entanglement Entropy in Separable Hilbert Spacs

In the lesson of quantum mechanics, we know that a quantum system in a Hilbert space can be characterized by a positive and Hermitian operator with unit trace. To make this idea more explicit, we give some definitions here, which are mostly based on the papers[7][8] by Lindblad.

2.1 Quantum System and The Definition of Entropy

Definition 2.1.1 *A Hilbert space is a complete inner product space* $(H, \langle, \rangle)$.

The reason of the use of Hilbert space is elaborated in the book "Mathematical Foundations of Quantum Mechanics" by J. von Neumann[12]. Such inner product is important because the transition possibility between two states shall be an observable which we can determine with experiment. Under this consideration, an Banach space which has only a norm is not enough for our use. Also, the completeness matters because we expect physical processes bring plausible states into plausible states, that is, if we have a string of operations in a states, we would hope that this operation will converges to a state which is still in the Hilbert space, which is in fact characterized by the completeness.

Definition 2.1.2 *An operator T on a Hilbert space H is a linear map on H. The space of operators on H is denoted by* $\mathrm{L}(H)$.

In order to ensure considered sums can be convergent, we can only sum a countable set of numbers, for sums of uncountable positive numbers diverge.

Definition 2.1.3 *A Hilbert space H is separable if there is a countable orthonormal set whose closure of span is H. In this case, such set is called the orthonormal, or O.N., basis of H.*

In fact, a Hilbert is separable if and only if there's a countable set of vectors with the close its span is the whole Hilbert space if and only if there is a countable set of vectors that is densed in the Hilbert space supposed the Hilbert space is over a second countable field. It shall be noted that the Fock space is not separable, hence, we cannot apply most of the theorems and results in separable Hilbert spaces directly in the calculation Quantum Field Theory. The seperability matters because it not only restricts the topology that the space of states can have and also that can we find a proper basis for an series expression of any states. Now, let's consider the trace of operators, which is important in quantum mechanics.

Definition 2.1.4 *The trace of a operator T on a separable Hilbert space H is the number $\sum_k \langle Te_k, e_k \rangle$, and denoted by $Tr(T)$, for some (and, hence, all) O.N. basis $\{e_k\}$ of H. Furthermore, an operator T on a separable Hilbert space H is said to be in the trace class $T(H)$ if $tr(T) < \infty$.*

Definition 2.1.5 *An operator T in a Hilbert space is called positive if $\langle Tx, x \rangle \geq 0$ for all x in H. In this case, we write $A \geq 0$. Furthermore, the set of positive elements in the trace class operators $T(H)$ is denoted by $T_+(H)$.*

The subset of $T_+(H)$ which contains only the element with trace 1 is the set of all physical states, or the space of density matrices, in a Hilbert space H. In the following context, we will define the most important concept in this article : entropy.

Definition 2.1.6 *For an operator A in $T_+(H)$, the operator-valued entropy is defined by $\hat{S}(A) = -A\log(A)$, which is still in $T_+(H)$. Def : For an operator A in $T_+(H)$, the entropy of A is the number Tr($\hat{S}$(A)).*

Definition 2.1.7 *Given λ in $(0,1)$, we define the operator-valued relative entropy by $\hat{S}_\lambda(A|B) = \lambda^{-1}[\hat{S}(\lambda A + (1-\lambda)B) - \lambda\hat{S}(A) - (1-\lambda)\hat{S}(B)]$. And the relative entropy $S(A|B)$ is defined by $lim_{\lambda\to 0}Tr(\hat{S}_\lambda(A|B))$, where the positivity is guaranteed by the concavity of entropy.*

Definition 2.1.8 *A state ρ is said to be a maximally entangled state on H if S(ρ) = $sup_{A\in T_+(H)}$ S(A).*

Since the entropy may diverge, the maximally entangled state may not be unique or even may not exist in infinite dimensional Hilbert Spaces.

To study all the possible operations on a quantum system characterized by a density operator ρ, the following concepts shall be introduced.

Definition 2.1.9 *A map Φ :T(H) $\to$ T(H) T is said to be trace-preserving if Tr(Φ(A)) = Tr(A) for all A in T(H).*

Definition 2.1.10 *A map $\Phi : T_+(H) \to \mathrm{L}(H)$ is said to be completely positive if $id_{\mathbb{F}^{k\times k}}\Phi(A) \geq 0$ for all A in $T_+(H)$.*

Definition 2.1.11 *Let (a, b) be an interval in $\mathbb{R}$. A map Φ :(a,b) $\times T_+(H) \to T_+(H)$ is said to be a dynamical map on a Hilbert space H if Φ(t, $\cdot$) is completely positive and trace-preserving $\forall t \in$ (a,b).*

A unique and well defined maximally entangled state exists when H is of finite dimensional,which is $\frac{1}{dim H}\mathbb{I}$. If an operation of states raises entropy, then we need the following definition.

Definition 2.1.12 *Given an interval (a,b) in (R), a linear dynamical map Φ : (a,b) $\times T_+(H) \to T_+(H)$ on a finite dimensional Hilbert space H is said to be unital if $\Phi(t,\mathbb{I}) = \mathbb{I}, \forall t \in (a,b)$.*

In the end of this section, since we are usually interested in multi-partied quantum systems, we shall introduce the idea of partial trace and reduced density operator.

Definition 2.1.13 *Let H_1 and H_2 be two separable Hilbert spaces with O.N. bases $\{\alpha_k\}$ and $\{\beta_k\}$ respectively and $T \in \mathrm{L}(H_1 \otimes H_2)$. We can represent T by matrix $T = \sum_{ijkl}(T)_{ijkl}\alpha_i \otimes \beta_j\langle\alpha_k \otimes \beta_l, \rangle$. The partial trace of T over H_2 is the operator $\sum_k(T)_{ikjk}\alpha_i\langle\alpha_k, \cdot\rangle$, denoted by $Tr_{H_2}(T)$, or ρ_{H_1}.*

It shall be noticed that the partial of a density operator on the tensor product of two Hilbert spaces over one of the Hilbert spaces is also a density operator on the remaining Hilbert space. Hence, the idea of reduced density operator arises.

Definition 2.1.14 *Let H_1 and H_2 be two separable Hilbert spaces and ρ a density operator in $H_1 \otimes H_2$. The reduced density operator of ρ on H_1 is the density operator $Tr_{H_2}(\rho)$.*

Finally, we can define the Entanglement Entropy on a product Hilbert space.

Definition 2.1.15 *Let H_1 and H_2 be two separable Hilbert spaces. Given a density operator ρ on a product Hilbert space $H_1 \otimes H_2$. The Entanglement Entropy of the state ρ on space H_1 is the number $S(Tr_{H_1}(\rho)) = -Tr(\rho_{H_1} \ln \rho_{H_1})$. In general, $S(Tr_{H_1}(\rho)) \neq S(Tr_{H_2}(\rho))$, but the equality holds if ρ is a pure state.*

2.2 Properties of Entropy and Mutual Information

In this section, we introduce some properties of entropy and relative entropy, such as Strong Subadditivity, monotonicity of mutual information, and the Second Law of thermodynamics.

The Strong Subadditivity of entropy, or SSA, concerns the inequalities among the entropies of a multi-partied system of a larger system .

Theorem 2.2.1 (*Strong Subadditivity*) *Let H_1, H_2, and H_3 be three Hilbert spaces. Given a density operator ρ on the Hilbert space $H_1 \otimes H_2 \otimes H_3$,then*

(1)$S(\rho) + S(\rho_{H_2}) \leq S(\rho_{H_1 \otimes H_2}) + S(\rho_{H_2 \otimes H_3})$.

(2)$S(\rho_{H_1}) + S(\rho_{H_2}) \leq S(\rho_{H_1 \otimes H_3}) + S(\rho_{H_2 \otimes H_3})$

For the proof, see.[6].

Corollary 2.2.1.1 *Taking $H_2 = \varnothing$, we have $S(\rho) \leq S(\rho_{H_1}) + S(\rho_{H_3})$ from* (1).

This allows us to define the mutual information.

Definition 2.2.1 *Let H_1 and H_2 be two separable Hilbert spaces. Given a density operator ρ on a product Hilbert space $H_1 \otimes H_2$. The Mutual Information of the state ρ between space H_1 and H_2 is the number $I(\rho_{H_1}|\rho_{H_2}) = S(\rho_{H_1}) + S(\rho_{H_2}) - S(\rho)$. It's clear that $I(\rho_{H_1}|\rho_{H_2}) = I(\rho_{H_2}|\rho_{H_1})$.*

Theorem 2.2.2 (*Lindblad 1974*) *If Φ is a completely positive trace-preserving linear map on $T_+(H)$ into itself, then for all A, B in $T_+(H)$, we have $S(\Phi A|\Phi B) \leq S(A|B)$.*

This property is usually called the monotonicity of relative entropy.

Corollary 2.2.2.1 *If Φ is a completely positive, trace-preserving, and unital linear map on $T_+(H)$ into itself, then we have $S(\Phi A) \geq S(A)$, $\forall A \in T_+(H)$.*

It follows by taking B = $\mathbb{I}$ in the theorem.

Corollary 2.2.2.2 (*Second law of thermodynamics*) *Let (a,b) be an interval in $\mathbb{R}$. If Phi : (a,b) $\times T_+(H) \to T_+(H)$ is a unital dynamical map on a Hilbert space H, then $S(\Phi(t, A)) \leq S(\Phi(s, A)), \forall A \in T_+(H), \forall t \leq s$ in (a, b).*

It follows from the previous corollary as we apply it with any two fixed times.

It shall be noted that a dynamical map satisfying the second law of thermodynamics has to be unital not only because the previous corollary shall hold but also because it must send the maximally entangled state to itself.

3 Entanglement Entropy in Quantum Field Theory

Since quantum fields have infinite degrees of freedom, its the geometric entropy, or the entropy due to geometric divisions of space, diverges to infinity. So, one must regularize the theory such that the entropy serves as a limit of this regularization. The most common regularizations can be lattice regularization, energy cutoff regularization, or the Replica Trick that we will frequently use in the following context. In this chapter, we will introduce the replica trick and some analytic and numerical results of entropy in quantum field theories.

3.1 The Replica Trick

The von Neumann entropy $S[\rho] = -Tr[\rho \ln \rho]$ is usually difficult to calculate for generic quantum field theory and infinite dimensional systems (quantum states form a infinite dimensional Hilbert space), partially because of the logarithm which causes obstacles in regularization. When employing the limit

$$x \log x = \frac{1}{n-1} \lim_{n \to} x^n$$

, we can calculate something like the entropy function.

The n-th Renyi entropy is defined as

$$S^n[\rho] = \frac{1}{1-n} Tr[\rho^n]$$

So, the von Neumann entropy is the limit of the Rényi entropy when n approaches 1. In fact, due to the Carlson's theorem on complex analysis, we can analytic continue the Rényi entropy of index "n" to that of index "z", where z is in a domain in the complex plane, supposed the "integer" Rényi entropy satisfies some conditions about its behaviour for large $|z|$. Such regularization of identifying von Neumann entropy as a limit of Rényi entropy is the so called Replica Trick, as its name suggests.

For the generic field operator eigenstate $|\Phi\rangle$ and $|\Psi\rangle$, the entries of ρ^n can be calculated as

$$\langle \Psi | \rho^n | \Phi \rangle = \int ... \int \mathcal{D}\Xi_1 \mathcal{D}\Xi_2 ... \mathcal{D}\Xi_{n-1} \langle \Psi | \rho | \Xi_1 \rangle \langle \Xi_1 | \rho | \Xi_2 \rangle ... \langle \Xi_{n-1} | \rho | \Phi \rangle$$

If the state is a thermal state with temperature β, we can do the Wick rotation on the Lagrangian of the field and get

$$S^n[\rho] = \frac{1}{Z^n(1-n)} \int \mathcal{D}\Xi_1 ... \mathcal{D}\Xi_n \delta(\Xi_n(\cdot, \beta) - \Xi_1(\cdot, 0)) ... \delta(\Xi_{n-1}(\cdot, \beta) - \Xi_n(\cdot, 0)) e^{-\sum_k S_E[\Xi_k]}$$

, where Z is the partition function of the theory, S_E is the Euclidean action, and δ is the delta function of fields, which one can see as product of the delta function on the field stress at different spatial points.

However, we can view the action $S_{n,E} = \sum_k S_E[\Xi_k]$ as an action on a n-copies of subsets of spacetime such that it equals to the original action on each subset. We now glue the n-copies together and call it R_n, which is then a manifold parametrized on a Lorentzian vector space. Thus, we can rewrite the Rényi entropy as

$$S^n[\rho] = \frac{1}{Z^n(1-n)} \langle \delta(\Xi_n(\cdot, n\beta) - \Xi_1(\cdot, 0)) ... \delta(\Xi_{n-1}(\cdot, \beta) - \Xi_n(\cdot, 0)) \rangle_{S_{n,E}, R_n}$$

To make the sense more concrete, we may see this in pictures. Geometrically, we can represent the square of the density matrix as an integral on the following manifold:

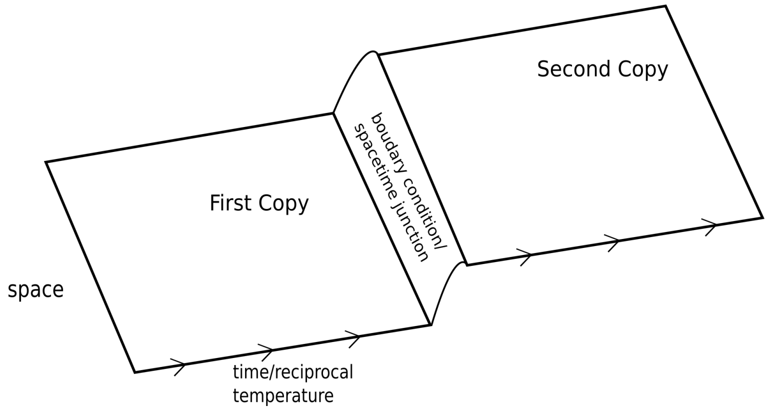


Figure 1: Replica Trick 1

,where the boundary condition is fixed to be eigenstates of the field operators. So, we can view the calculation of Rényi entropy as the expectation value of some observable on this kind of "replica band":

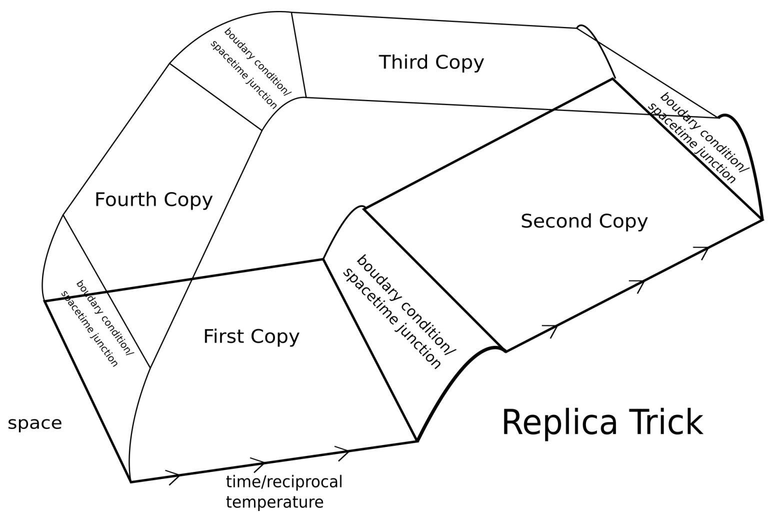


Figure 2: Replica Trick 2

So, it remains to figure out how to calculate the expectation value of the above "delta function", or any operator that would fixed the "conjunction condition".

In particular, for a 1+1 dimensional conformal field theory, we can view the underlying spacetime as a complex plane, thus such "replica band" can be seen as a complex manifold, which is locally parametrized as a domain in a complex plane. Thus here R_n is in fact a Riemann surface. This fact will be important when we consider the entropy of a 1+1 CFT in the following chapter.

However, if we consider subsystem as a union of disjoint intervals, then the topology of the replica manifold will be much more complicated, which requires more discussions.

3.2 C-Theorem and the Entropy C-Function

Another open problem in Quantum Field Theory called the existence of c-theorem in any dimension has been of theoretical interest for many years. In the scheme of renormalization group, the fixed points are usually thought as a system with scale invariance. The c-theorem puts it clear, by stating that for a quantum field theory, we can construct a c function as a function of the variables of the theory which grows monotonically under the RG flow and equals to the central charge of the theory describing the fixed point.

In 1986, Zomolodchikov [13] firstly proved a 1+1D version of it, the main idea of the proof is that we can construct the Collain-Symanzik equation which describes the flow of the correlation functions. Among them, the one which is of the energy momentum tensor together with its conservation law can provide us a function that monotonically grows under RG flow, and reaches to a value proportional to the two points correlation of the energy momentum tensor.

Furthermore, the energy momentum tensor flows to the holomorphic energy momentum tensor of the CFT describing the fixed points, thus, flows to the correlation function of the Virasoro algebra. With this properties, the c function flows to the central charge of the CFT.

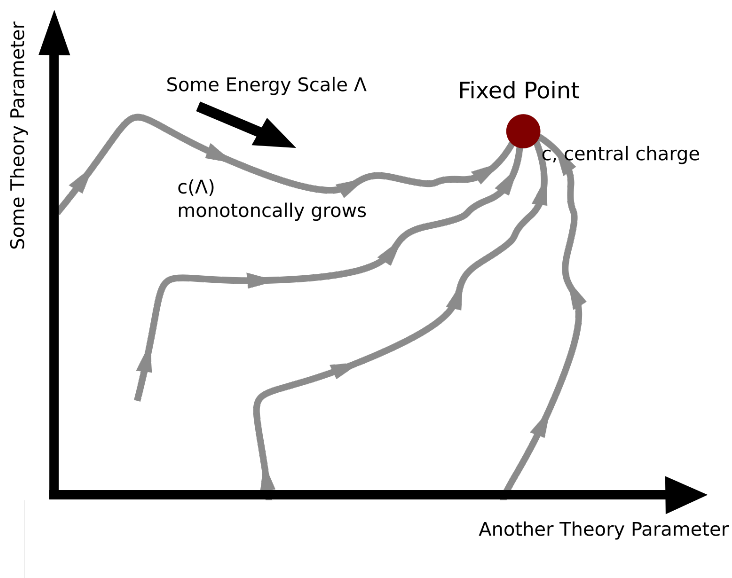


Figure 3: C-Theorem

However, the c-theorem in higher dimensional spacetime, or the a-theorem named by Cardy, is still not fully proved. Cardy himself once gave a construction by the Weyl anomaly by considering that the underlying spacetime is curved, but the monotonicity is not fully proved. But in the case of 4 dimension, the a theorem is proven by Solodukhin in 2013 [11]. The idea of Solodukin is also based on considering the space time can be curved so that we can construct a c function by the geometric invariants. With introducing dilaton in the theory such that the metric function of the Riemannian spacetime, Solodukhin proved that the dilaton entropy can be such c function which monotonically grows along the RG flow and flows to the central charge of a CFT.

Nevertheless, the c-theorem in arbitrary dimensional spacetime seems to be still an open problem. But we can still see that the deeper understanding of the entanglement entropy may provides us some ideas to prove a general c theorem, especially because of that we know the 2D CFT has entanglement entropy relating to its central charge.

4 Conformal Field Theory

4.1 Ward Identities of Energy Momentum Tensor

Classically, Noether's theorem, or Noether's construction of conserved charge, guarantees if there's a continuous symmetry of the action, there's a correspondent conserved charge. From the lesson of quantum field theory, we know that the invariance of the action under a symmetry not only implies the conservation law at the zero order, we can have further correspondent identities in any order. For example the U(1) symmetry of QED leads to

$$q_\mu \Gamma^{(3)\mu\nu}(p,q) = \Gamma^{(2)\nu}(p+q) - \Gamma^{(2)\nu}(p)$$

,where $\Gamma^{(n)\mu_1 ... \mu_n}$ is the nth vertex function in the momentum space.

In conformal field theory, symmetries of the system also implies to identities in any order. The holomorphic energy momentum tensor serves as the generator of spacetime translation in 1+1 CFT, thus, the correlation function of energy momentum tensor and other primary fields can be re-expressed by correlation function of these other fields and their derivatives. For any holomorphic $\epsilon(z)$, we have

$$\langle \int \frac{dz}{2\pi} \epsilon(z) T(z) \phi_1(\omega_1) ... \phi_n(\omega_n) \rangle$$

$$= \sum_{k=1}^{n} \langle \phi_1(\omega_1) ... (\int \frac{dz}{2\pi} \epsilon(z) T(z) \phi_k(\omega_k)) ... \phi_n(\omega_n) \rangle$$

$$= \sum_{k=1}^{n} \langle \phi_1(\omega_1) ... \delta_\epsilon \phi_k(\omega_k) ... \phi_n(\omega_n) \rangle$$

$$= \sum_{k=1}^{n} \int \frac{dz}{2\pi} \epsilon(z) \big(\frac{h_k}{(z-\omega_k)^2} + \frac{1}{z-\omega_k} \partial_{\omega_k} \big) \langle \phi_1(\omega_1) ... \phi_n(\omega_n) \rangle$$

The last equality arises from the contour integral presentation of infinitesimally translated primary fields. Since $\epsilon(z)$ is arbitrary, we, hence, have

$$\langle T(z) \phi_1(\omega_1) ... \phi_n(\omega_n) \rangle = \sum_{k=1}^{n} \big(\frac{h_k}{(z-\omega_k)^2} + \frac{1}{z-\omega_k} \partial_{\omega_k} \big) \langle \phi_1(\omega_1) ... \phi_n(\omega_n) \rangle$$

which is called the ward Identity (of the spacetime translation symmetry). This result will be very useful and significant in our later discussion.

4.2 The Twist Fields

A global symmetry means the symmetric operation can be done on all the space. A symmetry is called internal if the symmetric operation doesn't change the spatial point of the field when the operation acts in the field. Twist field exists when the theory has a global internal symmetry. In fact, if we know how the symmetry acts on fields of the system, we can construct twist fields according to the action. Suppose the symmetry form a group G, then for every g $\in$ G, we can construct another field $T_g(x)$ by

$$T_g(x)\phi(x) = g\phi(x)$$

If the symmetry is discrete and identifies with group $\mathbb{Z}_n$, then we can view the action of the symmetric as a change on the monodromy of the field operator, as the thesis [10] suggests : "Twist fields are local fields which have non-trivial monodromy." The n-folded symmetry of replica trick can serve as an example. Let σ be the group element in $\mathbb{Z}_n$ that transform i into i+1 mod n. Consider the twist field be defined by

$$\mathcal{T}_n(y)\phi_i(x) = \phi_{\sigma i}(x)\mathcal{T}_n(y) = \phi_{i+1 \mathrm{mod\ n}}(x)\mathcal{T}_n(y) \text{ if x} > y$$
$$\mathcal{T}_n(y)\phi_i(x) = \phi_i(x)\mathcal{T}_n(y) \text{ if x} < y$$

And the inverse,

$$\tilde{\mathcal{T}}_n(y)\phi_i(x) = \phi_{\sigma^{-1}i}(x)\tilde{\mathcal{T}}_n(y) = \phi_{i-1 \mathrm{mod\ n}}(x)\tilde{\mathcal{T}}_n(y) \text{ if x} < y$$
$$\tilde{\mathcal{T}}_n(y)\phi_i(x) = \phi_i(x)\tilde{\mathcal{T}}_n(y) \text{ if x} > y$$

In fact, $\tilde{\mathcal{T}}_n = \mathcal{T}_{-n}$. Such twist fields related to the replica structure is firstly proposed and named "branch point twist field" in [4].

Focused on a single interval (u,v) on the one dimensional real space, one can see that the field strength out of the interval is left on the same sheet, and only that in the interval is pushed or pulled into the nearest sheets. These twist fields push the field strength at different sheets together, since the field shall be kind of analytic and thus stiff, the field strength before and behind the interval are forced to match each others, which leads to the junction field strength matching condition that replica trick needs. However, this orbifold theoretical argument seems to get no strict proof, only left to be an argument.

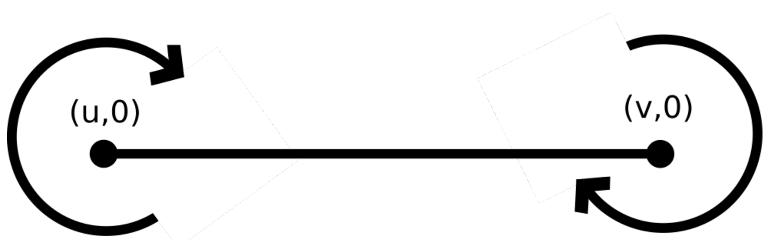


Figure 4: Branch Point Twist Fields

Under this argument, we can say that the partition function can be calculated as correlation function of twist fields at the points that defines the intervals.

$$\mathcal{Z}_{\mathcal{R}_{n,N}} \propto \langle \prod_{k=1}^{N} \mathcal{T}_n(u_k, 0)\tilde{\mathcal{T}}_n(v_k, 0)\rangle$$

Also, if we modify the Lagrangian used in the path integral such that the fields in the theory or the external fields couple to some operators of interest, we can see that the expectation value of the operators on the replica sheets can be expressed as their correlation functions with the twist fields on the n-copied (with no replica condition) sheets. The idnentification can be written as

$$\langle O_i(x, \tau; \text{i sheet})...\rangle_{\mathcal{Z}_{\mathcal{R}_{n,1}}} = \frac{\langle \mathcal{T}_n(u, 0)\tilde{\mathcal{T}}_n(v, 0)O_i(x, \tau; \text{i sheet})...\rangle_{\mathbb{C}}}{\langle \mathcal{T}_n(u, 0)\tilde{\mathcal{T}}_n(v, 0)\rangle_{\mathbb{C}}}$$

This important relation ends this chapter, and now we can see how the entanglement entropy in CFT is calculated.

5 Entanglement Entropy in Conformal Field Theory

5.1 A Single Interval Calculation

Due the idea that cancellation of twist fields is the boundary condition which replica trick needs, the problem of calculating the Rényi entropy of a interval (u,v) on complex plane now reduces to how to calculate the correlation functions of the twist fields, that is

$$Tr[\rho^n] = \frac{c_n}{1-n}\frac{\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle}{\langle \mathcal{T}_1(u,0)\tilde{\mathcal{T}}_1(v,0)\rangle^n}$$

,where c_n arises because the normalization constants of the expectation value of the twist fields of different n are not the same.

On a single interval, expectation values of product of operators on the replica Riemann surface can be expressed as

$$\langle O_i(x,\tau;\text{i sheet})...\rangle_{\mathcal{Z}_{\mathcal{R}_{n,1}}} = \frac{\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)O_i(x,\tau;\text{i sheet})...\rangle_{\mathbb{C}}}{\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle_{\mathbb{C}}}$$

If we want to calculate the correlation of Twist fields, we have to know what is their conformal dimension, which can be calculated via the Ward Identity evaluated on the Riemann surface $\mathcal{R})_{n,1}$. Consider the conformal transformation

$$\omega = f(z) = (\frac{z-u}{v-z})^{1/n}$$

which sends the Riemann surface into the complex plane, as the following graph shows.

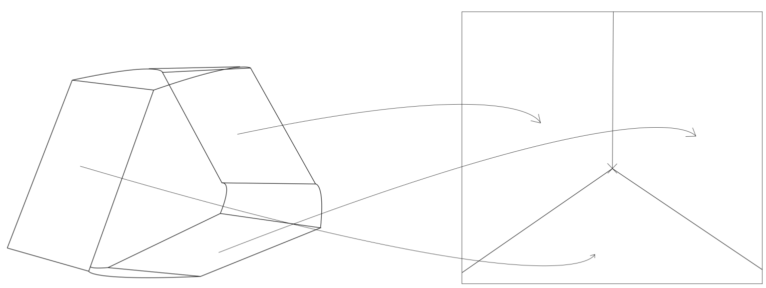


Figure 5: Conformal Transformation

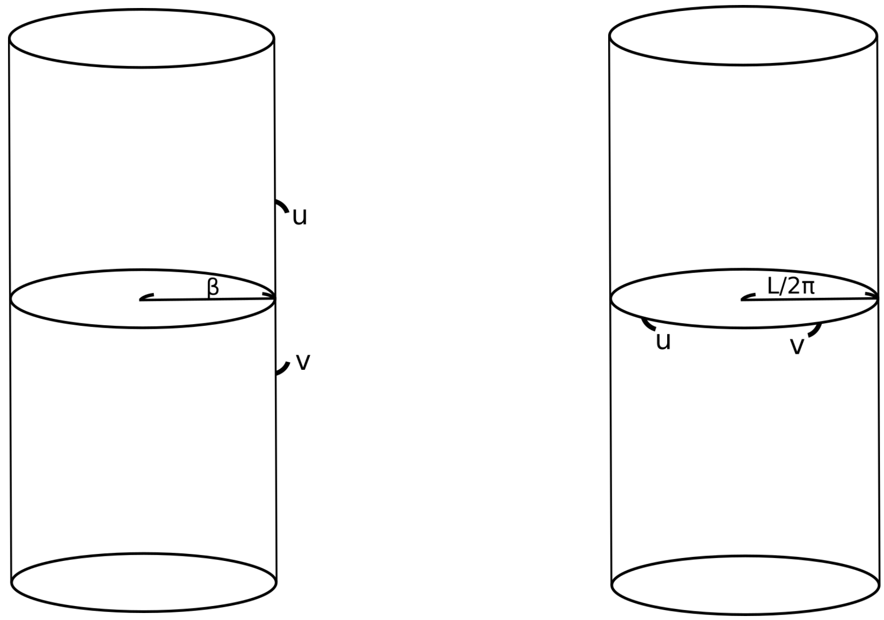
One shall notice that this conformal mapping preserves the topology. Thus, the expectation value of the holomorphic energy momentum tensor can be calculated by the Schwarzian as

$$\langle T(\omega)\rangle_{\mathcal{R}_{n,1}} = (\frac{dz}{d\omega})^2\langle T(z)\rangle_{\mathbb{C}} + \frac{c}{12}\{z,\omega\} = \frac{c(1-n^{-2})}{24}\frac{(v-u)^2}{(\omega-u)^2(\omega-v)^2}$$

On the other hand, by Ward Identity, we have

$$(\frac{1}{\omega-u}\frac{\partial}{\partial u} + \frac{h_{\mathcal{T}_n}}{(\omega-u)^2}\frac{1}{\omega-v}\frac{\partial}{\partial v} + \frac{h_{\tilde{\mathcal{T}}_n}}{(\omega-v)^2})\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle = \langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)T(\omega)\rangle.$$

Compare these two equations by writing $\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle = \|u-v\|^{-2d_n}$, then one can eventually find that $d_n = \frac{c}{12}(n-\frac{1}{n}) = \bar{d}_n$, which is the conformal dimension of the twist fields.



(a) Finite Temperature

(b) Finite Size

Figure 6: The conformal mapping for finite size or finite temperature system

As a result, up to a normalization c_n that cannot be solved until the exact theory is given, we have for a single interval A as a subsystem of the whole real line,

$$Tr[\rho_A^n] = c_n \left(\frac{v-u}{a}\right)^{-\frac{c}{6}(n-\frac{1}{n})}$$

,where a is a length scale that arises in the normalization. If we write u-v = l and $c_1' = \lim_{n\to 1} \frac{\log c_n}{n-1}$, then the von Neumann entropy of this subsystem A is

$$S_A = \frac{c}{3}\log\left(\frac{l}{a}\right) + c_1'$$

,which is consistent to the area law calculated by other methods, and one shall notice that in 1+1 dimensional spacetime, area law means a logarithm behaviour in the system size.

Using the conformal transformation

$$z = \frac{\beta}{2\pi}\log\omega$$

, which maps each sheet of $\mathcal{R}_{n,1}$ to the cylinder of infinite length and circumference β as figure 4(a) shows, we can transform the subsystem to a state of reciprocal temperature β. Then the conformal transformation of correlation function gives

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi l}{\beta}\right) + c_1'.$$

The height of the cylinder is set to be infinite as the size of the total system is infinite. If the size of the system is much more smaller than the temperature, then the limit is $\frac{c}{3}\log\frac{l}{a} + c_1'$ which is independent of the temperature, as a maximally entangled state shall be. On the other hand, if the size of the system is much more larger than the temperature, then one shall get the result of $\frac{\pi c}{3\beta}l + c_1'$, which is consistent with the third law of thermodynamics, that is, the entropy shall go to zero as the temperature goes to zero with c_1' may serve as a measurement of impurity.

On the other hand, if we want to set the total system as finite with length L and periodic boundary condition, we may just transform the Riemann surface into cylinder like figure 4(b). The height of the system now denote real time. The interval (u,v) is turned perpendicular to the axis, so not only β is substituted by L, we also need to put an $\sqrt{-1}$ in front of u-v. As a result, the entropy is

$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\frac{\pi l}{L}\right)$$

In fact, for any n in $\mathbb{N}$ we have

$$Tr\rho_A^n = c_n\left(\frac{L}{\pi a}\sin\frac{\pi l}{L}\right)^{-\frac{c}{6}(n-\frac{1}{n})}$$

If we now want not only the total system to obey periodic boundary condition but also the subsystem to be a thermal state, then we have to require the target space of the Riemann surface $\mathcal{R}_{n,1}$ to be a torus. Due to the constrain of content, we may not discuss CFT on torus here. One can find reference in the fifth lecture of [9]. The result has only been conducted for the case of massive Fermions, but not for general CFT yet.

Another system we may interest is the entanglement entropy in bCFT, or boundary Conformal Field Theory. In such system like what in figure 5, we only need one twist field at l to fix the boundary condition, thus, the we only have to calculate what $\langle\mathcal{T}(\omega)\rangle_{\mathcal{R}_{n,1}}$ is.

Consider the conformal mapping which maps the replica half plane to the whole $\mathbb{C}$

$$z = \left(\frac{\omega-\sqrt{-1}\ell}{\omega+\sqrt{-1}\ell}\right)^{\frac{1}{n}}$$

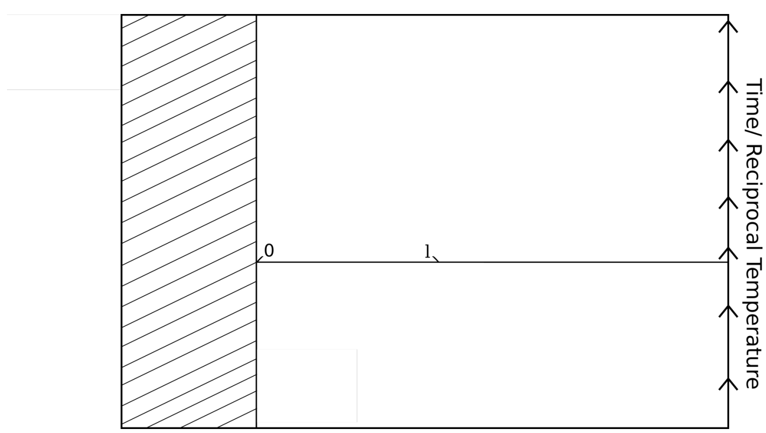


Figure 7: bCFT

The expectation value of the "single" twist field on the replica half plane is

$$\langle\mathcal{T}(\omega)\rangle_{\mathcal{R}_{n,1}} = \frac{c}{24}\left(1-\frac{1}{n^2}\right)\frac{4\ell^2}{(\omega^2+\ell^2)^2}$$

Thus, we have

$$Tr\rho_A^n = c_n\left(\frac{2\ell}{a}\right)^{\frac{c}{12}(n-\frac{1}{n})}$$

with the limit

$$S_A = \frac{c}{6}\log\frac{2\ell}{a} + c_1'$$

which is half of the entropy on the whole plane, and it again matches to other numerical results like DMRG. We can of course apply this idea to the case of the thermal or total system with boundary condition, and the result is still half of the whole one. So let's not repeat it again here.

5.2 Entanglement of domains as unions of disjoint intervals

According to what we know previously, the entanglement entropy of a union of two disjoint intervals in a total system of 1D real space can be expressed as product of twist fields. In particular, for a system of two intervals, the correlation function is a function of the cross ratios of the for points defining these intervals. In the paper [3], they discussed it in details. In the paper [1], for a quantum system of two intervals, we have

$$Tr\rho^n = c_n^2 \big(\frac{|u_1-u_2||v_1-v_2|}{|u_1-v_1||u_2-v_2||u_1-v_2||u_2-v_1|}\big)^{2d_n}\mathcal{F}_n\big(\frac{(u_1-v_1)(u_2-v_2)}{(u_1-u_2)(v_1-v_2)}\big)$$

and the function $\mathcal{F}$ of the cross ratio is theory dependent and can only be calculated case by case.

For general case N¿2, there's still no concrete result in the time of [1], but with the conformal invariance in mind, one can still write down

$$Tr\rho^n = c_N^2 \big(\frac{\prod_{j<k}(u_k-u_j)(v_k-v_j)}{\prod_{j,k}(u_k-u_j)}\big)^{\frac{c}{6}(n-\frac{1}{n})}\mathcal{F}_{n,N}(\text{cross ratios})$$

5.3 Entanglement Spectrum

Similar to the Jacobi Eigenvalue Algorithm, if we take the limit $\lim_{n\to\inf} Tr[\rho^n]$, then we can estimate the maximal eigenvalue. In fact, we have Tr$\rho^n \to \lambda_{max}^n$ as n $\to \infty$. If we employ the L'Hopital's law on the Renyi entropy with the limit taking n to infinity, then we have

$$S_A^{\infty} = -\lim_{n\to\infty}\frac{1}{n}\lambda^n \max = -\log\lambda_{\max}.$$

,where $\lambda_{\max}$ is the largest eigenvalue of ρ, that is, the probability of the most probable state.

Such value is called the "single copy entanglement", which is named so because it's like the entanglement entropy of a infinitely copied single system. Furthermore, if we know all the moments of the reduced density matrix, then it may be that we can regain the spectrum of ρ_A. This method is called the entanglement spectrum, which concerns whether can we extract the probability spectrum from moments of the density matrix, or similarly, the Renyi entropy.

This problem is not only considered on CFTs, we can ask this problem to any open quantum system. Here we simply introduce the results in [2]. Let's consider the function f defined by

$$f(z) = \frac{1}{\pi}\sum_{n=1}^{\infty} R_n z^{-n} = \sum_{n=1}^{\infty}\sum_{\lambda} P(\lambda)\big(\frac{\lambda}{z}\big)^n$$

,where R_n is the nth momentum of the density matrix, P(λ) is the probability of the density matrix to have eigenvalue λ, and $|z| > 1$. That is, we have the moments are exactly

$$R_n = \sum_{\lambda} P(\lambda)\lambda^n$$

Noting that the spectrum of a QFT is continuous, so we can rewrite it as

$$f(z) = \frac{1}{\pi}\int d\lambda P(z)\sum_{n=1}^{\infty}(\frac{\lambda}{z})^n \tag{1}$$
$$= \frac{1}{\pi}\int d\lambda P(z)(\frac{\lambda}{z})\sum_{n=0}^{\infty}(\frac{\lambda}{z})^n \tag{2}$$
$$= \frac{1}{\pi}\int d\lambda \frac{P(z)}{1-\frac{\lambda}{z}}(\frac{\lambda}{z}) \tag{3}$$
$$= \frac{1}{\pi}\int d\lambda \frac{\lambda P(z)}{z-\lambda} \tag{4}$$

If we write

$$R_\alpha = c_\alpha e^{-b(\alpha-\frac{1}{\alpha})}$$

, then one can express f as

$$f(z) = \frac{1}{\pi}\frac{b^k}{k!}\mathrm{Li}_k(\frac{e^{-b}}{z})$$

, where Li_k is the k-th polylogarithm function. Now, we can compare the integral and the sum, which reduces to

$$P(\lambda) = \delta(\lambda_{\max}-\lambda) + \frac{b\theta(\lambda_{\max}-\lambda)}{\lambda\sqrt{b\log(\lambda_{\max}/\lambda)}}\mathcal{I}_1(2\sqrt{b\log(\lambda_{\max}/\lambda)})$$

, where $\mathcal{I}_k$ is the modified Bessel function of the first kind, and b $= -\log\lambda_{\max}$. With this formula of P, we can calculate many non-local quantities .

The first quantity is the mean number of eigenvalues larger than λ

$$n(\lambda) = \int_{\lambda}^{\lambda_{\max}} d\lambda P(\lambda) = \mathcal{I}_0(2\sqrt{b\log(\lambda_{\max}/\lambda)})$$

, which diverges as λ reaches 0. This makes sense since QFTs have infinite degree of freedom. Another quantity that matters is the normalization condition, that is $\mathrm{Tr}\,\rho = 1$. This can be expressed as

$$1 = \int_0^{\lambda_{\max}} \lambda P(\lambda)\mathrm{d}\lambda$$

Also, the von Neumann entropy is

$$S_A = -\int_0^{\lambda_{\max}} \lambda\log\lambda P(\lambda)\mathrm{d}\lambda = -2\log\lambda_{\max}$$

, which is twice the single copy entanglement entropy.

6 Summary

Whether there's similar method to entanglement entropy of conformal field theory in higher dimensions is still an open problem to the writer's knowledge. A paper in 2014 [5] has discussed the generalization of twist operators in higher dimensions, but since the boundary of higher dimensional objects are closed in the sense of homology, unlike we only need two points in 1+1 dimension, we might have trouble in the use of twist operators to match the fields at the edge of different sheets.

In the sense of Shannon and von Neumann's realization, we now have deeply understand the theoretical foundation and practical meaning of the entanglement entropy, however, we do still not that understand how to calculate entanglement entropy in many system like blackholes. Concrete calculation of entanglement entropy of black hole can certainly help us understand not only the thermodynamics of it but also help us to solve the information paradox.

Entanglement entropy, as a non-local properties like Wilson loop, provides many different aspects of a system for us, such as the information stored in the system , and the geometric and topological properties the system has. Concrete calculation of entanglement entropy has provided, is providing, and will provide physicist the guidance and pathway to the realization of the physical worlds, through the time of Boltzmann, of us, and of the future. The importance of entanglement entropy can never be overestimated.

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References

[1] P. Calabrese and J. Cardy. Entanglement entropy and conformal field theory. *J. Phy. A: Mathematical and Theoretical*, 42(50):504005, 2009.

[2] Pasquale Calabrese and Alexandre Lefevre. Entanglement spectrum in one-dimensional systems. *Phys. Rev. A*, 78:032329, Sep 2008.

[3] Michele Caraglio and Ferdinando Gliozzi. Entanglement entropy and twist fields. *Journal of High Energy Physics*, 2008(11):076–076, Nov 2008.

[4] J.L. Cardy, O.A. Castro-Alvaredo, and B. Doyon. Form factors of branch-point twist fields in quantum integrable models and entanglement entropy. *J. Statist. Phys.*, 130:129–168, 2008.

[5] Ling-Yan Hung, Robert C. Myers, and Michael Smolkin. Twist operators in higher dimensions. *JHEP*, 10:178, 2014.

[6] H. Lieb and M. Ruskai. A fundamental property of quantum-mechanical entropy. *Phys. Rev. Lett*, Vol. 30, No. 10(434-436), 1972.

[7] Lindblad. Expectations and entropy inequalities for finite quantum systems. *Commun. math Phys*, Vol. 39(111-119), 1972.

[8] G. Lindblad. Completely positive maps and entropy inequalities. *math Phys*, Vol. 40(147-151), 1975.

[9] Joshua D. Qualls. *Lectures on Conformal Field Theory*. arXiv, 2016.

[10] James Silk. More general correlation functions of twist fields from ward identities in the massive dirac theory. *Journal of Physics A*, 2012.

[11] Sergey N. Solodukhin. The a-theorem and entanglement entropy. 4 2013.

[12] John von Neumann. *Mathematical Foundations of Quantum Mechanics*. Princeton University Press, 1932.

[13] A.B. Zamolodchikov. Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory. *JETP Lett.*, 43:730–732, 1986.