A Note on Symmetries and their Breaking

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Chapter 1

Introduction

Physics is largely about the symmetries.

This note is aimed to introduce some fundamental aspects of symmetry in the domain of physics and explain the general tool to understand them. Some applications of these tools are also introduced to make this note more meaningful and contextually complete.

In the first chapter, we discuss some consequence of symmetry in the classical and quantum system, that is, the Noether's theorem and the Wigner theorem of unitarity/anti-unitarity. Since the famous Noether "first" theorem can be found in any textbook of classical mechanics, we only briefly introduce it in a formal form and focus on the (forgotten) Noether's second theorem of symmetry, which involves in some differential equations on the off-shell equation of motion and the form of the conserved current at the off-shell level. This then can be useful in the quantum systems with symmetry. To instantiate this, we introduce the use of Noether's second theorem to derive the Ward-Takayanagi identity in the quantum field theory with background (BQFT) in the section of quantum symmetry.

In the first chapter we also introduce a fundamental proof of the Wigner's theorem, which is usually mentioned but not prove in a quantum physics lecture, for both separable and non-separable Hilbert space, but this section assume the reader has some background in the basic point set topology (e.g. the concept of dense sets). Nevertheless, the proof is basically based on linear algebra, and we try to make it as readible as possible. At the end of the first chapter, we then introduce again the power of symmetry in the constrain of theory, that is, the McGlinn theorem, the LOR theorem, and the famous Coleman-Mandula theorem. However, the proof of the last two theorem are complicated and, in fact, the former two theorems are special case of the CM theorem, whose proof can be found on the third volume of Weinberg, so it'd be relatively meaningless to introduce the proof of these two theorems. On the contrary, the proof of the McGlinn theorem is relatively easy, additionally, a fundamental step in the original proof and another modified proof we cite is lacked of detailed explaination, so we suitably complete the explaination of it and present the proof in a tidy version.

In the second section, we introduce the tool to understand the use of symmetry in quantum systems, that is, the projective representations and the central extensions of Lie groups and Lie algebra, which are generally omitted in a typical class of Lie algebra, which usually end at the introduce of root systems and Dedikind diagrams for the classification of semi-simple Lie algebra themselves and their the representations.

The almost the whole chapter can be viewed as a proof to explain why the projective representations of a semi-simple Lie group are in fact the ordinary representations of its universal covering group, which is again typically not mentioned in either Lie algebra class and quantum field theory lecture. In order to finish the proof, we firstly introduce the equivalance of projective representations and central extension by $U(1)$ for Lie groups, but in fact, we introduce the central extension and the related Lie group cohomology in a more general level for possible future usage.

After this, we then introduce the fact that every central extension of simply-connected Lie group is equivalent to a central extension of Lie algebra, thus, the classification of projective representations of Lie groups can be resorted to the study of the Lie algebra cohomology, which need the concept of Universal enveloping algebra and Casimir operators. Also, a proper discussion of the Casimir operators could be useful since the Lie algebra in fact has no such a product operation and we have to define it on the universal enveloping allgebra of it, which most of the quantum mechanics lecture does not involve and leaves the student with some confusion or misconception.

In detailed explaination, the first Lie algebra cohomology classifies the complete reducibility of representations of Lie algebra and the second Lie algebra cohomology classifies the central extensions of Lie algebras. With this classification, the two Whitehead's lemmas about the complete reducibility of semi-simple Lie algebras and the triviality of the semi-simple Lie algebra central extensions are properly explained. In this fashion, we then can explain why all the projective representations of a connected and simply-connected semisimple Lie groups are in fact trivial, and one only has to discuss when can an ordinary representations of the universal covering group of a semi-simple Lie group descend to the Lie group itself. At the end of this chapter, we introduce another application of Lie group cohomology in the construction of symmetry protected topological phases (SPT phase) via the some nonlinear σ model on a space-time lattice. Also, we discuss some properties of nonlinear σ model, but the content seems to be insufficient and is expected to be complete in the future.

The fourth chapter of this note is about the spontaneous symmetry breaking, we firstly introduce the definition and briefly explain the Nambu-Goldstone theorem with proof for the understanding of the third section. In the third section, we disucss the counting of the Nambu-Goldstone modes of different types and explain their difference in the dispersion relations from an effective field theory point of view. Actually, the dimension of the coset by quotienting the original group by the unbroken group is generally not the number of Nambu-Goldstone modes since the different generator of the broken symmetry may refer to the same mode. In nonrelativistic quantum field theory, we explain why the single counted modes have linear dispersion, as the Nambu-Goldstone modes in relativistic quantum field theories have, and why the doubly counted Nambu-Goldstone modes have a quadratic dispersion relation.

In the end of this note, we introduce the Mermin-Wanger-Coleman theorem, which argues that the continuous symmetry in 1-D and 2-D cannot be broken. The zero temperature part, also known as the Coleman theorem is proved. However, the general proof of the case with finite temperature is lacked, thus, we only show why it'd be generally true by the example of isotropic Heisenberg model via the Bogoliubov inequality. In fact, the author of this note is considering to give a general proof of the theorem with finite temperature via this inequality despite of this difficulty from system generality.

We assume the reader has sufficient background in quantum mechanics, quantum field theory, and basic knowledge of Lie algebra (e.g. definition and basic properties of semi-simple Lie algebras), and this note does not give a general review of them.

Chapter 2

Classical and Quantum Symmetries

2.1 Noether's Theorems

The relation between constant of motion and symmetry was an open problem before the early twentieth century, and it's not until Noether's famous paper "Invariant Variationspobleme" [1] that their correpondece was really realized. In fact, in Noether's original paper, there were two theorems proposed, and the first one is the well-known theorem about correpondece between symmetry and conserved current. The second theorem is mainly on the constrains on the off-shell equations of motion and the conserved current under the presence of some controlled symmetry.

Although the first theorem was typically introduced in proper textbooks of classical dynamics, the Noether's second theorem is typically omitted and not widely known. In this section, we'll briefly discuss the first theorem and then introduce the second theorem. In the later part of this note, the usage of the second theorem for the Ward identity will be then discussed.

2.1.1 Noether's First Theorem

Let $\mathcal{L}$ be any Lagrangian of some field ϕ on a differentiable manifold M. Here we do not assume ϕ to be the section of any specific type vectro bundle E to keep the generality. If $\mathcal{L}$ has a continuous symmetry, that is there is a one parameter differentiable map u_t on E such that $\mathcal{L}(u_t\phi) = \mathcal{L}(\phi), \forall t$ in some open interval.

If so, then we have

$$0 = \frac{d}{d\epsilon}\mathcal{L}(u_\epsilon\phi) = \mathcal{E}(\phi)u_{0*}(\phi) + \mathrm{Div}K \tag{2.1}$$

Here K is some vector field emerging from the integration by part to obtain the equations of motion $\mathcal{E}$. If now the system is on-shell, i.e. ϕ solves $\mathcal{E}(\phi) = 0$, then we must have $\mathrm{Div}K = 0$. From this, we get the result of the Noether's first theorem by contructing such a conserved current K, and here do not assume much detail of the Lagrangian and the field to show the generality of the theorem.

Typical examples of Noether's first theorem includes momenta/enegy momentum tensor conservation from translation invariance/presence of cyclic coordinates, angular momenta conservation from rotational invrariance, and (classically) conservation of chiral pseudo-current from chiral symmetry. Since these are generally discussed in proper textbooks, we'll omit these here and start to discuss the Noether's second theorem.

2.1.2 Noether's Second Theorem

In the Noether's first theorem, the equations of motion are in used to obtain the vanishing of divergence term. Thus one would naturally ask whether we could still obtain some information without the assumption that the field ϕ has to be on-shell.

In general, if we do not constrain the form of the transformation u_t, then it's difficult to obtain any off-shell information, however, in Noether's original paper, she proposed the following manner to control the symmetry transformation and obtain some off-shell information [2].

If we have for some arbitrary field λ such that the symmetry transformation $u_{\lambda,t}$ for some parameter t is controlled by

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} u_{\lambda,\epsilon}\phi = \mathcal{D}\lambda \tag{2.2}$$

for some differential operator $\mathcal{D}$ independent of λ.

Then from the fact of u us a symmetry of the Lagrangian gives us

$$0 = \frac{d}{d\epsilon}\mathcal{L}(u_\epsilon\phi) = \mathcal{E}(\phi)\mathcal{D}\lambda + \mathrm{Div}K. \tag{2.3}$$

By integration by parts, we have

$$\mathcal{E}(\phi)\mathcal{D}\lambda = \mathcal{D}^*\Big(\mathcal{E}(\phi)\Big)\lambda - \mathrm{Div}B. \tag{2.4}$$

Here $\mathcal{D}^*$ is the adjoint of $\mathcal{D}$. Consequently, we have

$$0 = \mathcal{D}^*\Big(\mathcal{E}(\phi)\Big)\lambda + \mathrm{Div}(K - B). \tag{2.5}$$

Since λ is arbitrary, we then have

$$\mathcal{D}^*\Big(\mathcal{E}(\phi)\Big)\lambda = 0, \tag{2.6}$$

and

$$\mathrm{Div}\Big(K - B\Big) = 0. \tag{2.7}$$

If addtionally we have the manifold M is a Riemannian manifold (with metric g) and the first de Rham cohomology class $\mathcal{H}^1(M) = 0$, then the last equation can be written as

$$K = B + g^\# \delta F. \tag{2.8}$$

Here $g^\#$ is the map sending differential forms to anti-symmetric tensors via metric, and F is some two-form whose existence is guranteed by the vanishing of first de Rham cohomology.

From this, one can control the form of equation of motion when the field is off-shell by a differential equation and contrain the on-shell conserved current by a vector field B and a two form F. And this is the Noether's second theorem.

For an example, if we have

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} u_{\lambda,\epsilon}\phi = a\lambda + b^\mu\partial_\mu\lambda + c^{\mu\nu}\partial_\mu\partial_\nu\lambda \tag{2.9}$$

which is similar to the local conformal transformation of the metric on flat space-time, then we have the equation of motion $\mathcal{E}$, at off-shell level, shall obey

$$a\mathcal{E} - \partial_\mu\Big(b^\mu\mathcal{E}\Big) + \partial_\mu\partial_\nu\Big(c^{\mu\nu}\mathcal{E}\Big) = 0. \tag{2.10}$$

2.2 Quantum Symmetry

Under what condition, an operator, say U, can be a symmetry of a quantum system?

First, such an operator has to be invertible. This is because we expect we can undo this operation and get the original state. As a result, these symmetry operators form a subgroup of the set of automorphisms on the Hilbert space.

Secondly, we have to figure out all the observables we can possibly have, and then discuss all the possible operations on the states which make these observables unchanged. If such an operation leads to no observale effect, then we can say this operation is a quantum symmetry. The only observable in a quantum system are the probabilities of observing any state, say $|\psi\rangle$, in another state, say $|\xi\rangle$, which is characterized by the square of their inner product $|\langle\xi|\psi\rangle|^2$ as the Born interpretation of states, and the expectation values of Hermitian oeprators. However, the expectation values can in fact be expressed by the probabilities in the following meaning.

Since $\hat{O}$ is an Hermitian operator, it's diagonalizable thus can be written as

$$\hat{O} = \sum_\alpha \lambda_\alpha |\alpha\rangle \langle\alpha| . \tag{2.11}$$

Here α is an index set denoting all the eigenvectors of the operator $\hat{O}$, and λ_α are the corresponding eigenvalues. In terms of the expansion, we have

$$\langle\psi| \hat{O} |\psi\rangle = \sum_\alpha \lambda_\alpha \langle\psi|\alpha\rangle \langle\alpha|\psi\rangle = \sum_\alpha \lambda_\alpha |\langle\alpha|\psi\rangle|^2. \tag{2.12}$$

Hence, we can sum over the measured values and their frequencies and get the expectation value in a quantum system as we do in statistics, which then implies that the only independent observable of a quantum system is the value $|\langle\psi|\xi\rangle|$.

As a result, we say an operator U is a symmetry if

$$|\langle U\psi|U\xi\rangle| = |\langle\psi| U^\dagger U |\xi\rangle| = |\langle\psi|\xi\rangle| \tag{2.13}$$

for all states $|\psi\rangle$ and $|\xi\rangle$ of the Hilbert space.

Note that a symmetry is different from a transition operator, ‘ where the unitary operation only acts on the kets rather than both of bras and kets.

Unitary and anti-unitary operators are both trivial symmetries, since they obey $U^\dagger U = UU^\dagger = \mathbb{I}$. Here anti-unitary operators are those operators obeying the former identity and anti-linearity, and anti-linear opearators are those operator anti-commuting whith the complex structure, or $\sqrt{-1}$, that is,

$$U(\lambda |\psi\rangle) = \lambda^* U |\psi\rangle , \tag{2.14}$$

$\forall$ scalars $\lambda \in \mathbb{C}$ and vectors $|\psi\rangle$ in the Hilbert space.

For example, the time reversion operator T, which reverses the time direction literally and obeys $T^2 = \mathbb{I}$, acts on the position and momentum operator as $TxT = x$ and $TpT = -p$. With the commutation realtion $[x, p] = \sqrt{-1}\mathbb{I}$, we have $T\sqrt{-1} = -\sqrt{-1}T$, and this implies T is anti-linear and consequently anti-Unitary. Clearly, one then may ask whether all the quantum symmetries are either unitary or anti-unitary.

In either a seperable or an inseparable Hilbert space, this answer is positive.

2.2.1 Wigner's Theorem

Theorem 2.2.1.1 *(Wigner 1931, Non-Bijective nor Separable by Gehér 2014 [3]) Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{P}_1$ be the set of unit vectors in $\mathcal{H}$. If L: $\mathcal{P}_1 \to \mathcal{P}_1$ be any continuous* *map that preseverves the transition probabilities i.e.* $|\langle L(u)|L(v)\rangle| = |\langle u|v\rangle|, \forall u, v \in \mathcal{P}_1$*, then* L *is either a unitary or an anti-unitary operator.*

Note that the theorem does not require L to be invertible.

$\langle$Proof of theorem 2.2.1.1 $\rangle$

Firstly, let's consider the separable case. Given a countable orthonormal basis of $\mathcal{H}$, say $\{e_j\}_1^{N_{\mathcal{H}}}$, where $N_{\mathcal{H}}$ is the dimension of the Hilbert space. Since $|\langle Le_i|Le_j\rangle| = |\langle e_i|e_j\rangle| = \delta_{ij}$, we have $\{Le_j\}$ is also an orthonormal set (may not be a basis) and spans the subspace, say, $\mathcal{H}'$.

Now we may consider the linear map $V : \mathcal{H} \to \mathcal{H}$ define by $Ve_j = Le_j$, then clearly V is an isometry on $\mathcal{H}$. Notice that we are not saying $V = L$ since L might not be linear. As a result, the map $V^\dagger \circ L$, which is also may not linear, satisfies

$$\langle V^\dagger Le_i|e_j\rangle = \langle Le_i|Ve_j\rangle = \langle e_i|V^\dagger Ve_j\rangle = \langle e_i|e_j\rangle = \delta_{ij}.$$

This implies each e_j is invariant under $V^\dagger \circ L$. Since for each $\omega \in \mathcal{H}$ we have the j-th component of $V^\dagger \circ L\omega$ in the $\{e_j\}$ basis is

$$|(V^\dagger L\omega)_j| = |\langle V^\dagger L\omega|e_j\rangle| = |\langle \omega|V^\dagger Le_j\rangle| = |\langle \omega|e_j\rangle| = |\omega_j|.$$

Hence, the modulus of the components of any vector is preserved, thus, $V^\dagger L$ must be invertiable. We now like to show $V^\dagger L$ is either the identity map or the identity anti-linear map. To understand how $\sqrt{-1}$ varies through L, consider the following definitions

$$V^\dagger L(\frac{1}{\sqrt{2}}e_j - \frac{1}{\sqrt{2}}e_{j+1}) := \frac{1}{\sqrt{2}}e_j - \delta_{j+1}\frac{1}{\sqrt{2}}e_{j+1},$$

and

$$V^\dagger L(\frac{1}{\sqrt{2}}e_j + \frac{\sqrt{-1}}{\sqrt{2}}e_{j+1}) := \frac{1}{\sqrt{2}}e_j + \epsilon_{j+1}\frac{1}{\sqrt{2}}e_{j+1}.$$

Hence, we have $|\epsilon_j| = |\delta_j| = 1$. However, from the requirement about L, we have

$$\sqrt{2} = |\langle e_j - e_{j+1}|e_j + \sqrt{-1}e_{j+1}\rangle| = |\langle e_j - \delta_{j+1}e_{j+1}|e_j + \sqrt{-1}\epsilon_{j+1}e_{j+1}\rangle| = |1 + \delta_{j+1}\epsilon^*_{j+1}|.$$

Consequently, we have $\delta_{j+1}\epsilon^*_{j+1} = \pm\sqrt{-1}$. Now consider the unitary/anti-unitary map U on $\mathcal{H}$ defined by $Ue_1 = e_1, Ue_j = (\Pi_{j=2}^k \delta_j)e_k (k > 1)$ if $\epsilon_2 = \sqrt{-1}\delta_2/\epsilon_2 = -\sqrt{-1}\delta_2$. If so, them the operator $T := U^\dagger \circ V^\dagger \circ L$ satisfies $T(\frac{1}{\sqrt{2}}e_j + \frac{\sqrt{-1}}{\sqrt{2}}e_{j+1}) := \frac{1}{\sqrt{2}}e_j \pm \frac{\sqrt{-1}}{\sqrt{2}}e_{j+1}$.

Now one may ask if it's possible for some j the $\pm$ gets flipped. Supposed this is true, then we can consider

$$T(v_{j-1}e_{j-1} + te_j + v_{j+1}e_{j+1}) = u_{j-1}e_{j-1} + se_j + u_{j+1}e_{j+1}$$

with varying t. Ths gives $v_{j-1} = u_{j-1}$ and $v_{j+1} = u^*_{j+1}$. Hence,

$$T(v_{j-1}e_{j-1} + te_j + v_{j+1}e_{j+1}) = v_{j-1}e_{j-1} + te_j + v^*_{j+1}e_{j+1}.$$

If so, we have

$$\frac{\sqrt{2}}{4} = \langle x = \frac{-1}{2}e_{j-1} + \frac{1}{2}e_j + \frac{1}{\sqrt{2}}e_{j+1})|y = \frac{\sqrt{-1}}{2}e_{j-1} + \frac{1}{2}e_j + \frac{\sqrt{-1}}{\sqrt{2}}e_{j+1}\rangle = \langle Tx|Ty\rangle = \frac{\sqrt{10}}{4}.$$

But this is impossible. Now what's only left to prove the theorem is correct on the whole $\mathcal{P}_1$ rather than some basis. Clearly, the set $D := \{v \in \mathcal{P}_1 : v_j \neq 0 \ \forall j\}$ is dense in $\mathcal{P}_1$. Now we then shall prove the set R defined by

$$R := \{e_j\}_j \cup \{\frac{1}{\sqrt{2}}(e_j - e_{j+1})\}_j \cup \{\frac{1}{\sqrt{2}}(e_j + \sqrt{-1}e_{j+1})\}_j$$

resolves D, that is, $x, y \in D$ satisfying $||x-z|| = ||y-z||\ \forall z \in R$ implies $x = y$. The proof is relatively easy. Let $v, w \in D$, then $||v-z|| = ||w-z||$ for all z implies $|v_j| = |w_j|, |v_j - v_{j+1}| = |w_j - w_{j+1}|, |v_j - \sqrt{-1}v_{j+1}| = |w_j - \sqrt{-1}w_{j+1}|$. Up to a global phase, we may assume $v_1 = w_1$, and this inductively implies $v_j = w_j$ for all j.

Now, since T is identity/anti-linear identity on R, it's also so on D. By continuity of L and the fact that D is dense in $\mathcal{P}_1$, we get L is either unitary or anti-unitary on $\mathcal{P}_1$.

In the non-separable case, choose a orthonormal basis $\{e_1, e_2\} \cup \{e_{\alpha,j} : \alpha \in \Lambda, j \in \mathbb{N}\}$ for some index set Λ. Also, we write $e_j = e_{\alpha,j}$ whenever $j = 1, 2$. Similar to the separable case, we can construct unitary operator V such that the basis is invariant under $V^\dagger L$. For each fixed α, L is be unitarily/anti-unitarily transformed in to the identiy map on the subspace $\mathcal{K}_\alpha$ spanned by $\{e_{\alpha,j}\}$ by the result in the separable case. Now we shall prove that it implies the transformed function $T = U^\dagger V^\dagger L$ is the identity map on the whole $\mathcal{P}_1$. For $Tv = w$, define $v_\alpha = \sum_j \langle v, e_{\alpha,j}\rangle e_{\alpha,j}$ and $w_\alpha = \sum_j \langle w, e_{\alpha,j}\rangle e_{\alpha,j}$. Then similarly, we have $|v_{\alpha,k}| = |w_{\alpha,j}|$ and $|\langle v, v_\alpha\rangle| = |\langle w, v_\alpha\rangle| = |\langle w_\alpha, v_\alpha\rangle|$. This implies $v_\alpha = \lambda_\alpha w_\alpha$ for some global phase λ_α and each α since by the previous result the each $v_\alpha andw_\alpha$ are non-zero and coincide. Furthermore, since we set up v_α in the manner that $e_j = e_{\alpha,j}$ whenever $j = 1, 2$, all the λ_α's must coincide. And this tells us the map T is idenity on a dense subset of $\mathcal{P}_1$ and thus the whole $\mathcal{P}_1$.

2.2.2 Ward-Takayanagi Identity

From the second Noether's theorem, the Ward-Takayanagi with the presence of boundary can be derived as in the paper [2]. In this section, we'll introduce this derivation.

The expectation value of a quantum field theory in the language of path integral is written as

$$\langle \phi_1(x_1)\dots\phi_n(x_n)\rangle_{\Psi_f,\Psi_0} = \frac{1}{Z}\int \mathcal{D}\phi \Psi_f^*(\phi_f)\Psi_i(\phi_i)\phi_1(x_1)\dots\phi_n(x_n)e^{\sqrt{-1}S[\phi]}. \tag{2.15}$$

Here $\Psi_{i/f}$ are the initial and final state of the theory (with a future and past boundary).

Under an infinitesimal local change $\delta\phi$ of the field ϕ, suppose the theory, in quantum level, is invariant, then we in general have the Ward-Takayanagi identity

$$\begin{aligned} 0 &= \delta\langle \phi_1(x_1)\dots\phi_n(x_n)\rangle_{\Psi_f,\Psi_0} \\ &= \langle \delta(\phi_1(x_1)\dots\phi_n(x_n))\rangle_{\Psi_f,\Psi_0} + \langle \phi_1(x_1)\dots\phi_n(x_n)\rangle_{\delta\Psi_f,\Psi_0} \\ &+ \langle \phi_1(x_1)\dots\phi_n(x_n)\rangle_{\Psi_f,\delta\Psi_0} + \sqrt{-1}\langle \phi_1(x_1)\dots\phi_n(x_n)\delta S\rangle_{\Psi_f,\Psi_0}. \end{aligned} \tag{2.16}$$

Now suppose we're considering a region R with boundary on a spacetime manifold M also with boundary and denote the intersection $\Sigma = R \cap \partial M$ and its boundary $\partial\Sigma = \partial R \cap \partial M$ is also not empty.

Since from the Noether's theory, we have the differential equation on the change of S for the equation of motion part. Suppose again the region where the field theory is define has vanishing first de Rham class, as we take the variation of the field to be controlled by a constant funtion on the region R and zero outside R, then we have (when off-shell)

$$\delta S = \int_\Sigma d^{D-1}x(n_\mu K^\mu) - \int_{\partial\Sigma} d^{D-2}x(s_{\mu\nu}F^{\mu\nu}). \tag{2.17}$$

In particular, without inserting other fields, we'd have

$$\int_\Sigma d^{D-1}x(n_\mu\langle K^\mu\rangle) - \int_{\partial\Sigma} d^{D-2}x(s_{\mu\nu}\langle F^{\mu\nu}\rangle) = -\sqrt{-1}\delta_R \langle \Psi_f|\Psi_i\rangle. \tag{2.18}$$

Thus the change of transition amplitude of a BQFT could cause some non-trivial current on the boundary at quantum level from the Ward-Takayanagi identity.

2.2.3 The Coleman-Mandula Theorem

A strong usage of symmetry is to constrain the possible theory.

For example, the CPT theorem is a strong theorem even proved in mathematically rigorous level(, namely from the Wightman axioms of QFT or the algebraic quantum field theory)[4]. The CP violation is general believed true due to the absence of right-handed particles in the electroweak interaction, as a result, the time reversal symmetry must be broken in pair to make the CPT symmetry valied, and that's a possibility to explain the arrow of time. (Yet another possibility is the locally monotonic increase of relative entropy.)

Another example takes place in supersymmetry. Discussing possible supermultiplets is usually the first step to study a supersymmetry theories without writing down the Lagrangian, however, there're too many supersymmetric theories, thus, the CPT symmetry is usually considered, which shall be correct in a local and relativistic quantum field theory. And this consideration of CPT pairs allows us to rule out large classes of $\mathcal{N}=3$ supersymmetry. (In fact, some $\mathcal{N}=3$ supergravity theories could still possible obey the CPT theorem.)

In this section, we'd like to introduce another strong theorem constraining the relation between space-time symmetry and internal symmetry, this theorem is called the Coleman-Mandula theorem[5]. It's a No-Go theorem about the possible form of the overall symmetry of a theorem.

In order to properly introduce the Coleman-Mandula theorem, we firstly introduce its previous work, the McGlinn theorem.

Theorem 2.2.3.1 *(McGlinn) Let L be the Poincaré algebra, M and P be the homogeneuous and translation part of L respectively, and A be any semi-simple internal symmetry alegebra. If E is any semi-direct product of A and L obeying $[A, M] = 0$, then we have $[A, P] = 0$, thus, A itself is a subalgebra of E and $E = A \oplus L$.*

⟨ Proof of 2.2.3.1, based on [6] ⟩

Let $\{A_i\}, \{M_{\mu\nu}\}$, and $\{P_\lambda\}$ be the generator of A, M, and P on some Hilbert space respectively. And let $U(a, \Lambda)$ be a unitary representation of the Poincaré group, then we have

$$U(0,\Lambda)^{-1} P_\lambda U(0,\Lambda) = \Lambda^\kappa_\lambda P_\kappa, \tag{2.19}$$

and

$$U(0,\Lambda)^{-1} M_{\mu\nu} U(0,\Lambda) = \Lambda^\alpha_\mu \Lambda^\beta_\nu M_{\alpha\beta}. \tag{2.20}$$

In genenral, we have

$$[A_i, A_j] = a^k_{ij} A_k + b^\lambda_{ij} P_\lambda + c^{\mu\nu}_{ij} M_{\mu\nu}, \tag{2.21}$$

which requires

$$b^\lambda_{ij} = \Lambda^\lambda_\kappa b^\kappa_{ij}, \tag{2.22}$$

and

$$c^{\mu\nu}_{ij} = \Lambda^\mu_\alpha \Lambda^\nu_\beta c^{\alpha\beta}_{ij}, \tag{2.23}$$

with

$$c^{\mu\nu}_{ij} = -c^{\nu\mu}_{ij}. \tag{2.24}$$

But this is impossible, since there's neither invariant 4-vector nor invariant second rank anti-symmetric tensor. This then gives

$$[A_i, A_j] = a^k_{ij} A_k, \tag{2.25}$$

which implies A itself forms a subalgebra of E.

By basically the same argument, we have

$$[A_i, P_\lambda] = d_i P_\lambda. \tag{2.26}$$

But from this equation and the Jacobi identity on $[A, P]$, we have

$$a_{ij}^s a_{sk}^p + a_{jk}^s a_{si}^p + a_{ki}^s a_{sj}^p = 0, \tag{2.27}$$

and

$$a_{ij}^l d_l = 0. \tag{2.28}$$

Let B_{ij} be the Killing form of A with respect to the basis of $\{A_i\}$, then since

$$B_{ij} = Tr(ad_{A_i} \circ ad_{A_j}) = \langle A_k, [A_i, [A_j, A_k]]\rangle = a_{im}^s a_{sj}^m, \tag{2.29}$$

we have

$$a_{ijk} := B_{is} a_{jk}^s = a_{im}^l a_{ls}^m a_{jk}^s = a_{im}^l a_{jn}^m a_{kl}^n + a_{in}^l a_{jl}^m a_{km}^n \tag{2.30}$$

is totally anti-symmetric.

Hence, if we assume a_l is nonzero and consider the bracket

$$\begin{aligned} [B^{ij} a_i A_j, A_k] &= B^{ij} d_i a_{jk}^m X_m \\ &= B^{ij} d_i a_{mjk} B^{mn} X_n \\ &= -B^{ij} d_i a_{jmk} B^{mn} X_n \\ &= -d_l a_{mk}^l B^{mn} X_n \\ &= 0, \end{aligned} \tag{2.31}$$

then we have $B^{ij} a_i A_j \neq 0$ (for B_{ij} being non-degenerate) itself spans an Abelian ideal in A, which is impossible if A is semi-simple. This then proves $a_l = 0$ by contradiction.

In fact, the paper [6] prove the theorem in a weaker condition that it only requires A to commute with the rotational part of M and leaves the boost part along, then we can still find a subalgebra A' in E being (Lie algebraically) isomorphic to A satisfying the McGlinn theorem.

To achieve the Coleman-Mandula theorem, it's partly necessary to introduce another previous work, the Lochlainn O'Raifeartaigh theorem, also known as the LOR theorem. However, to prove the LOR theorem, we first need the Levi decomposition theorem for general Lie algebras, and some prerequisite definitions are required.

A radical $R(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ is a solvable, i.e. the derived series of any algebra $\mathfrak{h}$

$$\mathfrak{h} \subseteq \mathfrak{h}_1 = [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}_2 = [\mathfrak{h}_1, \mathfrak{h}_1] \subseteq \dots\dots \tag{2.32}$$

arrives at the zero sublagebra for some finite step, ideal of $\mathfrak{g}$ with maximal dimension. In fact, the radical is unique and contains all the other solvable ideals of $\mathfrak{g}$ since the sum of two solvable ideals is again a solvable ideal. Also, one can see that $R(\mathfrak{g}) = 0$ if and only if $\mathfrak{g}$ is semi-simple, and thus $\mathfrak{g}/R(\mathfrak{g})$ is semi-simple. This property then leads to the Levi decomposition theorem, whose proof can be seen in any classical Lie algebra textbook. In fact, a proof generally uses induction on the dimension (for one dimensional Lie algebra the theorem is trivial) in the either cases $R(\mathfrak{g})$ is an Abelian ideal or it's not.

Theorem 2.2.3.2 *(Levi decomposition) For any finite dimensional Lie algebra $\mathfrak{g}$, there is a semi-simple algebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} \rtimes R(\mathfrak{g})$.*

$\mathfrak{s}$ is usually called the Levi factor or Levi subalgebra of $\mathfrak{g}$.

Theorem 2.2.3.3 *(Lochlainn O'Raifeartaigh) IF E is any (finite dimesional) Lie algebra with some Levi subalgebra $\mathfrak{s}$ and contains the Poincaré algebra L, whose homogeneuous part and translation part are denoted by M and P respectively. Then only the following four cases occur:*

$$\begin{aligned} &(1) P = R(E); \\ &(2) P \subsetneqq R(E), \ \textit{and } R(E) \textit{ is Abelian}; \\ &(3) P \subsetneqq R(E), \ \textit{but } R(E) \textit{ is not Abelian}; \\ &(4) P \cap R(E) = 0. \end{aligned} \tag{2.33}$$

In all the four cases, we have $R(E) \cap M = 0$.

$\langle$ Proof of 2.2.3.3, $\rangle$

See [7].

In fact, in this original work, O'Raifeartaigh further analyzed that the case (1) is essentially a direct sum if L and the internal symmetry, (2) is introducing translation for extra-dimension, (3) could forbid Hermitian conjugacy being defined for elements in $R(E)$ but not P in nontrivial representations, and (4) is equivalent to imbedding L into some simple Lie algebra.

Now we finally shall introduce to the Coleman-Mandula theorem.

Theorem 2.2.3.4 *(Coleman-Mandula) For every relativistic quantum field theory satisfying*

(1) *For any M, there're only finite types of particles with mass smaller than M*;

(2) *Any two-particle state undergoes some reaction at almost all energies*;

(2) *The amplitudes for elastic two-body scattering are analytic function at almost energies and angles*,

the Lie algebra of the S-matrix symmetry group is the direct sum of the Poincaré algebra and some semi-simple Lie algebra of the internal symmetry.

For the proof, see page 12., the third volume of Weinberg's Quantum Theory of Fields [8].

In order to find some loophole in theorem 2.2.3.4, the supersymmetry considers extra generators, called supercharges, as some spin representation elements rather than ordinary ones.

This section is partly based on [9].

Chapter 3

Projective Representations and Central Extensions

In quantum systems, all the states are in fact defined up to a global phase, since the probabilities and expectation values are independent of this global phase.

As a result, in quantum mechanics if one want to discuss how a group can act on a Hilbert space, that is, the representation of the group, then one cannot just discuss the unitart and anti-unitary representations from Winger's theorem, and should discuss those such representations defined up to a global phase, which are called the projective representations.

3.1 Projective Representations and Multipliers

A Lie group representation of a Lie Group G is a Lie group homomorphism $\phi : G \to GL(V)$ from G to the general linear group on a (complex) vector space V. The dimension of the representation is defined as the dimension of the vector space V. If the vector space is further equipped with an inner porduct, then ϕ is called a unitary representation if the image of ϕ are the unitary subgroup of $GL(V)$.

Similarly, a Lie algebra representation of a Lie algebra $\underline{g}$ is a Lie algebra homomorphism $\phi : \underline{g} \to M(V)$ from $\underline{g}$ to the matrix algebra $M(V)$, which is also the Lie algebra of $GL(V)$, on a (complex) vector space V. The dimension of the representation is also defined as the dimension of the vector space V. If the vector space is further equipped with an inner porduct, then ϕ is called a unitary representation if the image of ϕ are the anti-Hermitian subalgebra of $M(V)$. In physics, we usually multiply $\sqrt{-1}$ for a Lie algebra representation to make it Hermitian.

Since the Lie algebra of a Lie group can be idenitified as the tangent space of a Lie group at its origin, every Lie group homomorphism induces a Lie algebra homomorphism by the differential map. However, the converse may not be correct. In fact, if the underlying Lie group is simply connected, then evergy Lie group homomorphism can be lifted to a Lie group homomorphism. The correpondece between Lie algebras and simply-connected Lie groups are the so-called Lie group-Lie algebra correpondece.

By "up to a global phase" we mean the Lie group structure is preseved up to some phase function $h : G \times G \to U(1)$, that is,

$$\phi(g_1)\phi(g_2) = h(g_1, g_2)\phi(g_1 g_2) \tag{3.1}$$

for all $g_1, g_2 \in G$. Such h can't be arbitrary, since firstly for $g_1 = e$, we have $\phi(g) = h(e, g)\phi(g)$ thus $h(e, g) = 1$ for all $g \in G$. Similarly, for g, g^{-1} we have $\phi(gg^{-1}) = h(g, g^{-1})\phi(g)\phi(g)^{-1}$ thus $h(g, g^{-1})$=1.

Also, associativity implies

$$\begin{aligned}\phi(g_1g_2g_3) &= h(g_1g_2, g_3)\phi(g_1g_2)\phi(g_3) = h(g_1, g_2)h(g_1g_2, g_3)\phi(g_1)\phi(g_2)\phi(g_3) \\ &= h(g_1, g_2g_3)\phi(g_1)\phi(g_2g_3) = h(g_1, g_2g_3)h(g_2, g_3)\phi(g_1)\phi(g_2)\phi(g_3).\end{aligned} \tag{3.2}$$

Hence, it requires

$$h(g_1, g_2)h(g_1g_2, g_3) = h(g_1, g_2g_3)h(g_2, g_3). \tag{3.3}$$

In fact by putting $g_1 = g, g_2 = g^{-1}$ and $g_3 = h$ we have

$$h(g, g^{-1})h(e, h) = h(g, g^{-1}h)h(g^{-1}, h). \tag{3.4}$$

So here we say that a map $h : G \times G \to U(1)$ is a multiplier of G if it satisfies $h(e, g) = h(g, e) = h(g, g^{-1}) = 1, \forall g \in G$ and (3.3).

Let $f : G \to U(1)$ be a function, such phase functions provide trivial multipliers via defining

$$h(g_1, g_2) = \frac{f(g_1)f(g_2)}{f(g_1g_2)}, \tag{3.5}$$

because

$$\begin{aligned}h(g_1, g_2)h(g_1g_2, g_3) &= \Big(\frac{f(g_1)f(g_2)}{f(g_1g_2)}\Big)\Big(\frac{f(g_1g_2)f(g_3)}{f(g_1g_2g_3)}\Big) \\ &= \frac{f(g_1)f(g_2)f(g_3)}{f(g_1g_2g_3)} \\ &= \Big(\frac{f(g_1)f(g_2g_3)}{f(g_1g_2g_3)}\Big)\Big(\frac{f(g_2)f(g_3)}{f(g_2g_3)}\Big) \\ &= h(g_1, g_2g_3)h(g_2, g_3).\end{aligned} \tag{3.6}$$

In fact, for any ordinary representation ϕ, $\pi(g) := f(g)\phi(g)$ defines a trivial projective representation. Every projective representation constructed in this trivial manner is called exact. By this, we say that two projective representations ϕ_1 and ϕ_2 are equivalent if they are related by a phase function $f : G \to U(1)$ s.t. $\phi_2(g)U = f(g)U\phi_2(g)$ for some unitary operator U.

In fact, the multipliers of a Lie group forms an Abelian group $M(G)$. The idenity of this group is the constant 1-map, denoted by m_1, over $G\times G$, which is clearly a multiplier. And the binary action over M(G) is defined as follows: if h_1 and h_2 are two multipliers, then $h_1 \cdot h_2(g, h) = h_1(g, h)h_2(g, h)$ via pointwise mupltiplication over $G \times G$ satisfies

$$h_1 \cdot h_2(e, g) = h_1(e, g)h_2(e, g) = 1 \cdot 1 = 1, \tag{3.7}$$

and

$$\begin{aligned}h_1 \cdot h_2(g_1g_2, g_3) &= h_1(g_1g_2, g_3)h_2(g_1g_2, g_3) \\ &= h_1(g_1, g_2g_3)h_2(g_1, g_2g_3) \\ &= h_1 \cdot h_2(g_1, g_2g_3)\end{aligned} \tag{3.8}$$

Thus the multiplication is closed in M(G). The complex conjugation of a multiplier is its inverse since their product is the map m_1 over $G \times G$. As a result, $M(G)$ does form an Abelian group. Furthermore, the exact multipliers, denoted by $M_0(G)$, forms a subgroup of $M(G)$. Since the all of them are Abelian groups, the quotient group $M(G)/M_0(G)$ is available and characterizes the nontrivial multipliers, i.e. the nontrivial projective representations.

Here we provide an example. If G is compact, then we have a bi-invariant (i.e. Haar) measure on G, which defines the Hilbert space of squared-integrable functions $L^2(G)$. For each $m \in M(G)$, $\phi : G \to U(L^2(G))$ defined by

$$\Big(\phi(g)f\Big)(h) := \overline{m(h^{-1}, g)}f(g^{-1}h) \tag{3.9}$$

gives a projective representation on $L^2(G)$.

3.2 Central Extensions

The projective representations can in fact be viewed as some product between the Lie group G and $U(1)$. For any multiplier h, this is realized by defining

$$(g_1, a) \cdot (g_2, b) = (g_1 g_2, \overline{h(g_1, g_2)} ab), \tag{3.10}$$

then we have $H = G \times_h U(1)$ is some extension of G with effect of the projective representation. Also, the number of projective representations are characterized by the number of such extensions. And this point of view provides some method for us to analyze the projective representations since we understand Lie groups more.

The above discussion can in fact be generalized from $U(1)$ to any Abelian group with a G-action i.e. a G-module. Since the extended part are central elements of the extended Lie group, the extended Lie groups are so-called the central extension of G.

A Central Extension of a connected Lie group G by an Abelian Lie group T is a short exact sequece

$$1 \to T \to H \to G \to 1 \tag{3.11}$$

for some connected Lie group H. Similarly, A Central Extension of a Lie algebra $\underline{g}$ by an Abelian Lie algebra $\underline{t}$ is a short exact sequece

$$0 \to \underline{t} \to \underline{h} \to \underline{g} \to 0 \tag{3.12}$$

for some Lie algebra $\underline{h}$. Clearly, Lie group central extensions induce Lie algebra ones, in this case, we call the induced Lie algebra central extension an associated central extension.

Here one would naturally wonder whether all central extensions of Lie algebra can be lifted to central extensions of simply-connected Lie groups. It turns out that this is correct, but we firstly require the below lemma.

Lemma 3.2.0.1 *(Malčev) Let K be a closed connected normal subgroup of H and $r : H \to H/K$ be the quotient map, then there's a smooth map $c : H/K \to H$ s.t. $r(c(t)) = t$ for all t in H/K. Consequently, $\xi : K \times H/K \to H$ defined by $(k, t) \longmapsto c(t)k$ is a homeomorphism.*

For the proof, see [10]. In fact, this note largely refers to this book.

Theorem 3.2.0.1 *Let G and T be connected Lie groups with $\underline{g}$ and $\underline{t}$ being their Lie algebra respectively. If G is simply connected and T is Abelian, then there is a 1-1 correpondece between the equivalence class of central extensions of G by T and the equivalence class of central extensions of $\underline{g}$ by $\underline{t}$, which sends each central extension to exactly the associated ones.*

⟨ Proof of theorem 3.2.0.1 ⟩

We firstly consider the case where T is also simply connected and the central extension

$$0 \to \underline{t} \xrightarrow{\alpha} \underline{h} \xrightarrow{\beta} \underline{g} \to 0. \tag{3.13}$$

By Lie's third theorem, there's a simply connected and connected Lie group H correponding to $\underline{h}$. Since T is simply connected, there's a homomorphism $T \xrightarrow{i} H$ s.t. $i_* = \alpha$ by the universal covering property. Similarly, we have a homomorphism $H \xrightarrow{j} G$ s.t. $j_* = \beta$.

As a result, $i(T)$ is a Lie subgroup of H with Lie algebra $\alpha(\underline{t})$ as an ideal of $\underline{h}$. By [11], $i(T)$ is a closed Lie subgroup of H. Since H is simply connected, then by the previous lemma, so is $i(T)$. This proves i is an isomorphism. Also, $i(T)$ must be the connected component containing the identity of $ker(j)$. If $i(T) \neq ker(j)$m then $H/ker(j)$ is a covering space of G, which is impossible. Hence, $i(T) = ker(j)$ and

$$1 \to T \xrightarrow{i} H \xrightarrow{j} G \to 1 \tag{3.14}$$

is an exact sequence.

Under this construction, two simply connected Lie group central extensions are equvalent if and only if so are their associated ones.

Now, if T is not simply connected, let $\tilde{T}$ be its universal covering group with covering map ε. We may then consider a central extension

$$1 \to \tilde{T} \xrightarrow{\tilde{i}} \tilde{H} \xrightarrow{\tilde{j}} G \to 1 \tag{3.15}$$

of G by $\tilde{T}$, then $\tilde{H}$ must be simply connected. Consider $H = \tilde{H}/\tilde{i}(ker(\varepsilon))$ and δ be the quotient map $\tilde{H} \to H$, which has a discrete kernel. Since $ker\delta \subseteq \tilde{i}(\tilde{T})$, $\exists!$ homomorphism $j : H \to G$ s.t. $\tilde{j} = j \circ \delta$ (universal lifting property). Since $ker\delta = \tilde{i}(ker\varepsilon)$, $\exists!$ homomorphism $i : T \to H$ s.t. $\delta \circ \tilde{i} = i \circ \varepsilon$ and i is injective. It's then clear that $ker(j) = i(T)$, thus we have the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \tilde{T} \arrow[r, "\tilde{i}"] \arrow[d, "\varepsilon"] & \tilde{H} \arrow[r, "\tilde{j}"] \arrow[d, "\delta"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & T \arrow[r, "i"] & H \arrow[r, "j"] & G \arrow[r] & 1
\end{tikzcd} \tag{3.16}$$

In particular, we have the central extension $1 \to T \xrightarrow{i} H \xrightarrow{j} G \to 1$. Conversely if we have such a central extension of G by T, let $\tilde{T}$ and $\tilde{H}$ be the universal covering group of T and H with covering map ε and δ respectively, then $\exists!$ homomorphism $\tilde{i} : \tilde{T} \to \tilde{H}$ s.t. $\delta \circ \tilde{i} = i \circ \varepsilon$ by the universal lifting property. Write $\tilde{j} = j \circ \delta$, since G is simply connected, $ker(\tilde{j})$ is simply connected by the previous lemma. It then follows that $ker(\tilde{j}) = \tilde{i}(\tilde{T})$, hence, $\tilde{i}(\tilde{T})$ is simply connected.

However, since $\tilde{T}$ is also simply connected, $\tilde{i}$ must be injective. As a result,

$$1 \to \tilde{T} \xrightarrow{\tilde{i}} \tilde{H} \xrightarrow{\tilde{j}} G \to 1 \tag{3.17}$$

is a central extension of G by $\tilde{T}$. Moreover, we again have the previous commutative diagram and $ker\delta = \tilde{i}(ker\varepsilon)$. Thus there's a natural correpondece betweeb central extension of G by T and $\tilde{T}$. Diagram chasing shows that this correpondece respects equivalence, thus the theorem is proved.

With this theorem, understanding central extensions of a simply connected Lie group is equivalent to undestand the central extension of its Lie algebra. Furthermore, for a generic Lie group, we may study the central extension of its universal cover and see whether the central extension can descend to the original group or not, that is, whether it's just a lift of a central extension.

3.3 Lie Group Cohomology

In order to study the Lie group central extension, the second group cohomology with $U(1)$ coefficient is usually used. However, for the possible future use, here we'll discuss the general Lie group cohomology.

Let M be a G-module, that is, a module with a G-action on it, then for each $r \in \mathbb{N}$, the homogeneous r-cochain of G with coefficient in M is the set $C^r(G; M) := \{\phi : G^{r+1} \to M | \phi$ is conti. and G-equivariant$\}$. By G-equivariance we mean

$$g\phi(g_0, \ldots, g_r) = \phi(gg_0, \ldots, gg_r) \tag{3.18}$$

for any element $g, g_0, \ldots, g_r \in G$.

For each $r \in \mathbb{N}$, we define the boundary map $d^r : C^\cdot(G; M) \to C^{\cdot+1}(G; M)$ by

$$\begin{aligned} d^r\phi(g_1, \ldots, g_{r+1}) := {} & g_1\phi(g_2, \ldots, g_{r+1}) \\ & + \sum_{j=1}^{r} (-1)^j \phi(g_1, \ldots, g_j g_{j+1}, \ldots, g_{r+1}) + (-1)^{r+1}\phi(g_1, \ldots, g_r). \end{aligned} \tag{3.19}$$

An element in $C^r(G;M)$ is called exact if it's in $B^r(G;M) := d^{r-1}C^{r-1}(G;M)$.

An element in $C^r(G;M)$ is called closed if it's in $Z^r(G;M) := \{\phi \in C^r(G;M) | d^r\phi = 0\}$.

Since $d^2 = 0$, the chain complex

$$0 \to C^1(G;M) \xrightarrow{d^1} C^2(G;M) \xrightarrow{d^2} \ldots\ldots \tag{3.20}$$

induces the cohomology groups $H^r(G;M) := Z^r(G;M)/B^r(G;M)$ for each $r \in \mathbb{N} - \{0\}$ called the r-th group cohomology of G with coefficient in M.

Let's consider some examples.

If $M = U(1)$ and G acts trivially on M, we identify the product on $U(1)$ and the summation as an Abelian group, then $\phi \in Z^2(G;M)$ satisfies

$$1 = \frac{\phi(g_2, g_3)\phi(g_1, g_2g_3)}{\phi(g_1g_2, g_3)\phi(g_1, g_2)}. \tag{3.21}$$

This is exactly the condition for ϕ to be a multiplier. On the other hand, if $\phi \in B^2(G;M)$, then there some $f \in C^1 G; M$ such that

$$\phi(g_1, g_2) = d^1 f(g_1, g_2) = \frac{f(g_2)f(g_1)}{f(g_1g_2)}. \tag{3.22}$$

This is exactly the condition for ϕ to be an exact multiplier. As a result, we have

$$M(G)/M_0(G) = Z^2(G;U(1))/B^2(G;U(1)) = H^2(G;U(1)). \tag{3.23}$$

For generic G-module M, $\phi \in Z^1(G;M)$ implies $g_1\phi(g_2) = \phi(g_1g_2)$, thus $\phi \in Hom_G(G, M)$ the G-invariant homomorphism from G on to M.

The group cohomology in fact characterizes the central extensions, to see this consider the central extension of G by M

$$1 \to M \to E \xrightarrow{\pi} G \to 1. \tag{3.24}$$

Pick a section $S : G \to E$, we then set $g \cdot m = s(g)ms(g)^{-1}$ for all $g \in G$ and $m \in M$. Since M is Abelian, this defines a G-action on M for $\pi(s(g)ms(g)^{-1}) = \pi(m) = e \in G$. For $s(g_1)s(g_2)$ and $s(g_1g_2)$ both are mapped to g_1g_2 under π, they differ only by an element $\phi(g_1, g_2) \in M$. That is,

$$s(g_1g_2) = \phi(g_1, g_2)s(g_1)s(g_2). \tag{3.25}$$

The associativity in E gives

$$\begin{aligned} s(g_1g_2g_3) &= \Big(\phi(g_1g_2, g_3) + \phi(g_1, g_2)\Big)s(g_1)s(g_2)s(g_3) \\ &= \Big(\phi(g_1, g_2g_3) + s(g_1)\phi(g_2, g_3)s(g_1)^{-1}\Big)s(g_1)s(g_2)s(g_3) \end{aligned} \tag{3.26}$$

Therefore, ϕ shall satisfies

$$\phi(g_1g_2, g_3) + \phi(g_1, g_2) - \phi(g_1, g_2g_3) + g_1 \cdot \phi(g_2, g_3) = 0, \tag{3.27}$$

which is equivalent to that $\phi \in Z^2(G;M)$. A change in the choice of section is just an G-invariant homomorphism, so the class of $[\phi] \in H^2(G;M)$ is independent of the choice of a section. In this manner, $H^2(G;M)$ classifies the isomorphism classes of central extension of G by M with the adjoint action on M.

A short exact sequence of G-modules $0 \to M' \to M \to M'' \to 0$ gives a long exact sequence

$$0 \to H^0(G;M') \to H^0(G;M) \to H^0(G;M'') \to H^1(G;M') \to \ldots, \tag{3.28}$$

as a morphism of short exact sequence induces a morphism of long exact sequence.

Theorem 3.3.0.1 *Let G be a connected Lie group and M be a G-module, then the map $\chi \in Hom(\pi_1(G), M) \mapsto [m_\chi] \in H^2(G; M)$ define by fixing a section $s : G \to \tilde{G}$ to the universal covering group of G and $m_\chi(g,h) := \chi(s(h)^{-1}s(g)^{-1}s(gh))$ is a homomorphism, and this homomorphism is independent of the choice of the section s. Furthermore, this homomorphism is injective if $Hom(\tilde{G}, M)$ is trivial and surjective if $H^2(\tilde{G}; M)$ is trivial.*

If we have $Hom(\tilde{G}, M) = 0$ and $H^2(\tilde{G}; M) = 0$, then $H^2(G; M)$ and $Hom(\pi(G), M)$ is isomorphic. If M is a Lie group with Lie algebra $\underline{(m)}$, then $Hom(\tilde{G}, M) = Hom(\underline{(g)}, \underline{(m)})$. In this case that G is semi-simple, consequently, so is $\tilde{G}$, $Hom(\underline{g}, \underline{m})$ must be trivial and the above homomorphism is surjective. Also, $H^2(G; M) = 0 \iff$ all the central extension of $\underline{g}$ by $\underline{m}$ is exact. As a result, it'll suffice to study $Hom(\underline{g}, \underline{m})$ and "$H^2(\underline{g}, \underline{m})$".

In fact, $H^2(\underline{g}; \mathbb{R}) = 0$ if $\underline{g}$ is semi-simple as we'll prove in the later section. So for $G = SO(3)$, $\pi_1(G) = \mathbb{Z}_2$, then $H^2(G; M) = Hom(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. As a result, the nontrivial projective representation of $SO(3)$ concerns a "minus sign". They are the half-integer or Fermion part "representations" of $SO(3)$.

$\langle$ Proof of thm 3.3.0.1 $\rangle$

Since $\pi(s(h^{-1})s(g^{-1})s(gh)) = h^{-1}g^{-1}gh = e \in G$, we have $s(h^{-1})s(g^{-1})s(gh) \in \pi_1(G)$ and m_χ is well-defined. Observe that $\pi(g^{-1} \circ s \circ \pi(g)) = \pi(g^{-1})\pi(g) = e$, we can define $f_\chi : \tilde{G} \to M$ by $f_\chi(g) = \chi(g^{-1} \circ s \circ \pi(g))$. Then for $g, h \in G$,

$$\begin{aligned} m_\chi(\pi(g), \pi(h)) &= \chi(s(\pi(h))^{-1}s(\pi(g))^{-1}s(\pi(gh))) \\ &= \chi(s(\pi(h))^{-1}hh^{-1}s(\pi(g))^{-1}gg^{-1}gh(gh)^{-1}s(\pi(gh))) \\ &= \chi((h^{-1}s(\pi(h)))^{-1}h^{-1}(g^{-1}s(\pi(g)))^{-1}g^{-1}h((gh)^{-1}s(\pi(gh)))) \\ &= f_\chi(gh) - f_\chi(g) - f_\chi(h) \end{aligned} \tag{3.29}$$

Hence, m_χ can be lifted as an exact multiplier of $\tilde{G}$ and it's really a multiplier. This proves $\chi \mapsto m_\chi$ is a homomorphism.

Let m_χ^* be the multiplier on G obtained by choosing another section $t : G \to \tilde{G}$. Since both s and t are sections of π, there is a $u : G \to \pi_1(G)$ s.t. $t(g) = u(g)s(g)$ for all $g \in G$. Since

$$\begin{aligned} \chi(t(h)^{-1}t(g)^{-1}t(gh)) &= \chi(s(h)^{-1}u(h)^{-1}s(g)^{-1}u(g)^{-1}u(gh)s(gh)) \\ &= \frac{\chi(u(gh))}{\chi(u(g))\chi(u(h))} m_\chi(g,h), \end{aligned} \tag{3.30}$$

we have $[m_\chi^*] = [m_\chi]$.

To prove the injectivity with $Hom(\pi_1(G), M)$ trivial, let $m_\chi \in B^2(G; M)$. That is, $\exists f : G \to M$ s.t. $m_\chi(gh) = f(gh) - f(g) - f(h)$, for all $g, h \in G$. From this, we deduce that

$$(f \circ \pi - f_\chi)(gh) = (f \circ \pi - f_\chi)(g) + (f \circ \pi - f_\chi)(h). \tag{3.31}$$

for all $g, h \in \tilde{G}$. However, since $Hom(\pi_1(G), M)$ trivial, $f \circ \pi \equiv f_\chi$. But $f \circ \pi$ is a constant on $ker(\pi)$ and $f_\chi(z) = \chi(z)^{-1}$ for all $z \in ker(\pi)$, we have χ is also a constant on $ker(\pi)$, thus a trivial character of $ker(\pi)$. As a result, $[m_\chi] = 0$ implies χ is trivial.

On the other hand, to see the surjectivity from $H^2(\tilde{G}, M) = 0$, let $m \in Z^2(G; M)$ and define $\tilde{m} : \tilde{G} \times \tilde{G} \to M$ by $\tilde{m}(g,h) = m(\pi(g), \pi(h))$, then $\tilde{m} \in Z^2(G; M) = B^2(G; M)$ since by assumption $H^2(\tilde{G}, M) = 0$. Thus there's some $f : \tilde{G} \to M$ s.t. $m(\pi(g), \pi(h)) = \frac{f(gh)}{f(g)f(h)}$ for all $g, h \in \tilde{G}$.

This shows that $f|_{\pi_1(G)}$ is a character of $ker(\pi)$, say χ^{-1}. Hence, $f(gh) = f(g)\chi(h)^{-1}$ for all $x \in \tilde{G}$ and $h \in ker(\pi)$. Define $g : G \to M$ by $g = f \circ s$ by some choice of section s, then

$f(s(h)z) = g(h)\chi(z)^{-1}$ for all $h \in G$ and $z \in ker(\pi)$. As a result,

$$\begin{aligned} m(u,v) &= m(\pi \circ s(u), \pi \circ s(v)) \\ &= \frac{f(s(u)s(v))}{f(s(u))f(s(v))} \\ &= \frac{f(s(uv)s(uv)^{-1}s(u)s(v))}{f(s(u))f(s(v))} \\ &= \chi^{-1}(s(uv)^{-1}s(u)s(v))\frac{g(uv)}{g(u)g(v)} \\ &= \frac{g(uv)}{g(u)g(v)} m_\chi(u,v) \end{aligned} \tag{3.32}$$

This shows $[m] = [m_\chi]$ and the surjectivity.

From the above proof, we can then define the following concepts.

Let $\tilde{G}$ and π be as in the previous theorem and β be an orifinary unitary representation of $\tilde{G}$. We say that β is of pure type if there's a character χ of $\ker \pi$ s.t. $\beta(z) = \chi(z)\mathbb{1}$ for all $z \in ker(\pi)$. We may also say β is pure of type χ. Notice that β is irreducible implies β is of pure type by the Schur's lemma.

3.4 The Universal Enveloping Algebra and Casimir Operators

Since there's Lie bracket but no multiplication on Lie algebras since Lie algebras are defined as the Tangent space of a Lie group at the origin, or equivalently, the space of left-invariant vector fields, if we want to define operator like "$\mathbf{S}^2$" which is proportional to the identity or in the "center" of the Lie algebra ("$= S(S+1)\mathbb{1}$") for each irreducible representation of the Lie algebra $so(3)$, we have to formally define the multiplication.

Denote $\otimes V = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ for any vector space V.

The universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is $u(\mathfrak{g}) = \otimes\mathfrak{g}/\{X \otimes Y - Y \otimes X - [X,Y]\}$, defined by quotienting the tensor algebra by a two-sided ideal.

Theorem 3.4.0.1 *Suppose A is a associated algebra with identity and $j : \mathfrak{g} \to A$ is a linear map s.t. $j([X,Y]) = j(X)j(Y) - j(Y)j(X)$ for all $X, Y \in \mathfrak{g}$, then $\exists!$ algebra homomorphism $\phi : u(\mathfrak{g}) \to A$ s.t. $\phi(0) = 0$ and $\phi \circ i \equiv j$, where i in the inclusion $\mathfrak{g} \hookrightarrow u(\mathfrak{g})$.*

This is proved simply by the universal property of tensor product.

Since such a lift exists, every representation of Lie algebra can be viewed as a $u(\mathfrak{g})$-module by lifting the representation to $u(\mathfrak{g})$. In the case that $\mathfrak{g}$ is Abelian, i.e. $[\mathfrak{g}, \mathfrak{g}] = 0$, the two-sided ideal is generated by simply $X \otimes Y - Y \otimes X$, hence, $u(\mathfrak{g}) = Sym(\mathfrak{g})$, the symmetry algebra generated by $\mathfrak{g}$. An immediate question is whether we generalize this identification.

For any Lie algebra $\mathfrak{g}$, we write $u_n(\mathfrak{g})$ to be the subspace of $u(\mathfrak{g})$ generated by tensor product of at most n elements of $\mathfrak{g}$, thus, we have $u_0(\mathfrak{g}) \subset u_1(\mathfrak{g}) \subset u_3(\mathfrak{g}) \subset \ldots\ldots$ and $u_n(\mathfrak{g})u_m(\mathfrak{g}) \subset u_{n+m}(\mathfrak{g})$. We then can define $gr_n(\mathfrak{g}) = u_n(\mathfrak{g})/u_{n-1}(\mathfrak{g})$ and $gru(\mathfrak{g}) = \oplus_{n=0}^{\infty} gr_n u(\mathfrak{g})$ with the multiplication $gr_n u(\mathfrak{g}) \times gr_m u(\mathfrak{g}) \to gr_{n+m} u(\mathfrak{g})$ induced from $u(\mathfrak{g})$.

Theorem 3.4.0.2 *$gru(\mathfrak{g})$ is a commutative algebra, thus there's a unique algebra homomorphism $Sym(\mathfrak{g}) \to gru(\mathfrak{g})$.*

By the quotient $u_n(\mathfrak{g}) \to gru(\mathfrak{g})$, it can be readily shown.

Theorem 3.4.0.3 *(Poincaré-Birkhoff-Witt) The isomorphism $Sym(\mathfrak{g}) \to gru(\mathfrak{g})$ is in fact an isomorphism.*

Lemma 3.4.0.1 *The center $Z(u(\mathfrak{g}))$ of the universal enveloping algebra is equal to the centralizer $C_{u(\mathfrak{g})}(\mathfrak{g})$ of the (cannonically embedded) Lie algebra $\mathfrak{g}$.*

⟨ Proof of lemma 3.4.0.1 ⟩

The ($\Rightarrow$) side is trivial. For ($\Leftarrow$), the Poincaré-Birkhoff-Witt theorem implies that any element commuting with a basis of $\mathfrak{g}$ commutes with any basis of $u(\mathfrak{g})$.

Theorem 3.4.0.4 *(Gol'fand) Let $\{T_i\}$ be a basis of a finite generated Lie algebra $\mathfrak{g}$ with Lie bracket $[T_i, T_j] = f_{ij}^k T_k$. The Weyl-ordered polynomial given by*

$$P_W(T) = \sum \Pi^{i_1 \ldots i_n} T_{i_1} \ldots T_{i_n} \tag{3.33}$$

(with completely symmetric contravariant components) is in $Z(u(\mathfrak{g})) \iff$

$$f_{kl}^{i_1} \Pi^{l i_2 \ldots i_n} + f_{kl}^{i_2} \Pi^{i_1 l i_3 \ldots i_n} + \cdots + f_{kl}^{i_n} \Pi^{i_1 \ldots i_{n-1} l} = 0. \tag{3.34}$$

As a result, a Weyl-ordered polynomial polynomial $P_W(T) \in Z(u(\mathfrak{g}))$ iff it corresponding symmetric tensor Π is invariant under the adjoint action.

⟨ Proof of theorem 3.4.0.4 ⟩

Simply calculate $[P_W(T), T_i]$ for all i.

The Casimir operators of a finite-dimensional Lie algebra $\mathfrak{g}$ are a distinguished basis of the center $Z(u(\mathfrak{g}))$ of the universal enveloping algebra made of the homogeneous polynomials in $\{T_i\}$. The degree of the homogeneous polynomial defining the corresponding Casimir operator is called the order of the Casimir operator.

For example, since the Killing form $K(X,Y) := Tr(ad_X ad_Y)$ of a semi-simple Lie algebra is non-degenerate, its inverse defines the quadratic Casimir operator $C_2 = \sum_{i,j} (K(T^a, T^b))_{ij}^{-1} T^i T^j$.

Theorem 3.4.0.5 *Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra of rank r over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then $Z(u(\mathfrak{g})) \cong \mathbb{F}[C^i]$, the polynomial algebra over $\mathbb{F}$ in r-variables $\{C^i\}$. As a result, the number of algebraically independent Casimir operators is equal to $rank(\mathfrak{g})$.*

(The proof is omitted.)

Let $f : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$, then it can be uniquely lifted tp a representation of $u(\mathfrak{g})$, say $\tilde{f}$. If additioanlly f is faithful i.e. injective, then we define the Casimir endomorphism of f to be $C_f = \tilde{f}(C_2) = \sum_{i,j} K_{ij} f(T^i) f(T^j)$, where K_{ij} is the matrix of the Killing form in the same basis $\{T^i\}$ s.t. $Tr(ad_{T^i} ad_{T^j}) = \delta_{ij}$.

Lemma 3.4.0.2 *$C_f \neq 0$, and if f is irreducible, then $C$is non-singular.*

⟨ Proof of 3.4.0.2 ⟩

$TrC_f = \sum_{i,j} K_{ij} Tr(f(T^i) f(T^j)) = \delta_i^i = rank(\mathfrak{g}) > 0$, so C_f cannot be zero.

Since f is irreducible and $C_f \in End_{\mathfrak{g}}(V)$, by Schur's lemma, C_f must be an automorphism.

3.5 Lie Algebra Cohomology

Let $\mathfrak{g}$ be a Lie algebra and V be a vector field both over the same $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $f : \mathfrak{g} \to End(V) = \mathfrak{gl}(V)$ be a $\mathfrak{g}$-action on V, that is, a representation of $\mathfrak{g}$. A r-linear map $\phi : \mathfrak{g}^r \to V$ is called a V-cochain of dimension r of $\mathfrak{g}$ if ϕ is alternating i.e. $\phi(X_1, \ldots, X_r) = 0$ if $X_i = X_j$ for some $1 \leq i < j \leq r$. The $\mathbb{F}$-vector space of all V-cochains of dimension r of $\mathfrak{g}$ with representation f is denoted by $C^r(\mathfrak{g}; V, f)$, for all $r = 1, \ldots\ldots$, and we conventionally let $C^0(\mathfrak{g}; V, f) = V$ with the trivial action.

For each $r \in \mathbb{N}$ and $X \in \mathfrak{g}$, we define $d^r : C^r(\mathfrak{g}; V, f) \to C^{r+1}(\mathfrak{g}; V, f)$ by $d\phi(X) = f(X)\phi$ for $r = 0$, and

$$\begin{aligned} d\phi(X_1, \ldots, X_{r+1}) = &\sum_{j=1}^{r+1} (-1)^{j+1} f(X_i)\phi(X_1, \ldots, \hat{X}_j, \ldots, X_{r+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \phi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{r+1}). \end{aligned} \tag{3.35}$$

For each $r \in \mathbb{N}$ and $X \in \mathfrak{g}$, we define $L_X \in End(C^r(\mathfrak{g}; V, f))$ by $L_X\phi = f(X)\phi$ for $r = 0$, and

$$\begin{aligned} L_X\phi(X_1, \ldots, X_r) = &f(X)\phi(X_1, \ldots, X_r) \\ &- \sum_{j=1}^{r} \phi(X_1, \ldots, X_{j-1}, [X, X_j], X_{j+1}, \ldots, X_r) \end{aligned} \tag{3.36}$$

For each $r \in \mathbb{N}$ and $X \in \mathfrak{g}$, we define $\iota_X : C^r(\mathfrak{g}; V, f) \to C^{r-1}(\mathfrak{g}; V, f)$ by $\iota_X\phi = 0$ for $r = 0$, and

$$\iota_X\phi(X_1, \ldots, X_{r-1}) = \phi(X, X_1, \ldots, X_{r-1}). \tag{3.37}$$

Theorem 3.5.0.1 *For each $r \in \mathbb{N}$ and $X, Y \in \mathfrak{g}$, we have*

$$\begin{aligned} &1.\ L_X = \iota_X d + d\iota_X, \\ &2.\ \iota_[X, Y] = L_X\iota_Y - L_Y\iota_X, \\ &3.\ L_[X, Y] = L_X L_Y - L_Y L_X, \\ &4.\ dL_X = L_X d,\ \textit{thus,} \\ &5.\ d^2 = 0. \end{aligned} \tag{3.38}$$

One would notice that the boundary map d in Lie algebra cohomology is consistent with that in the Lie group cohomology by taking differential, but one shall ve aware that they're both different from the singular cohomology or the de Rham cohomology on a Lie group, although we can define similar operation on Lie algebra cohomology and the de Rham cohomology as above.

3.5.1 The Second Lie Algebra Cohomology and Central Extensions

Now we'd like to discuss the relation between Lie algebra cohomology and the central extensions of a Lie algebra. If we still want to use the former consideration of using conjugate action by G on T to get a central extension, then we shall consider the following definition.

Let

$$0 \to V \hookrightarrow E \xrightarrow{\pi} \mathfrak{g} \to 0 \tag{3.39}$$

be a central extension of $\mathfrak{g}$ by an Abelian Lie algebra V. By picking up a section $s : \mathfrak{g} \to E$, we define $f_V : \mathfrak{g} \to End(V)$ by $f_V(X)v = [s(X), v]$ for all $X \in \mathfrak{g}$ and $v \in V$.

Proposition 3.5.1.1 *f_V is independent of the choice of section, and f_V is a representation of V, which makes V a $u(\mathfrak{g})$-module.*

⟨ The proof of proposition 3.5.1.1 ⟩

We first prove the independence.

Let s, s' be two sections from $\mathfrak{g}$ to E. Since $\pi \circ (s - s') = 0$, we have $s - s' : \mathfrak{g} \to V = ker(\pi)$. Hence, $[s(X) - s(Y), v] = 0$ for all $X, Y \in \mathfrak{g}$ and $v \in V$. This proves f_V defined by different sections give the same endomorphism on V, as in the Lie group case.

To prove it's a representation, consider $X, Y \in \mathfrak{g}$, $v \in V$, and

$$\begin{aligned}(f_V(X)f_V(Y) - f_V(Y)f_V(X))v &= [s(X), [s(Y), v]] - [s(Y), [s(X), v]] \\ &= [[s(X), s(Y)], v].\end{aligned} \tag{3.40}$$

On the other hand, since $\pi([s(X), s(Y)] - s([X, Y])) = 0$, we have $[s(X), s(Y)] - s([X, Y]) \in V$, this makes

$$\begin{aligned}[[s(X), s(Y)], v] &= [s([X, Y], v] \\ &= f_V([X, Y])v\end{aligned} \tag{3.41}$$

Proposition 3.5.1.2 *Follow the settings in the previous proposition, and define $g_s : \mathfrak{g} \times \mathfrak{g} \to V$ by $g_s(X, Y) = [s(X), s(Y)] - s([X, Y])$, then $g_s \in Z^2(\mathfrak{g}; V, f_V)$.*

⟨ The proof of proposition 3.5.1.2 ⟩ Direct calculation gives

$$\begin{aligned}dg_s(X, Y, Z) = &f_V(X)g_s(Y, Z) + f_V(Y)g_s(Z, X) + f_V(Z)g_s(X, Y) \\ &- g_s([X, Y], Z] - g_s([Y, Z], X) - g_s([Z, X], Y) \\ = &[s(X), [s(Y), s(Z)]] - [s(X), s([Y, Z])] \\ &+ [s(Y), [s(Z), s(X)]] - [s(Y), s([Z, X])] \\ &+ [s(Z), [s(X), s(Y)]] - [s(Z), s([X, Y])] \\ &- [s([X, Y]), s(Z)] + s([[X, Y], Z]) \\ &- [s([Y, Z]), s(X)] + s([[Y, Z], X]) \\ &- [s([Z, X]), s(Y)] + s([[Z, X], Y]) \\ = &[s(X), [s(X), s(Z)]] + [s(Y), [s(Z), s(X)]] + [s(Z), [s(X), s(Y)]] \\ &- [s([X, Y]), s(Z)] - [s([Y, Z]), s(X)] - [s([Z, X]), s(Y)] \\ = &0\end{aligned} \tag{3.42}$$

by Bianchi identity.

Proposition 3.5.1.3 *Follow the settings in the previous proposition, and let s_1, s_2 be two sections, then $g_{s_1} - g_{s_2} \in B^2(\mathfrak{g}; V, f_V)$.*

From this proposition, we then prove the previous claim: different central extensions of $\mathfrak{g}$ by V are classified by $H^2(\mathfrak{g}; V, f_V)$.

⟨ The proof of proposition 3.5.1.3 ⟩

Let $h = s_1 - s_2$, since $\pi \circ h = 0$, $h \in C^1(\mathfrak{g}; V, f_V)$, and

$$\begin{aligned}g_{s_1}(X, Y) &= [s_2(X) + h(X), s_2(Y) + h(Y)] - s_2([X, Y]) - h([X, Y]) \\ &= [s_2(X), s_2(Y)] - s_2([X, Y]) + [s_2(X), h(Y)] + [h(X), s_2(Y)] - h([X, Y]) \\ &= g_{s_2}(X, Y) - f_V(Y)h(X) + f_V(X)h(Y) - h([X, Y]) \\ &= g_{s_2}(X, Y) + dh(X, Y)\end{aligned} \tag{3.43}$$

From these propositions, we then arrive at the following definition.

A central extension

$$0 \to V \hookrightarrow E \xrightarrow{\pi} \mathfrak{g} \to 0 \tag{3.44}$$

is said belonging to $\{\mathfrak{g}, V\}$ if V is an Abelian ideal of E and V is a $u(\mathfrak{g})$-module acts by f_V defined previously. Two central extensions

$$0 \to V \hookrightarrow E_1 \xrightarrow{\pi_1} \mathfrak{g} \to 0 \tag{3.45}$$

and

$$0 \to V \hookrightarrow E_2 \xrightarrow{\pi_2} \mathfrak{g} \to 0 \tag{3.46}$$

belonging to $\{\mathfrak{g}, V\}$ are said to be equivalent if there is an isomorphism $\theta : E_1 \to E_2$ s.t. $\theta|_V = id_V$ and $\pi_2 = \pi_1 \circ \theta$. Furthermore, we denote the set of equivalent classes of central extensions belonging to $\{\mathfrak{g}, V\}$ by $|\mathfrak{g}, V|$.

Lemma 3.5.1.1 *For every $g \in Z^2(\mathfrak{g}; V, f_V)$, there's a central extension*

$$0 \to V \hookrightarrow E \xrightarrow{\pi} \mathfrak{g} \to 0 \tag{3.47}$$

s.t. $g = g_s$ for some section $s : \mathfrak{g} \to E$.

⟨ The proof of lemma 3.5.1.1 ⟩

On $E = \mathfrak{g} \times V$, define a bracket on E by

$$[(X_1, v_1), (X_2, v_2)] = ([X_1, X_2], f_V(X_1)v_2 - f_V(X_2)v_1 + g(X_1, X_2)). \tag{3.48}$$

This makes E a Lie algebra with Abelian ideal V.

By defining the section $s : \mathfrak{g} \to E$ by $s(X) = (X, 0)$, we have

$$[s(X), s(Y)] = ([X, Y], g(X, Y)) = s([X, Y]) + g(X, Y) \tag{3.49}$$

and prove the lemma.

Theorem 3.5.1.1 *There's a 1-1 correspondence between $|\mathfrak{g}, V|$ and $H^2(\mathfrak{g}; V, f_V)$.*

⟨ The proof of theorem 3.5.1.1 ⟩

Pick a presentative

$$0 \to V \hookrightarrow E \xrightarrow{\pi} \mathfrak{g} \to 0 \tag{3.50}$$

of an element in $|\mathfrak{g}, V|$, and a section $s : \mathfrak{g} \to E$.

Then we map this element of $|\mathfrak{g}, V|$ to $g_s \in Z^2(\mathfrak{g}; V, f_V)$. Another central extension corresponds to this element in $|\mathfrak{g}, V|$ has a difference in $B^2(\mathfrak{g}; V, f_V)$ by proposition 3.5.1.3, hence, this map is well-defined and in fact onto by lemma 3.5.1.1.

To prove the injectivity, let g_1, g_2 be correspondent to sections s_1, s_2 of central extension

$$0 \to V \hookrightarrow E_i \xrightarrow{\pi_i} \mathfrak{g} \to 0 \tag{3.51}$$

for $i = 1, 2$, with $g_1 - g_2$ being an element in $B^2(\mathfrak{g}; V, f_V)$, say $g_1 - g_2 = dh$ for some $h \in C^1(\mathfrak{g}; V, f_V)$.

It suffices to prove these two central extensions are equivalent.

Note that $E_i = s_i(\mathfrak{g}) \oplus V$. Define $\theta : E_1 \to E_2$ by

$$\theta(s_1(X) + v) := s_2(X) + h(X) + v, \tag{3.52}$$

for all $X \in \mathfrak{g}$ and $v \in V$, then clearly θ is linear and $\theta|_V = id_V$. If $s_2(X) + h(X) + v = 0$, then $s_2(X) = 0$ and $X = \pi_2 \circ s_2(X) = 0$, which implies $v = 0$. This proves θ is injective.

On the other hand, for $X + v \in E_2$, define $X_1 = \pi_2(X)$ and $v_1 = v - h(X_1)$, we have

$$\theta(s_1(X_1) + v_1) = s_2(\pi_2(X)) + h(X_1) + v - h(X_1) = X + v, \tag{3.53}$$

which proves θ is also onto.

Also, simple calculation gives

$$\begin{aligned}\pi_1(s_1(X) + v) &= X \\ &= \pi_2(s_2(X)) \\ &= \pi_2(s_2(X) + h(X) + v) \\ &= \pi_2 \circ \theta(s_2(X) + v),\end{aligned} \tag{3.54}$$

that is, $\pi_1 = \pi_2 \circ \theta$. Then it remains to prove θ is a Lie algebra homomorphism. Let $X, Y \in \mathfrak{g}$ and $u, v \in V$.

$$\begin{aligned}\theta([s_1(X) + u, s_1(Y), v]) &= \theta([s_1(X), s_1(y)] + f_V(X)v - f_V(Y)u) \\ &= \theta(s_1([X,Y]) + g_1(X,Y)) + f_V(X)v - f_V(Y)u \\ &= s_2([X,Y]) + h([X,Y]) + g_1(X,Y) + f_V(X)v - f_V(Y)u \\ &= [s_2(X), s_(Y)] + g_1(X,Y) - g_2(X,Y) \\ &\quad + h([X,Y]) + f_V(X)v - f_V(Y)u\end{aligned} \tag{3.55}$$

But we also have

$$\begin{aligned}g_1(X,Y) - g_2(X,Y) &= dh(X,Y) \\ &= f_V(X)h(Y) - f_V(Y)h(X) - h([X,Y]),\end{aligned} \tag{3.56}$$

therefore,

$$\begin{aligned}\theta([s_1(X) + u, s_1(Y), v]) &= [s_2(X), s_2(Y)] + f_V(X)(h(Y) + v) - f_V(Y)(h(X) + u) \\ &= [s_2(X) + h(X) + u, s_2(Y) + h(Y) + v] \\ &= [\theta(s_1(X) + u), \theta(s_1(Y) + v)].\end{aligned} \tag{3.57}$$

From this correspondence, we call the class in $H^2(\mathfrak{g}; V, f_V)$ representing some central extension the factor cocycle of this central extension. A central extension is said to be splittable if its factor cocycle is a boundary, hence, we can take $g = 0$ in this case, and the section s is then an isomorphism of Lie algebra.

Clearly, $H^2(\mathfrak{g}; V, f_V) = 0 \iff$ all the central extensions are splittable.

In the case where the Lie group G is simply connected, then $H^2(G, M) =$ equivalence classes of central extensions of G by M with adjoint action $=$ equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{m}$ with adjoint action $= H^2(\mathfrak{g}; V, f_V) = 0$.

Every projective representation of G can be lifted to the projective representation on its universal cover $\tilde{G}$ by

$$\phi \in Hom(G, GL(V)) \mapsto \tilde{\phi} := \phi \circ \pi \in Hom(\tilde{G}, GL(V)), \tag{3.58}$$

which is then classified by calculating $H^2(\mathfrak{g}; \mathbb{R} = \mathfrak{u}(1), f_{\mathbb{R}} = id_{\mathbb{R}})$. So the main issue is whether a central extension of $\tilde{G}$ by $U(1)$ can descend to one of G or not.

3.5.2 The First Lie Algebra Cohomology and Complete Reducibility

In this section, we'd like to introduce the usage of the first Lie algebra cohomology.

Let

$$0 \to V \to E \underset{s_1}{\overset{\pi}{\rightleftarrows}} \mathfrak{g} \to 0 \tag{3.59}$$

be a central extension and s_1 be some section. Suppose s_2 is another section $\mathfrak{g} \to E$, then $\pi \circ (s_1 - s_2) = 0$. This implies $\omega := s_1 - s_2 \in Hom(\mathfrak{g}, V) = C^1(\mathfrak{g}; V, f_V)$.

If both s_1 and s_2 respects the Lie algebra bracket, then for all $X, Y \in \mathfrak{g}$,

$$\begin{aligned} s_2([X,Y]) + \omega([X,Y]) &= s_1([X,Y]) \\ &= [s_1(X), s_1(Y)] \\ &= [s_2(X) + \omega(X), s_2(Y) + \omega(Y)] \\ &= s_2([X,Y]) + f_V(X)\omega(Y) - f_V(Y)\omega(X), \end{aligned} \tag{3.60}$$

thus,

$$\omega([X,Y]) = f_V(X)\omega(Y) - f_V(Y)\omega(X). \tag{3.61}$$

An element $\omega \in C^1(\mathfrak{g}; V, f_V)$ is called a derivation if it obeys (3.61). However, since

$$d\omega(X,Y) = f_V(X)\omega(Y) - f_V(Y)\omega(X) - \omega([X,Y]), \tag{3.62}$$

the set of derivation is in fact $Z^1(\mathfrak{g}; V, f_V)$. s_2 preserves the bracket if and only if $d\omega = 0$, in this case, we say that s_2 is a Lie algebra splitting.

Certainly, $\omega(\cdot) := f_V(\cdot)v$ for some $v \in V$ is a trivial kind of derivation for

$$\begin{aligned} \omega([X,Y]) &= f_V([X,Y])v \\ &= [s_1([X,Y]), v] \\ &= [f_V(X), f_V(Y)]v \\ &= f_V(X)\omega(Y) - f_V(Y)\omega(X), \end{aligned} \tag{3.63}$$

thus $d\omega$ is identically zero. In fact, $\omega(X) = [s_1(X), v] = dv(s_1(X))$, ω is a coboundary. A derivation ω is said to be inner if $\omega(\cdot) = f_V(\cdot)v$ for some $v \in V$, equivalently, if $\omega \in B^1(\mathfrak{g}; V, f_V)$.

Sometimes $Z^1(\mathfrak{g}; V, f_V)$ is denoted by $Der(\mathfrak{g})$, and $B^1(\mathfrak{g}; V, f_V)$ is denoted by $IDer(\mathfrak{g})$.

As a result, $H^1(\mathfrak{g}; V, f_V) = Z^1(\mathfrak{g}; V, f_V)/B^1(\mathfrak{g}; V, f_V) = Der(\mathfrak{g})/IDer(\mathfrak{g})$ classifies the Lie algebra splittings up to a difference of an inner derivation, thus, all possible sections $\mathfrak{g} \to E$.

Another aspect of Lie algebra $H^1(\mathfrak{g}; V, f_V)$ can reveal is the complete reducibility.

If $f : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ s.t. f is invariant on some proper subspace $U \subset V$ i.e. $f(\mathfrak{g})U \subseteq U$, then we'd ask whether f is a complete reducible representation or not, that is, there exists another subsapce $W \subset V$ s.t. $V = U \oplus W$ and $f(\mathfrak{g})W \subseteq W$. If so, then f can be viewed as a direct some of two independent representations.

In general, if U is such a $\mathfrak{g}$-invariant subspace and W is any complement of U, then we can define $f_1 = f|_U, f_2 = f|_W$, and $g : \mathfrak{g} \to Hom(W, U)$ such that for all $X \in \mathfrak{g}, u \in U, w \in W$, we have

$$f(X)\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} f_1(X) & g(X) \\ 0 & f_2(X) \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}. \tag{3.64}$$

If so, we have

$$\begin{aligned} [f(X), f(Y)] &= \left[\begin{pmatrix} f_1(X) & g(X) \\ 0 & f_2(X) \end{pmatrix}, \begin{pmatrix} f_1(Y) & g(Y) \\ 0 & f_2(Y) \end{pmatrix} \right] \\ &= \begin{pmatrix} [f_1(X), f_1(Y)] & f_1(X)g(Y) + g(X)f_2(Y) - f_1(Y)g(X) - g(Y)f_2(X) \\ 0 & [f_2(X), f_2(Y)] \end{pmatrix} \end{aligned} \tag{3.65}$$

This observation gives the following theorem.

Theorem 3.5.2.1 *f is a representation if and only if so are f_1, f_2 and*

$$f_1(X)g(Y) + g(X)f_2(Y) - f_1(Y)g(X) - g(Y)f_2(X) - g([X,Y]) = 0. \tag{3.66}$$

We now may wonder when we could find a subspace W' such that $f(\mathfrak{g})W' \subseteq W'$, so we'd like to see how we can adjust W.

Let $w' \in W'$, then $\exists! u \in U, w \in W$s.t. $w' = u + w$. Since $w' \mapsto w$ is 1-1, the map $h : w \mapsto u$ is well-defined. In this way, $w' = (id_W + h)w \Rightarrow W' = (1 + h)W$. Consequently, any change of complementary subspace of U is characterized by $h \in Hom(W, U)$.

Proposition 3.5.2.1 *Given $h \in Hom(W, U)$, $(id_W + h)W$ is $\mathfrak{g}$-invariant $\iff$ $g(X) = h \circ f_2(X) - f_1(X) \circ h$ for all $X \in \mathfrak{g}$.*

⟨ Proof of proposition 3.5.2.1 ⟩ To have

$$f(X) \begin{pmatrix} id_U & h \\ 0 & id_W \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} f_1(X) & f_1(X)h + g(X) \\ 0 & f_2(X) \end{pmatrix} = \begin{pmatrix} f_1(X)u + hf_2(X)w \\ f_2(X)w \end{pmatrix}, \tag{3.67}$$

it requires $g = h \circ f_2 - f_1 \circ h$.

Proposition 3.5.2.2 *The map $F : \mathfrak{g} \to End(Hom(W, U))$ defined by*

$$F(X)s = f_1(X)s - sf_2(X), \tag{3.68}$$

for all $X \in \mathfrak{g}, s \in Hom(W, U)$, is a representation of $\mathfrak{g}$.

⟨ Proof of proposition 3.5.2.2 ⟩

Direct calculation gives

$$\begin{aligned} F([X, Y])s &= f_1([X, Y])s - sf_2([X, Y]) \\ &= [f_1(X), f_2(Y)]s - s[f_2(X), f_2(Y)] \\ &= f_1(X)f_2(Y)s + f_1(Y)sf_2(X)s \\ &\quad - f_1(X)sf_2(Y)s - sf_2(Y)f_2(X) - (X \leftrightarrow Y) \\ &= F(X)(f_1(Y)s - sf_2(Y)) - (X \leftrightarrow Y) \\ &= [F(X), F(Y)]. \end{aligned} \tag{3.69}$$

Proposition 3.5.2.3 *Given $g \in Hom(\mathfrak{g}, Hom(W, U))$, g satisfies the criteria in proposition 3.5.2.1 $\iff$ $g \in Z^1(\mathfrak{g}; Hom(U, W), F)$.*

⟨ Proof of proposition 3.5.2.3 ⟩

This is because

$$\begin{aligned} dg(X, Y) &= F(X)g(Y) - F(Y)g(X) - g([X, Y]) \\ &= f_1(X)g(Y) - g(Y)f_2(X) - f(Y)g(X) + g(X)f_2(Y) - g([X, Y]). \end{aligned} \tag{3.70}$$

Proposition 3.5.2.4 *Given $g \in Hom(\mathfrak{g}, Hom(W, U))$, then $\exists h \in Hom(W, U)$ s.t. $h \circ f_2(X) - f_1(X) \circ h = g(X)$ for all $X \in \mathfrak{g}$ $\iff$ $g \in B^1(\mathfrak{g}; Hom(U, W), F)$.*

⟨ Proof of proposition 3.5.2.4 ⟩

For $h \in Hom(W, U), dh(X) = F(X)h = f_1(X) \circ h - h \circ f_2(X)$.

From the above propositions, we then can summarize the following theorem.

Theorem 3.5.2.2 *Let $f : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. If U is a $\mathfrak{g}$-invariant subspace of V, then $F(X)s = f|_U(X)s - sf|_{V/U}(X)$ defines a representation of $\mathfrak{g}$ on $Hom(V/U, U)$. If $H^1(\mathfrak{g}; Hom(U, V/U), F) = 0$, then there's a choice W for V/U s.t. W is $\mathfrak{g}$-invariant.*

In other words, f is a completely reducible representation.

3.5.3 Two Whitehead's Lemmas

In this section, we fix $\mathfrak{g}$ to be a semi-simple Lie algebra of finite dimension n over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Lemma 3.5.3.1 *Let $f : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}, K \in \mathfrak{gl}(V)$ and $\phi \in C^r(\mathfrak{g}; V, f)$, then*

(1) $K\phi \in C^r(\mathfrak{g}; V, f)$

(2) $$((dK - Kd)\phi)(X_1, \ldots, X_{r+1}) = \sum_{j=1}^{r+1} (-1)^{j+1} [f(X_j), K] \phi(X_1, \ldots, \hat{X}_j, \ldots, X_{r+1}) \qquad (3.71)$$

(3) *If $Kf(X) = f(X)K, \forall X \in \mathfrak{g}$, then $d \circ K = K \circ d$*

⟨ Proof of lemma 3.5.3.1 ⟩

(1) is obvious from definition, and (2) is trivial for $r = 0$. For $r > 0$, observe

$$\begin{aligned}(Kd\phi) = &\sum_j (-1)^{j+1} K f(X_j) \phi(\ldots, \hat{X}^j, \ldots) \\ &+ \sum_{i<j} (-1)^{i+j} K \phi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots),\end{aligned} \qquad (3.72)$$

and

$$\begin{aligned}(dK\phi) = &\sum_j (-1)^{j+1} f(X_j) K \phi(\ldots, \hat{X}^j, \ldots) \\ &+ \sum_{i<j} (-1)^{i+j} K \phi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots).\end{aligned} \qquad (3.73)$$

Then (2) is clearly true, and (3) follows immediately from (2).

Lemma 3.5.3.2 *(Whitehead) If $f : \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible representation of $\mathfrak{g}$ and $f \neq 0$, then $H^r(\mathfrak{g}; V, f) = 0$ for all $r \in \mathbb{N}$.*

⟨ Proof of lemma 3.5.3.2 ⟩

Denote $\mathfrak{g}_2 = ker(f)$, then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ for some ideal $\mathfrak{g}_1$ of $\mathfrak{g}$ s.t. $f|_{\mathfrak{g}_1}$ is faithful.

Let $\{F^i\}$ and $\{E_i\}$ be a basis and its dual basis of $\mathfrak{g}_1$ with respect to the Killing form. Since f is irreducible, we have $C_{f|_{\mathfrak{g}_1}}$ non-singular by lemma 3.4.0.2. Since $[C_f, f(X)] = 0$ for all $X \in \mathfrak{g}_1$ and $f(X) = 0$ for all $X \in \mathfrak{g}_2$, we have $[f(X), C_{f|_{\mathfrak{g}_1}}] = 0$ for all $X \in \mathfrak{g}$.

For simplicity, write $C = C_{f|_{\mathfrak{g}_1}}$.

Obviously, $H^0(\mathfrak{g}; V, f) = 0$. For $r \geq 1$ and $\phi \in Z^r(\mathfrak{g}; V, f)$, we define $h_\phi = \sum_i f(E^i) \iota_{F_i} \phi \in C^{r-1}(\mathfrak{g}; V.f)$ by the Killing form, then we have

$$dh = \sum_i (df(E_i) - f(E_i)d) \iota_{F^i} \phi + \sum_i f(E_i) d\iota_{F^i} \phi. \qquad (3.74)$$

Since $d\phi = 0$, $d\iota_{F^i}\phi = L_{F^i}\phi$. By lemma 3.5.3.1 for $K = f(E_i)$, we have

$$\begin{aligned}dh(X_1, \ldots, X_r) = &\sum_{i=1}^{n} \sum_{j=1}^{r} (-1)^{j+1} [f(X_j), f(E_i)] \iota_{F^i} \phi(X_1, \ldots, \hat{X}_j, \ldots, X_r) \\ &+ \sum_{i=1}^{n} f(E_i) f(F^i) \phi(X_1, \ldots, X_r) \\ &- \sum_i \sum_{k=1}^{r} (-1)^{j+1} f(F^i) \phi(X_1, \ldots, X_{j-1}, [E_i, X_j], X_{j+1}, \ldots, X_r)\end{aligned} \qquad (3.75)$$

Write $[X_j, E_i] = \sum_s a^s{}_{ji} E_s$, then $[X_j, F^i] = \sum_s a_{sj}{}^i F^s$ and

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{r}(-1)^{j+1}[f(X_j), f(E_i)]\iota_{F^i}\phi(X_1, \dots, \hat{X}_j, \dots, X_r) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{s=1}^{n}(-1)^{j+1}a^s{}_{ji} f(E_s)\phi(X_1, \dots, X_{j-1}, F^i, X_{j+1} \dots, X_r).
\end{aligned} \tag{3.76}$$

On the other hand,

$$\begin{aligned}
&\sum_{i}\sum_{k=1}^{r}(-1)^{j+1} f(F^i)\phi(X_1, \dots, X_{j-1}, [E_i, X_j], X_{j+1}, \dots, X_r) \\
&= \sum_{i}\sum_{k=1}^{r}\sum_{s=1}^{n}(-1)^{j+1}a^s{}_{ji} f(F^i)\phi(X_1, \dots, X_{j-1}, E_s, X_{j+1}, \dots, X_r) \\
&= \sum_{i}\sum_{k=1}^{r}\sum_{s=1}^{n}(-1)^{j+1}a_{sj}{}^i f(E_i)\phi(X_1, \dots, X_{j-1}, F^s, X_{j+1}, \dots, X_r) \\
&= \sum_{s}\sum_{k=1}^{r}\sum_{i=1}^{n}(-1)^{j+1}a_{ij}{}^s f(E_s)\phi(X_1, \dots, X_{j-1}, F^i, X_{j+1}, \dots, X_r).
\end{aligned} \tag{3.77}$$

Then we have

$$dh(X_1, \dots, X_r) = \sum_{i=1}^{n} f(E_i) f(F^i)\phi(X_1, \dots, X_r) = C\phi(X_1, \dots, X_r). \tag{3.78}$$

Apply lemma 3.5.3.1 again with $K = C^{-1}$, we have $\phi = C^{-1}dh = d(C^{-1}h)$, hence, $\phi \in B^r(\mathfrak{g}; V, f)$ and $H^r(\mathfrak{g}; V, f) = 0$.

Lemma 3.5.3.3 *(Whitehead's first lemma) For every representation $f : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$, we have $H^1(\mathfrak{g}; V, f) = 0$.*

$\langle$ Proof of lemma 3.5.3.3 $\rangle$

If $f = 0$, then $H^1(\mathfrak{g}; V, f) = 0$ because for $\phi \in Z^1(\mathfrak{g}; V, f)$

$$0 = d\phi(X, Y) = f(X)\phi(Y) - f(Y)\phi(X) - \phi([X, Y]) = -\phi([X, Y]), \tag{3.79}$$

for all $X, Y \in \mathfrak{g}$. However, since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, $\phi \equiv 0$.

Now we'll do induction on the dimension of the representation.

If f is irreducible, the from lemma 3.5.3.2, it's true. Also, if $dimV = 1$, then f is irreducible.

Suppose now f is reducible and U is a proper $\mathfrak{g}$-invariant subspace of V i.e. $0 \subsetneqq U \subsetneqq V$, then $f|_U, f|_{V/U}$ are representations of $\mathfrak{g}$. Let $p : V \to V/U$ be the quotient map, then

$$pf(X)v = f|_{V/U}(X)pv \tag{3.80}$$

for all $X \in \mathfrak{g}, v \in V$, thus, for $\phi \in Z^1(\mathfrak{g}; V, f)$, $p\phi \in C^1(\mathfrak{g}; V/U, f|_{V/U})$. Observe that

$$\begin{aligned}
d(p\phi)(X, Y) &= f|_{V/U}(X)p\phi(Y) - f|_{V/U}(Y)p\phi(X) - p\phi([X, Y]) \\
&= p(f(X)\phi(Y) - f(Y)\phi(X) - \phi([X, Y])) \\
&= 0.
\end{aligned} \tag{3.81}$$

We have $p\phi \in Z^1(\mathfrak{g}; V/U, f|_{V/U})$. Since V/U has dimension smaller than V, by induction hypothesis, $p\phi \in B^1(\mathfrak{g}; V/U, f|_{V/U})$, therefore, there exists some $\tilde{v} \in V/U = C^0(\mathfrak{g}; V/U, f|_{V/U})$ such that

$$f_{V/U}(X)\tilde{v} = d\tilde{v}(X) = p\phi(X). \tag{3.82}$$

Pick $v \in V$ s.t. $pv = \tilde{v}$, then

$$\begin{aligned} pf(X)v &= f|_{V/U} pv \\ &= f|_{V/U} \tilde{v} \\ &= p\phi(X). \end{aligned} \tag{3.83}$$

Hence, $p(f(X)v - \phi(X)) = 0$ and $h(X) := \phi(X) - f(X)v \in U$. Since $dh = d\phi - d^2 v = 0$, we have $h \in Z^1(\mathfrak{g}; V, f)$. On the other hand, U is $\mathfrak{g}$-invariant, thus, $h \in Z^1(\mathfrak{g}; U, f|_U)$, again by induction hypothesis, we have $h \in B^1(\mathfrak{g}; U, f|_U)$. That is, there exists $u \in U$ such that $h = du$.

As a result, $\phi = h + dv = d(u + v) \in B^1(\mathfrak{g}; V, f)$.

From this lemma and the theorem of complete reducibility, we then have the theorem.

Theorem 3.5.3.1 *Every representation of a semi-simple Lie algebra is completely reducible.*

Lemma 3.5.3.4 *(Whitehead's second lemma) If $f : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of a semi-simple Lie algebra $\mathfrak{g}$, then we have $H^2(\mathfrak{g}; V, f) = 0$.*

⟨ Proof of lemma 3.5.3.4 ⟩

By the completely reducibility, it suffices to prove for irreducible representations.

But this has been done in 3.5.3.2.

3.6 Projective Representations of Semi-Simple Lie Groups

This section is mainly based on [12].

For a connected, simply-connected and semi-simple Lie group, from the theorem 3.2.0.1 and Whitehead's second lemma 3.5.3.4, we have $H^2(G, M) \cong H^2(\mathfrak{g}; \mathfrak{m}, ad)$. As a result, all the projective representations of simply-connected semi-simple Lie groups in fact arise from exact multipliers.

Additionally, for semi-simple Lie group G, its universal covering group $\tilde{G}$ satisfies $Hom(\tilde{G}, M) \cong Hom(\mathfrak{g}, \mathfrak{m}) = 0$ and $H^2(\tilde{G}, M) = 0$. Hence the homomorphism in 3.3.0.1 is an isomorphism.

Furthermore, there's a 1-1 correspondence between $Hom(\pi_1(G), M)$ and $H^2(G, M)$. In particular, we have 1-1 correspondence

$$H^2(G; U(1)) \leftrightarrow Hom(\pi_1(G), U(1)) = \widehat{\pi_1(G)}, \tag{3.84}$$

between projective representations of G and the Pontrjagin dual of the fundamental group of G.

As a result, we have the following theorem as summary.

Theorem 3.6.0.1 *Let G be a connected semi-simple Lie group and $\tilde{G}$ be its universal covering group. Then there is a natural bijection between the equivalence classes of projective unitary representations of G and the equivalence classes of ordinary unitary representations of pure type of $\tilde{G}$. Under this bijection, for each character χ of the kernel of the covering map, $ker(\pi)$, the projective representations of G with multiplier m_χ corresponds to the ordinary unitary representations of pure type χ and vice versa.*

Furthermore, the irreducible projective representations of G corresponds to the irreducible representations of $\tilde{G}$, and vice versa.

⟨ Proof of 3.6.0.1 ⟩

For each multiplier m on G, we denote $\tilde{m} = m \circ \pi$ as the lift of m to $\tilde{G}$.

Let α be any projective representations of G, say with multiplier m. By theorem 3.3.0.1, $m = m_\chi$ for some character χ of $ker(\pi)$. Let $\tilde{\alpha} = \alpha \circ \pi$ be any projective representation of $\tilde{G}$ obtained by lifting α.

Since $H^2(\tilde{G}, U(1)) = 0$,

$$\tilde{m}(g,h) = \frac{f_\chi(gh)}{f_\chi(g) f_\chi(h)}, \tag{3.85}$$

where $f_\chi(g) = \chi(g^{-1} \circ s \circ \pi(g))$ for some irrelevent choice of section s.

Define β on $\tilde{G}$ by $\beta(g) := \frac{1}{f_\chi(g)} \tilde{\alpha}(g)$, then β is an ordinary representation of pure type χ. The map $\alpha \mapsto \beta$ is clearly one to one, additionally, if α is irreducible, then so is β, and vice versa.

To see the surjectivity, fix ordinary representation β of $\tilde{G}$ which is trivial on $ker(\pi)$, then there is a well-defined projective representation α of G s.t. $\tilde{\alpha} = \alpha \circ \pi$.

Clearly, this map $\beta \to \alpha$ is the inverse of the previous map $\alpha \to \beta$.

Here we provide two examples.

$SO(3)$ has universal covering group $SU(2)$ with $ker(\pi) = \mathbb{Z}_2$. Since $Hom(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$, the possible characters of $\pi_1(G) = ker(\pi)$ form $\mathbb{Z}_2$. And they are determined by $f_\chi(-\mathbb{1}) = \pm \mathbb{1}$, i.e. $f_\chi($ rotation with angle $2\pi) = \pm \mathbb{1}$.

The Möbius group of biholomorphisms on an open disc D is defined by

$$M\ddot{o}b := \{\phi_{\alpha,\beta} : \alpha \in T, \beta \in D, \text{ and } \phi_{\alpha,\beta}(z) = \alpha \frac{\beta - z}{1 - \beta^* z}, z \in D\} \tag{3.86}$$

$M\ddot{o}b$ is topologized via identification with $T \times D$ and is a connected semi-simple Lie group. Algebraically, $M\ddot{o}b \cong PSL(2,\mathbb{R}) \cong PSU(1,1)$. Define $n : M\ddot{o}b \times M\ddot{o}b \to \mathbb{Z}$ by

$$n(\phi_1^{-1}, \phi_2^{-1}) = \frac{1}{2\pi}\Big(arg((\phi_1\phi_1)'(0)) - arg(\phi_1'(0)) - arg(\phi_2'(0))\Big) \tag{3.87}$$

Then for every $w \in T$, we can define $m_w : M\ddot{o}b \times M\ddot{o}b \to T$ by $m_w(\phi_1, \phi_2) = w^{n(\phi_1,\phi_2)}$.

This is the $1-1$ correspondence betweeb w and m_w.

3.7 An Application: Nonlinear-Sigma Models

An significant application of the Lie group cohomology is the study of nonlinear-sigma models (NLσM) with Lie group target space on a space-time lattice and the classification of the symmetry protected topological phases (SPT phases). The following content is mainly based on [13], [14], [15], [16], [17], [18] for some treatment of loops integrations, and the 2015 talk "SPT order and algebraic toplology" by Xioa-Gang Wen.

We firstly introduce the general settings of a NLσMs and their properties, and we'll then discuss the case where the target space is a Lie group.

A nonliner σ mode on a d-dimensional (pseudo)-Riemannian manifold (M, h) called the parameter space is given by a differentiable map $\phi : M \to N$ to another R-dimensional vector space N called the target space with a vector dependent positive-definite two rank tensor g on N and a potential V, bounded below nonlinear function on N, with the action

$$S = \Lambda \int_M \sqrt{|h|} d^d x \Big(\frac{1}{2} h^{\mu\nu}(x) g_{ab}(\phi) \partial_\mu \phi^a \partial_\nu \phi^b - V(\phi)\Big). \tag{3.88}$$

The field ϕ^a in a NLσM can also be viewed as the coordinates function of a manifold with metric $g_{av}(\phi)$, for example, in the string theorem M is a tow dimensional world sheet imbedded into the spacetime manifold, but with a Lorentzian metric g bringing negatively-normed tachyon states into the theory, which shall be eliminated by the BRST operator during quantization.

The loops calculation is generally difficult due to the nonlinear essence. However, as the dimension of the parameter space is 2, since the existence of the Riemannian normal coordinates, the one-loop effective action for a free theory ($V = 0$) can be calculated as follows.

Up to a redefinition of field, we can assume the field ϕ^a are locally inertial coordinates at some point ϕ_0^k, where the metric is locally the Euclidean metric δ_{ab} to the order $O(\phi^2)$. Let $y^a = \phi^a - \phi_0^a$, then

$$g_{ab}(\phi^k) = \delta_{ab} - \frac{1}{3}R_{acbd}(\phi_0^k)y^c y^d - \frac{1}{6}D_e R_{acbd}y^e y^c y^d + O(y^4). \tag{3.89}$$

Here D_e and R are the covariant derivative and the Riemannian curvature given by the metric g, respectively. As a result, we have the action to be

$$S = \Lambda \int_M \sqrt{|h|}d^d x\Big(\frac{1}{2}h^{\mu\nu}(x)\Big(\partial_\mu y^a \partial_\nu y_a - \frac{1}{3}R_{acbd}(\partial_\mu y^a)(\partial_\nu y^b)y^c y^d\Big) + O(y^5)\Big). \tag{3.90}$$

In the scheme of dimensional regularization, only the logarithmical divergent terms contribute to the renormalization, and when the dimension of M is 2, the contraction $\langle \partial y \partial y \rangle$ gives quadratic divergence and can be dropped. Differently, the term $\langle y^a y^b \rangle$ gives logarithmical divergence since

$$\lim_{x_1 \to x_2} \langle y^a(x_1)y^b(x_2)\rangle = \pi\delta^{ab} \lim_{x_1 \to x_2} \int \frac{d^{2+\epsilon}k}{(2\pi)^{2+\epsilon}} \frac{e^{\sqrt{-1}k(x_1-x_2)}}{k^2} \approx \frac{\delta^{ab}}{2\epsilon}, \tag{3.91}$$

in the dimension $2+\epsilon$. Hence, the one loop effective action is given by

$$S_{1-\text{loop}} = -\frac{\Lambda}{12\epsilon}\int_M \sqrt{|h|}d^d x h^{\mu\nu}(x)R_{ab}(\partial_\mu y^a)(\partial_\nu y^b). \tag{3.92}$$

With multplying the coupling constant by μ^ϵ to make the action dimensionless in dimensional regularization, this consequently gives us the beta function as

$$\beta_{ab}(y^a) := \mu\frac{d}{d\mu}g_{ab}(y^a) = -\frac{1}{4\pi}R_{ab}(y^a). \tag{3.93}$$

Hence, at one loop level, the fix points of the 2-dimensional NLσM are those Ricci flat ones, and in the case of V being a Kähler manifold, this involves the Calabi-Yau manifold due to the Ricci-flatness. About the higher loops corrections and the insertion of torsion, see [17] chapter 2.

Now when the target space being a compact Lie group, the NLσM of common interest is given by the action

$$S = \Lambda \int_M Tr(g^{-1}\partial_\mu g)^2 = -\Lambda \int_M Tr(\partial_\mu g \partial^\mu g^{-1}). \tag{3.94}$$

for $dg^{-1} = -g^{-1}dgg^{-1}$. Since the Maurer-Cartan 1-form is given by $\omega_g = (L_{g^{-1}})_* = g^{-1}dg$ for matrix groups, this action is in fact

$$S = \Lambda \int_M Tr(\phi^*\omega_g(h_\flat \phi^*(\omega_g))). \tag{3.95}$$

Here $\phi : M \to G$ is a differentiable map into the (usually) compact Lie group G, thus, ϕ^* is a pullback from the forms on G to M, which then made a vector field by $h_\flat$. As we believe the renormalization flow shall bring Λ to 0^+, in this limit, we shall have the contributive (Euclidean) action shall give unit partition function, that is

$$e^{-S[g]} = 1. \tag{3.96}$$

Also, under a constant change of $g \mapsto hg$ for any $h \in G$, this NLσM is unchanged, thus, has a global symmetry.

To discuss the NLσM on a spacetime lattice with Lie group target space, we firstly introduce the idea of L-type local bosonic quantum system in d-dimensional spacetime, which is described by the following settings

- A triangulization of M with a branching structure.

- A set of indices for all the cells (vertices $\{v_i\}$, edges $\{e_{ij}\}$, triangles $\{s_{ijk}\} \ldots$).

- complex numbers for the partition function contribution given by the cells (W_v, A_e, C_s such that the partition function is

$$Z = \sum_{(v,e,s,\ldots)} \prod_{v=\text{vertices}} W_v \prod_{e=\text{edges}} A_v \prod_{s=\text{triangles}} C_s. \tag{3.97}$$

The L-types local bosonic quantum systme generally are in short-range entangled states, thus, they could serve as examples for symmetry protected topological/trivial phases (SPT phases). In fact, it's a conjecture that a system with $Z(M) = 1$ for all closed and orientable space-time M has a trivial topological order.

Now we'd like to consider the example of 1+2D spacetime lattice described by $W_v = 1$, A_e also being trivial, and $C = \nu(g_1, g_2, g_3) \in U(1)$, for $g's$ representating the embedded vetices from the space-time lattice. And the NLσM is said to have a symmetry if

$$\nu(g_1, g_2, g_3) = \nu(hg_1, hg_2, hg_3), \tag{3.98}$$

for any element h in G, which is inspired by the symmetry in the continuum NLσM with Lie group target space. This is equivalent to the equivariant condition in the definition of Lie group chohomology. If we denote s_{ijk} for the orientation of the vertice given by (g_i, g_j, g_k), we then have the partition function to be

$$e^{-S} = \prod v^{s_{ijk}}(g_i, g_j, g_k). \tag{3.99}$$

The simplest case of a tetrahedron (g_0, g_1, g_2, g_3) then gives the two cycle condition if we want every contributive partition fucntion to be 1, since

$$\prod v^{s_{ijk}}(g_i, g_j, g_k) = \nu(g_1, g_2, g_3)\nu(g_0, g_1, g_3)\nu(g_0, g_2, g_3)^{-1}\nu(g_0, g_1, g_2)^{-1} = 1. \tag{3.100}$$

Also, the 2-cocycle can be changed by

$$\tilde{\nu}(g_0, g_1, g_2) = \frac{\beta(g_1, g_2)\beta(g_0, g_1)}{\beta(g_0, g_2)}\nu(g_0, g_1, g_2) \tag{3.101}$$

but still solve the unity equation. In this situation, we say $\tilde{\nu}$ and ν are equivalent since they only differ in a redefinition of partition function contribution on the edges.

From the analogy of the above discussion and the Lie group cohomology we've seen before, we then know the equivalent classes of these ν are classified by $H^2(G; U(1))$, if we denote π_0 as the set of connected components in the space of solutions, then we have

$$H^2(G, U(1)) = \pi_0(\text{space of the solutions}). \tag{3.102}$$

The above condition can be generalized to higher dimension similar models, but we omit them here and notifiy the readers the Lie group cohomology cannot classified all the SPT phases.

For a more general study of SPT phases, it will require the usage of G-crossed braided tensor category (in 2d), see [19].

Chapter 4

Spontaneous Symmetry Breaking

4.1 Degenerate Vacua and Free Energies

For convenience, we will use the path integral formalism in the following sections and assume the readers are familiar with the textbook content about quantum theory of fields.

In the following context, let $\phi(x)$ be a multiplet of fields, and $S[\phi]$ be an action of any field theory. If we'd lile to calculate the correlation function of the theory, we may consider the partition function, or the generating function, defined by

$$Z[J] = \frac{\int \mathcal{D}\phi \ e^{\sqrt{-1}(S[\phi]+\int d^D x J(x)\phi(x))}}{\int \mathcal{D}\phi \ e^{\sqrt{-1}S[\phi]}}. \tag{4.1}$$

by putting an external current-field coupling term. And if we take functional derivative upon the external current, we get the expectation values. In order to differentiate the case with or without the presence of this external current, we write $\langle \mathcal{O} \rangle_J$ for the expectation value of any operator $\mathcal{O}$ with the presence of the external current $J(x)$, otherwise, we just write $\langle \mathcal{O} \rangle$.

Since the generating function gives both connected and disconnected diagrams, but the disconnected diagrams may gives faraway contributions such as $\langle \mathcal{O}_1(x) \rangle \langle \mathcal{O}_2(y) \rangle$ with x and y being far aparted, it's then convenient to introduce the free energy

$$F[J] = -\sqrt{-1} \log Z[J], \tag{4.2}$$

so that only the local distributions, i.e. the connected diagrams, in the correlation function are considered.

With the presence of the external current $J(x)$, the vacuum Expectation value of the fields multiplet could be given by

$$\phi_J(x) := \langle \phi(x) \rangle_J = \frac{\delta F[J]}{\delta J(x)} \tag{4.3}$$

And by taking again the derivative with respect to the external current, we have the identity

$$\frac{\delta \phi_J(x)}{\delta J(y)} = \frac{\delta^2 F[J]}{\delta J(x) \delta J(y)}_J = \langle \phi(x)\phi(y) \rangle_J = G_J(x,y). \tag{4.4}$$

Here we denote the G_J as the correlation function with the presence of external current. Thus as $J = 0$ it reduces to the normal correlation function.

It'd be convenient to introduce the effective action via Lagendre tranformation:

$$\Gamma[\phi_J] = F[J] - \int d^D x \ J(x)\phi_J(x). \tag{4.5}$$

As a result, we get this identity relating the variation of the effective action and the external current:

$$\frac{\delta\Gamma[\phi_J]}{\delta\phi_J(x)} = \int d^D y \frac{\delta F[J]}{\delta J(y)} \frac{\delta J(y)}{\delta\phi_J(x)} - \int d^D y \left(\frac{\delta J(y)}{\delta\phi_J(x)} \phi_J(y) + J(y)\delta(x-y) \right) = -J(x). \tag{4.6}$$

As $J(x) = 0$, $\phi(x)_J$ reduces to the expectation value of $\phi(x)$, thus, it satisties

$$\frac{\delta\Gamma[\phi_J]}{\delta\phi_J(x)}\Big|_{\phi(x)_J=\langle\phi(x)\rangle} = 0. \tag{4.7}$$

This gives the equation of motion with the quantum correction and defines the vecuum expectation values as the extreme points of the effective action.

Also, by the functional chain rule, we have

$$\int d^D y \frac{\delta^2\Gamma[\phi_J]}{\delta\phi_J(x)\delta\phi_J(y)} \frac{\delta\phi_J(y)}{\delta J(z)} = -\delta(x-z). \tag{4.8}$$

This implies

$$\int d^D y \frac{\delta^2\Gamma[\phi_J]}{\delta\phi_J(x)\delta\phi_J(y)} G(x,y) = -\delta(x-z). \tag{4.9}$$

Using Fourier transformation and assuming $G(x,y) = G(x-y)$ by translational invariance, we then get

$$\Gamma(k) := \frac{1}{\sqrt{2\pi}} \int d^D x e^{-\sqrt{-1}kx} \frac{\delta^2\Gamma[\phi_J]}{\delta\phi_J(x)\delta\phi_J(0)} = G(k)^{-1}. \tag{4.10}$$

This implies the second derivatives of the effective action are the inverse of the correlation functions. Also, the higher derivatives presents the inverse of the vertexes, which we omit here.

If R is a symmetry acting on the field configuration space, that is, the action satisfies $S[\phi] = S[R\phi]$, then we may expect $\Gamma[\phi] = \Gamma[R\phi]$, which leaves the physical observables (e.g. scattering amplitudes) invariant. If $R\phi \neq \phi$, then the vacuum, defined as a state whose expectation values of the fields minimize the effective potential, is degenerate. As a result, the symmetry of the available choices of vacua, say H, is merely a subgroup of the symmetry of the Lagrangian, say G, and the equivalent choice of vacuum is characterized by G/H. If so, we say that the theory is spontaneous broken.

Similarly, we can define free energy and Landau freee energy for equilibrium system, and define spontaneous symmetry breaking in the same manner.

Here we discuss a formal example of the spontaneous symmetry breaking in D+1 dimension space-time. Suppose Φ is the set of minimum of the effective action, or up to a shift, the zeros of the potential part of effective action, then to make the integral finite, we must have $\phi(x \to \infty)$ in Φ, define $\tilde{\phi} : S^{D-2} \to \Phi$ as $\hat{n} \mapsto \lim_{||x|| \to \infty} \phi(x = ||x||\hat{n}) \in \Phi$. If the symmetry operations on this set of zeros, say G, are transitive, then we can pick up a direction $\hat{h}$ and use a map $g(\hat{n})$ to send $\hat{n}$ to $g(\hat{n})\tilde{\phi}(\hat{x}) = \tilde{\phi}(\hat{n})$. However, G might has some subgroup stablized on $\phi(\hat{x})$, say $H \subseteq G$, then the homotopically equivalence class of maps $[S^{D-1} : G/H] = \pi_{D-1}(G/H)$ characterizes topologically inequivalent vacua of the effective action. Note that the definition is meaningful only if G/H can be equipped with the induced quotient topology as a topological space, for example, G is a Lie group and H is a closed subgroup which makes coset still a manifold.

4.2 The Nambu-Goldstone Theorem

Let $U = e^{\sqrt{-1}\alpha Q}$ be a one parameter symmetry transformation generated by Q and parametrized by α commuting with the Hamiltonian. A state ρ is said to break the symmetry if there exists any operator Φ such that

$$Tr\Big([Q,\Phi]\rho\Big) \neq 0. \tag{4.11}$$

Otherwise, we say that the symmetry is unbroken. Also, we call the operator $\mathcal{O} := [Q, \Phi]$ the Order Parameter Operator and $O := Tr([Q, \Phi]\rho)$ the Order Parameter. Since we have $Tr([Q, \Phi]\rho) = Tr(Q\Phi\rho - \Phi Q\rho) = Tr(\Phi\rho Q - \Phi Q\rho) = Tr(\Phi[\rho, Q])$, the commutator $[\rho, Q] = 0$ sufficiently implies the symmetry is unbroken.

In fact, we have

$$\frac{d}{d\alpha}\Big|_{\alpha=0} Tr\Big(\Phi U\rho U^{-1}\Big) = \sqrt{-1}O. \tag{4.12}$$

Thus the order parameter is the expectation value of Φ on the state ρ transformed by U.

In the case that ρ is a pure state, e.g. the ground state/vacuum of a Hamiltonian with no ground state degeneracy. The definition for some pure state $|\psi\rangle$ then reduces to

$$\langle\psi|\,[Q,\Phi]\,|\psi\rangle \neq 0. \tag{4.13}$$

It was speculated that wether broken continuous symmetry H, a sub-Lie-group of the Lie group G, could give rise to massless excitation in the spectrum or not since along the remaining symmetry H since there should be no gap in these direcitons. These modes are called the Nambu-Goldstone (NG) modes.

There are several conditions that a quantum system has to meet to have NG modes, to start with, it firstly requires a course-grained level translational invariace in the interested state and the Hamiltonian, so that we have the momentum k as a good quantum number, which allows as to have a complete set of eigenstates of the Hamiltonian H, say $\{|n, k\rangle\}$, with some extra quantum number n to differentiate the momentum degeneracy. In this manner, we denote the energy of the state $|n, k\rangle$ as $E_n(k)$.

This complete set of eigenstates then gives us the resolution of the identity as

$$\mathbb{1} = \sum_n \int \frac{d^D k}{(2\pi)^D} \,|n,k\rangle\,\langle n,k|\,. \tag{4.14}$$

Also, with the continuous symmetry, we have a conserved current $j^\mu(x, t)$, which leads to the Noether charge $Q_R = \int_{||x||<R} d^D x j^0(x, t)$ in the ball with radius R. As R approaches to the infinity, we'd hope that it converges to the conserved Noether charge Q, and this is the second assumption that there's no long-ranged interaction so that the commutator $[j(x, t), \Phi]$ decays fast enough as $||x|| \to \infty$. The third assumption is then the symmetry (at quantum level) is generated by the Noether charge Q (as a operator).

Inserting this into the definition of broken symmetrym then we have

$$\langle\psi|\,[Q,\Phi]\,|\psi\rangle_R := \sum_n \int\int_{||x||<R} \frac{d^D k d^D x}{(2\pi)^D}\Big(\,\langle\psi|\,j^0(x,t)\,|n,k\rangle\,\langle n,k|\,\Phi\,|\psi\rangle - c.c\Big) \tag{4.15}$$

But from translation operation we have

$$j^0(x,t) = e^{-\sqrt{-1}(tH-xP)} j^0(0,0) e^{\sqrt{-1}(tH-xP)}. \tag{4.16}$$

But P and H act on $|\psi\rangle$ trivially and act on $|n, k\rangle$ as an eigenstate with eigenvalues k and$E_n(k)$. Defining

$$\delta_R(k) = \int d^D x e^{-\sqrt{-1}kx}, \tag{4.17}$$

then we have

$$
\begin{aligned}
0 \neq & \lim_{R\to\infty} \sum_n \int \int_{||x||<R} \frac{d^D k d^D x}{(2\pi)^D} \Big(\langle\psi| j^0(x,t) |n,k\rangle \langle n,k| \Phi |\psi\rangle - c.c \Big) \\
= & \lim_{R\to\infty} \sum_n \int d^D k \delta_R(k) \Big(e^{\sqrt{-1}E_n(k)t} \langle\psi| j^0(0,0) |n,k\rangle \langle n,k| \Phi |\psi\rangle - c.c \Big)
\end{aligned}
\tag{4.18}
$$

As R approaches infinity, the distribution $\delta_R(k)$ approaches to the Dirac delta function at momentum zero, however, the order parameter shall approach to a non-zero constant to have a symmetry broken state, which implies there must be some $|n,k\rangle$ making the integrand non-zero near $k=0$. This tells us there's some state excited by $j^0(0,0)$ and Φ around $k=0$.

Now, if we have Φ and the state ψ are in fact time independent, then we have

$$
\begin{aligned}
0 = & \partial_t \langle [Q,\Phi] \rangle \\
= & \lim_{R\to\infty} \sum_n \int d^D k \delta_R(k) \Big(\sqrt{-1}E_n(k) e^{\sqrt{-1}E_n(k)t} \langle\psi| j^0(0,0) |n,k\rangle \langle n,k| \Phi |\psi\rangle - c.c \Big)
\end{aligned}
\tag{4.19}
$$

But since we have the previous mode around $k=0$ gives a non-zero term, $E_n(k)$ shall go to zero as k goes to zero, otherwise the integrand cannot be zero. This is then the so-called Nambu-Goldstone theorem.

Theorem 4.2.0.1 *(Nambu-Goldstone) If a global and continuous symmetry is spotaneously broken in the absence of long-ranged interactions, and leaving some translational symmetry in intact, then there's a mode in the spectrum whose energy vanishes as its momentum approaches zero.*

4.3 The Number of Nambu-Goldstone Bosons

Now, one may wonder what if the order parameter operator is just another (time independent) Noether current. Let's say we have Noether charge Q_a, Q_b and Noether current j_a^μ, j_b^μ, so we have their commutators as

$$
\begin{aligned}
\langle [Q_a, j_b^0(x)] \rangle &= \int d^D y \langle [j_a^0(y), j_b^0(x)] \rangle \\
&= \int d^D \delta(x-y) \sum_c \sqrt{-1} f_{ab}^c \langle j_c^0(y) \rangle \\
&= \sum_c \sqrt{-1} f_{ab}^c \langle j_c^0(x) \rangle \\
&= \langle [j_a^0(x), Q_b] \rangle
\end{aligned}
\tag{4.20}
$$

As a result, the Noether charge Q_a and Q_b shall give rise to the same NG modes, this kind of modes are called the type-B Nambu-Goldstone modes, and the oridinary modes are then called the e type-A Nambu-Goldstone modes.

In order to determine the number of different Nambu-Goldstone modes, the Watanabe-Brauner (WB) matrix is defined as

$$
M_{ab} = -\sqrt{-1} \langle\psi| [Q_a, j_b^0(x)] |\psi\rangle, \tag{4.21}
$$

which is a matrix of dimension $\dim(G/H)$.

The kernel of the WB matrix characterizes of the NG modes whose order parameter operator are not another Noether current, as a result, we have $n_A = \dim(G/H) - \mathrm{rank}(M)$. Since the number of type-B modes are double counted in the matrix M, we have $n_B = \frac{1}{2}\mathrm{rank}(M)$.

Thus we then have an important question to answer: can we distinguish type-A and type-B NG modes by their properties?

In the relativistic case, since all the particles shall obeys $E^2 = m^2 + p^2$ as P^2 is the Casimir operator of the Poincaré algebra, we must have the linear dispersion relation $E \propto ||k||$ for NG modes. As a result, we could focus on the non-relativistic case for the dispersion relation of the NG modes.

Papers [20][21][22] discover that (at most of the cases) type-A NG modes have linear dispersion relation and type-B NG modes have quadratic dispersion relation. In the remaining part of this seciton, we'll introduce the effective field theory method provided by [21].

Let G be a symmetry group spotaneously broken to a closed Lie subgroup H, then the coset G/H is still a manifold and characterizes the degenerate groud state of the system. With the choice of one element in the coset as the ground state, we select a local coordinates $\{\pi_a\}$ around this point in G/H (a=1,......,dim$G-$dim$H$), and we dentoe $\partial_a = \frac{\partial}{\partial \pi_a}$.

With the G-action acting on G/H, we denote have the fundamental vector field generated by the Lie algebra: $\mathbf{h}_i := h_i^a \partial_a$, $i = 1, \ldots\ldots \mathrm{dim}G$. Clearly, the fundamental vector satifies

$$[\mathbf{h}_i, \mathbf{h}_j] = f_{ij}^k \mathbf{h}_k. \tag{4.22}$$

To describe the excitations propagating on G/H, it suffices to understand how theories with field $\pi^a(x)$ on the (non-Lorentzian) space-time fluatuates from the origin we pick.

With low energy condition and space isotropy, in the non-relativistic case the high derivative terms contributes negligibly and the odd-spatially-derivatives terms cannot appears. This then constrain the Lagragian as

$$\mathcal{L} = c_a(\pi)\dot{\pi}^a + \frac{1}{2}\overline{g}_{ab}(\pi)\dot{\pi}^a\dot{\pi}^b - \frac{1}{2}g_{ab}(\pi)\partial_r\pi^a\partial_r\pi^b + O(\partial_t^3, \partial_t\partial_r^2, \partial_r^4), \tag{4.23}$$

for spatial index r, and some G-invariant symmetric tensor $\overline{g}$ and g, that is,

$$\partial_c g_{ab} h_i^c + g_{ac}\partial_b h_i^c + g_{cb}\partial_a h_i^c = 0. \tag{4.24}$$

Under the infinitesimal transformation $\delta\pi^a = h_i^a\theta^i$, we have

$$\begin{aligned}
\delta\mathcal{L} &= (\partial_b c_a)\dot{\pi}^a\delta\pi^b - (\partial_b c_a)\dot{\pi}^b\delta\pi^a + \partial_t\Big(c_a(\pi)\delta\pi^a\Big) \\
&+ \frac{1}{2}\partial_c\overline{g}_{ab}(\pi)\dot{\pi}^a\dot{\pi}^b\delta\pi^c + \overline{g}_{ab}(\pi)\dot{\pi}^a\delta\dot{\pi}^b \\
&- \frac{1}{2}\partial_c g_{ab}(\pi)\partial_r\pi^a\partial_r\pi^b\delta\pi^c - g_{ab}(\pi)\partial_r\pi^a\partial_r\delta\pi^b \\
&= (\partial_b c_a - \partial_a c_b)\dot{\pi}^a\delta\pi^b + \partial_t\Big(c_a(\pi)\delta\pi^a\Big) \\
&+ \overline{g}_{ab}(\pi)\dot{\pi}^a\delta\dot{\pi}^b - \overline{g}_{ac}\partial_b(\delta\pi^c)\dot{\pi}^a\dot{\pi}^b \\
&- g_{ab}(\pi)\partial_r\pi^a\partial_r\delta\dot{\pi}^b + g_{ac}\partial_b(\delta\pi^c)\partial_r\pi^a\partial_r\pi^b \\
&= \partial_t\Big(e_i + c_a(\pi)h_i^a\Big)\theta^i.
\end{aligned} \tag{4.25}$$

if and only if

$$(\partial_b c_a - \partial_a c_b)h_i^b = \partial_a e_i \tag{4.26}$$

for some vector field e_i. This gives us the Noether current

$$j_i^0 = e_i - \overline{g}_{ab}h_i^a\dot{\pi}^b. \tag{4.27}$$

By the time invariace of the ground state, we have

$$e_i(0) = \langle j_i^0 \rangle. \tag{4.28}$$

Hence, by definition we have

$$M_{ij} = \langle [Q_i, j_j^0] \rangle = h_i^a \partial_a e_j = (\partial_b c_a - \partial_a c_b) h_i^b h_j^a. \tag{4.29}$$

Now if we expand the single time derivaitve term of the Lagragiang around the origin as a Taylor expansion, i.e.

$$c_a(\pi) = c_a(0) + \partial_b c_a(0)\pi^b + O(\pi^2), \tag{4.30}$$

and write $\partial_b c_a(0) = A_{ab} + S_{ab}$ as aniti-symmetric and symmetric parts, then we'll have

$$c_a(\pi)\dot{\pi}^a = \partial_t \Big(c_a(0)\pi^a + \frac{1}{2} S_{ab}\pi^a\pi^b \Big) + A_{ab}\dot{\pi}^a\pi^b. \tag{4.31}$$

Now at the origin of the chosen point in G/H, if we restrict on the index i on the index of a, then the matrices $X_b^a := h_b^a(\pi = 0)$ are invertiable since the broken generators shall from a basis pf the tangent space at the origin.

This allows us to redefine $\tilde{\pi}^a := (X^{-1})_b^a \pi^b$, and by solving (4.29) we have

$$2X_b^a X_d^c A_{ac} = \rho_{bd} \tag{4.32}$$

and

$$c_a(\pi)\dot{\pi}^a = \frac{1}{2}\rho_{ab}\dot{\tilde{\pi}}^a\tilde{\pi}^b \tag{4.33}$$

mod total derivatives and up to higher order terms.

Since ρ_{ab} is real and anti-symmetric, up to an orthogonal transformation, we can then write

$$\rho = \mathrm{diag}\Big(M_1, \ldots, M_{\mathrm{rank}(\rho)}, 0, \ldots, 0 \Big), \tag{4.34}$$

where

$$M_\alpha = \begin{pmatrix} 0 & \lambda_\alpha \\ -\lambda_\alpha & 0 \end{pmatrix}. \tag{4.35}$$

Hence,

$$c_a(\pi)\dot{\pi}^a = \sum_m \frac{1}{2}\lambda_m \Big(\dot{\tilde{\pi}}^{2\alpha}\tilde{\pi}^{2\alpha-1} - \dot{\tilde{\pi}}^{2\alpha-1}\tilde{\pi}^{2\alpha} \Big). \tag{4.36}$$

And we also use this orthogonal transformation to redefine g and $\overline{g}$, as a result, in the Fourier space we'd have these $\frac{1}{2}\mathrm{rank}(\rho)$ modes to have linear ω correspond to k^2, and other NG modes starts with quadratic term ω^2 corresponding to k^2 term in the Lagragian.

Therefore, type-A NG modes shall have linear dispersion and type-B NG modes shall have quadratic dispersion in the non-relativistic case.

Heisenberg ferro/anti-ferromagnet model in fact serves as an good example of this classification of different types NG modes.

By choosing z as the unbroken axis, we have the rotation symmetry generated by $S^x and S^y$ is spotaneously broken. Also, the Watanabe-Brauner matrix then identifies with the expectation value of S^z, or magnetization. In anti-ferromagnetic side, the magnetization is trivial thus the WB matrix is vanishing, and this corresponds to the two independent spin wave with linear dispersion as two type-A NG modes.

On the other hand, with ferromagnetic order the magnetization is nontrivial, thus, so is the MB matrix, which corresponds to the single type-B mode with quadratic dispersion.

4.4 The Mermin-Wagner-Coleman Theorem

From the previous section, we learnt that the Nambu-Goldstone modes have either linear or quadratic disperion relation around the low energy region $k = 0$. As a result, the correlation function of the NG modes around the IR region is well-determined, and if we would like it to have to IR divergence, there's then a minimum of the dimension for a qunatum system to meet spontaneous symmetry breaking and get NG modes.

In the case of relativistic quantum field theory, since the form of Lagragian is well controlled, Sidney Coleman [23] provided a general proof that in dimenion 2, there cannot be Nambu-Goldstone modes. In the first part of the section, we'll introduce this proof.

It seems that it lack of a general proof for statistical system on the limit of dimension to have NG modes, nevertheless, in second part of this section, we'll introduce the idea in statistical system with finite temperature.

For conveniece, we firstly define

$$k_\pm = k_0 \pm k_1, \tag{4.37}$$

so that the boost in 2 dimension Minkowski space-time is given by

$$k_\pm \to e^{\pm\alpha} k_\pm. \tag{4.38}$$

Also, we define peaked test function as test functions with compact support such that

$$f(x) \leq f(y) \text{ if } x/y \geq 1. \tag{4.39}$$

From this, for any positive Lorentz-invariant distribution $F(k_+, k_-)$ and positve number λ, we have that

$$\int dk_- f(\lambda k_-) F(k_+, k_-) \tag{4.40}$$

is a positive and monotonically decreasing function of λ, hence, under the limit $\lambda \to \infty$, the integral exists and is still positive. On the other hand, λ can be viewed as a boost, thus, we have for any test function $g(k_+)$, the integral

$$\lim_{\lambda\to\infty} \int dk_+ dk_- f(\lambda k_-) F(k_+, k_-) g(k_+) \tag{4.41}$$

has the same value with $g(e^\alpha k_+)$ by the Lorentz invariance of F. And the only possibility is

$$\lim_{\lambda\to\infty} \int dk_- f(\lambda k_-) F(k_+, k_-) = c\delta(k_+), \tag{4.42}$$

for some positive number c.

Let $j_\mu(x)$ be the Noether current corresponding to the breaking symmetry and $\phi(x)$ the time independent order parameter field, then we can denote the following correlation functions as

$$F(k) = \int d^2x e^{\sqrt{-1}kx} \langle \phi(x)\phi(0) \rangle, \tag{4.43}$$

$$F_\mu(k) = \int d^2x e^{\sqrt{-1}kx} \langle j_\mu(x)\phi(0) \rangle, \tag{4.44}$$

and

$$F_{\mu\nu}(k) = \int d^2x e^{\sqrt{-1}kx} \langle j_\mu(x) j_\nu(0) \rangle. \tag{4.45}$$

The conservation of Noether current implies

$$k^\mu F_\mu(k) = 0 \tag{4.46}$$

and restricts the forms of F_μ as

$$F_\mu(k) = \sigma k_\mu \delta(k^2)\theta(k_0) + \epsilon_{\mu\nu} k^\nu \rho(k^2)\theta(k_0), \tag{4.47}$$

for some number σ and scalar distribution $\rho(k^2)$. So we have

$$\langle j_\mu(x)\phi(0)\rangle = \frac{1}{(2\pi)^2} \int d^2k e^{-\sqrt{-1}kx} F_\mu(k). \tag{4.48}$$

By definition, we have the order parameter as

$$\langle [Q, \phi(0)] \rangle = \sqrt{-1} \int dx_1 \langle [j_0(x_0, x_1), \phi(0)] \rangle = \frac{\sqrt{-1}\sigma}{4\pi} \tag{4.49}$$

Now for the state vector

$$\int d^2x h(x) \Big(a j_0(x) + b\phi(x) \Big) |0\rangle \tag{4.50}$$

given by some test function $h(x)$ must have a positive norm, hence, F and F_{00} are both positive distribution, and with $\tilde{h}(k)$ denoting the Fourier dual of h, we also have the inequality

$$\Big(\int d^2k F(k) |\tilde{h}(k)|^2 \Big) \Big(\int d^2k F_{00}(k) |\tilde{h}(k)|^2 \Big) \geq \Big| \int d^2k F_0(k) |\tilde{h}(k)|^2 \Big|^2 \tag{4.51}$$

Choosing

$$\tilde{h}(k) = f(\lambda k_-) g(k_+) + f(\lambda k_+) g(k_-), \tag{4.52}$$

then we have

$$\int d^2k F_0(k) |\tilde{h}(k)|^2 = \sigma |f(0)|^2 \int dk_+ |g(k_+)|^2, \tag{4.53}$$

which is independent of λ.

However, by the lemma, as λ goes to infinity, we have $\int d^2k F(k)|\tilde{h}(k)|^2$ goes to zero since the two terms in $\tilde{h}$ no longer share common support. On the other hand, since $F_0 0$ is a positive distribution, the related integral is monotonically decreasing as a function of λ, and this require the left-hand side of the inequality to be zero, as a result, we must have

$$\sigma = 0. \tag{4.54}$$

And this implies the symmetry is not broken by definition.

In finite temperature, we will follows the original proof[24] and firstly prove the Bogoliubov inequality. To prove this identity, we firstly consider a complete set of energy eigenstates $\{|n\rangle\}$ with energy E_n corresponding to Hamiltonian H and the semi-positive bilinear form of operators defined by

$$(A, B) = \sum_{n \neq m} \langle n| A^\dagger |m\rangle \langle m| B |n\rangle \frac{W_m - W_n}{E_m - E_n}, \tag{4.55}$$

where

$$W_n = \frac{e^{-\beta E_n}}{Tre^{-\beta H}}. \tag{4.56}$$

For semi-positive bilinear forms, the Schwarz inequality is still valid, that is,

$$|(A, B)|^2 \leq (A, A)(B, B). \tag{4.57}$$

Choose B to be $[C^\dagger, H]$, then we have

$$\begin{aligned}(A,B) &= \sum_{n\neq m} \langle n| A^\dagger |m\rangle \langle m| [C^\dagger, H] |n\rangle \frac{W_m - W_n}{E_m - E_n} \\ &= \sum_{n\neq m} \langle n| A^\dagger |m\rangle \langle m| C^\dagger |n\rangle \Big(W_m - W_n\Big) \\ &= \sum_m \langle m| C^\dagger A^\dagger |m\rangle W_m - \sum_n \langle n| A^\dagger C^\dagger |n\rangle W_n \\ &= \langle [C^\dagger, A^\dagger]\rangle.\end{aligned} \tag{4.58}$$

So we also have

$$(B,B) = \langle [C^\dagger, [C^\dagger, H]^\dagger]\rangle = \langle [C^\dagger, [H, C]]\rangle. \tag{4.59}$$

On the other hand, we have

$$\begin{aligned}0 &< \frac{W_n - W_m}{E_n - E_m} \\ &= \frac{1}{Z}\Big(\frac{e^{-\beta E_m} + e^{-\beta E_n}}{E_n - E_m}\Big)\Big(\frac{e^{-\beta E_m} - e^{-\beta E_n}}{e^{-\beta E_m} + e^{-\beta E_n}}\Big) \\ &= \frac{W_n + W_m}{E_n - W_m} tanh(\frac{\beta}{2}(E_n - E_m)) \\ &\leq \frac{\beta}{2}(W_n + W_m),\end{aligned} \tag{4.60}$$

by $tanh(x) < x$ for $x > 0$. Hence,

$$\begin{aligned}(A,A) &= \sum_{n\neq m} \langle n| A^\dagger |m\rangle \langle m| A |n\rangle \frac{W_m - W_n}{E_m - E_n} \\ &< \frac{\beta}{2}\sum_{n,m} \langle n| A^\dagger |m\rangle \langle m| A |n\rangle (W_n + W_m) \\ &= \frac{\beta}{2}\sum_n \langle n| (A^\dagger A + AA^\dagger) |n\rangle W_n \\ &= \frac{\beta}{2}\langle \{A, A^\dagger\}\rangle.\end{aligned} \tag{4.61}$$

So from Schwarz inequality, we have

$$\frac{\beta}{2}\langle \{A, A^\dagger\}\rangle \langle [[C,H], C^\dagger]\rangle \geq (A,A)(B,B) \geq |(A,B)|^2 = |\langle [C,A]\rangle|^2. \tag{4.62}$$

Let's now consider the isotropic Heisenberg model

$$H = -\sum_{i,j} J_{ij}\mathbf{S_i}\mathbf{S_j} - b\sum_i S_i^z, \tag{4.63}$$

of spin S, and we are're interested to study the magnetization

$$M_S(\beta) = \lim_{b\to 0}\langle \sum_i S_i^z\rangle_{\beta,b}. \tag{4.64}$$

Firstly, we substitute A and C as the ladder operators in momentum basis, that is

$$A = S_{-k}^- = \sum_i S_i^- e^{\sqrt{-1}kR_i}, \tag{4.65}$$

and

$$C = S_k^+ = \sum_i S_i^+ e^{-\sqrt{-1}kR_i}, \tag{4.66}$$

thus,

$$[S_{k_1}^+, S_{k_2}^-] = \sum_{i,j} [S_i^+, S_j^-] e^{-\sqrt{-1}(k_1 R_i + k_2 R_j)} = 2\hbar \sum S_i^z e^{-\sqrt{-1}(k_1+k_2)R_i} = 2\hbar S_{k_1+k_2}^z, \tag{4.67}$$

and

$$[S_{k_1}^z, S_{k_2}^\pm] = \sum_{i,j} [S_i^z, S_j^\pm] e^{-\sqrt{-1}(k_1 R_i + k_2 R_j)} = \pm\hbar \sum S_i^\pm e^{-\sqrt{-1}(k_1+k_2)R_i} = \pm\hbar S_{k_1+k_2}^\pm. \tag{4.68}$$

Firstly, we have

$$\begin{aligned} \langle [C, A] \rangle &= \langle [S_k^+, S_{-K}^-] \rangle \\ &= 2\hbar \langle S_0^z \rangle \\ &= 2\hbar \langle \sum_i S_i^z \rangle \\ &= 2\hbar M_S(\beta, b), \end{aligned} \tag{4.69}$$

and

$$\begin{aligned} \sum_k \langle \{A, A^\dagger\} \rangle &= \sum_k \langle \{S_{-k}^-, S_k^+\} \rangle \\ &= \sum_k \sum_{i,j} e^{\sqrt{-1}k(R_i - R_j)} \langle (S_i^- S_j^+ + S_i^+ S_j^-) \rangle \\ &= 2N \sum_i \langle (S_i^x S_i^x + S_i^y S_i^y) \rangle \\ &\leq 2N \sum_i \langle \mathbf{S}_i^2 \rangle \\ &= 2\hbar^2 N^2 S(S+1). \end{aligned} \tag{4.70}$$

Since

$$\langle [[C, H], C^\dagger] \rangle = \sum_{i,j} e^{\sqrt{-1}k(R_i - R_j)} \langle [[S_i^+, H], S_j^-] \rangle, \tag{4.71}$$

we first calculate

$$[S_i^+, H] = -\hbar \sum_m J_{mi} (2 S_m^+ S_i^z - S_m^z S_i^+ - S_i^+ S_m^z) + \hbar b S_i^+ \tag{4.72}$$

so

$$\begin{aligned} [[S_i^+, H], S_j^-] = 2\hbar^2 \sum_m J_{mj} \delta_{ij} (S_m^+ S_j^- + 2 S_m^z S_j^z) \\ - 2\hbar^2 J_{ij} (S_i^+ S_j^- + 2 S_i^z S_j^z) + 2\hbar^2 b \delta_{ij} S_j^z. \end{aligned} \tag{4.73}$$

Hence, we have

$$\langle [[C, H], C^\dagger] \rangle = 2\hbar^2 b \langle \sum_i S_i^z \rangle + 2\hbar^2 \sum_{ij} J_{ij} \Big(1 - e^{-\sqrt{-1}k(R_i - R_j)}\Big) \langle S_i^+ S_j^- + 2 S_i^z S_j^z \rangle. \tag{4.74}$$

Since we have (B, B) must be real, we can add it with its complex conjugate, this gives us

$$\begin{aligned} \langle [[C, H], C^\dagger] \rangle &= 4\hbar^2 b \langle \sum_i S_i^z \rangle + 4\hbar^2 \sum_{ij} J_{ij} \Big(1 - \cos(k(R_i - R_j))\Big) \langle \mathbf{S}_i \mathbf{S}_j + S_i^z S_j^z \rangle \\ &\leq 4\hbar^2 |b M_S(\beta, b)| + 4\hbar^2 \sum_{ij} \frac{J_{ij} k^2}{2} |R_i - R_j|^2 \hbar^2 \Big(S(S+1) + S^2\Big) \end{aligned} \tag{4.75}$$

If we define

$$Q = \frac{1}{N} \sum_{ij} \frac{J_{ij} k^2}{2} |R_i - R_j|^2, \tag{4.76}$$

which shall converges if J_{ij} decays fast enough for large $|i-j|$, then

$$\langle [[C, H], C^\dagger] \rangle \leq 4\hbar^2 |b M_S(\beta, b)| + 4Nk^2\hbar^4 Q S(S+1). \tag{4.77}$$

By the Bogoliubov inequality, we have

$$\begin{aligned}
\beta\hbar^2 N^2 S(S+1) &\geq \frac{\beta}{2} \sum_k \langle \{A, A^\dagger\} \rangle \\
&\geq \sum_k \frac{|\langle [C, A] \rangle|^2}{\langle [[C, H], C^\dagger] \rangle} \\
&\geq \sum_k \frac{4\hbar^2 M_S(\beta, b)^2}{4\hbar^2 |b M_S(\beta, b)| + 4Nk^2\hbar^4 Q S(S+1)} \\
&\geq \sum_k \frac{M_S(\beta, b)^2}{|b M_S(\beta, b)| + Nk^2\hbar^2 Q S(S+1)}
\end{aligned} \tag{4.78}$$

In the dynamical limit, we have

$$S(S+1) \geq \frac{m_S(\beta, b)^2 \Omega(D)}{\beta} \int_0^{k_0} \frac{k^{D-1} dk}{|b m_S(\beta, b)| + k^2\hbar^2 Q S(S+1)} \tag{4.79}$$

for some constant Ω depends only on D.

Since $m_S = M_S/N$ is bounded, we must have $m_S(\beta, b = 0) = 0$ provided that $D \leq 2$, otherwise as $b = 0$

$$S(S+1) \geq \frac{m_S(\beta, 0)^2 \Omega(D)}{\beta\hbar^2 Q S(S+1)} \int_0^{k_0} k^{D-3} dk = \infty, \tag{4.80}$$

that is, the IR divergence will distropy the order.

In fact, since from the Nambu-Goldstone theorem we expect that the Goldstone modes are well-defined in low wave-number i.e. IR region, the correlation function, either of type-A or type-B, will face the IR divergence if the spatial dimension is equal or smaller than 2, thus, we expect the continuous symmetry cannot be broken in $D = 1, 2$.

This is the so-called Mermin-Wagner theorem.

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