Homomorphism - generalization of an isomorphism.

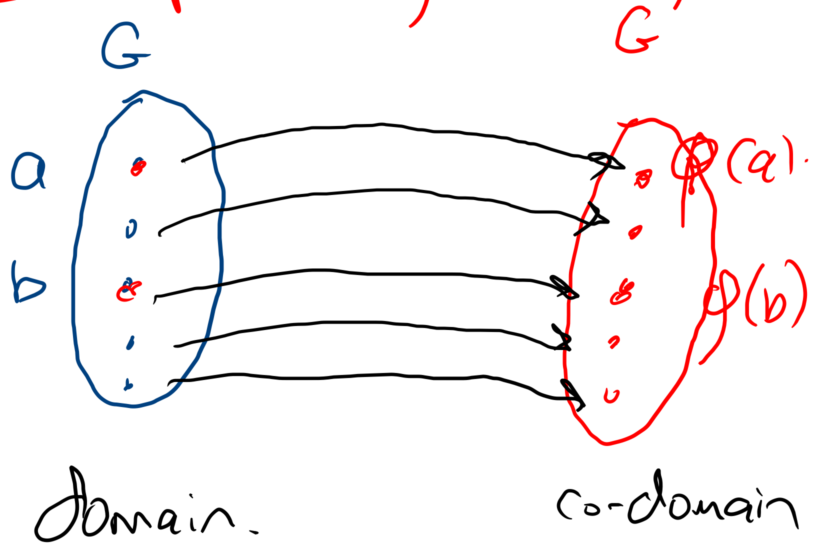
Isomorphism: φ

1.) Bijection - i) injective (one-to-one).

ii) surjective (onto).

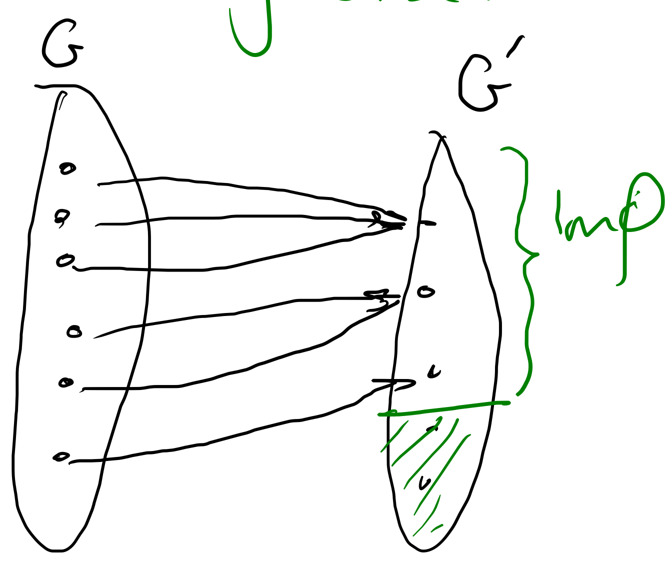
2.) Compatibility cond^n.

$$\boxed{\varphi(a \circ_G b) = \varphi(a) \circ_{G'} \varphi(b)}$$

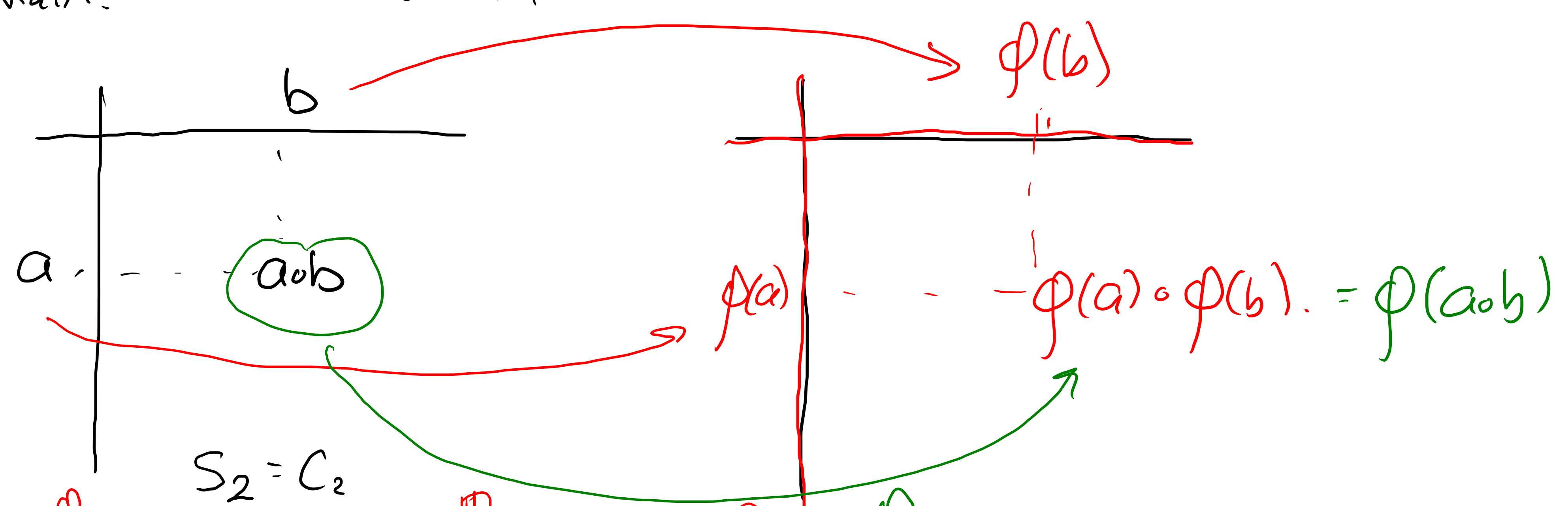


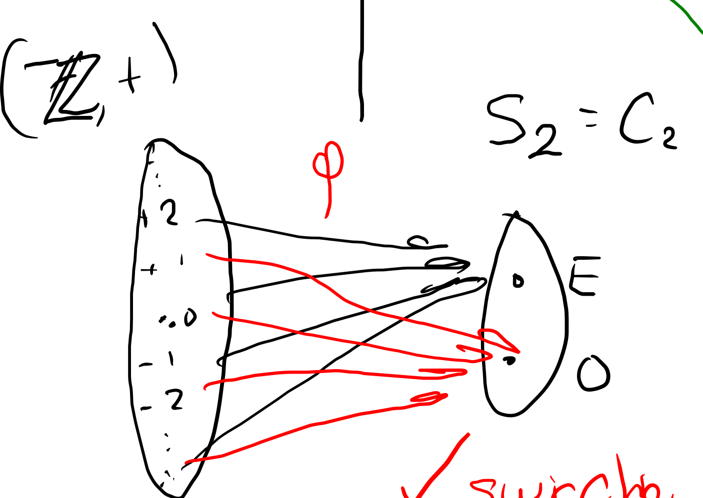
1.) Not a bijection

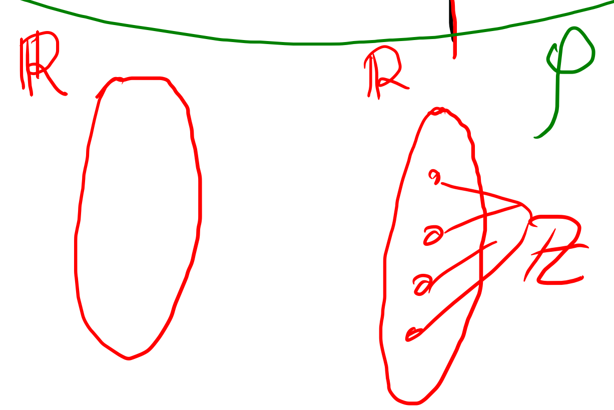
$\to$ not injective.



2.) not surjective







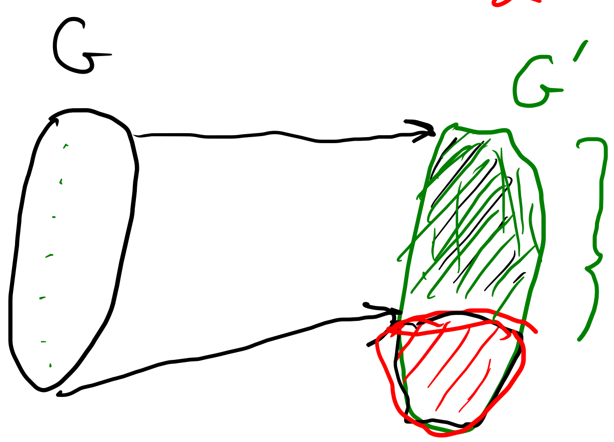
Homomorphism: 1) Keeps the compatibility condⁿ

$$\phi(a \circ b) = \phi(a) \circ \phi(b)$$

2) No longer bijective.

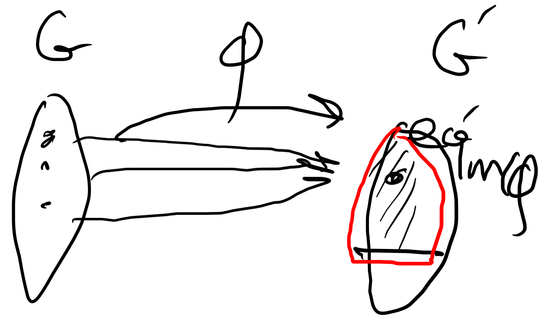
$$\boxed{\phi : G \longrightarrow \mathrm{Im}(\phi)}$$

$$G \longrightarrow G'$$



$$\mathrm{Im}(\phi) \equiv \{ \phi(g) \mid g \in G \}$$

Claim: If ϕ is not surjective it can be reduced to a homo$^{\underline{m}}$ on the image of ϕ, $\mathrm{Im}(\phi)$.
$\rightarrow$ $\mathrm{Im}(\phi)$ is a group in it's own right & satisfies the compatability cond$^{\underline{n}}$.



2 Required Facts:

1.) indentity maps to identity.

Proof: TRICK: For any elt x in a group where $\boxed{x \circ x = x}$ holds. then $x = e_G$.

$$x^{-1} \circ x \circ x = x^{-1} \circ x$$

$$\boxed{x = e_G}$$

Want to show $\phi(e_G) = e_{G'}$.

$$\phi(e_G) \circ \phi(e_G) = \phi(e_G \circ e_G) \quad \text{compt}^{y} \text{ cond.}$$
$$= \phi(e_G).$$

$$\boxed{\phi(e_G) = e'_G}$$

<u>Fact</u>: Inverses map to inverses.

Need to show: $\underset{\text{c.c.}}{\phi(a) \circ \phi(a^{-1})} = e_{G'}$

$$\Rightarrow \phi(a^{-1}) = [\phi(a)]^{-1}$$

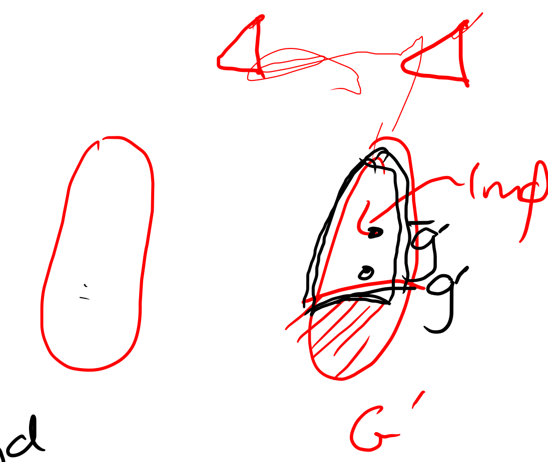
$$\Rightarrow \phi(a) \circ \phi(a^{-1}) = \phi(a \circ a^{-1})$$

$$= \phi(e_G)$$

$$\boxed{\phi(a) \circ \phi(a^{-1}) = e_{G'}}$$

$\therefore$ $\phi(a^{-1})$ in G' is the inverse of $\phi(a)$.

Claim: $\mathrm{Im}(\varphi)$ is a subgroup of G': $\boxed{\mathrm{Im}(\varphi)} < G'$.



Recall def: $\mathrm{Im}\,\varphi \equiv \{\varphi(g) \mid g \in G\}$.

1.) Identity exist: YES $\because$ $\varphi(e_G) = e_{G'}$.

2.) Associacity ✓ YES

3.) Say $a' = \varphi(a)$, then $a' \in \mathrm{Im}\,\varphi$. Inverse ✓

Show $a'^{-1} \in \mathrm{Im}(\varphi)$.

But $a'^{-1} = \varphi(a^{-1})$.

$\therefore a'^{-1} \in \mathrm{Im}\,\varphi$ $\because$ $a^{-1} \in G$.

4.) Closure:

To show given g' and $\bar{g}' \in \mathrm{Im}(\varphi)$.

then $g' \circ \bar{g}' \in \mathrm{Im}\,\varphi$.

$\therefore \exists\ g$ & $\bar{g} \in G$ s.t. $g' = \varphi(g)$

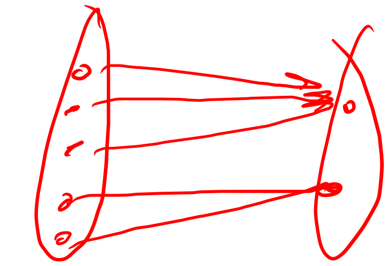
$\bar{g}' = \varphi(\bar{g})$.

Then $g' \circ \bar{g}' = \varphi(g) \circ \varphi(\bar{g})$.

$= \varphi(\underbrace{g \circ \bar{g}}_{G})$ c.c.

$\therefore \boxed{g' \circ \bar{g}' \in \mathrm{Im}\,\varphi}$.

Case 2: Failure for ϕ to be injective (not one-to-one)



Example: $\phi : (\mathbb{Z}, +) \longrightarrow S_2$

Recall $\mathbb{Z} = \{0, +1, -1, +2, -2, \dots\}$

$+$	y
x	$x+y$

$\xrightarrow{\phi}$

$S_2 = \{+1, -1\}$ under $\times$

$\times$	$+1$	-1
$+1$	$+1$	-1
-1	-1	$+1$

$\circ$	E	O
E	E	O
O	O	E

$+_2$	0	1
0	0	1
1	1	0

$\circ$	e	τ
e	e	τ
τ	τ	e

$\cong$ $\cong$

Define: $\phi(x) = \begin{cases} +1 & \text{if } x \text{ even} \\ -1 & \text{if } x \text{ odd.} \end{cases}$

Check compatability cond^n: $\phi(x+y) = \phi(x) \times \phi(y) \quad \forall\, x, y \in \mathbb{Z}$

Case 1: x & y both even:

$\phi(x+y) = +1$ ✓

$\times \quad \left.\begin{array}{l} \phi(x) = +1 \\ \phi(y) = +1 \end{array}\right\} = +1$

Case 2: x & y both odd.

$\phi(x+y) = +1$ ✓

$\left.\begin{array}{l} \phi(x) = -1 \\ \phi(y) = -1 \end{array}\right\} = +1$

Case 3: One odd one even.

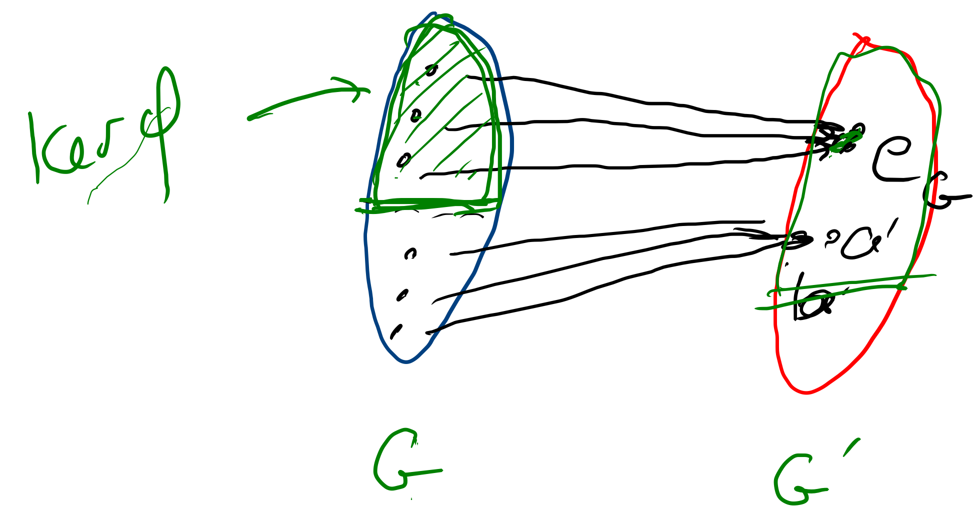
$\phi(x+y) = -1$ ✓

$x =$ odd, $y =$ even

$\left.\begin{array}{l} \phi(x) = -1 \\ \phi(y) = +1 \end{array}\right\} = -1$

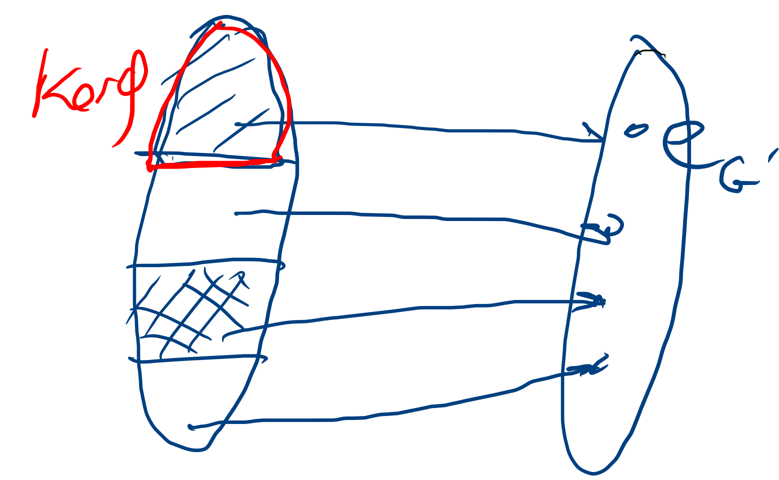
Kernel of a homomorphism.

Def^n: $\ker\phi = \{g \in G \mid \phi(g) = e_{G'}\}$.



Claim: the kernel $\ker\phi$ is a subgroup of G,

$\ker\phi < G$.



1. Assoc. ✓
2. Identity: Is e_G in $\ker\phi$?

   YES $\because$ $\phi(e_G) = e_{G'}$ ✓

3. Closure: To show if a and $b \in \ker\phi$ then $a \circ b \in \ker\phi$.

   $\phi(a \circ b) = e_{G'}$ if $\phi(a) = e_{G'}$ and $\phi(b) = e_{G'}$

But by C.C. $\phi(a \circ b) = \phi(a) \circ \phi(b)$ $\therefore$ $a \circ b \in \ker\phi$.

$= e_{G'} \circ e_{G'}$

$= e_{G'}$

4.) Inverse ✓: <u>If $\varphi(a) = e_{G'}$</u> then $\varphi(a^{-1}) = e_{G'}$

$$\boxed{\begin{aligned} &\varphi(a^{-1}) = \varphi(a)^{-1} \\ &\text{If } \varphi(a) = e_{G'}. \\ &\therefore \varphi(a)^{-1} = e_{G'} \\ &\therefore \varphi(a^{-1}) = e_{G'} \\ &\therefore a^{-1} \in \ker\varphi. \end{aligned}}$$