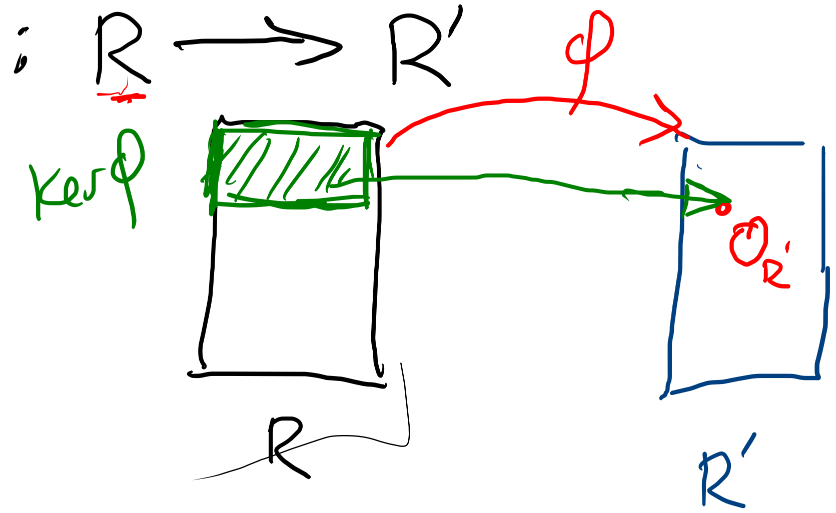
Dec. 4/2022 - Ideals in Rings

→ • analogous to normal subgroups in groups.

• recall normal subgroup were kernels of homomorphisms

Def^n: kernel of a ring homomorphism, $\varphi : R \longrightarrow R'$

$$\ker \varphi = \{ a \in R \mid \varphi(a) = 0_{R'} \}$$



Properties of ker φ

① It's a subgroup of R under +

i) - since R is already an Abelian group under add^n

→ then ker φ is an Abelian subgroup of R ✓

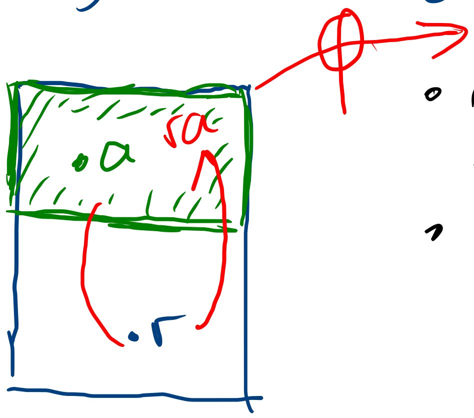
ii — it's a normal subgroup under +.

→ R is Albeian group under +.

② What mult^n? — for the kernel.

$\ker\varphi = \{a \in R \mid \varphi(r) = 0_{R'}\}$

It's closed under mult^n wrt the larger group R.



R- comm've.

- pick an elt $a \in \ker\varphi$
- " " " $r \in R$

Then: $ar \in \ker\varphi$

claim $\boxed{ra \in \ker\varphi}$

} $\ker\varphi$ acts as an absorber under mult^n

Why? Where does ar get mapped to?

$$\varphi(ra) = \varphi(r) \cdot_{R'} \varphi(a)$$
$$= \varphi(r) \cdot 0_{R'}$$
$$= 0_{R'}$$

$ra \in \ker\varphi$

$\boxed{\ker\varphi \text{ is closed under mult}^n.}$

③ Kernel is not in general a subring.

- because it usually doesn't contain 1_R.

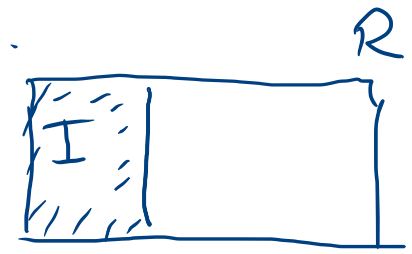
Why?

Where does $1_R \xrightarrow{\varphi} 1_{R'} \neq 0_{R'}$

Def^n of an Ideal

- defined in terms of being a mult^n absorber.
- $I \subseteq R$ with properties:

① Additive subgroup – Abelian ✓



② Closed under mult^n ✓

$\forall a \in I$ and $r \in R$

$ar \in I$

$ra \in I$

Note: $I \trianglelefteq R$

Example:

Pick $R = \mathbb{Z}$

$I = 3\mathbb{Z} = \{\dots, -6, -3, 0, 3, 6, \overset{ar}{9}, \dots\}$

Claim: I is an ideal of $\mathbb{Z}$.

$n\mathbb{Z} < \mathbb{Z}$

Pick $r = 5$

$3\mathbb{Z} = (3)$

$= \{3 \cdot n \mid n \in \mathbb{Z}\}$

$a = 3 \cdot n$ where $n \in \mathbb{Z}$

$r \in \mathbb{Z}$

$a \cdot r = 3(n \cdot r)$

$\Downarrow$

$a \cdot r \in I$

Example of How to Construct Ideals:

① Principle Ideals: - generated by a single elt $a \in R$

- choose an elt $a \in R$.
- denote the ideal generated by r as $(a) \triangleleft R$.

$\rightarrow \boxed{(a) \cong \{ar \mid r \in R\} \cong aR}$

Claim (a) is an ideal. → 3 req's to be an ideal.

i) (a) is a subset R.

2) Addive subgroup of R.

- closure - take two typical elts of (a):

$ar_1 + ar_2 = a(r_1 + r_2) \quad r_1, r_2 \in R$

$= ar_3$ → def^n of being in (a)

- Identity: is 0_R in (a)?
- Inverse: pick $ar \in (a)$

Is the addive in (a)?

→ What's the addive inverse of ar $\overset{-1}{\Rightarrow} a(-r)$.

Why: $ar + a(-r) = a(r + (-r))$

$= a0$

$= 0$

$\therefore a(-r)$ is the inverse of ar.

$\in (a)$

- associve under + ✓.

3) Closed under mult^{n}?

- pick an elt of (a): call it $a \cdot r$ for some $r \in R$
- mult. by another $r' \in R$.
- where is $[ar] \cdot r' = arr'$

$$= a(rr')$$
$$= ar'' \in (a).$$

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② Two special cases – trivial but important down the road.

✓ - zero ideal

- unit ideal.

- zero ideal $\equiv (0) = \{0\}$.

$$= \{0 \cdot r \mid r \in R\}.$$
$$= \{0 \mid r \in R\}$$
$$= \{0\}. \longrightarrow \text{ideal}$$

- unit ideal: $(1) = \{1 \cdot r \mid r \in R\}$

$= \{r \mid r \in R\}$.

$\boxed{(1) = R}$

$\boxed{(0) = \{0\}}$ ← proper ideal.

- recall a unit in a ring → an elt with a multi've inverse.

• Claim $(u) = R$ where u is a unit in R.

$(u) = \{u \cdot r \mid r \in R\}$.

$\Rightarrow u \cdot u^{-1} = 1$.

$\therefore$ we know $1 \in (u)$.

- recall any elt of an ideal mult'd by any elt in the ring has to be (where?) in ideal.

$\therefore$ $1 \cdot r$ has to be in the ideal

→ R → $(u) = (1) = R$.

Example concrete.

① • Pick ring $R = \mathbb{Z}[x]$ = {set of all finite poly's in x with integer coeff's}

$= \{a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n \mid a_i \in \mathbb{Z},\ n \text{ is finite}\}$.

• consider the PI generated by x: (x)

$(x) = \{x \cdot f(x) \mid f(x) \in \mathbb{Z}[x]\}$

$= \{x \cdot (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)\}$

$$\boxed{(x) = \{a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_n x^{n+1}\}}$$

← all poly's in $\mathbb{Z}[x]$ where no constant term.

② PI: $(x-2) = \{(x-2) \cdot f(x) \mid f(x) = \mathbb{Z}[x]\}$

x^2+1 · $f(x)$

= {all poly's with $(x-2)$ as factor.

= all poly's with a root of 2.

(x^2-3) = set of all poly's with x^2-3 as factor

= set of all poly's with roots of $\pm\sqrt{3}$.

Say we're interested in (solving) poly's with roots $\pm i$.

(x^2+1). ~~PI~~.

$x = i$

$x^2 = -1$

$x^2 + 1 = 0$