# Acceleration in SR & the Rindler Metric (0)

① Co-ord Systems & geometry

② Postulates 1. & 2.

③ Galilean transf^n's

④ Ex. of $F=ma$ same in IMF's. under $t=t'$ assumption.

⑤ Consequence of $c$ finite $\rightarrow$ Lorentz transformation.

⑥ Proper time & 4-velocities. & acceleration.

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# SR – Basic Kinematics.

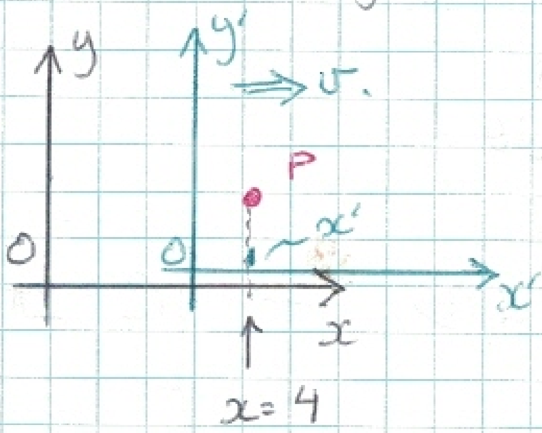
Two assumptions: ① Laws of physics in uniformly moving reference frames are the same.

② The speed of light in all UMRF is the same.

① Galileo's idea of a ship in harbour or at sea

→ someone in an enclosed cabin can't tell
→ the juggling test.

②




• say frames overlap at $t=0$

Then:

$$\boxed{\begin{aligned} x'(t') &= x - vt \\ y'(t') &= y \\ t' &= t \end{aligned}}$$

Newton's Law: $F = ma$ in $O$.

In $O'$: $F' = ma'$

$$= m\frac{d^2 x'(t)}{dt'}$$

$$= m\frac{d^2}{dt^2}(x(t) - vt)$$

$$= m\frac{d^2}{dt^2}x(t)$$

$$= ma$$

$$F = ma$$

$$= m\frac{d^2 x(t)}{dt^2}$$

$$= m\frac{d^2}{dt^2}(x'(t') + vt)$$

$$= m\frac{d^2}{dt'^2}(x'(t') + vt')$$

$$= m\frac{d^2}{dt'^2}x'(t')$$

$$= ma'$$

$$= F'$$

$\boxed{F' = ma'}$ in $O'$.

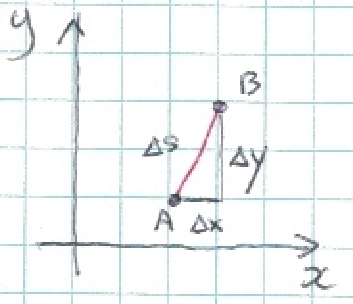
• Also conservation of momentum:

$$\sum_i m_i v_i = \text{const.} \iff \sum_i m_i v_i' = \text{const.}$$

Different?

# Geometry & Coordinate Systems

## Euclidean Geometry of the Plane:

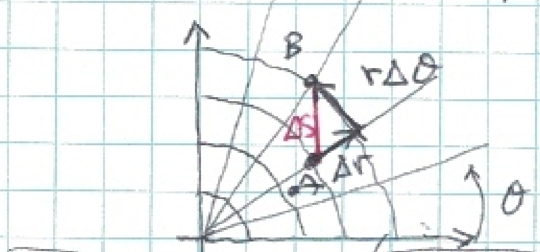



- co-ord system assigns unique labels to points.
- we want to be able to measure a distance btwn pts.

$$\Rightarrow |\overline{AB}|^2 \equiv \Delta s^2 = \Delta x^2 + \Delta y^2.$$

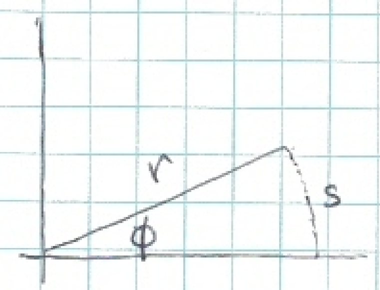
- infinitessimally this is $\boxed{ds^2 = dx^2 + dy^2}$ ← called the metric of this particular geometry in this particular co-ord sys.

## Polar co-ords of plane:




Now $\Delta s^2 = \Delta r^2 + (r\Delta\theta)^2$

$$\boxed{ds^2 = dr^2 + r^2 d\theta^2}$$




Note: • It's the same geometry (of a flat plane), just a different co-ord system.

Recall an angle in radians is <u>defined</u> as $\phi = \frac{s}{r}$

$\Rightarrow$ arclength $\boxed{s = r\phi}$

And for a given r, $\Delta s = r\Delta\phi$

- ds is the same line element $\overline{AB}$'s length in both co-ord systems.

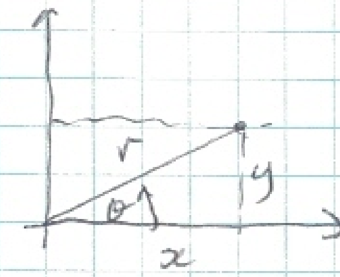
$\boxed{\text{IT'S AN INVARIANT}}$

More formally we think of it as a coord system transformation:

$(r,\theta) \leftrightarrow (x,y)$

Where

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}} \quad (*)$$





$\frac{x}{r} = \cos\theta$

$\frac{y}{r} = \sin\theta$

$\therefore\ dx(r,\theta) = \frac{\partial x(r,\theta)}{\partial r} dr + \frac{\partial x(r,\theta)}{\partial \theta} d\theta$

$dx = \cos\theta\, dr - r\sin\theta\, d\theta.$

And $\rightarrow$ $dy = \sin\theta\, dr + r\cos\theta\, d\theta.$

$dx^2 = \cos^2\theta\, dr^2 - 2r\sin\theta\cos\theta\, dr\,d\theta + r^2\sin^2\theta\, d\theta^2$

$dy^2 = \sin^2\theta\, dr^2 + 2r\sin\theta\cos\theta\, dr\,d\theta + r^2\cos^2 d\theta^2$

$ds^2 = dx^2 + dy^2 = (\cos^2 + \sin^2)\, dr^2 + r^2(\sin^2 + \cos^2)\, d\theta^2$

$$\boxed{ds^2 = dr^2 + r^2 d\theta^2}$$ ← Still the flat Euclidean plane!

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Note: The transᵃ $(*)$ is a motivated (Cartⁿ vs. Polar coords) transformation.

What about an arbitrary $\begin{cases} x = f(u,v) \\ y = g(u,v) \end{cases}$ ← ? Still Flat

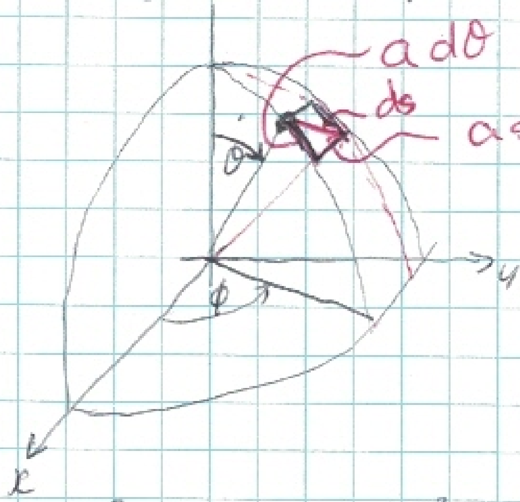
Still: $ds^2 \equiv dx^2 + dy^2 \Rightarrow$ where $dx = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$

$dy = \partial g/\partial u\, du + \partial g/\partial v\, dv$

~~Ex.~~ Problem 2.7 Hartle.

# Non-Euclidean (not flat) Geometries.

Geometry of the Sphere: radius $a$, spherical co-ords.




$$\boxed{ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta\, d\phi}$$

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

OR $\boxed{ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2}$

After normalizing to $a = 1$.

Cf.
$$\left.\begin{aligned} ds^2 &= dx^2 + dy^2 \\ ds^2 &= dr^2 + r^2 d\theta^2 \end{aligned}\right\} \text{Flat}$$

$\Rightarrow$ GOTO 5 (back of pg. 1)

Final example of co-ord transfo's.

Parabolic Coords:

- transf$^n$ $\begin{cases} x = \mu\nu & ① \\ y = \frac{1}{2}(\mu^2 - \nu^2) & ② \end{cases}$

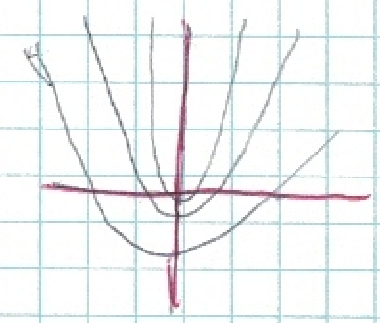
- what do the co-ord lines look like?

  → lines of constant ~~$x = \mu_0$~~ ~~or~~ ~~$x = \mu_0$~~ ~~or~~ $\nu = \nu_0$

  ~~then ① → $\mu = \frac{x}{\nu}$ $\xrightarrow{②}$ $y = \frac{1}{2}($~~

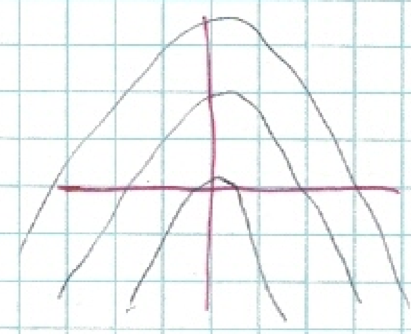
  then ① → $\mu = \frac{x}{\nu_0}$ $\xrightarrow{②}$ $y = \frac{1}{2}\left(\frac{x^2}{\nu_0^2} - \nu_0^2\right)$

$$\boxed{y = \frac{1}{2\nu_0^2}x^2 - \frac{\nu_0^2}{2}}$$

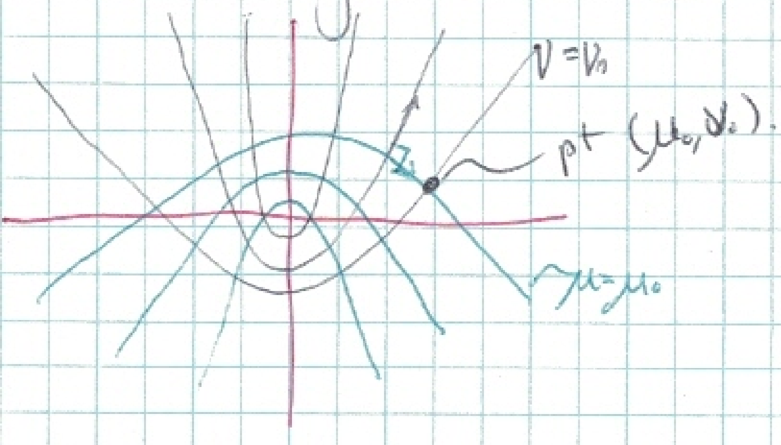


- Similarly for const. $\mu = \mu_0$

  ① → $\nu = \frac{x}{\mu_0}$ $\xrightarrow{②}$ $y = \frac{1}{2}\left(\mu^2 - \frac{x^2}{\mu_0^2}\right)$



⇒ together the co-ord lines form a co-ord grid over the (flat) Euclidean plane.




Notes: ① Can show the lines are orthogonal.

② Line elt:

$dx = \mu d\nu + \nu d\mu$

$dy = \mu d\mu - \nu d\nu$

$ds^2 = (\mu^2 + \nu^2)(d\mu^2 + d\nu^2)$.

- we want to synchronize clocks in the moving frame.
- if light emitted towards B in the middle of $\overline{AC}$ from $AC$ hit B at the same time the the moving observer will say they we emitted at the same time.
- so let a light beam be emitted from A ($x=x'=0$) at time $t=0$.

Need to find:

① Where (i.e, what $(x_B, t_B)$) does the light from A hit B. ←event E

② Given the $(x_B, t_B)$ co-ords of E, when must the light from C have been emitted → $(x_c, t_c)$.

⇒ This corresponds to $t'=0$ in the moving frame.

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- set $c=1$, light speed.
- light from event A follows the line $\boxed{x=t}$
- B is folling the line $\boxed{x=vt+1}$
- E is their intersection: $t=vt+1$

$$\Rightarrow t_B = \frac{1}{1-v}$$

$$x_B = \frac{1}{1-v}$$

- backwards light (slope $-1$) has the eq$^n$ $x=-t+\text{const}$.
  $x+t=\text{const}$.
- but $x_B+t_B = \frac{2}{1-v}$.
- C follows the eq$^n$ $x_c = vt_c+2$ →

So, solve

$$\begin{cases} x_c + t_c = \dfrac{2}{1-v} \\ x_c - v t_c = 2 \quad (*) \end{cases}$$

$$\ominus \quad t_c + v t_c = \frac{2}{1-v} - 2$$

$$t_c(1+v) = \frac{2 - 2(1-v)}{1-v}$$

$$\boxed{t_c = \frac{2v}{1-v^2}}$$

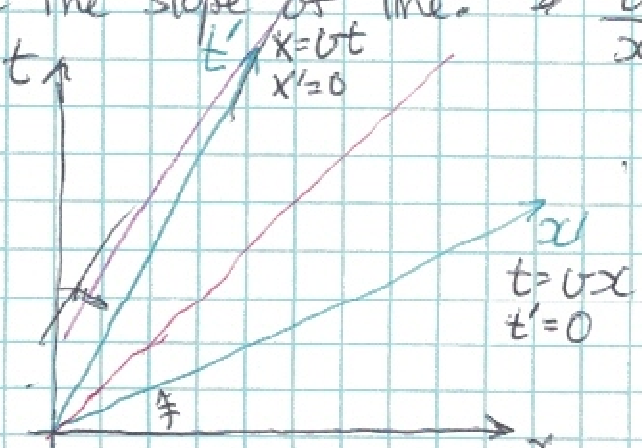
Subs' into $(*)$: $x_c - \dfrac{2v^2}{1-v^2} = 2$

$$x_c = \frac{2(1-v^2) + 2v^2}{1-v^2}$$

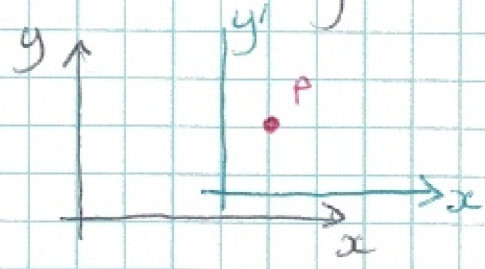
$$= \frac{2 - 2v^2 + 2v^2}{1-v^2}$$

$$\boxed{x_c = \frac{2}{1-v^2}}$$

Note the slope of line: $\dfrac{t_c}{x_c} = v \Rightarrow t = vx$ is the $t'=0$ ax



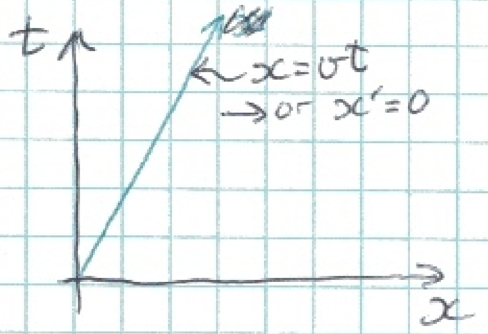
Back to moving reference frames.




We have the Galilean transf^ns.

$$\begin{cases} x' = x - vt \\ t' = t \end{cases}$$

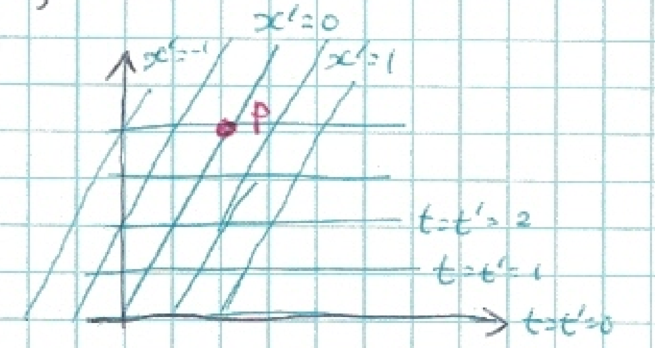
- these are co-ord transf^ns.




Note $x=0$ is just the t-ax

$\rightarrow$ $x'=0$ is the $t'$-axis

So, the Galilean transf^n of co-ords has a co-ord grid:




- ~~P ⇒ {x = 2, t = 4}~~

$P \Rightarrow \begin{Bmatrix} t = 4 \\ x = 2 \end{Bmatrix}$ $\quad$ $t, x$: $(4, 2)$

$\begin{Bmatrix} t' = 4 \\ x' = 0 \end{Bmatrix}$ $\quad$ $t', x'$: $(4, 0)$

What is $v$ here? (Ans. ½).

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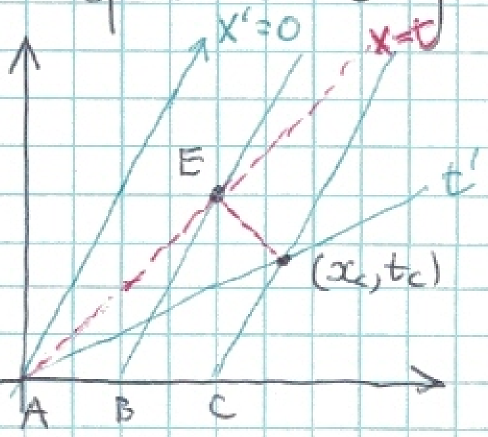
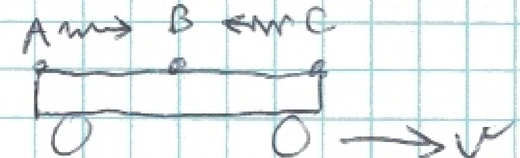
Now we add the assumption that light travels at the same speed in every IRF. (inertial ref. frame).

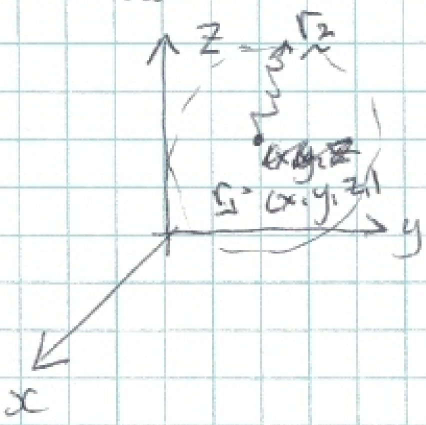



$\Rightarrow$ Go to 10b

- What happens when we insist the speed of light is the same in all F of R?

- consider stationary frame S & light emitted from pt. $(x_1 y_1 z_1)$ at time $t_1$. which arrives at pt $(x_2 y_2 z_2)$ at time $t_2$.




- it's traveled at speed c & so distance $c(t_2 - t_1)$.

- but this distance also equals

$$\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 - (z_2-z_1)^2}.$$

∴ we have the relation.

$$(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - c^2(t_2-t_1)^2 = 0.$$

- in a moving frame K′ it sees the same events in ′ co-ords & ∴ light also travels at c it has the relation.

$$(x_2'-x_1')^2 + (y_2'-y_1')^2 + (z_2'-z_1')^2 - c^2(t_2'-t_1')^2 = 0$$

- define a new quantity called the <u>interval</u> between <u>any</u> two events (not necessarily connected by a light ray).

$$s_{12}^2 \equiv c^2(t_2-t_1)^2 - (x_2-x_1)^2 - (y_2-y_1)^2 - (z_2-z_1^2).$$

- clearly if the interval is zero in one frame it is zero in any other.

- we can write it as.

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2.$$

or infinitesimaly.

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

- it is a metric $\longrightarrow$ we want to find the relationship of $ds^2$ btw frames

START
- Sp ① if $ds^2 = 0$ in one inertial frame $ds'^2 = 0$ in any other.

② $\therefore$ they are proportional to each other.

$ds^2 = A \cdot ds'^2$ from some number $A$, where

③ $A = A(v)$ depends only on the mag. of velocity btwn frames

- it cannot depend on pos$^n$ or time $\because$ homogeneity of s-t.
- it " " " dir$^n$ of velocity $\because$ ~~iso~~ isotropy of space

④ Consider 3 systems $K$, $K_1$ ($\bar{c}$ vel. $V_1$), $K_2$ ($\bar{c}$ vel. $V_2$).

Then

$$\begin{cases} ds^2 = A(v_1)\, ds_1^2 \\ ds^2 = A(v_2)\, ds_2^2 \end{cases}$$

& $ds_2^2 = A(v_{12})\, ds_1^2$ $\bar{c}$ $v_{12}$ rel. velocity of $K_1$ wrt $K_2$

Divide: $1 = \dfrac{A(v_1)}{A(v_2)} \dfrac{ds_1^2}{ds_2^2}$

$$\therefore ds_2^2 = \frac{A(v_1)}{A(v_2)} ds_1^2 = A(v_{12})\, ds_1^2.$$

$\therefore \dfrac{A(v_1)}{A(v_2)} = A(v_{12})$. ← notice: RHS depends on $v_1, v_2$ and the relative angle btwn the two $\theta$.

LHS does not depend on the angle.

$\therefore \boxed{A = \text{const}}$

(cf separation of variables.

$$\frac{X'(x)}{X(x)} = \frac{Y'(y)}{Y(y)} \Rightarrow \text{LHS=RHS = const}$$

$\therefore$ LHS $= 1 = A$ (on RHS).

$\therefore \boxed{ds^2 = ds'^2}$ for infinitesimal intervals.

$\therefore S^2 = S'^2$ for finite intervals $\rightarrow$ $\boxed{\text{AN INVARIANT}}$

i.e. $c^2\Delta t^2 - \Delta\vec{x}^2 = c^2\Delta t'^2 - \Delta\vec{x}'^2$ for all inertial frames.

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Proper Time (Dilation).

- suppose a clock in the $S'$-coords is moving relative to $S$.
- in infinitesimal time $dt$ in the $S$-system it moves distance

$$\sqrt{dx^2 + dy^2 + dz^2} = |d\vec{x}|$$

- in $S'$ it hasn't moved at all in time $dt'$

$$\therefore ds^2 = c^2 dt'^2 = c^2 dt^2 - |d\vec{x}|^2$$

$$\therefore dt'^2 = \cancel{\tfrac{1}{c^2}(c^2 dt} \; dt^2 - \frac{|d\vec{x}|^2}{c^2}$$

$$dt' = dt\sqrt{1 - \frac{1}{c^2}\frac{|d\vec{x}|^2}{dt^2}} \Rightarrow \boxed{dt' = dt\sqrt{1 - \frac{v^2}{c^2}}} \;(*)$$

- integrating $t_2' - t_1' = \int_{t=t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}}\, dt$ (This holds for non-uniform motion, too.)

This is called proper time, $\tau$.

The time on a clock moving wrt a stationary frame.

- note $\Delta\tau \leq \Delta t$. $\rightarrow$ moving clocks run slow.
- it's also <u>invariant</u> $\because$ $ds^2 = c^2 dt'^2 = c^2 d\tau^2$ in the moving frame.

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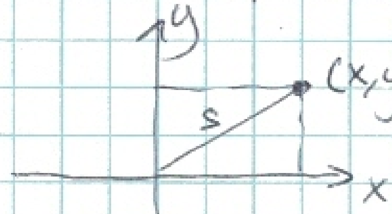
<u>Relation Between $(x,t)$ & $(x',t')$ given $v$.</u>

Recall Euclidean case:

$$ds^2 = dx^2 + dy^2$$

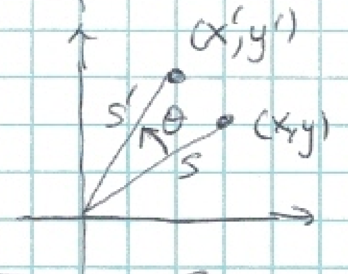
- look at distance from origin




$s^2 = x^2 + y^2$.

- linear transformations that preserve this distance are rotations.

Note: $s'^2 = s^2$




- use the rot$^n$ matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\hookrightarrow$ $x' = cx - sy$
$y' = sx + cy$.

Trig. Identity:

$\cos^2\theta + \sin^2\theta = 1$

$\text{ch}^2\theta - \text{sh}^2\theta = 1$

(Hyperbolic)

Check invariance:

$$x'^2 + y'^2 = c^2x^2 - \cancel{2xy \cdot cs} + s^2y^2 + s^2x^2 + \cancel{2xy \cdot cs} + c^2y^2$$

$$= (c^2 + s^2)x^2 + (s^2 + c^2)y^2 = x^2 + y^2.$$

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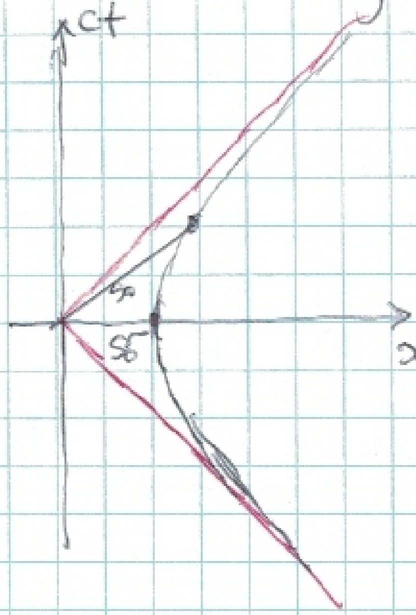
in Minkowski space we look for something similar, but the minus sign changes things a bit.




- now we look for a linear transformation that preserves

$$s^2 = c^2t^2 - x^2$$

- note for a given $s = s_0$ this describes a hyperbola.

- in the Euclidean case we had a nice cancellation of the middle terms. + $\boxed{\sin^2 + \cos^2 = 1}$ identity

- that won't work here because of the $(-)$ in the metric.
- but hyperbolic $\sinh = sh$ & $\cosh = ch$ have the identity

$$\boxed{ch^2\theta - sh^2\theta = 1} \longleftarrow \text{a } \underline{\text{minus}}!$$

- try it . . .

$$\begin{bmatrix} x' \\ ct' \end{bmatrix} = \begin{bmatrix} ch\theta & sh\theta \\ sh\theta & ch\theta \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \Rightarrow \boxed{\begin{aligned} x' &= ch \cdot x + sh \cdot ct \\ ct' &= sh x + ch \cdot ct \end{aligned}} \quad (*)$$

$$\begin{aligned}
\Rightarrow c^2t'^2 - x'^2 &= (sh\,x + ch \cdot ct)^2 - (ch \cdot x + sh \cdot ct)^2 \\
&= sh^2 x^2 + \cancel{2 sh \cdot ch \cdot xct}^{\,0} + ch^2 c^2 t^2 \\
&\quad - ch^2 x^2 - \cancel{2\, ch\, sh \cdot x \cdot ct}^{\,0} - sh^2 c^2 t^2 \\
&= (\cancel{sh^2 - ch^2})^{-1} x^2 + (\cancel{ch^2 - sh^2})^{1} c^2 t^2 \\
&= c^2t^2 - x^2
\end{aligned}$$

✓

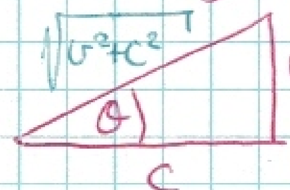
- the angle $\theta$ can only depend on $v$, the rel. velocity of $S$ & $S'$.
- consider the motion of the origin of $S'$ as it moves through $S$.
- it has co-ord $x'=0$. What does it look like in $S$?

$$x'=0 \Rightarrow \quad x = ct' \cdot sh(\theta)$$
$$ct = ct' \cdot ch(\theta).$$

- dividing: $\frac{x}{ct} = \cancel{th}\ \tanh(\theta)$.

or $\boxed{th(\theta) = \frac{v}{c}}$

**Recall trig.**:




- hypotenus $= \sqrt{v^2+c^2}$

$\therefore \sin\theta = \frac{v}{\sqrt{v^2+c^2}} = \frac{v/c}{\sqrt{1+v^2/c^2}}$

$\cos\theta = \frac{c}{\sqrt{v^2+c^2}} = \frac{1}{\sqrt{1+v^2/c^2}}$

- using analogous trig identities:

$$\sinh(\theta) = \frac{v/c}{\sqrt{1-v^2/c^2}}$$

$$\cosh(\theta) = \frac{1}{\sqrt{1-v^2/c^2}}$$

Sub'ing into (*) pg. ~~13~~ 14 we get the Lorentz transformations:

$$\boxed{x' = \frac{x - v \cdot t}{\sqrt{1-v^2/c^2}} \qquad t' = \frac{t - v/c^2 \cdot x}{\sqrt{1-v^2/c^2}}}$$

• do a check using proper time, $t'$.

→ a clock stationary in the S' system sitting at the ~~origin~~ pt → $x'_0$.

→ setting $v \longrightarrow -v$ to get $t$ in terms of $t'$:

$$t = \frac{t' + v/c^2 \, x'_0}{\sqrt{1 - v^2/c^2}}$$

• consider two time instances of the clock, $t_1$ & $t_2$.

$$\Rightarrow \quad t_1 = \frac{t_1' + v/c^2 \, x'_0}{\sqrt{1 - v^2/c^2}} \qquad t_2 = \frac{t_2' + v/c^2 \, x_0}{\sqrt{1 - v^2/c^2}}.$$

$$\therefore \quad t_2 - t_1 = \boxed{\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}}$$

⇒ agrees $\bar{c}$ (*) on pg 12.

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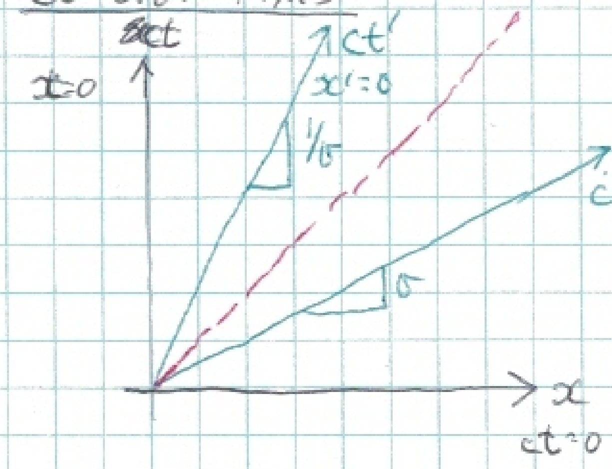
Co-ord Axis




Recall coord transforms:

$$\boxed{\begin{aligned} x &= x' \cdot ch(\theta) + ct' \cdot sh(\theta) \\ ct' &= x' sh(\theta) + ct' ch(\theta) \end{aligned}}$$

① $ct'$-axis ⇒ $x' = 0$

$$\therefore \quad x = ct' \, sh(\theta)$$
$$ct = ct' \, ch(\theta).$$

$$S_1 \rightarrow \frac{ct}{x} = \frac{ch(\theta)}{sh(\theta)} = \frac{1}{th(\theta)} = \frac{c}{v}$$

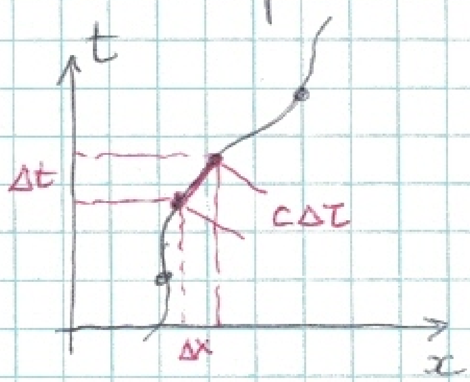
② $x'$-axis ⇒ $ct' = 0$.

$$\left.\begin{aligned} x &= x' ch(\theta) \\ ct &= ct' sh(\theta) \end{aligned}\right\} \quad \frac{ct}{x} = \frac{sh(\theta)}{ch(\theta)} = th(\theta) = v/c$$

## 4-Vectors

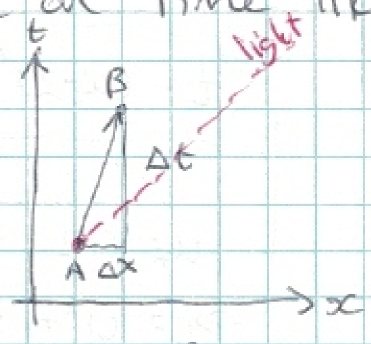
- an event is specified by a time & 3-dim pos$^{n}$ which we've packaged in a 4-dim vector:

- idea of the path of a particle moving thru space-time




- particles move on time-like trajectories.

That is,




- the (invariant) proper time elapse in moving from A→B is:

$$\Delta\tau^2 = \Delta t^2 - \Delta x^2$$

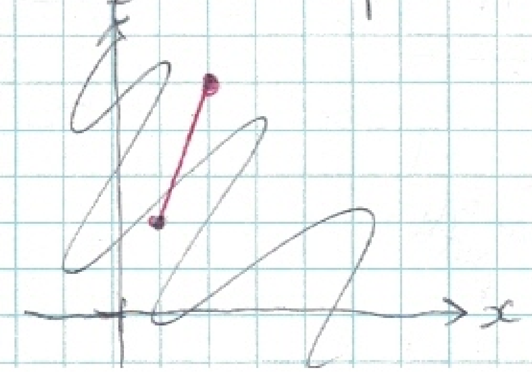
Note ~~$\Delta t^2$~~ $\Delta\tau^2 > 0$ if $\Delta t^2 > \Delta x^2$, i.e., the slope of $\overrightarrow{AB}$ is greater than 1 $\Rightarrow$ $\frac{\Delta t}{\Delta x} > 1$

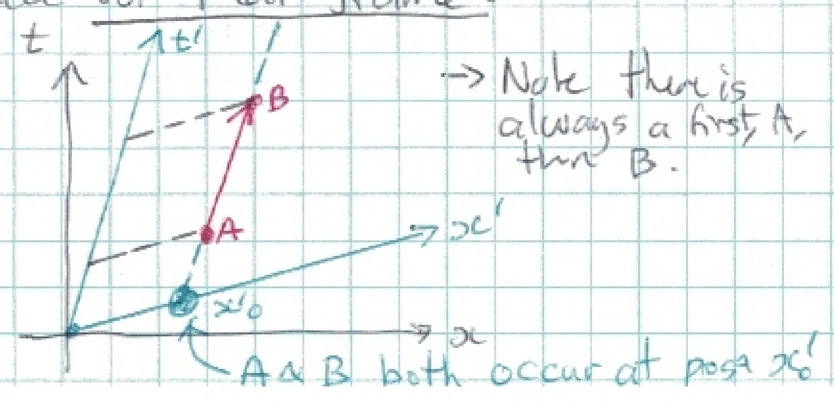
Or $\frac{\Delta x}{\Delta t} < 1$

$|v_{AB}| < 1$ is less than the speed of light.

Significance of time-like sep$^{\underline{n}}$?

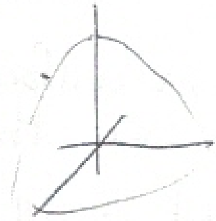
$\Rightarrow$ There always exists a moving frame s.t. they are at the same point in space for that frame?

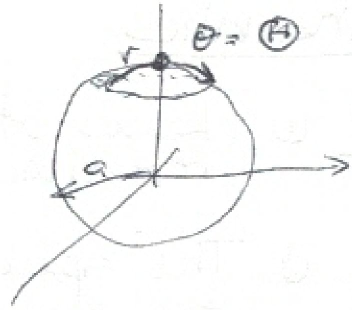



→ Note there is always a first, A, then B.

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta \, d\phi$$






Circum.: $\theta = \Theta$ fixed $: C = \oint ds = \int_0^{2\pi} a \sin \Theta \, d\phi = a \sin\Theta \int_0^{2\pi} d\phi = a \sin\Theta \cdot (2\pi - 0)$

$\downarrow$

$d\theta = 0$

$$\boxed{C = 2\pi a \sin\Theta}$$

Radius: $\phi$ = fixed $r = \int_{\theta=0}^{\theta=\Theta} a \, d\theta$

$\rightarrow d\phi = 0$

$\boxed{r = a\Theta} \longrightarrow \boxed{C = 2\pi a \cdot \sin\frac{r}{a}}$

---

$$\pi' = \frac{C}{2r} = \frac{2\pi a \sin\Theta}{2 \cdot a\Theta} = \pi \frac{\sin\Theta}{\Theta}$$

pseudo-$\pi$. d

$$= \pi \frac{\sin\frac{r}{a}}{\frac{r}{a}}$$

$\Downarrow$

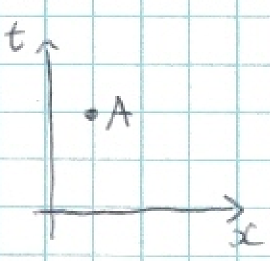
$$\boxed{\pi' = \pi \boxed{\frac{\sin\Theta}{\Theta}}}$$

Non-Euclidean $\pi$.

Note: $\lim_{\Theta \to 0} \frac{\sin\Theta}{\Theta} = 1$, $\Theta = \frac{r}{a}$.

$\therefore \boxed{\lim_{\substack{r\to 0 \\ \text{or } a\to\infty}} \pi' = \pi}$ $\rightarrow$ Euclidean Plane.

• the motion of a particle can be thought of as a sequence of small time-like intervals.




• we can write $\underset{\sim}{A} = a^t \underset{\sim}{e}_t + a^x e_x + a^y e_y + a^z e_z$

basis vectors

• so $\underset{\sim}{A}$ is rep'd by the 4-tuple $(a^t, a^x, a^y, a^z)$,

OR $(a^0, a^1, a^2, a^3)$ or $(a^0, \vec{a})$.

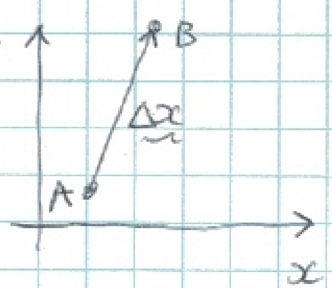
spatial 3-dim part

scalar 1-dim part.

• so $\underset{\sim}{A} = \sum_{\alpha=0}^{3} a^\alpha \underset{\sim}{e}_\alpha$

or $= a^\alpha \underset{\sim}{e}_\alpha$ where we leave the sum$^n$ implied by duplicated upper & lower indices.

• we can have displacement 4-vectors.




$\underset{\sim}{\Delta x} = (t_B - t_A, x_B - x_A, y_B - y_A, z_B - z_A)$

OR, write as: $\boxed{\Delta x^\alpha = x_B^\alpha - x_A^\alpha}$

• scalar (inner product).

$\underset{\sim}{a}\cdot\underset{\sim}{b} = (a^\alpha \underset{\sim}{e}_\alpha)\cdot(b^\beta \underset{\sim}{e}_\beta)$

$= a^\alpha b^\alpha \underbrace{(\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta)}$ ← double sum.

metric

Define this quantity as $\boxed{\eta_{\alpha\beta} = \underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta}$

So $\boxed{\underset{\sim}{a}\cdot\underset{\sim}{b} = \eta_{\alpha\beta}\, a^\alpha b^\beta}$

- recall $\Delta s^2 = \Delta t^2 - \Delta \vec{x}^2$

$$= \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2.$$

OR $\Delta s^2 = \Delta \underset{\sim}{x} \cdot \Delta \underset{\sim}{x}$

$$= \eta_{\alpha\beta} \Delta x^\alpha \Delta x^\beta$$

where $\eta_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & \emptyset & \\ & & -1 & \\ \emptyset & & & -1 \end{pmatrix} \longrightarrow$ the Minkowski metric

- infinitessimally, $\boxed{ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta}$

---

~~Preservation of Orthogonality Under~~ Lorentz Boosts

Recall the Lorentz transforms:

$$\left.\begin{aligned} t' &= \frac{t - xv}{\sqrt{1-v^2}} \\ x' &= \frac{x - vt}{\sqrt{1-v^2}} \\ y' &= y \\ z' &= z \end{aligned}\right\} \quad \boxed{\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z. \end{aligned}}$$

So a general 4-vector $\underset{\sim}{A} = (a^t a^x a^y a^z)$ transforms to $\underset{\sim}{A} = (a^{t'} a^{x'} a^{y'} a^{z'})$ in the new basis (same $\underset{\sim}{A}$!)

when
$$\begin{aligned} a^{t'} &= \gamma(a^t - va^x) \\ a^{x'} &= \gamma(a^x - va^t) \\ a^{y'} &= a^y \\ a^{z'} &= a^z. \end{aligned}$$

## Example of Invariance of Inner Product.

Say $a^{\alpha} = (1, 0, 0, 0)$

$b^{\beta} = (0, 1, 0, 0)$.

Then $\underline{a} \cdot \underline{b} = \eta_{\alpha\beta} a^{\alpha} b^{\beta} = 0 \rightarrow$ their orthogonal.

In a primed system we have.

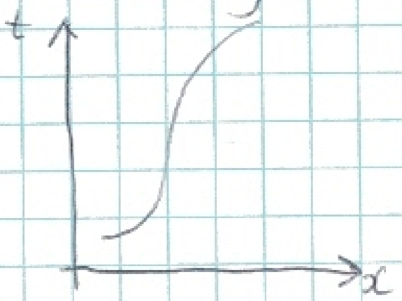
$a^{\alpha'} = (\gamma, -v\gamma, 0, 0)$.

$b^{\alpha'} = (-v\gamma, \gamma, 0, 0)$.

$$\eta_{\alpha\beta} a^{\alpha'} b^{\beta'} = \gamma(-v\gamma) - (-v\gamma)\gamma + 0 + 0$$

$= 0$. ✓ $\Rightarrow$ orthogonal in ' system.

## Parameterizing Motion in Space-Time.




• we could write a particle's path as

$\underset{\sim}{x}(t) : \mathbb{R} \longrightarrow \mathbb{R}^3$

$t \longmapsto (x(t), y(t), z(t)) = \underset{\sim}{x}(t)$.

• but $t$ is different from frame to frame as is $\underset{\sim}{x}$.

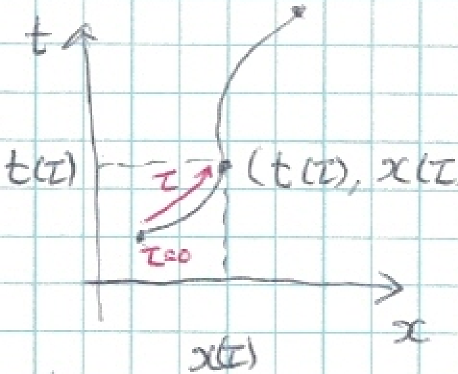
• better to parameterize the path using something that is invariant between frames

$\Rightarrow$ use <u>proper time</u> – the time of a clock moving $\bar{c}$ the particle.

– this is just the arc length $\rightarrow$ Recall $\boxed{d\tau^2 = dt^2 - dx^2}$

$\Rightarrow$ clocks are devices that measure distance along time-like world lines

- So $x^\alpha = x^\alpha(\tau)$. $\longrightarrow$ $x^\alpha = (x^0(\tau), x^1(\tau))$ in 2-d
$$= (t(\tau), x(\tau))$$




So we define the 4-velocity as
$$\frac{d}{d\tau} x^\alpha \equiv u^\alpha$$

Components:

- now $\frac{d\cancel{x}^0}{d\tau} = \frac{d}{d\tau} t$

$$\boxed{u^0 = \gamma} \longrightarrow \text{recall } \Delta t = \gamma \Delta \tau$$

- $\frac{dx^1}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau}$
$$= \gamma v_x \rightsquigarrow \text{ordinary 3-velocity}$$

- similarly $\frac{dx^2}{d\tau} = \frac{dy}{dt}\frac{dt}{d\tau} = \gamma v_y$
$$\frac{dx^3}{d\tau} = \gamma v_z.$$

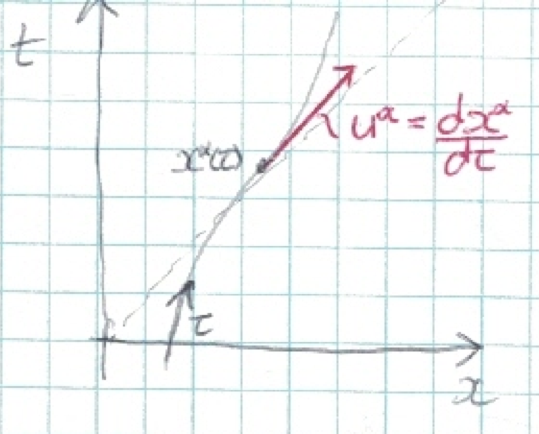
So 4-velocity is given by $\boxed{u^\alpha = (\gamma, \gamma\vec{v}) = (\gamma, \gamma v_x, \gamma v_y, \gamma v_z)}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$ $\gamma(1, \vec{v})$

- note when stationary in $(x,y,z) \rightarrow \vec{v} = 0$ & $\gamma = 1$
$$\boxed{u^\alpha = (1,0,0,0)}$$

---

→ note: this is the rest frame of the moving particle
So the 4-velocity in that frame is the time axis for a momentary co-moving frame.




• what is the magnitude of $u^\alpha$? $\Rightarrow$ $\underset{\sim}{u} \cdot \underset{\sim}{u} = |\underset{\sim}{u}|^2$.

• recall $\cancel{d\tau^2}$ inner product $\underset{\sim}{a} \cdot \underset{\sim}{b} = \eta_{\alpha\beta} a^\alpha b^\beta$

So $\underset{\sim}{u} \cdot \underset{\sim}{u} = \eta_{\alpha\beta} u^\alpha u^\beta$

$$\boxed{|\underset{\sim}{u}|^2 = \eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}}$$

But we have the invariant $d\tau^2 = -\eta_{\alpha\beta}\, dx^\alpha dx^\beta$

So $\frac{d\tau^2}{d\tau^2} = -\eta_{\alpha\beta} \frac{dx^\alpha dx^\beta}{d\tau^2}$

$$\boxed{-1 = \eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}}$$

$\therefore$ $\boxed{|\underset{\sim}{u}|^2 = -1}$ Always time-like.

---

You can also see this by $u^\alpha = (\gamma, \gamma v_x)$ (e.g. 1-1 space)

$\eta_{\alpha\beta} u^\alpha u^\beta = \frac{-1}{\sqrt{1-v_x^2}} + \frac{v_x}{\sqrt{1-v_x^2}} \cdot v_x$

$=$

$$\eta_{\alpha\beta} U^\alpha U^\beta = -\gamma^2 + \gamma^2 v_x^2$$

$$= \frac{-1}{1-V_x^2} + \frac{1}{1-V_x^2} \cdot V_x^2$$

$$= \frac{-(1-V_x^2)}{1-V_x^2}$$

$$= -1 \checkmark$$

Need

$U^\alpha = (\gamma, \gamma v_x)$

$$\eta = \begin{pmatrix} -1 & & & \emptyset \\ & 1 & & \\ & & 1 & \\ \emptyset & & & 1 \end{pmatrix}$$

---

## 4-Acc$^{n}$

Similarly we can define the 4-acc$^{n}$.

$$\underset{\sim}{a}(\tau) = \frac{d\underset{\sim}{u}}{d\tau}$$

Note that since $\underset{\sim}{u} \cdot \underset{\sim}{u} = -1$ and diff'ing both sides:

$$\frac{d}{d\tau}(\underset{\sim}{u} \cdot \underset{\sim}{u}) = \frac{d}{d\tau} -1$$

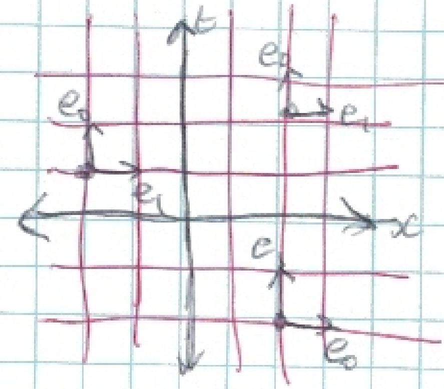
$$\underset{\sim}{u} \cdot \frac{d\underset{\sim}{u}}{d\tau} + \frac{d\underset{\sim}{u}}{d\tau} \cdot \underset{\sim}{u} = 0$$

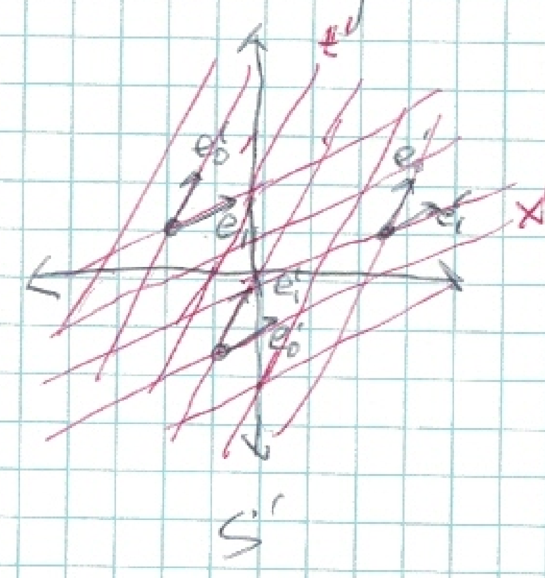
equal by symm. of i.p.

$$2\underset{\sim}{u} \cdot \frac{d\underset{\sim}{u}}{d\tau} = 0$$

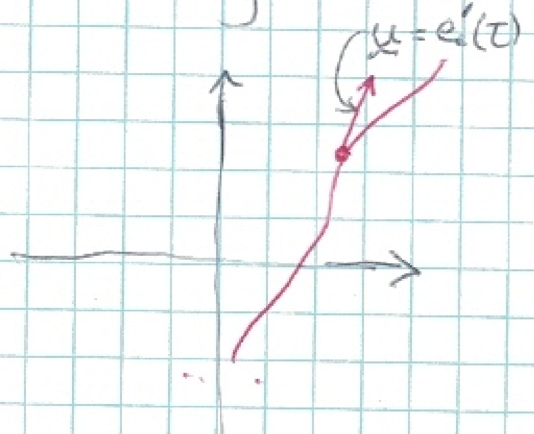
$\therefore \boxed{\underset{\sim}{u} \cdot \underset{\sim}{a} = 0} \longrightarrow$ the 4-vel. & 4-acc$^{n}$ are always orthogonal.

Recall the co-ord frames of a stationary & interial frame




- the co-ord system's basis are fixed for const. veloc.
- when the velocity varies under acc$^{c}$ define the instantaneous rest frame $S'$ for which the moving observer is momentarily at rest.




- so the velocity vector in general defines a local time axis.
- so $\boxed{\underset{\sim}{e}_0'(\tau) = \hat{\underset{\sim}{u}}(\tau)}$.
- in $(t, x)$ 1+1 space we can the the orthogonal $\underset{\sim}{a}(\tau)$ to be the spatial basis vector at that pt.

$$\boxed{\underset{\sim}{e}_1'(\tau) = \hat{\underset{\sim}{a}}(\tau)}$$

- note that at a particular point the observer's 4-velocity is $(1, \vec{0}) = \underset{\sim}{u}' (= u'^{\mu})$
- since the 4-acc$^n$ is $\perp$ it must have the form

$$a'^{\mu} = \underset{\sim}{a} = (0, \vec{\underset{\sim}{a}})$$

$\rightarrow$ $\vec{a}$ is the proper (particle's momentary rest frame) acc$^n$.

where $\vec{a} = \left( \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \right)$

$= \frac{d}{dt} (\vec{v}_x, \vec{v}_y \vec{v}_z)$

$= \frac{d\vec{v}}{dt}$