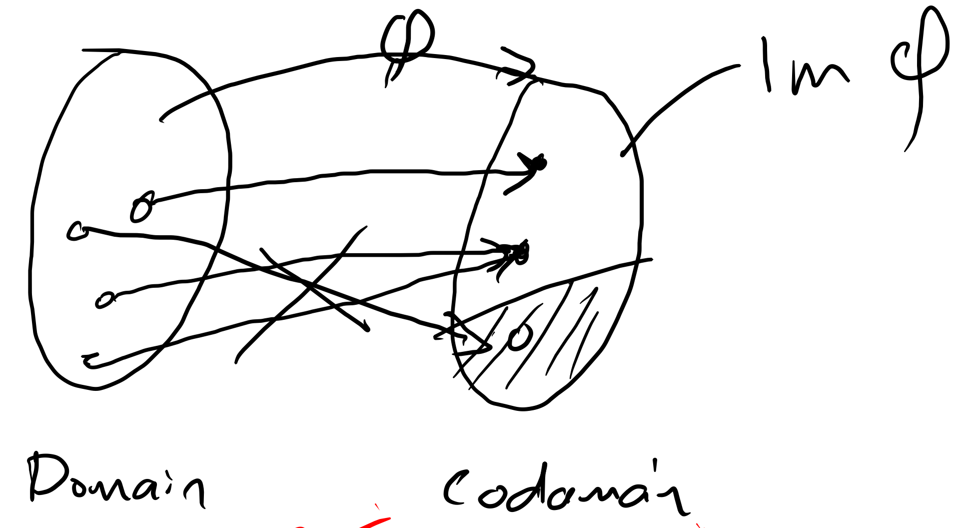
Ring Homomorphisms

- to start: isomorphisms

one to one & onto

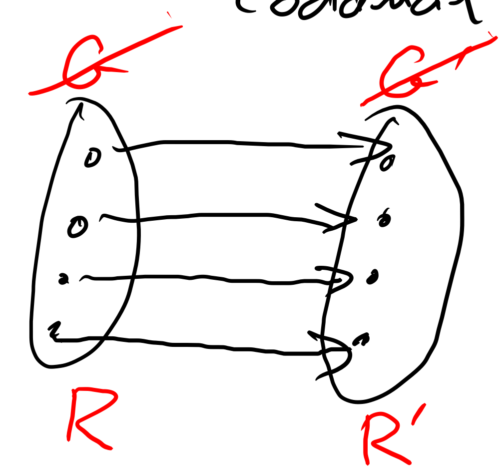
① bijective mapping
  - injective, one-to-one
  - onto, surjection



② compatability cond^n.

0_R ① $\varphi(a +_R b) = \varphi(a) +_{R'} \varphi(b)$ ←

1_R ② $\varphi(a \times_R b) = \varphi(a) \times_{R'} \varphi(b)$



$\forall a, b \in R$

Claim: $\varphi(0_R) = 0_{R'}$

Proof: $\varphi(a+0) = \varphi(a) + \varphi(0)$ ← isom^n. c.c.

$$\varphi(a) = \varphi(a) + \varphi(0)$$

$$\varphi(a) + -\varphi(a) = \cancel{\varphi(a)} + \varphi(0) + -\cancel{\varphi(a)}$$

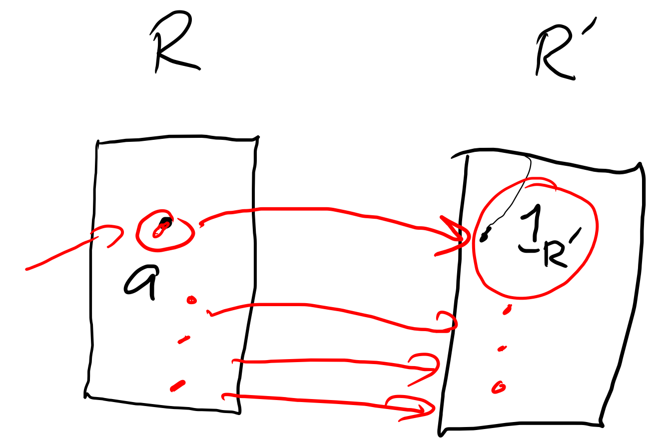
$$\boxed{0_{R'} = \varphi(0_R)}$$

Claim: $\phi(1_R) = 1_{R'}$ for isomorphism.

Proof: We know $\phi(a \times b) = \phi(a) \times \phi(b)$

$$\therefore \phi(a \times 1) = \underbrace{\phi(a)}_{\in R'} \times \phi(1)$$

$$\phi(a) = \phi(a) \times \phi(1)$$



$\exists\, a$ s.t. $\phi(a) = 1_{R'}$

Choose a s.t. $\phi(a) = 1_{R'}$

Using that a, then

$$1_{R'} = 1_{R'} \times \phi(1_R)$$

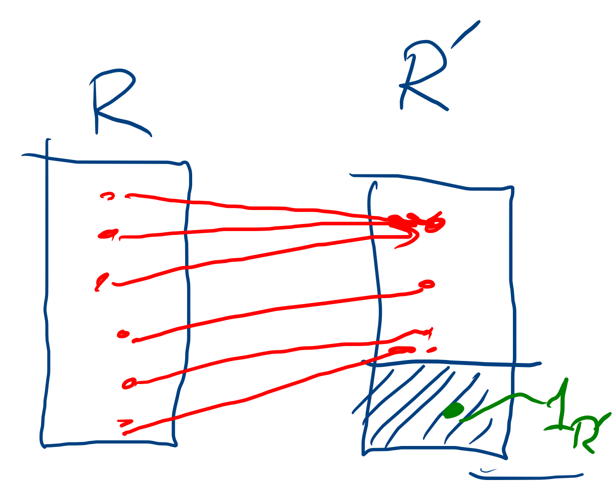
$$\boxed{1_{R'} = \phi(1_R)}$$

Homomorphisms:

① $\phi \to$ not bijective

② add'n compatability cond'ns mul'n

③ Require that $\phi(1_R) = 1_{R'}$



2. Examples of Ring Homomorphism:

(A) Evaluation homo^n of polynomial rings → high school substitution.

- given a ring R, form the polynomial ring $R[x]$.
  - set of polynomials in x with coefficient from R.

$$R[x] = \{\underbrace{a_0 + a_1 x + \cdots a_n x^n} \mid n \geq 0,\ a_i \in R\}.$$

$$\underline{R[x]} = \{\underbrace{\sum_{i=0}^{n} a_i x^i} \mid a_i \in R,\ n \geq 0\}$$

Add^n: $\underset{\underset{\underline{R[x]}}{\wedge}}{a(x)} \underset{R[x]}{+} b(x) = \sum_i a_i x^i \underset{R[x]}{+} \sum_i b_i x^i$

$$= \sum_i (a_i \underset{R}{+} b_i) x^i$$

Mult^n: $a(x) \times b(x) = \left(\sum_i a_i x^i\right) \underset{\underline{R[x]}}{\times} \left(\sum_i b_i x^i\right)$

$$= \sum_{i,j} (a_i \underset{R}{\times} b_j) x^{i+j}$$

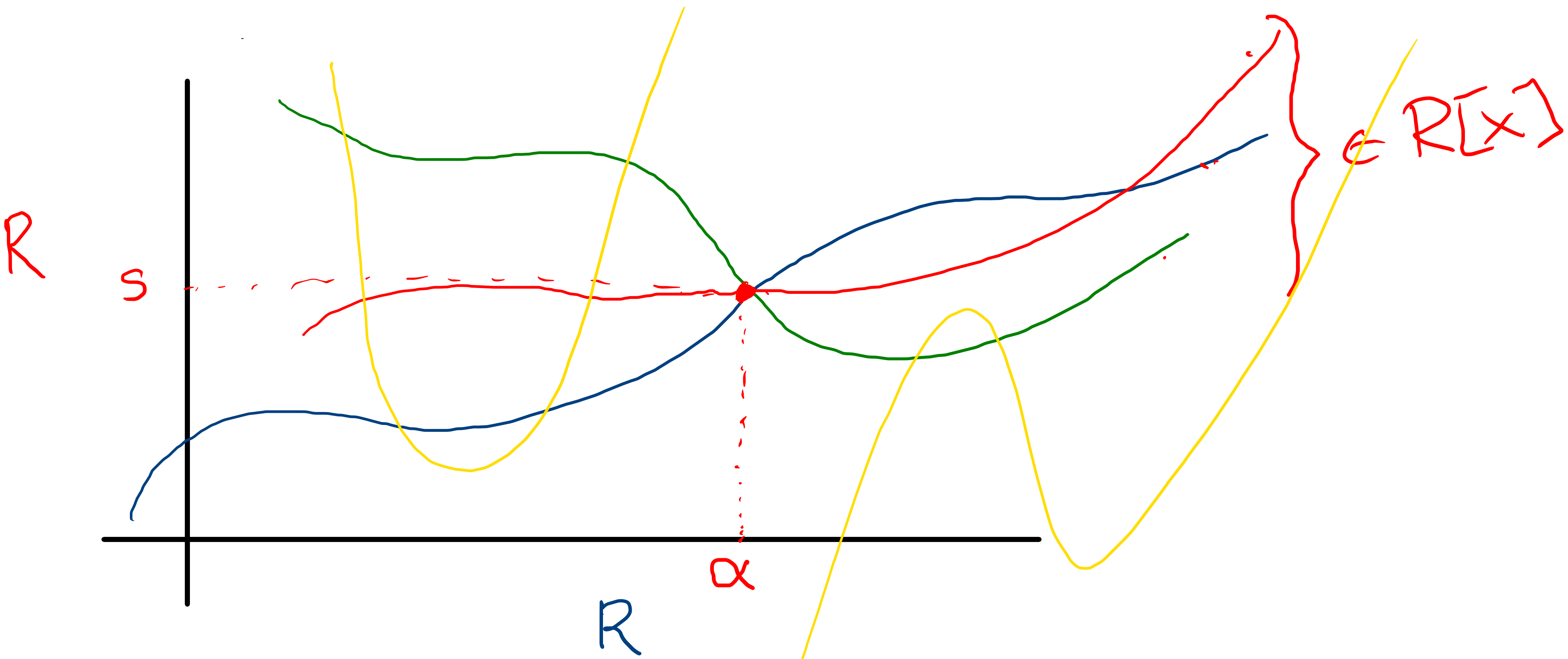
• define the evaluation homo^m:

$R \xrightarrow{\text{cont}} R[x] \xrightarrow{\varphi} R$

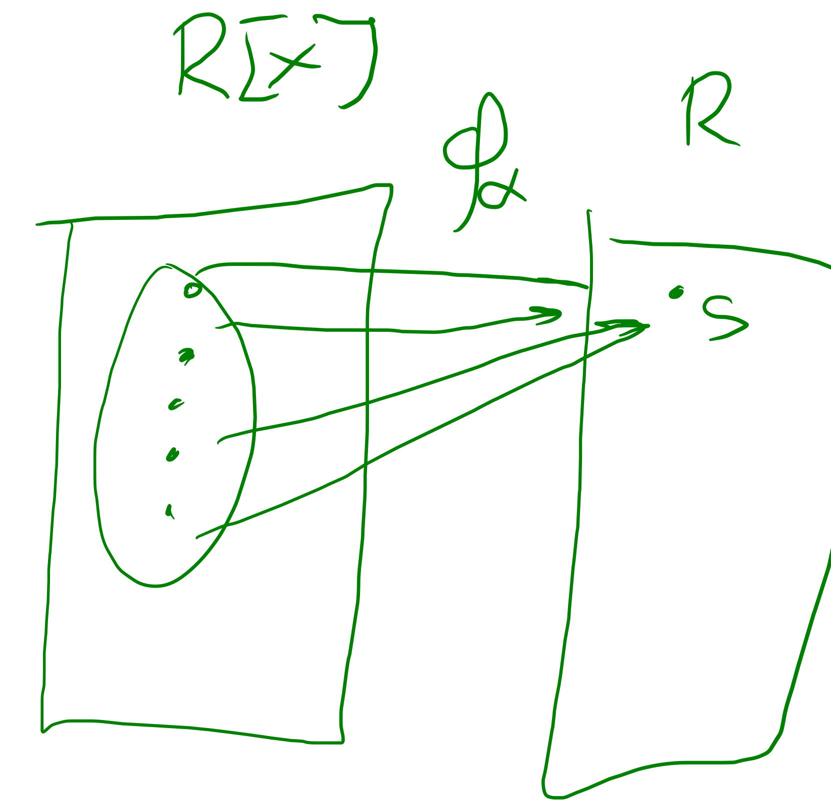
$$\varphi_\alpha : R[x] \longrightarrow R$$

$$\sum_i a_i x^i \longmapsto \sum_i \underbrace{a_i}_{R} \underbrace{\times}_{R} \alpha^i \in R, \quad \alpha \in R$$

→ Not bijective!



$$\varphi_\alpha(p(x)) = s$$



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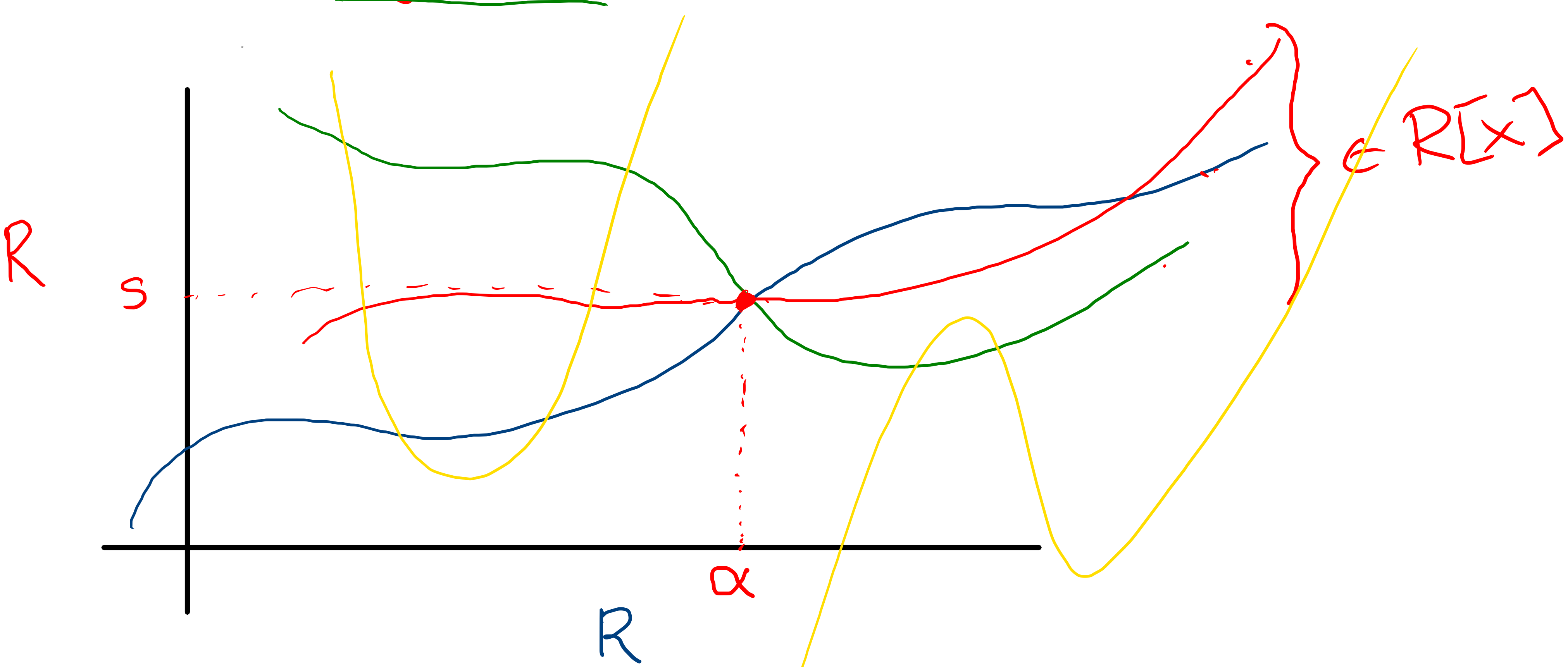

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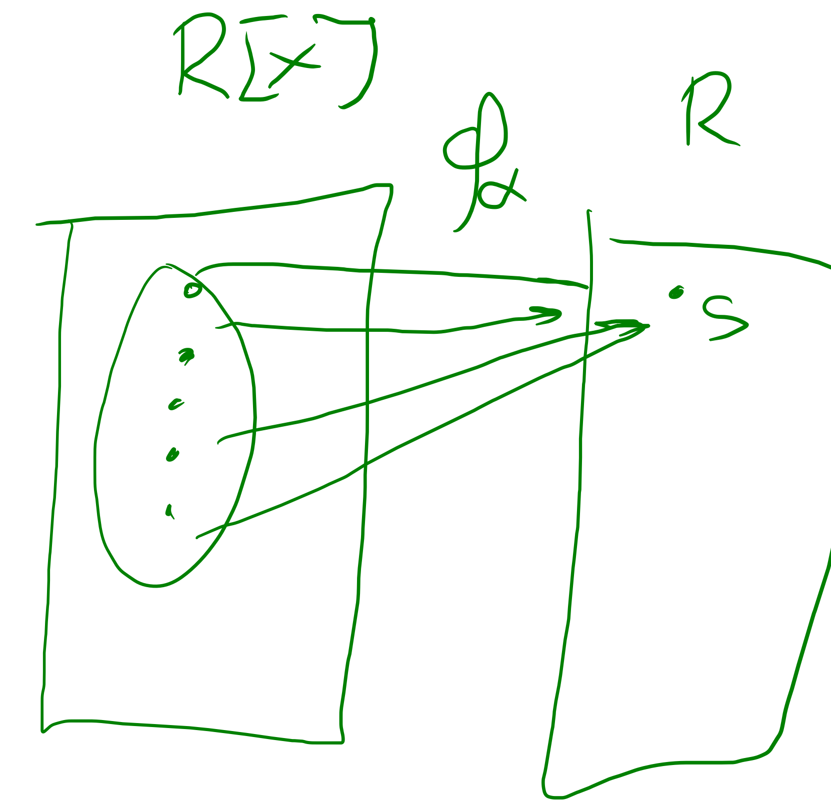

Claim: $\varphi_\alpha : R[x] \to R$ is a homom^{n}.

To show: ① c.c.'s for add^{n}, mult^{n}

② $\varphi(1_{R[x]}) = 1_R$

① Add^{n}: $\varphi_\alpha(a(x) +_{R[x]} b(x)) \overset{?}{=} \varphi_\alpha(a(x)) +_R \varphi_\alpha(b(x))$

LHS: $\varphi_\alpha(a(x) + b(x))$

$= \varphi_\alpha\left(\sum_i a_i x^i + \sum_i b_i x^i\right)$

$= \sum_i a_i \alpha^i + \sum_i b_i \alpha^i$

RHS: $\varphi_\alpha(a(x)) + \varphi_\alpha(b(x))$

$= \varphi_\alpha\left(\sum_i a_i x^i\right) + \varphi_\alpha\left(\sum_i b_i x^i\right)$

$= \sum_i a_i \alpha^i + \sum_i b_i \alpha^i$

② Mult^{n}:

③ Does $\varphi_\alpha(1_{R[x]}) = 1_R$?

LHS $\varphi_\alpha(1_R x^0)$

$= 1_R \times \alpha^0$

$= 1_R$

(3) The integers are part of every ring:
There is always a homo^{a} that maps the integers into any ring: $\mathbb{Z} \xrightarrow{\varphi} R$.

- define a $\varphi: \mathbb{Z} \longrightarrow R$ using the constraints of a homo^{a}.

$$\boxed{\begin{array}{l} ①\ \varphi(0_{\mathbb{Z}}) \longrightarrow 0_R \\ ②\ \varphi(1_{\mathbb{Z}}) \longrightarrow 1_R \\ ③\ \varphi(a +_{\mathbb{Z}} b) \longrightarrow \varphi(a) +_R \varphi(b) \\ ④\ \varphi(a \times_{\mathbb{Z}} b) \longrightarrow \varphi(a) \times_R \varphi(b). \end{array}}$$

$\Rightarrow$ force what every other elt in the integers are mapped to.

- we have $\phi(0)=0$

  $\phi(1_{\mathbb{Z}}) = 1_R$
- what would $\underline{a} \in \mathbb{Z}_{>0}$ be mapped.
- write $a = \underbrace{1_{\mathbb{Z}} +_{\mathbb{Z}} 1 + 1 + \cdots + 1}_{a\text{-times.}}$

$$\phi(a) = \phi(\underbrace{1+1+1+\cdots+1}_{a\text{-times.}})$$

$$= \underbrace{\phi(1_{\mathbb{Z}}) +_R \phi(1_{\mathbb{Z}}) +_R + \cdots + \phi(1_{\mathbb{Z}})}_{a\text{-times.}}$$

$$\boxed{\phi(\underline{a}) = \underbrace{1_R +_R 1_R + \cdots + 1_R}_{a\text{-times.}}}$$

$$\phi(-1_{\mathbb{Z}}) = -1_R$$

$$\phi(-1_{\mathbb{Z}}) + 1_R =$$

$$= \phi(-1) + \phi(1)$$

$$= \phi(-1+1)$$

$$= \phi(0)$$

$$= 0_R = \phi(-1) + 1_R$$

$$\boxed{\phi(-1) = -1}$$

a-time.

$$\phi(\underline{-a}) = \phi(-1_{\mathbb{Z}} + -1 + \cdots + -1)$$

$$= \phi(-1) + \phi(-1) + \cdots + \phi(-1)$$

$$= \underbrace{-1_R + -1_R + \cdots + -1_R}_{a\text{-times.}}$$

We have the mapping/relabling pal.

$$\varphi : \mathbb{Z}^{\mathrm{d}} \longrightarrow \mathbb{R}$$

$$a \longmapsto \underbrace{1_{\mathbb{R}} +_{\mathbb{R}} 1_{\mathbb{R}} + \cdots + 1_{\mathbb{R}}}_{}\quad - \text{a-times}$$

$$-a \longmapsto -1 +_{\mathbb{R}} -1 + -1_{\mathbb{R}} \cdots + -1 \quad - \text{a-times}$$

C.C.'s for add^n, mult^n: $\boxed{\varphi(a+b) = \varphi(a) + \varphi(b)}$ ✓

• Assume $a, b > 0$

LHS: $\varphi(a+b)$

$$= \varphi\Big(\big[\underbrace{1 + 1 + \cdots + 1}_{\text{a-times}}\big] + \big[\underbrace{1 + \cdots + 1}_{\text{b-times}}\big]\Big)$$

$$= \varphi(\underbrace{1 + \cdots + 1}_{\text{a+b-times.}})$$

$$= \underbrace{\varphi(1) +_{\mathbb{R}} \varphi(1) + \cdots \varphi(1)}_{\text{a+b-times.}}$$

RHS: $\varphi(a) + \varphi(b)$.

$$= \varphi(\underbrace{1 + \cdots + 1}_{\text{a-times}}) + \varphi(\underbrace{1 + 1 + \cdots + 1}_{\text{b-times}})$$

$$= \underbrace{\varphi(1) + \cdots + \varphi(1)}_{\text{a-times.}} + \underbrace{\varphi(1) + \cdots + \varphi(1)}_{\text{b-times}}$$

$$= \underbrace{\varphi(1) + \cdots + \varphi(1)}_{\text{a+b-times.}}$$ ✓