Note on Solving the Two Mass Oscillator

Wayne Dam

Physics Cafe

February 22, 2021

Uncoupling the Equations by Change of Coordinates

- The coupled equations for a two mass harmonic oscillator were shown to be,

$$\ddot{x}_1 = -2Kx_1 + Kx_2 \tag{1}$$
$$\ddot{x}_2 = Kx_1 - 2Kx_2. \tag{2}$$

  Here I've absorbed the $\frac{1}{m}$ factor into K for notational convenience. I.e., I've redefined $K \leftarrow \frac{K}{m}$ from the original talk.

- Noting that the right hand side is linear in x_1 and x_2, i.e., no powers like x_1^3 or cross multiplication of variables like $x_1 x_2$, we can write this equation in vector-matrix form,

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -2K & K \\ K & -2K \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{3}$$

- As noted in the talk, the matrix is symmetric with eigenvalues and eigenvectors

$$\lambda_1 = -K \qquad u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4}$$
$$\lambda_1 = -3K \qquad u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5}$$

- Recall from the September talks on eigenvalues and eigenvectors a symmetric matrix, Ω, can always be rewritten as a product involving the eigenvalues and eigenvectors,

$$\Omega = U \Lambda U^T. \tag{6}$$

  The unitary matrix U has the eigenvectors as its columns and the diagonal matrix, Λ, the corresponding eigenvalues. Note also that because it is unitary it's inverse is equal to its transpose, $U^{-1} = U^T$.

- So we can write

$$\begin{pmatrix} -2K & K \\ K & -2K \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -K & 0 \\ 0 & -3K \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{7}$$

- Our vector-matrix differential equation can then be written in the expanded form

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -K & 0 \\ 0 & -3K \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{8}$$

- We multiply on the left on both sides by U^T to cancel the left U in the eigen-decomposition of Ω, giving us,

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} = \begin{pmatrix}-K & 0\\ 0 & -3K\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}, \tag{9}$$

or

$$\begin{pmatrix}\frac{\ddot{x}_1+\ddot{x}_2}{\sqrt{2}}\\ \frac{\ddot{x}_1-\ddot{x}_2}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}-K & 0\\ 0 & -3K\end{pmatrix}\begin{pmatrix}\frac{x_1+x_2}{\sqrt{2}}\\ \frac{x_1-x_2}{\sqrt{2}}\end{pmatrix} \tag{10}$$

- Defining the new variables

$$y_1 = \frac{x_1 + x_2}{\sqrt{2}} \tag{11}$$

$$y_2 = \frac{x_1 - x_2}{\sqrt{2}} \tag{12}$$

we can write the simplified decoupled system of equation as

$$\begin{pmatrix}\ddot{y}_1\\ \ddot{y}_2\end{pmatrix} = \begin{pmatrix}-K & 0\\ 0 & -3K\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}, \tag{13}$$

or

$$\ddot{y}_1 = -Ky_1 \tag{14}$$

$$\ddot{y}_2 = -3Ky_2. \tag{15}$$

Each equation can be solved as a separate simple harmonic oscillator.