Group Cayley Table: Example of Order 4 Group

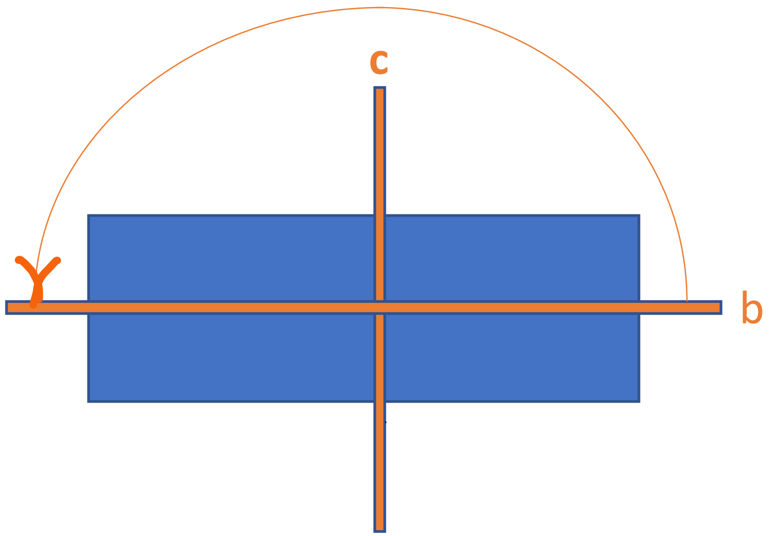
•	**e**	**a**	**b**	**c**
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Group Cayley Table: Is the symmetry group of a rectangle.

•	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

a: 180 deg rotation

b: Reflection about b axis.



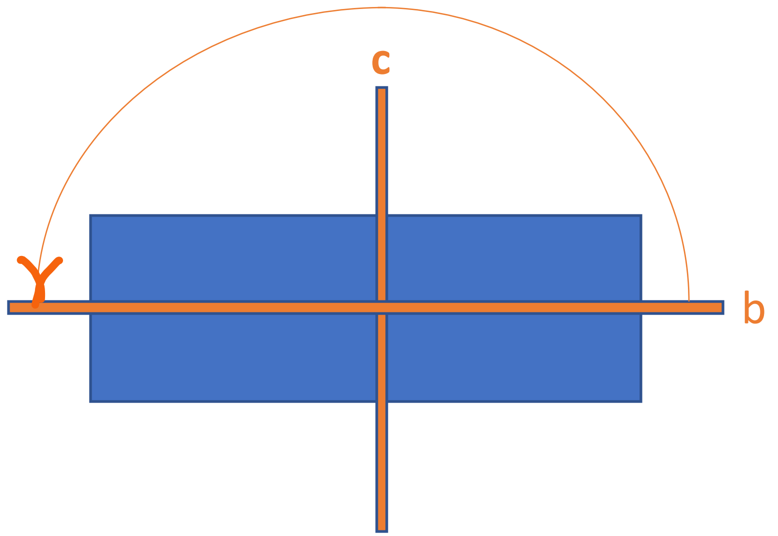
c: Reflection about a axis.

Group Cayley Table: Symmetry group of a rectangle.

•	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

a: 180 deg rotation

b: Reflection about b axis.



c: Reflection about a axis.

EXAMPLE 4.5.4

The table for $\mathbf{Z}_2$ using the operation $+$:

(a)

	0	1
0	0	1
1	1	0

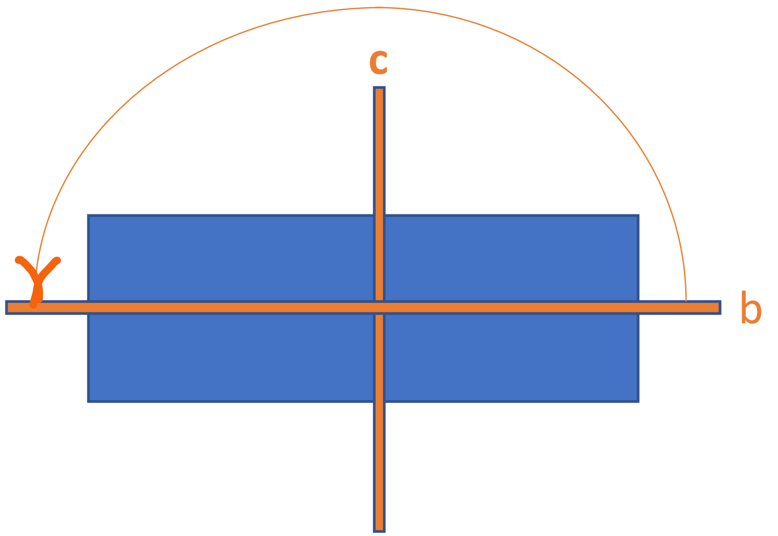
(b

Group Cayley Table: Symmetry group of a rectangle.

•	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

a: 180 deg rotation

b: Reflection about b axis.



c: Reflection about a axis.

Also isomorphic to Z_2 x Z_2

EXAMPLE 11.1.3

Figure 11.5a shows the group $\mathbf{Z}_2 \times \mathbf{Z}_2$

(a)

	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(0, 0)	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(1, 0)	(1, 0)	(0, 0)	(1, 1)	(0, 1)
(0, 1)	(0, 1)	(1, 1)	(0, 0)	(1, 0)
(1, 1)	(1, 1)	(0, 1)	(1, 0)	(0, 0)

$$(a,b) \bullet (x,y) = (a \bullet x, b \bullet y)$$

Note each row is a permutation of the group elements.

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\underline{G}$

Note each row is a permutation of the group elements.

•	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

•	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$$\begin{pmatrix} e & a & b & c \\ \hline e & a & b & c \end{pmatrix} = \pi_e \qquad \leftarrow G$$

Let's make the permutation explicit.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

$x \in G$

•	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

G

e	a	b	c
e	a	b	c

$= \pi_e$

e	a	b	c
a	e	c	b

G

$= \pi_a$

e	a	b	c
b	c	e	a

$= \pi_b$

e	a	b	c
c	b	a	e

$= \pi_c$

Give each permutation a name, indexed by the row group element.

$$\pi_b(x) = bx$$
$$\pi_b(e) = be = b$$
$$\pi_b(a) = ba = c$$
$$\pi_b(b) = \quad = e$$
$$\pi_b(c) = \quad = a$$

G

$$\{e, a, b, c\} \longrightarrow \{\pi_e, \pi_a, \pi_b, \pi_c\}$$

group elts

permutations

∘	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$$\begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix} = \pi_e$$

$$\begin{pmatrix} e & a & b & c \\ a & e & c & b \end{pmatrix} = \pi_a$$

$$\begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix} = \pi_b$$

$$\begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix} = \pi_c$$

Abstract Group: is a set of elements.

& an operation ∘.

satisfies

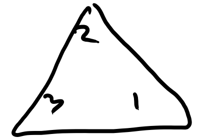
1. identity
2. closure
3. associative
4. inverses

$e \Rightarrow \pi_e$
$a \Rightarrow \pi_a$
$b \Rightarrow \pi_b$
$c \Rightarrow \pi_c$

Make the association (I.e., bijective map) between Group elements and their row permutation.

$$\begin{pmatrix} e\,a\,b\,c \\ a\,e\,c\,b \end{pmatrix} = \begin{pmatrix} e\,c\,b\,a \\ a\,b\,c\,e \end{pmatrix}$$

a



roots of polynomial.

Cayley's Theorem: Look at the corresponding table of permutations.

G

•	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

∘	π_e	π_a	π_b	π_c
π_e	π_e	π_a	π_b	π_c
π_a	π_a	π_e	π_c	π_b
π_b	π_b	π_c	π_e	π_a
π_c	π_c	π_b	π_a	π_e

H

Just suppose:

~ Cayley's Th$^{\underline{m}}$

A Group

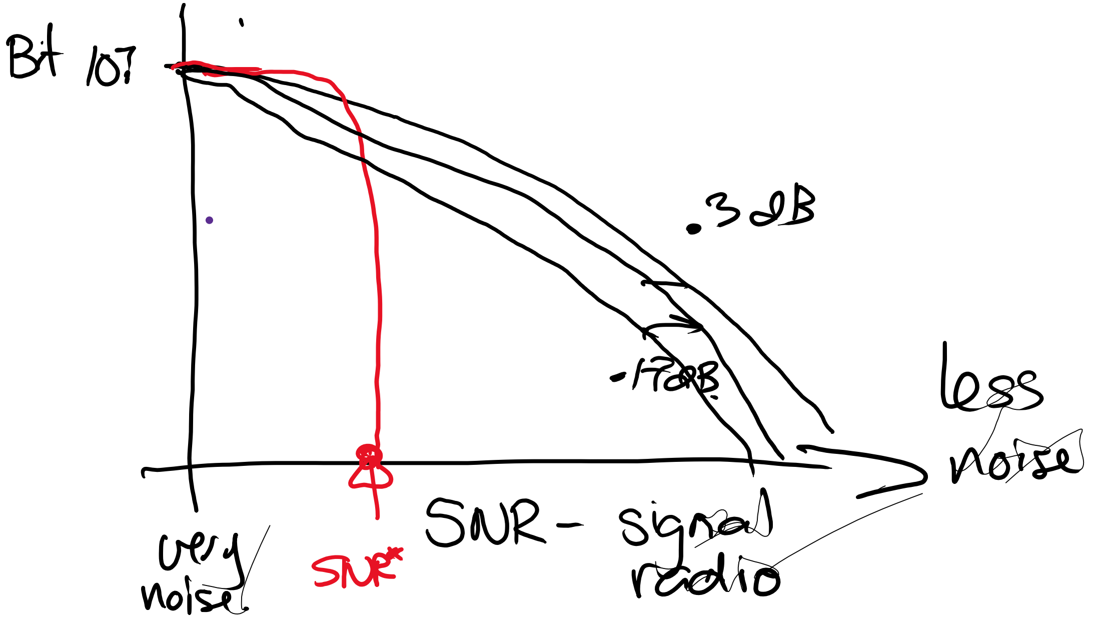
G

$e \to \pi_e$
$a \to \pi_a$
$b \to \pi_b$
$c \to \pi_c$

Is This A Group?

Shannon
- information

1947 ~



Cayley's Theorem:

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

f

$\cong$

composition of permutations

$\circ$	π_e	π_a	π_b	π_c
π_e	π_e	π_a	π_b	π_c
π_a	π_a	π_e	π_c	π_b
π_b	π_b	π_c	π_e	π_a
π_c	π_c	π_b	π_a	π_e

$\circ$	$\begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix}$	$\begin{pmatrix} e & a & b & c \\ a & e & c & b \end{pmatrix}$	$\begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix}$	$\begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix}$
$\begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix}$	π_e	π_a	π_b	π_c
$\begin{pmatrix} e & a & b & c \\ a & e & c & b \end{pmatrix}$	π_a	π_e	$\begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix}$	π_b
$\begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix}$	π_b	π_c	π_e	π_a
$\begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix}$	π_c	π_b	π_a	π_e

The abstract group table and the table of corresponding permutations are ***isomorphic***.

$f: a \longrightarrow \pi_a$

Cayley's Theorem: $a \bullet b = c$

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\circ$	π_e	π_a	π_b	π_c
π_e	π_e	π_a	π_b	π_c
π_a	π_a	π_e	π_c	π_b
π_b	π_b	π_c	π_e	π_a
π_c	π_c	π_b	π_a	π_e

$\boxed{a \bullet b = c}$

Cayley's Theorem: $a \bullet b = c \iff \pi_a \circ \pi_b = \pi_c$

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\circ$	π_e	π_a	π_b	π_c
π_e	π_e	π_a	π_b	π_c
π_a	π_a	π_e	π_c	π_b
π_b	π_b	π_c	π_e	π_a
π_c	π_c	π_b	π_a	π_e

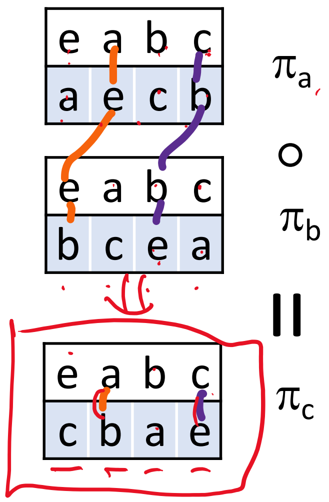
$a \bullet b = c$

$\pi_a \circ \pi_b = \pi_c$

Cayley's Theorem: $a \bullet b = c \iff \pi_a \circ \pi_b = \pi_c$

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\circ$	π_e	π_a	π_b	π_c
π_e	π_e	π_a	π_b	π_c
π_a	π_a	π_e	π_c	π_b
π_b	π_b	π_c	π_e	π_a
π_c	π_c	π_b	π_a	π_e



$a \bullet b = c$

$\pi_a \circ \pi_b = \pi_c$

$\pi_a \circ \pi_b = \pi_c$

Cayley's Theorem: $a \bullet b = c \quad \Leftrightarrow \quad \pi_a \circ \pi_b = \pi_c$

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\circ$	π_e	π_a	π_b	π_c
π_e	π_e	π_a	π_b	π_c
π_a	π_a	π_e	π_c	π_b
π_b	π_b	π_c	π_e	π_a
π_c	π_c	π_b	π_a	π_e

e	a	b	c
a	e	c	b

π_a

$\circ$

e	a	b	c
b	c	e	a

π_b

$=$

e	a	b	c
c	b	a	e

π_c

$a \bullet b = c$

$\pi_a \circ \pi_b = \pi_c$

$\pi_a \circ \pi_b = \pi_c$

$\pi_{a \bullet b} = \pi_a \circ \pi_b \quad <=====> \quad f(a \bullet b) = f(a) \circ f(b)$

This and next two slides are just scratch slides.

Cayley's Theorem:

•	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\circ$	π_e	π_a	π_b	π_c
π_e	π_e	π_a	π_b	π_c
π_a	π_a	π_e	π_c	π_b
π_b	π_b	π_c	π_e	π_a
π_c	π_c	π_b	π_a	π_e

e	a	b	c
a	e	c	b

π_a

$\circ$

e	a	b	c
b	c	e	a

π_b

$=$

e	a	b	c
c	b	a	e

π_c

$\circ$	e a b c / e a b c	e a b c / a e c b	e a b c / b c e a	e a b c / c b a e
e a b c / e a b c	π_e	π_a	π_b	π_c
e a b c / a e c b	π_a	π_e	e a b c / c b a e	π_b
e a b c / b c e a	π_b	π_c	π_e	π_a
e a b c / c b a e	π_c	π_b	π_a	π_e

e	a	b	c
e	a	b	c

e	a	b	c
a	e	c	b

e	a	b	c
b	c	e	a

e	a	b	c
c	b	a	e

e a b c / e a b c $= \pi_e$

e a b c / a e c b $= \pi_a$

e a b c / b c e a $= \pi_b$

e a b c / c b a e $= \pi_c$

Cayley's Theorem: $a \bullet b = c \Leftrightarrow \pi_a \circ \pi_b = \pi_c$

$\circ$	π_e	π_a	π_b	π_c
π_e	π_e	π_a	π_b	π_c
π_a	π_a	π_e	π_c	π_b
π_b	π_b	π_c	π_e	π_a
π_c	π_c	π_b	π_a	π_e

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$$\begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix} = \pi_e$$

$$\begin{pmatrix} e & a & b & c \\ a & e & c & b \end{pmatrix} = \pi_a$$

$$\begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix} = \pi_b$$

$$\begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix} = \pi_c$$

$e \Rightarrow \pi_e$
$a \Rightarrow \pi_a$
$b \Rightarrow \pi_b$
$c \Rightarrow \pi_c$

$\begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix}$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$= G = (e,a,b,c)$

$= e \bullet G = e \bullet (e,a,b,c)$

$= a \bullet G = a \bullet (e,a,b,c)$: $x \Rightarrow a \bullet x$

$= c \bullet G = c \bullet (e,a,b,c)$: $x \Rightarrow c \bullet x$