DGT: Thm 35 #1

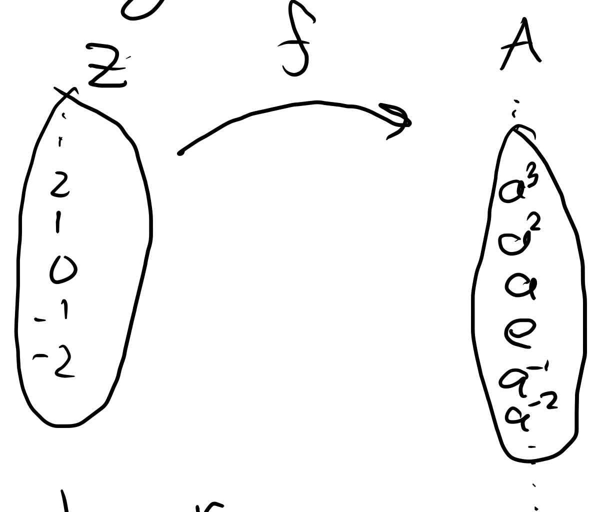
∘ finite cyclic grp $\cong (\mathbb{Z}_n, +)$ ✓

• Every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$. $\cong$

<u>Proof</u>: Let A be an ∞-cyclic group with a generator a.

$A = \langle a \rangle$

• Let $\boxed{\begin{aligned} f: \mathbb{Z} &\longrightarrow A \\ r &\longmapsto a^r \end{aligned}}$ $\boxed{f(r) = a^r}$



Is f bijective?

① Surjective: pick an arbitrary elt in A, call it a^r.
Is there a corresponding elt in $\mathbb{Z}$?
YES: r

② <u>Injective</u>: If $f(r) = f(s)$ does $r = s$ necessarily.

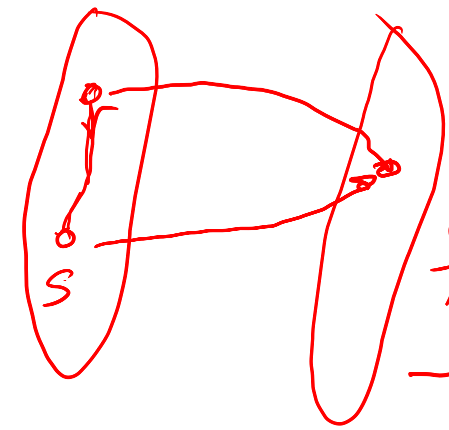
If $a^r = a^s$ does $r = s$?

If $a^r = a^s \longrightarrow a^r a^{-s} = e$

$\boxed{a^{r-s} = e}$

$\therefore$ A is ∞ then $r - s = 0$

$\therefore$ $\boxed{r = s}$



If $f(r) = f(s)$
$\longrightarrow s = r$?

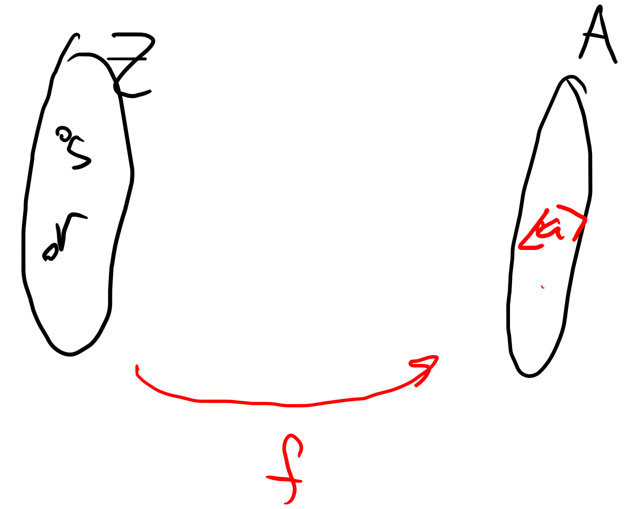
Is f compatable?

Check if $f(r+s) = f(r) \circ_A f(s)$

LHS: $f(r+s) = a^{r+s} = a^r \circ_A a^s$

RHS $= f(r) \circ_A f(s) = a^r \circ_A a^s$

LHS = RHS ✓



$f(r) = a^r$

$$\boxed{f(x) = a^x}$$

Z_n

Cayley's Th^m:

Every abstract group is isomorphic to a group of permutations.

Proof: From the discussion we define the map:

$$\pi_a : G \longrightarrow G \qquad \boxed{\pi_a(x) = ax} \checkmark$$
$$x \longmapsto ax$$

$a \longrightarrow \pi_a$

$b \longrightarrow \pi_b$

(A) Claim $\boxed{\pi_a(x) \text{ is bijective}}$: ① injective ② surjective.

Injective: If $\pi_a(x) = \pi_a(y) \longrightarrow x = y$?//

If $ax = ay$

then $\cancel{a^{-1}}ax = \cancel{a^{-1}}ay$

$\therefore x = y \quad \therefore$ injective.

Surjective: For all $y \in G$. does there exist an elt in G s.t. $\pi_a(x) = y$?

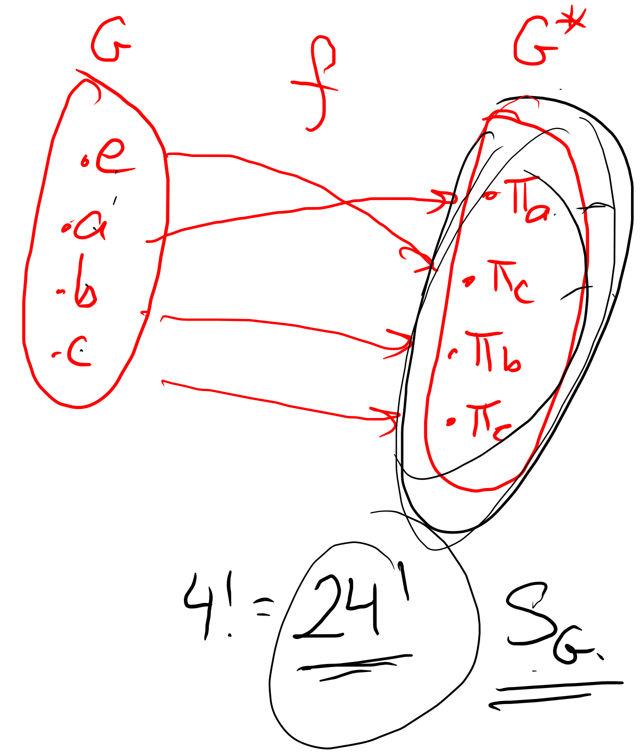
If $\pi_a(x) = y$

then $ax = y$

$\rightarrow \boxed{x = a^{-1}y}$

Now define $G^* = \{\pi_a \mid a \in G\} \subseteq S_G$

Claim: G^* is a group in itself under composition of perm^ns.



1. Assoc.: YES

2. Identity: YES

3. Closure: $\pi_a \circ \pi_b = \pi_{\_}$

$$(\pi_a \circ \pi_b)(x) = \pi_a \circ \pi_b(x)$$
$$= \pi_a(bx)$$
$$= abx$$
$$= \pi_{a \cdot b}(x)$$
$$= \pi_c(x)$$

$\pi_a \circ \pi_b = \boxed{\pi_c}$ where $c \in G$.

Remember: $\pi_a(x) = ax$

4. Inverse:

Claim: $\boxed{\pi_{a^{-1}} \text{ is the inverse of } \pi_a}$

If true then $\pi_{a^{-1}} \circ \pi_a = \pi_e$

$$\pi_{a^{-1}} \circ \pi_a(x) = \pi_{a^{-1}}(ax) = a^{-1}ax = x = \pi_e(x) \checkmark$$

ⓒ Claim: G is $\cong$ to G^*.

Use: $\boxed{f : a \longmapsto \pi_a .}$ — bijection:

Injective: If $f(a) = f(b)$ does $a = b$?

If $\pi_a = \pi_b$ does $a = b$.

If $\pi_a(x) = \pi_b(x)$ does $a = b$?

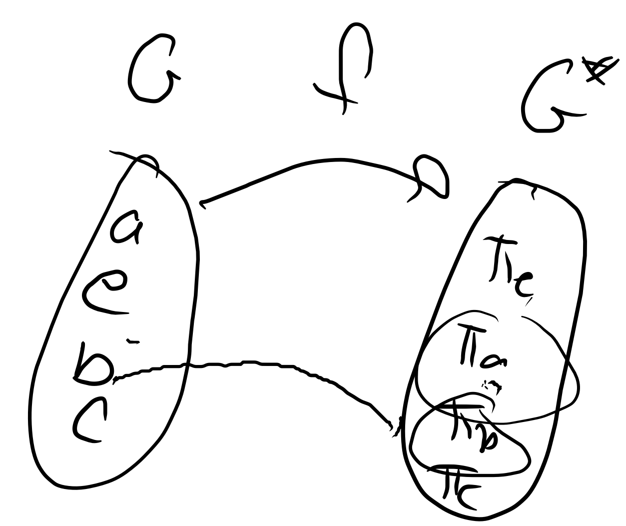
If $\underbrace{ax = bx}$ does $\underline{a = b}$?

$a\cancel{x x^{-1}} = b\cancel{x x^{-1}}$

$\therefore a = b$ ✓

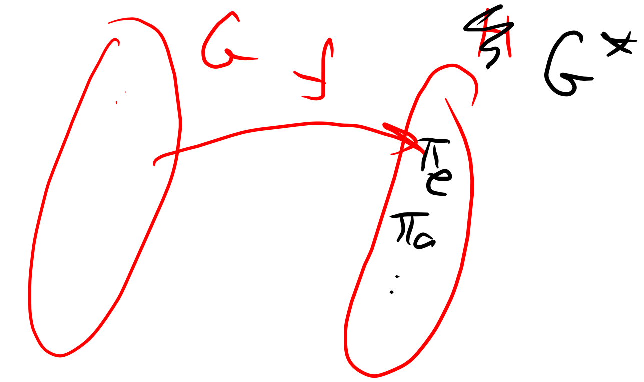
Surjective: For any elt in G^* is there a corresponding elt in G?

YES by definition., from inspection.



Compatibility:

Does $\boxed{f(a \circ_G b) = f(a) \circ_{G^*} f(b)}$?



LHS $= f(a \circ_G b)$ RHS $= \pi_a \circ_{G^*} \pi_b$

$= \pi_{a \circ_G b} \longleftrightarrow = \pi_{a \circ_G b}$

$\vdots$

$f: x \longrightarrow \pi_x$

Given an abstract group, define the function

$f: \overset{\in G}{x} \longrightarrow \pi_x \rightsquigarrow G^*$

Then $\boxed{G \cong G^*}$