Group

$G = \{S, \circ\}$

1) Closure $\forall a, b \in G,\ a \circ b \in G$
2) Associative $(a \circ b) \circ c = a \circ (b \circ c) = a \circ b \circ c$
3) Identity $\forall a \in G\ \exists e \in G$ s.t. $a \circ e = e \circ a = a$
4) Inverse $\forall a \in G\ \exists a^{-1} \in G$ s.t. $aa^{-1} = a^{-1}a = e$

$AB = CA$

Shown:
- single identity elt in a group.
- for each elt a there is unique inverse a^{-1}, i.e. unique.

What other properties:

Ch4. Thm 1. p.37

$R^* = \{x \in R \mid x \neq 0\}$

Given $a, b, c \in G$

Claim: $ab = ac \longrightarrow b = c$ ✓

If $\cancel{a}b = \cancel{a}c$ then $b = c$.

$ab = ac$

$(a^{-1}a)b = (a^{-1}a)c$

$eb = ec$

$\boxed{b = c}$

$AB = AC \rightarrow (A^{-1}A)B = (A^{-1}A)C$

$IB = IC$

$B = C$

If $ba = ca \rightarrow b = c$

$b(aa^{-1}) = c(aa^{-1})$

$be = ce$

$\boxed{b = c}$

If $aba^{-1} = caa^{-1}$

does $b = c$?

$\rightarrow ab = ca$

$ba = ca$

$b\cancel{a}a^{-1} = c\cancel{a}a^{-1} = b = c$

Thm 2 p.37

$$\boxed{\begin{matrix} ab = e \\ ba = e \end{matrix}}$$

Given $a, b \in G$

Claim: If $ab = e$ then $a = b^{-1}$ and $b = a^{-1}$

Proof:

$$ab = e$$

$$a^{-1}(ab) = a^{-1}(e)$$

$$(\cancel{a^{-1}a})b = a^{-1}e$$

$$eb = a^{-1}e$$

$$\boxed{b = a^{-1}}$$

Thm 3. p.38 - Shoes & socks theorem

Claim: $\boxed{(ab)^{-1} = b^{-1}a^{-1}}$

shoes sock socks shoes

$\Rightarrow (a_1 a_2 \cdots a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1}$

If this is true, expect $(ab)(b^{-1}a^{-1}) = e$

lhs $= abb^{-1}a^{-1} = a(bb^{-1})a^{-1}$

$= aea^{-1}$

$= aa^{-1}$

$= e$ ✓

$\therefore (ab)^{-1} = b^{-1}a^{-1}$

Thm 3 ii) : $(a^{-1})^{-1} = a$

Use similar argument.

If lhs is inverse of rhs. expect:

$$(a)(a^{-1}) = e$$?

LHS= $(aa^{-1}) = e$ = RHS.

– Proof

– Count example.

B) 1. If $x^2 = e$ then $x = e$? $x^2 = x \circ x$ **FALSE**

$x^2 = e$

$\underbrace{x^{-1}x}_{e}x = x^{-1}e$

$\boxed{x = x^{-1}}$

2. $x^2 = a^2$ then $x = a$? Check $x^2 = 4$ in $\langle Q^*, x\rangle$. **FALSE**

$x = \pm 2$.

3) $(ab)^2 = a^2b^2$?

LHS= $(ab)(ab) = abab \overset{?}{=} aabb$ (RHS) **FALSE**

abab.

4) If $x^2 = x$ then $x = e$? **TRUE**.

$\underbrace{x^{-1}x}_{e}x = x^{-1}x$

$\Rightarrow \boxed{x = e}$ = RHS.

$\langle Q^*, \times\rangle = 1$

$\boxed{1^2 = 1}$

5) For $x \in G$ there is some $y \in G$ st. $x = y^2$.

$\Rightarrow$ Every elt has a square root.

Counter example.

6) For any two elts x & $y \in G$ there is an elt $z \in G$ s.t. $y = xz$.

$x^{-1}y = x^{-1}xz$

$\boxed{z = x^{-1}y}$

C.) Elements which Commute:

Given two elts $a, b \in G$ s.t. $\underbrace{ab = ba}$, prove:

Claims

1. a^{-1} & b^{-1} commute

$$ab = ba \dashrightarrow a^{-1}b^{-1} = b^{-1}a^{-1}$$

$$(ab)^{-1} = b^{-1}a^{-1}$$

$$= (ba)^{-1} \quad \text{commuting}$$

$$\boxed{b^{-1}a^{-1} = a^{-1}b^{-1}} \quad \text{SoS.}$$

2. Claim: a & b^{-1} commute.

To show $ab^{-1} = b^{-1}a$

$$\cancel{ab^{-1}b = b^{-1}ab}$$

$$\cancel{a = b^{-1}ab}$$

$\therefore ab^{-1} =$

$$ab = ba$$

$$\underbrace{a = bab^{-1}}$$

$$=$$

$\therefore$ $b^{-1}a = b^{-1}bab^{-1}$

$$\boxed{b^{-1}a = ab^{-1}}$$

comm.

a & b^{-1} commute.

To Show $ab^{-1} = b^{-1}a$

Start: $ab = ba$

$$a\cancel{bb^{-1}} = bab^{-1}$$

$$\boxed{a = bab^{-1}}$$

$$b^{-1}a = b^{-1}(bab^{-1})$$

$$= b^{-1}bab^{-1}$$

$$= ab^{-1}$$

6.) Show $ab = ba$ iff $aba^{-1} = b$

↳ if and only if.

① If P then Q : $P \longrightarrow Q$

② If Q then P $Q \longrightarrow P$

If P then Q

If $\overline{Q}$ then $\overline{P}$

Contrapositive.

If: $\longleftarrow$ ✓ If $aba^{-1} = b$ the $ab = ba$

$aba^{-1} = b$

$ab\cancel{a^{-1}a} = ba$

$ab = ba$.

OF: $\longrightarrow$ ✓

If $ab = ba$ then $aba^{-1} = b$

$ab = ba$

$aba^{-1} = b\cancel{aa^{-1}}$

$\boxed{aba^{-1} = b}$

F) Constructing Small Groups

Proof: ① $\boxed{a^2 = a \longrightarrow a = e}$

② $\boxed{ab = a \longrightarrow b = e}$

③ $\boxed{ab = b \longrightarrow a = e}$

Group:

$\circ$	a	b	c
a	~~$a \circ a$~~		~~$a \circ c$~~
b			
c			

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2) Show why every row of a group table contains each elt once.

$\circ$	b_1	b_2
a	x	x

Suppose elt x appears twice in row a

Says that: $ab_1 = x$

and $ab_2 = x$

$\Longrightarrow \boxed{ab_1 = ab_2}$

$\cancel{a^{-1}a}b_1 = \cancel{a^{-1}a}b_2$

$\boxed{b_1 = b_2}$

$S = \{I\ V\ H\ D\}$

	I	V	H	D
V	o	o	o	o

n^{n^2}

$n=3$ 19,000

$\downarrow$

1

$n=4$: 2

4×10^9

3) There is one group of 3 elts.

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

1. $a^2 = a \rightarrow a = e$.

if $ab = b \rightarrow$

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Z_3 = addition mod 3.

Z_2 = addition of integers mod 2.

0	0	1
0	0	1
1	1	0

	e	a
e	e	a
a	a	e

	α	β	γ	δ
α				
β				
γ				
γ				

Isomorphism