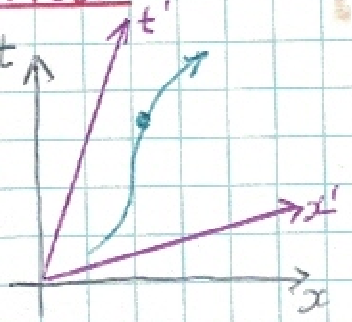
Finally - Relativistic Constant Acceleration

Review:



- a particle moves thru space-time
- we can describe its path as a function $x(t)$ wrt some inertial reference frame O.
- but it's equally correct to describe its coordinates in another frame O' as $x'(t')$.

- (x,t) & (x',t') are related by the Lorentz transformatio[n]

$$\boxed{\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \end{aligned}}$$

where $c \equiv 1$ & $\gamma = \frac{1}{\sqrt{1-v^2}}$.

- which frame you use is arbitrary & it would be nice to have a standard, canonical or <u>PROPER</u> way of doing this all inertial frames can agree on, i.e., one $X(T)$ function.

- recall the idea of invariants from before (e.g. bouncin[g] light beam on a moving boxcar).

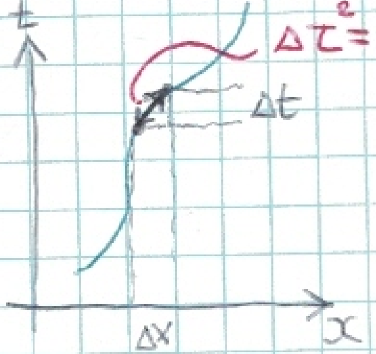
① The Interval $\Delta S^2 = \Delta x^2 - c\Delta t^2$

or

② Proper Time $\Delta \tau^2 = c\Delta t^2 - \Delta x^2$ → note $\Delta\tau^2 = -\Delta S^2$

- use whichever is convenient.

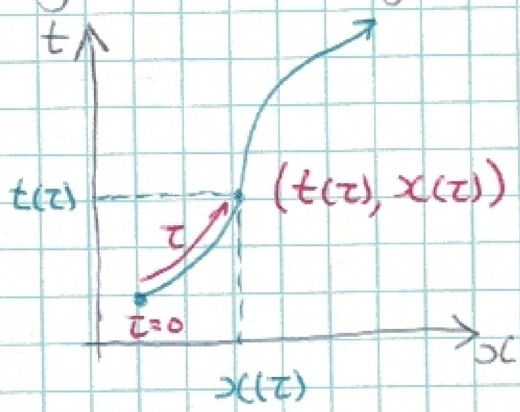
- all inertial frames "see" the same proper time.



(setting $c=1$ from now on).

- $\Delta\tau$ (or $-\Delta s$) acts as an arclength along the path, or world line of the particle.
- it's just the time measured by the traveller's watch as he moves about.

- parameterizing the co-ords (x,y,z) of points on a curve by the arclength along the curve is standard.



- we can write the curve as

$$x^\alpha = (x^0(\tau), x(\tau)) = (t(\tau), x(\tau)) = x^\alpha(\tau).$$

DEFINITION OF 4-VELOCITY

- the velocity of the particle is just the rate of change of its position wrt time.
- using proper time this define the particle's 4-velocity.

$$\boxed{u^\alpha \equiv \frac{dx^\alpha}{d\tau}}$$

→ this vector is tangent to the curve

- now find the components of the velocity 4-vector in familiar terms.

1. - recall we showed that between inertial frames that $\Delta t = \gamma \Delta \tau$

   - so $\frac{\Delta t}{\Delta \tau} = \gamma$

   or $\frac{dt}{d\tau} = \gamma$

   or $\boxed{\frac{dx^0}{d\tau} = \gamma}$ ← time component

$\Rightarrow$ Why? The clock in the moving frame is stationary in the moving frame.

$\therefore \Delta x' = 0$

So $c^2\Delta\tau^2 - \cancelto{0}{\Delta x'^2} = c^2\Delta t^2 - \Delta x^2$

$c^2\Delta\tau = \sqrt{\Delta t^2 - \frac{\Delta x^2}{c^2}}$

$\Delta\tau = \sqrt{1 - \frac{\Delta x^2}{\Delta t^2}\frac{1}{c^2}} \cdot \Delta t$

$\Delta\tau = \sqrt{1 - v^2/c^2} \cdot \Delta t$

$\boxed{\Delta\tau = \frac{1}{\gamma}\Delta t}$ ✓

2. $\frac{dx'}{d\tau} = \frac{dx}{dt} \cdot \frac{dt}{d\tau}$

   $\boxed{u^x = v_x \cdot \gamma}$ ← x-component

3. Similarly $\boxed{\begin{matrix} u^y = \gamma v_y \\ u^z = \gamma v_z \end{matrix}}$

$\Rightarrow$ Putting the pieces together we get the 4-velocity

$$\boxed{u^\alpha = (\gamma, \gamma\vec{v}) = \gamma(1, v^x, v^y, v^z) = \gamma(1, \vec{v})}$$

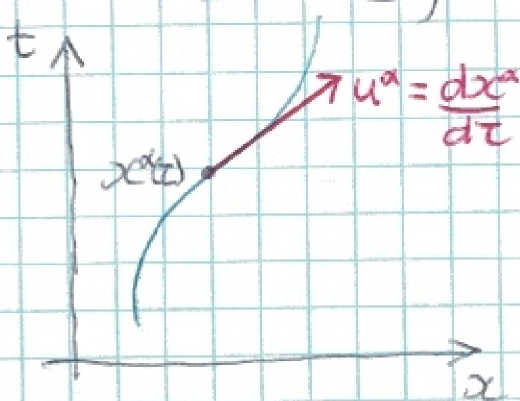
- when stationary in (x,y,z) then $\vec{v} = 0$ & $\gamma = 1$

  $\therefore u^\alpha = (1,0,0,0)$

NOTE! This is the rest frame of the moving particle.

$\therefore$ the 4-velocity is the time-axis for a frame (momentarily) co-moving (at rest) with the particle.

- this is a first step in defining a local co-ordinate system for an arbitrarily moving particle.



MAGNITUDE OF 4-VELOCITY

: recall the Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta \quad \text{where } \eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \emptyset \\ & 1 & & \\ & & 1 & \\ \emptyset & & & 1 \end{pmatrix}$$

- the general inner product of two 4-vectors, $\underset{\sim}{a}, \underset{\sim}{b}$, in M.S. is given by

$$\boxed{\underset{\sim}{a} \cdot \underset{\sim}{b} = \eta_{\alpha\beta}\, a^\alpha b^\beta} \quad \text{and} \quad \boxed{|\underset{\sim}{a}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a}}$$

- So for the 4-velocity,

$$|\underset{\sim}{u}|^2 = \underset{\sim}{u} \cdot \underset{\sim}{u}$$

$$\boxed{|\underset{\sim}{u}|^2 = \eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}}$$

- but look at the metric,

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$-d\tau^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$d\tau^2 = -\eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$\frac{d\tau^2}{d\tau^2} = -\eta_{\alpha\beta} \frac{dx^\alpha dx^\beta}{d\tau^2}$$

$$1 = -\eta_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

substitute.

$$\therefore \boxed{\eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = -1}$$

$$\therefore \boxed{|\underset{\sim}{u}|^2 = -1}$$ ← note the negative makes the vector time-like, as expected

- and it's already normalized to a unit.

Independent Check

- we showed the 4-velocity has the form:

$$u^\alpha = (\gamma, \gamma v_x)$$ ← In Mink. 1-1 space.

- here $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

- so $|\underset{\sim}{u}|^2 = \eta_{\alpha\beta} u^\alpha u^\beta$

$$= \underset{\eta_{tt}}{(-1)} \cdot \underset{u^t}{(\gamma)}^2 + \underset{\eta_{xx}}{(1)} \cdot \underset{u^x}{(\gamma v_x)}^2$$

$$= -1 \cdot \left(\frac{1}{\sqrt{1-v_x^2}}\right)^2 + \left(\frac{1}{\sqrt{1-v_x^2}} \cdot v_x\right)^2$$

$$= \frac{-1}{1-v_x} + \frac{v_x^2}{1-v_x^2}$$

$$= -\frac{(1-v_x^2)}{1-v_x^2}$$

$\therefore \boxed{|\underset{\sim}{u}|^2 = -1}$ ✓ — consistency checks.

4-ACCELERATION

- similarly we can define the 4-acc^{n}.

$$\boxed{\underset{\sim}{a}(\tau) = \frac{d\underset{\sim}{u}}{d\tau}}$$

- but we have $\underset{\sim}{u} \cdot \underset{\sim}{u} = 1$
- diff' both sides

$$\frac{d}{d\tau}(\underset{\sim}{u} \cdot \underset{\sim}{u}) = \frac{d}{d\tau}(-1).$$

So $\underset{\sim}{u} \cdot \dfrac{d\underset{\sim}{u}}{d\tau} + \dfrac{d\underset{\sim}{u}}{d\tau} \cdot \underset{\sim}{u} = 0$

$\rightarrow 2\underset{\sim}{u} \cdot \dfrac{d\underset{\sim}{u}}{d\tau} = 0$

or $\boxed{\underset{\sim}{u} \cdot \underset{\sim}{a} = 0}$

$\therefore$ the 4-velocity & 4-acc^n are always orthogonal.

PROPER ACC^N

- in the moving & accelerating observers stationary frame, his velocity is alway

$$\boxed{\underset{\sim}{u}' = (1, \vec{0})}$$

- since his acc^n is alway orthogonal, the observer's frame's acc^n must have the form

$$\boxed{a' = (0, \vec{a})}$$

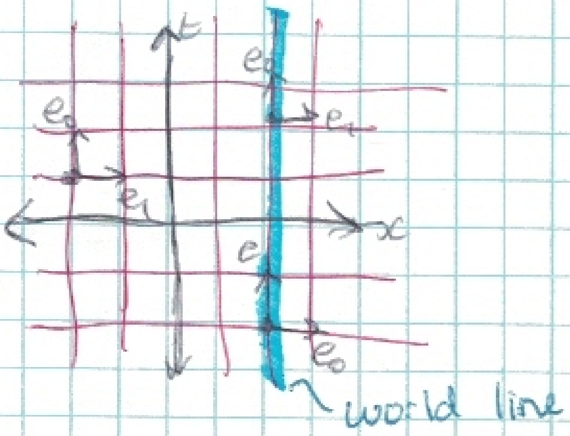
where $\vec{a} = \left(\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}, \dfrac{d^2z}{dt^2}\right)_{\text{in co-moving frame}}$

$= \dfrac{d}{dt}(v_x, v_y, v_z)\Big|_{\text{in cmf.}}$

$$\boxed{\vec{a} = \dfrac{d}{dt}\vec{v}\Big|_{\text{in cmf}}}$$

$\leftarrow$ this is called the PROPER acc^n

Recall the co-ord frames of a stationary & interial frame



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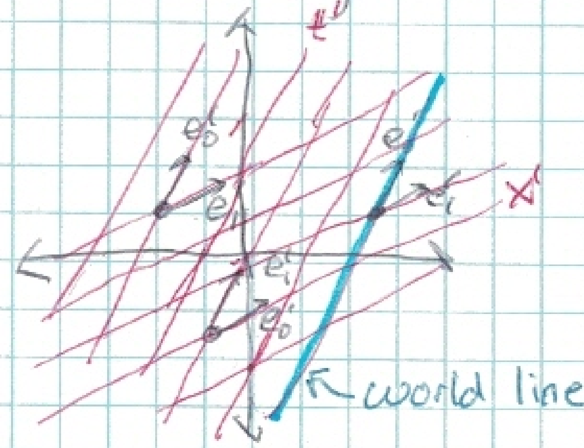
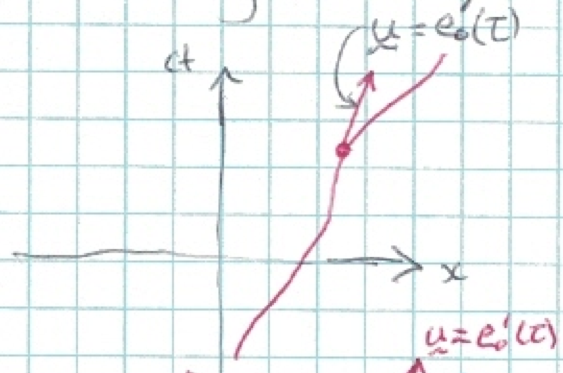
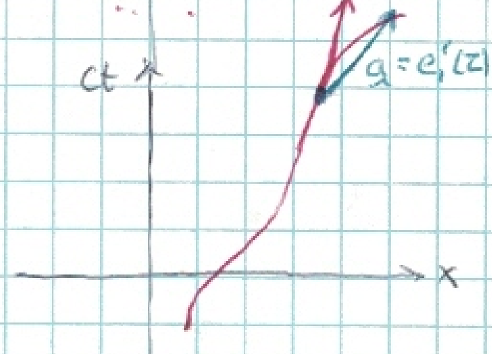


S'

* - the co-ord system's basis are fixed for const. veloc
- when the velocity varies under acc^c define the instantaneous rest frame S' for which the moving observer is momentarily at rest.



- so the velocity vector in general defines a local time axis.
- so $\boxed{\underset{\sim}{e}_0'(\tau) = \hat{\underset{\sim}{u}}(\tau)}$.



- in (t, x) 1-1 space we can the the orthogonal $\underset{\sim}{a}(\tau)$ to be the spatial basis vector at that pt

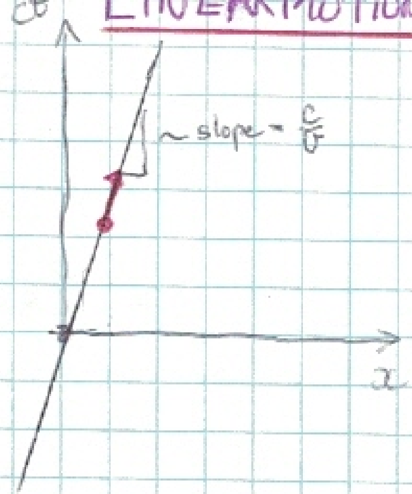
$$\boxed{\underset{\sim}{e}_1'(\tau) = \hat{\underset{\sim}{a}}(\tau)}$$

* This forms a (momentarily inertial) co-moving frame along the moving observer's world line *

Hartley p5.3

Simple Linear Motion - const. vel (7b)

① LINEAR MOTION



here $V = \frac{1}{3}c$

- $\gamma = \frac{1}{\sqrt{1-V^2}}$
- $\Delta\tau = \frac{1}{\gamma}\Delta t$
- ∴ go thru the origin we have

$$\tau = \frac{t}{\gamma} \rightarrow \boxed{t = \gamma\tau}$$

- set $c=1$ so velocity is the fraction of c (here $V = 1/3$).
- line's equation $y = mx + b$.

$$\Longrightarrow \quad t = \frac{1}{v}x + 0$$

$$t = \frac{x}{v}$$

$$x = vt$$

$$\boxed{x = v\gamma\tau}$$

- parametrically

$$\boxed{\begin{aligned} t(\tau) &= \gamma\tau \\ x(\tau) &= v\gamma\tau \end{aligned}}$$

$$\underset{\sim}{r}(\tau) = (\gamma\tau, v\gamma\tau)$$

$$\boxed{\underset{\sim}{r}(\tau) = \gamma\tau(1, v)}$$

Check:

$v = \frac{3}{5}$

$\gamma = \frac{5}{4}$

$\underset{\sim}{r}(\tau=4) = \frac{5}{\cancel{4}} \cdot \cancel{4}\left(1, \frac{3}{5}\right)$

$\boxed{\underset{\sim}{r}(4) = (5, 3)}$ ✓

② What is the "4"-velocity?

$$\frac{d\underset{\sim}{r}}{d\tau} = \frac{d}{d\tau}(\gamma\tau, v\gamma\tau).$$

$$\boxed{\underset{\sim}{u} = \gamma(1, v)}$$ ✓

$\Longrightarrow$ also the moving observers time axis basis vect.

CONSTANT ACC^{N} – Parametric Eq^{n}'s.

- suppose we have constant proper acc^{n} as experienced by the moving observer.
- In the observer's frame $\underset{\sim}{a} = (0, a)$ so $|\underset{\sim}{a}|^2 = a^2$

In general then, for any frame,

$$\boxed{\begin{aligned} |\underset{\sim}{a}|^2 = a^\mu a_\mu &= a^2 \\ u^\mu a_\mu &= 0 \\ u^\mu u^\mu &= -1 \end{aligned}}$$

← constant acc^{n}

← orthogonality of velocity & acc^{n}

← unit magnitude of time-like veloc

- recalling $x^\mu y_\mu = \eta_{\mu\nu} x^\mu y^\mu$ for inner products of vectors, where $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- we can expand the equations as.

$$\boxed{\begin{aligned} &① \quad -a^0 a^0 + a^1 a^1 = a^2 \\ &② \quad -u^0 a^0 + u^1 a^1 = 0 \\ &③ \quad -u^0 u^0 + a^1 a^1 = a^2 \end{aligned}}$$

← set of coupled 2'nd order polynomial eq^{n}'s.

- solve for the two acc^{n}'s in terms of the two velocitie
- can solve these to show:

$$\boxed{\begin{aligned} a^0 &= a u^1 \\ a^1 &= a u^0 \end{aligned}}$$

← use Maple, Mathematic to do.

– substituting into eq'ns ①, ②, ③ verifies the sol^{n}.

- So we now have a pair of coupled first order ordinary differential eq^{n}s to solve:

$$\boxed{\begin{aligned} \frac{du^0}{d\tau} &= a u^1 \\ \frac{du^1}{d\tau} &= a u^0 \end{aligned}}$$

$\Rightarrow$ impose initial conditions:

$t(\tau = 0) = 0$

$x(\tau = 0) = \frac{1}{a}$

– will be seen to be convenient.

- these have the solutions:

$$\boxed{\begin{aligned} t(\tau) &= \frac{1}{a}\sinh(a\tau) \\ x(\tau) &= \frac{1}{a}\cosh(a\tau) \end{aligned}}$$

$\leftarrow$ again, use a computer algebra package.

$\leftarrow$ check with substituting using $\cosh(x)' = \sinh(x)$
$\sinh(x)' = \cosh(x)$.

SOME INTERESTING POINTS.

① It's a hyperbola

- we have $x^2 - t^2 = \frac{1}{a^2}\cosh^2(a\tau) - \frac{1}{a^2}\sinh^2(a\tau)$

$$= \frac{1}{a^2}\underbrace{\left(\cosh^2(a\tau) - \sinh^2(a\tau)\right)}_{1}$$

$$\boxed{x^2 - t^2 = \frac{1}{a^2}}$$

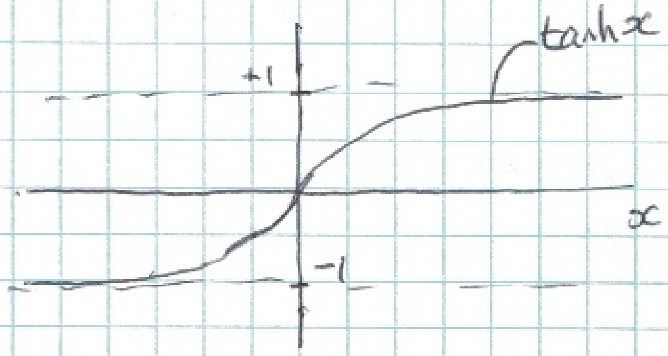
② Co-ordinate velocity only <u>approaches</u> c.

$$v_x = \frac{dx}{dt} = \frac{dx}{d\tau}\cdot\frac{d\tau}{d}$$

$$= \frac{dx}{d\tau}\Big/\frac{dt}{d\tau}$$

$$= \frac{\sinh(a\tau)}{\cosh(a\tau)}.$$

$$\boxed{v_x = \tanh(a\tau)}$$



③ Lines of constant τ are radial.

- we have $\dfrac{t(\tau)}{x(\tau)} = \dfrac{1/a\,\sinh(a\tau)}{1/a\,\cosh(a\tau)}.$

$$\therefore \frac{t}{x} = \tanh(a\tau).$$

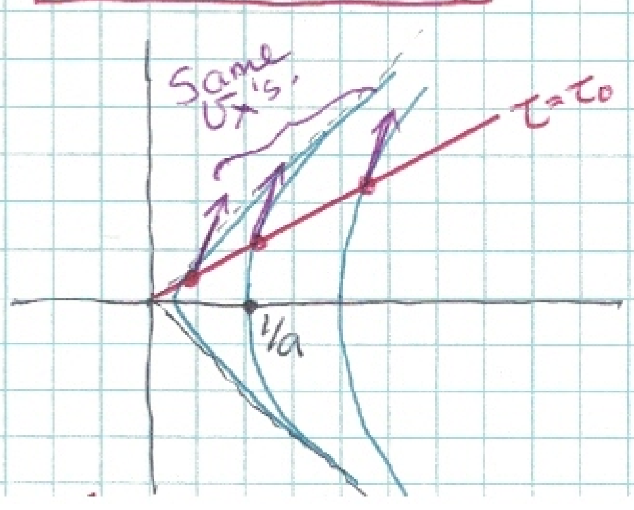
$$\boxed{t = \tanh(a\tau)\cdot x}$$

→ for a given proper time $\tau = \tau_0$:

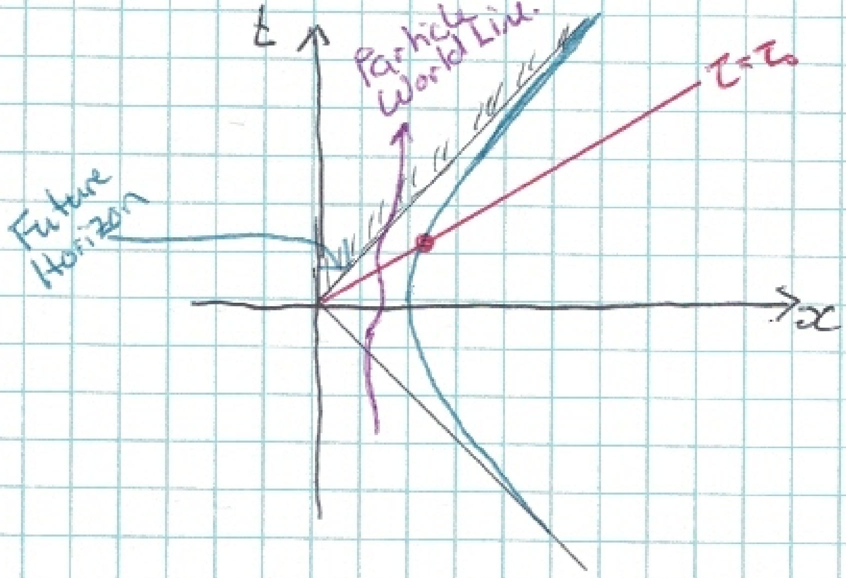
$$t = \tanh(a\tau_0)\cdot x$$

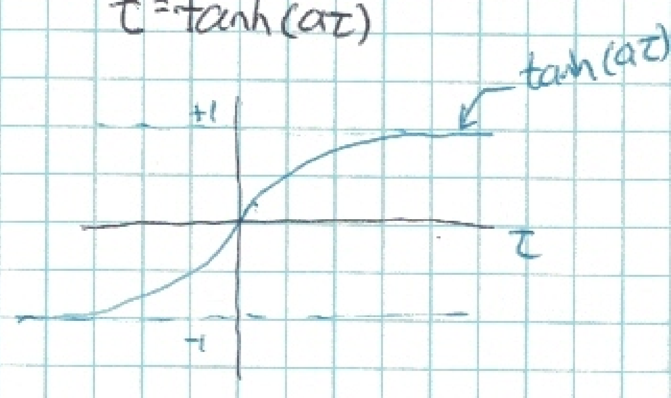
constant for given $\tau = \tau_0$

④ Picture.



⑤ There is an event horizon.





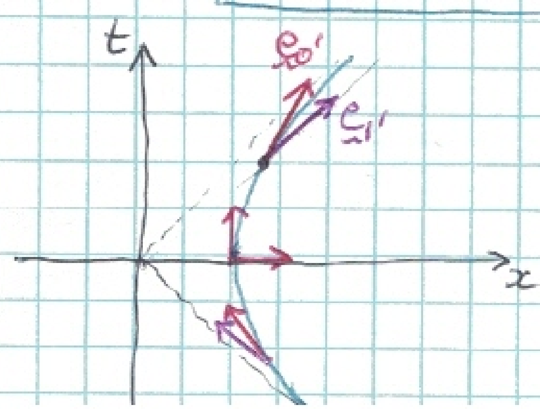
- to the inertial observer the particle just moves along normally.
- the accelerated observer never sees the particle cross the future horizon
  → $\boxed{\text{it will slow and slow as } \tau \to \infty \text{ and freeze at the point it crosses the horizon.}}$

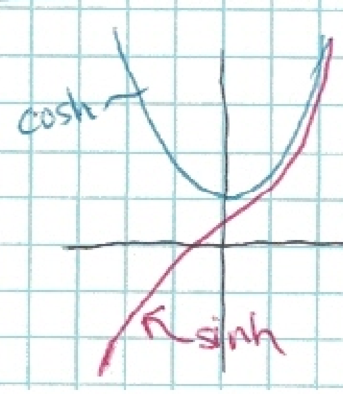
⑥ Local accelerating observer co-ords.

- position: $\underset{\sim}{x}(\tau) = \frac{1}{a}(\sinh(a\tau), \cosh(a\tau))$

  velocity: $\underset{\sim}{u}(\tau) = (\cosh(a\tau), \sinh(a\tau)) \Rightarrow \hat{\underset{\sim}{e}}_{0'}$ local time basis

  acc$^{\underline{n}}$: $\underset{\sim}{a}(\tau) = a(\sinh(a\tau), \cosh(a\tau)) \Rightarrow \hat{\underset{\sim}{e}}_{1'} = (\mathrm{sh}(a\tau), \mathrm{ch}(a\tau))$ local spatial basis.

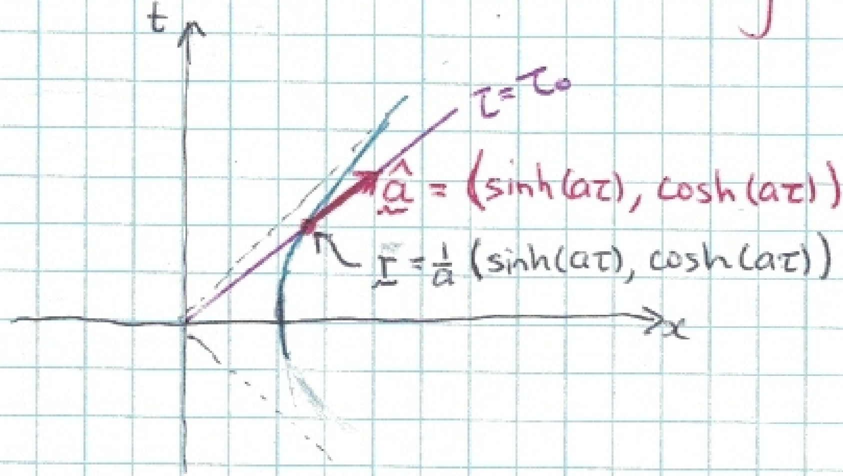




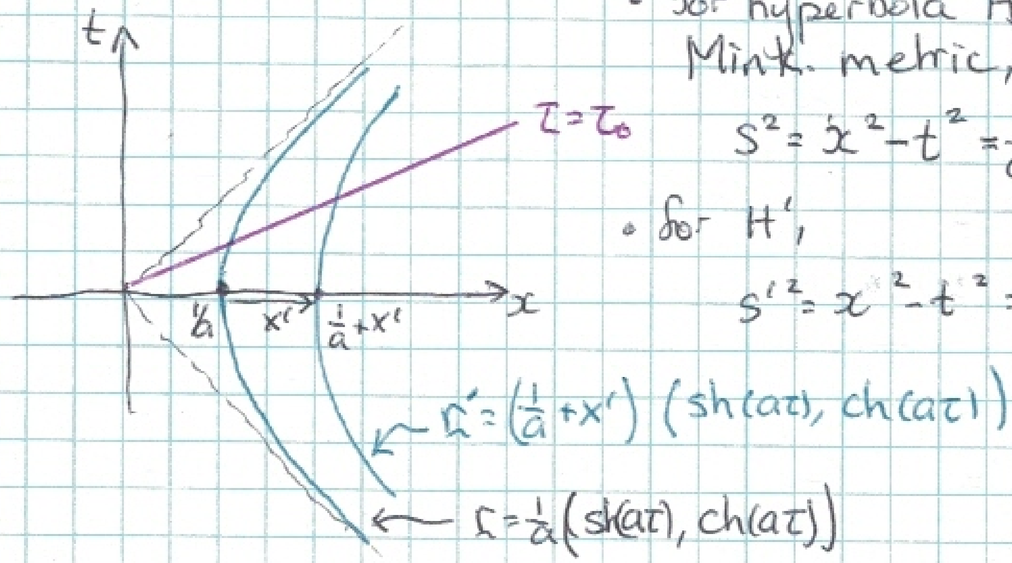
LOCAL Coord' SYSTEM OF AN ACC'd OBSERVER.

POINTS:

① Unit acceleration vectors lie along lines of constant τ.



② In the Minkowski metric, hyperbolas a separated by a constant distance.



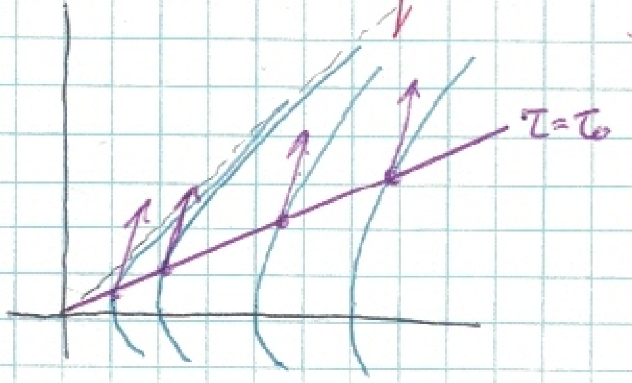
- for hyperbola H, under the Mink. metric,

$$s^2 = x^2 - t^2 = \frac{1}{a^2}$$

- for H',

$$s'^2 = x^2 - t^2 = \frac{1}{a^2} + x'$$

③ The 4-velocities are equal along lines of constant τ.



- as shown on pg. 10,

$$v_x = \frac{dx}{dt} = \tanh(a\tau).$$

- all the points move rigidly!

LOCAL COORD'S

- for a rocket acc'ing $\bar{c}$ acc^{n} a, define a spatial co-ord x' as the distance from the floor (the origin).
- define time as the proper time measured by the rocket, $t' \equiv \tau$.
- the relationship btwn the acc'ing local co-ord's and inertial co-ords, (x,t), are:

$$\boxed{\begin{aligned} t &= \left(\frac{1}{a} + x'\right)\sinh(at') \\ x &= \left(\frac{1}{a} + x'\right)\cosh(at') \end{aligned}}$$

METRIC

- now calculate the metric $ds^2 = dt^2 - dx^2$ in terms of the local co-ords:

$$\boxed{\begin{aligned} dt &= \sinh(at')\,dx' + (1+ax')\cosh(at')\,dt' \\ dx &= \cosh(at')\,dx' + (1+ax')\sinh(at')\,dt' \end{aligned}}$$

$$\Rightarrow dt^2 - dx^2 = s^2 \cdot dx'^2 + (1+ax')^2 c^2 dt'^2 + \cancelto{0}{2(1+ax')s \cdot c \cdot dx'dt'}$$
$$- c^2 \cdot dx'^2 - (1+ax')^2 s^2 dt'^2 - \cancelto{0}{2(1+ax') \cdot s \cdot c \cdot dx'dt'}$$

$$\boxed{\boxed{ds^2 = (1+ax')^2 dt'^2 - dx'^2}}$$

Recall:

$\cosh^2\theta - \sinh^2\theta = 1$

NOTE: This is still Minkowski space!

Transformation of metric:

- for convenience define the new variables:

$$\boxed{\begin{aligned}\rho &= \frac{1}{a} + x' \\ \varphi &= a t'\end{aligned}} \quad \Longrightarrow \quad \begin{aligned} dx' &= d\rho \\ dt' &= \frac{1}{a} d\varphi . \end{aligned}$$

- now $ds^2 = \cancel{a^2} \rho^2 \frac{1}{\cancel{a^2}} d\varphi^2 - d\rho^2$

$$\boxed{ds^2 = \rho^2 d\varphi^2 - d\rho^2}$$

← The Rindler metric.

- compare $\bar{c}$ Euclidean space metric:

	Cartesian	Polar
Euclidean	$ds^2 = dx^2 + dy^2$	$ds^2 = r^2 d\theta^2 + dr^2$
Minkowsi	$ds^2 = dt^2 - dx^2$	$ds^2 = \rho^2 d\varphi^2 - d\rho^2$

Rindler

Christophel Symbols of Metric

- standard calculations show:

$$\boxed{\begin{aligned} \Gamma^{\varphi}_{\varphi\rho} &= \Gamma^{\varphi}_{\rho\varphi} = \frac{1}{\rho} \\ \Gamma^{\rho}_{\varphi\varphi} &= \rho \end{aligned}}$$

Geodesic Eq$^{\underline{n}}$:

- geodesic eq$^{\underline{n}}$: $\boxed{\dfrac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\dfrac{dx^\beta}{d\tau}\dfrac{dx^\gamma}{d\tau} = 0}$

NOTE: τ is proper time along the geodesic

where $\Gamma^\varphi_{\varphi\rho} = \Gamma^\varphi_{\rho\varphi} = \dfrac{1}{\rho}$

$\Gamma^\rho_{\varphi\varphi} = \rho$

- substituting we get

$$\boxed{\begin{aligned} &a.)\quad \ddot{\varphi} + \frac{2}{\rho}\dot{\varphi}\dot{\rho} = 0 \\ &b.)\quad \ddot{\rho} + \rho(\dot{\varphi})^2 = 0 \end{aligned}}$$

Solving the Geodesic Equations.

- solve for the functions $\varphi(\tau), \rho(\tau)$ OR $\rho(\varphi)$ – eliminate the τ.
- the metric provides a third relationship

$$d\tau^2 = \rho^2 d\varphi^2 - d\rho^2$$

→ divide by ds^2 on the left & right hand side.

$$1 = \rho^2\left(\frac{d\varphi}{d\tau}\right)^2 - \left(\frac{d\rho}{d\tau}\right)^2$$

c) $\boxed{1 = \rho^2(\dot{\varphi})^2 - (\dot{\rho})^2}$

(16)

- a) $\rightarrow \quad \dfrac{d^2\varphi}{d\tau^2} + \dfrac{2}{\rho}\dfrac{d\varphi}{d\tau}\dfrac{d\rho}{d\tau} = 0$

Rewrite as:

$$\frac{1}{\rho^2}\frac{d}{d\tau}\left(\rho^2\frac{d\varphi}{d\tau}\right) = 0$$

$$\therefore \frac{d}{d\tau}\left(\rho^2\frac{d\varphi}{d\tau}\right) = 0$$

d) $\therefore \boxed{\rho^2\dfrac{d\varphi}{d\tau} = k}$ - a constant.

- Insert d) into c) : $1 = \rho^2\left(\dfrac{d\varphi}{d\tau}\right)^2 - \left(\dfrac{d\rho}{d\tau}\right)^2$

$$1 = \frac{1}{\rho^2}\left(\rho^2\dot{\varphi}\right)^2 - \dot{\rho}^2$$

$$\therefore 1 = \frac{k^2}{\rho^2} - \dot{\rho}^2$$

$$\Rightarrow \boxed{\frac{d\rho}{d\tau} = \sqrt{\frac{k^2}{\rho^2} - 1}}$$

- from d) $\boxed{\dfrac{d\varphi}{d\tau} = \dfrac{k}{\rho^2}}$

Eliminate the τ-dependence by dividing:

$$\frac{d\varphi}{d\rho} = \frac{\dfrac{d\varphi}{ds}}{\dfrac{d\rho}{ds}}$$

$$\frac{d\varphi}{d\rho} = \frac{k}{\rho^2}\Big/\left(\frac{k^2}{\rho^2} - 1\right)^{1/2}.$$

So $$\boxed{\frac{d\psi}{d\rho} = \frac{k}{\rho^2}\left(\frac{k^2}{\rho^2} - 1\right)^{-1/2}}$$

- solve by integration.

$$\int_{\psi} d\psi = \int_{\rho} \frac{k}{\rho^2}\left(\frac{k^2}{\rho^2} - 1\right)^{-1/2} d\rho.$$

$$\Rightarrow \quad \psi - \psi_0 = -\cosh^{-1}\left(\frac{k}{\rho}\right)$$

$$\cosh(\psi_0 - \psi) = \frac{k}{\rho}.$$

So: $$\boxed{\rho \cosh(\psi - \psi_0) = k}$$ ← Geodesic Eq^{n} in Rindler co-ords.

Geodesic In Inertial (x,t) Coord's

- expand the cosh fcn:

$$\rho\left(\operatorname{ch}\psi \cdot \operatorname{ch}\psi_0 - \operatorname{sh}\psi \operatorname{sh}\psi_0\right) = k$$

$$\operatorname{ch}\psi_0 \cdot x - \operatorname{sh}\psi_0 \cdot t = k$$

$$x = \left(\frac{\operatorname{sh}\psi_0}{\operatorname{ch}\psi_0}\right) t + \frac{k}{\operatorname{ch}\psi_0}$$

$$\boxed{x = \tanh(\psi_0) \cdot t + \frac{k}{\operatorname{ch}\psi_0}}$$

→ Compare to the eq^{n} of a line:

$$x = \underbrace{m}_{\text{slope}} t + \underbrace{b}_{\text{y-intercept}}$$

$\Rightarrow$ a line as expected.

Physical Meaning of ψ_0, k?

- We have $x = \tanh(\psi_0)\cdot t + \dfrac{k}{\cosh(\psi_0)}$

$$\therefore \frac{dx}{dt} = \tanh(\psi_0).$$

OR $\boxed{\begin{aligned}\tanh\psi_0 &= v_x \\ \psi_0 &= \tanh^{-1}(v_x)\end{aligned}}$ $\Rightarrow$ the velocity of the moving particle.

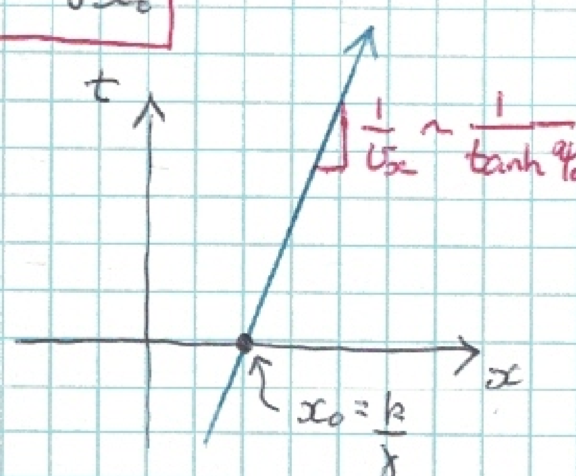
- also $x(t=0) = \boxed{\dfrac{k}{\cosh\psi_0} \equiv x_0}$ – the x-intercept.
- But $\cosh(\psi_0) = \cosh\left(\tanh^{-1}(v_x)\right)$

$$= \frac{1}{\sqrt{1-v_x^2}}$$

$$= \gamma$$! The Lorentz factor $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$

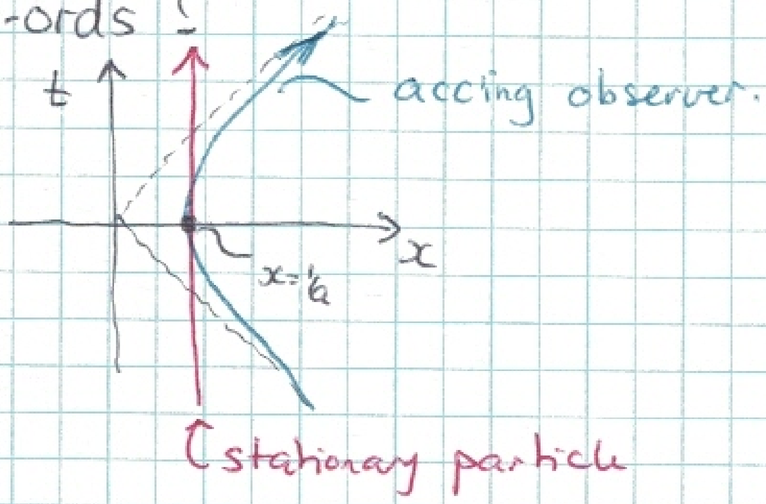
$$\therefore \boxed{k = \gamma x_0}$$

Picture



What Does The Acc'ng Observer "SEE"?

- suppose the acc'ing observer is moving $\bar{c}$ acc'n a & a particle is placed at $x = \frac{1}{a}$ & stationary.
- what does the observer measure in his local co-ords?



- the particle's velocity is $v_x = 0 \Rightarrow \gamma = 1$ & $\varphi_0 = 0$

It's x intercept is $x_0 = \frac{1}{a}$

- it's eq^n of motion in Rindler co-ords is

$$\boxed{\rho \cosh \varphi = \frac{1}{a}}$$

where $\rho = \frac{1}{a} + x'$
$\varphi = at'$

- so $\left(\frac{1}{a} + x'\right) \cosh(at') = \frac{1}{a}$

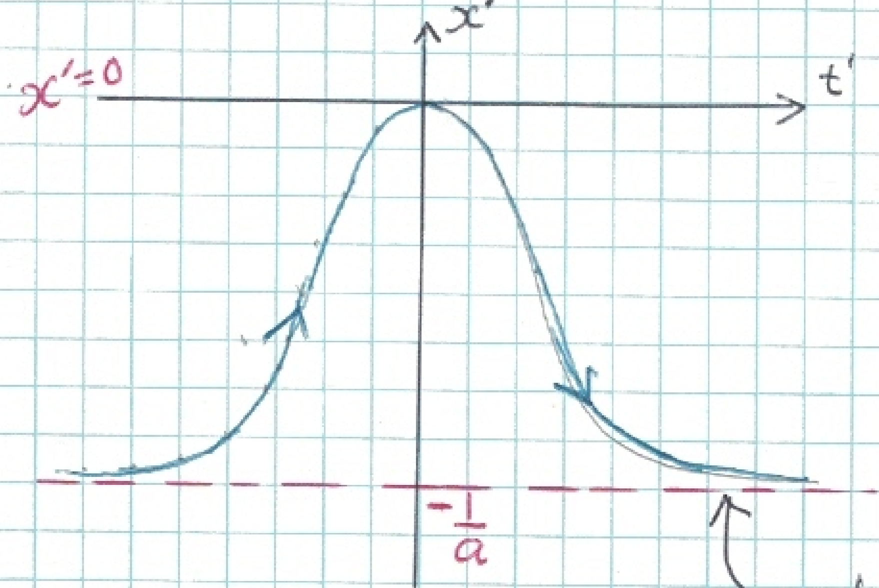
- or

$$\boxed{x' = \frac{1}{a \cosh(at')} - \frac{1}{a}}$$

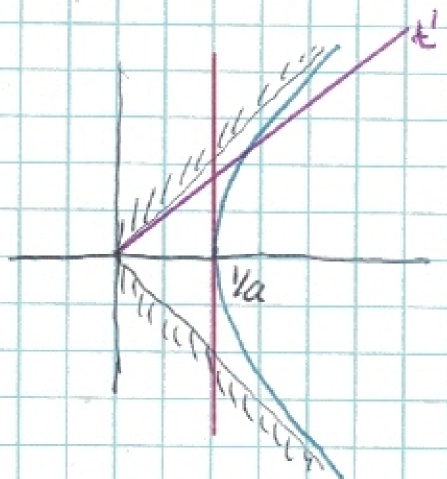
NOTE ① At $t' = 0 \Rightarrow x' = 0$ as expected

② As $t' \to \pm\infty \Rightarrow x' = -\frac{1}{a}$ ⇐ suprise!

Plot of Stationary Particle in Acc'd Coord's



a horizon!



Expansion Near $t'=0$

- Taylor expanding $x' = \frac{1}{a\cdot\cosh(at')} - \frac{1}{a}$ about $t'=0$:

$$\boxed{x' = \underbrace{-\frac{1}{2}at'^2}_{} + \frac{5}{24}a^3 t'^4 + \mathcal{O}(t'^6)}$$

classical result!