QFT by Lancaster and Blundell
Sections 4.4 and 4.5

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Physics Cafe

July 4, 2021

Outline

4.4 Two Particles

- We now consider the case of two particles interacting with each other.
- The second-quantized two particle operator $\hat{A}$ is given by

$$\hat{A} = \sum_{\alpha\beta\gamma\delta} A_{\alpha\beta\gamma\delta} \hat{a}^{\dagger}_{\alpha} \hat{a}^{\dagger}_{\beta} \hat{a}_{\gamma} \hat{a}_{\delta} \tag{1}$$

and the matrix elements are given by

$$A_{\alpha\beta\gamma\delta} = \langle \alpha\beta | \hat{A} | \gamma\delta \rangle \tag{2}$$

- Note the operators are written with the annihilation operators to the right and the creation operators on the left.
  This is known as *Normal Ordering* and is convenient for calculations.
- For example it ensures the operator has zero vacuum expectation value: $\langle 0 | A | 0 \rangle = 0$.

- Recall the decomposition of an observable operator, A, in terms of its eigenvalues and eigenvectors,

$$A = \sum_{\alpha} |\alpha\rangle\langle\alpha|, \tag{3}$$

where we are indexing the eigenvectors by the eigenvalues themselves.
The eigenvalues are the outcomes of measurements associated with the Hermitian operator A and the eigenvectors the resulting states after a measurement.

- Then we can write a function of the operator as,

$$f(A) = \sum_{\alpha} f(\alpha)|\alpha\rangle\langle\alpha|. \tag{4}$$

We covered this last fall.

- Writing this in our new language of creation and destruction operators of the eigenstates, we then have,

$$f(A) = \sum_{\alpha} f(\alpha)\hat{a}^{\dagger}_{\alpha}\hat{a}_{\alpha}. \tag{5}$$

- So for a scalar potential, $V(x)$, as a function of position its corresponding operator is,

$$\hat{V} = \int_x V(x) \cdot \hat{\psi}^\dagger(x)\hat{\psi}(x). \tag{6}$$

- We often will be dealing with particles that interact by a scalar potential that is a function of positions, for example, two charged particles interacting by a Coulomb field.

  The two particle potential in that case is given by,

$$V(x, y) = \frac{Q}{4\pi\epsilon_0}\frac{1}{|x-y|} = V(x-y). \tag{7}$$

  Notice it is a function of the *difference* of the two position variables.

- Recall the single particle operator of a scalar potential that is a function of position.
  Why? $\Longrightarrow$ Recall eigen-decomposition of operators.

- In the case of two position variables we need to integrate over both variables now. So,

$$\hat{V} = \frac{1}{2} \int_{xy} V(x, y) \cdot \hat{\psi}^{\dagger}(x)\hat{\psi}^{\dagger}(y)\hat{\psi}(y)\hat{\psi}(x). \tag{8}$$

- Notice in addition to the normal ordering of the creation/annihilation operators the variables are in the (x, y, y, x).
  This is again for convenience and avoids a self energy term that would appear in the case of fermions.

- We now follow the usual routine of translating it into a momentum representation. This turns out to give a very interesting interpretation of two particles interacting via a scalar potential.

- Recall the Fourier mode expansions of the creation and annihilation operators for particles at a position x,

$$\hat{\psi}^\dagger(x) = \sum_p e^{-ipx} \hat{a}_p^\dagger \tag{9}$$

$$\hat{\psi}(x) = \sum_p e^{ipx} \hat{a}_p \tag{10}$$

- Substitute this into our potential operator expression,

$$\hat{V} = \frac{1}{2} \int_{xy} V(x-y) \cdot \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(y) \hat{\psi}(y) \hat{\psi}(x) \tag{11}$$

$$= \frac{1}{2} \int_{xy} \sum_{p_1 p_2 p_3 p_4} e^{-ip_1 x} e^{-ip_2 y} e^{ip_3 y} e^{ip_4 x} V(x-y) \hat{a}_{p_1}^\dagger \hat{a}_{p_2}^\dagger \hat{a}_{p_3} \hat{a}_{p_4} \tag{12}$$

(13)

- We now rewrite to introduce an $x - y$ in the exponentials to deal with the $V(x-y)$.

$$\hat{V} = \frac{1}{2} \sum_{p_1 p_2 p_3 p_4} \int_{xy} V(x-y) e^{i(p_4 - p_1)(x-y)} \cdot e^{i(-p_1 - p_2 + p_3 + p_4)y} \hat{a}_{p_1}^\dagger \hat{a}_{p_2}^\dagger \hat{a}_{p_3} \hat{a}_{p_4}. \tag{14}$$

- Defining the new variable $z = x - y$ we can rewrite the integrals as,

$$\hat{V} = \frac{1}{2} \sum_{p_1 p_2 p_3 p_4} \hat{a}^\dagger_{p_1} \hat{a}^\dagger_{p_2} \hat{a}_{p_3} \hat{a}_{p_4} \int_z V(z) e^{i(p_4 - p_1)z} \cdot \int_y e^{i(-p_1 - p_2 + p_3 + p_4)y}. \tag{15}$$

- Recall the last integral is just a delta function in terms of momenta in the exponential,

$$\int_y e^{i(-p_1 - p_2 + p_3 + p_4)y} = \delta_{-p_1 - p_2 + p_3 + p_4}. \tag{16}$$

  That is it equals zero every except when $-p_1 - p_2 + p_3 + p_4 = 0$ and it evaluates to 1.

- This allows us to eliminate one of the momenta, setting $p_4 = p_1 + p_2 - p_3$. This gives us

$$\hat{V} = \frac{1}{2} \sum_{p_1 p_2 p_3} \hat{a}^\dagger_{p_1} \hat{a}^\dagger_{p_2} \hat{a}_{p_3} \hat{a}_{p_1 + p_2 - p_3} \int_z V(z) e^{-i(p_3 - p_2)z} \tag{17}$$

- And the last integral is just the Fourier transform of the potential function,

$$\int_z V(z) e^{-i(p_3 - p_2)z} = \tilde{V}(p_3 - p_2). \tag{18}$$

- Substituting this into our operator equation we have,

$$\hat{V} = \frac{1}{2} \sum_{p_1 p_2 p_3} \tilde{V}(p_3 - p_2) \hat{a}^\dagger_{p_1} \hat{a}^\dagger_{p_2} \hat{a}_{p_3} \hat{a}_{p_1 + p_2 - p_3} \tag{19}$$

- As a next to last step in interpreting this interaction operator we set $q = p_3 - p_2$ and eliminate p_3,

$$\hat{V} = \frac{1}{2} \sum_{p_1 p_2 q} \tilde{V}(q) \hat{a}^\dagger_{p_1} \hat{a}^\dagger_{p_2} \hat{a}_{p_2 + q} \hat{a}_{p_1 - q} \tag{20}$$

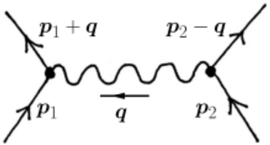
- Finally, we reindex the sum, subtracting q from p_2 and adding it to p_1.

- This gives the final two-particle interaction expression,

$$\boxed{\hat{V} = \frac{1}{2} \sum_{p_1 p_2 q} \tilde{V}(q) \hat{a}^\dagger_{p_1 + q} \hat{a}^\dagger_{p_2 - q} \hat{a}_{p_2} \hat{a}_{p_1}} \tag{21}$$

- We can interpret this expression diagrammatically as a scattering problem in momentum space – A Feynman diagram.

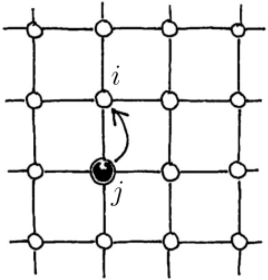
$$\boxed{\hat{V} = \frac{1}{2} \sum_{p_1 p_2 q} \tilde{V}(q) \hat{a}^\dagger_{p_1+q} \hat{a}^\dagger_{p_2-q} \hat{a}_{p_2} \hat{a}_{p_1}} \tag{22}$$



- A particle comes in with momentum p_2.
- It sends out a force-carrying particle with momentum q, reducing its final momentum to $p_2 - q$.
- The force-carrying particle is absorbed by a second particle which ends up with a final momentum of $p_1 + q$.
- Notice momentum is conserved.

4.5 The Hubbard Model, introduction

- Recall from last week the tight binding model of particles in a discrete lattice which are allowed to hop from one site to a another.



- Going from site j to site i is represented by the operator $\hat{c}_i^\dagger \hat{c}_j$ and lowers the system's energy by $-t_{ij}$.

- Assuming only nearest neighbour hopping with a constant energy saving per hop, the Hamiltonian of the system is given by the sum over all possible sites i and nearest neighbours per site, $i + \tau$,

$$\hat{H}_{tb} = -t \sum_{i\tau} \hat{c}^\dagger_{i+\tau} \hat{c}_i. \tag{23}$$

- In the *Hubbard model* we extend the Tight Binding model which only has kinetic energy terms to include a repulsive force.
  This models the two body repulsive force that electrons would experience from each other on a lattice.

- As a gross simplification we assume the lattice sites are far enough apart that we need to only include electrons on the same site.

- Thus the two particle potential operator

$$\hat{V} = \frac{1}{2} \sum_{ijkl} V_{ijkl} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l \tag{24}$$

  becomes

$$\hat{V} = \frac{1}{2} \sum_{i} V_{iiii} \hat{c}_i^\dagger \hat{c}_i^\dagger \hat{c}_i \hat{c}_i \tag{25}$$

$$\equiv U \sum_{i} \hat{c}_i^\dagger \hat{c}_i^\dagger \hat{c}_i \hat{c}_i \tag{26}$$

- Note that $U > 0$ so two particles on the same site raises the potential energy.

- The Hamiltonian of the Hubbard Model is then,

$$\boxed{\hat{H}_h = -t \sum_{i\tau} \hat{c}^\dagger_{i+\tau} \hat{c}_i + U \sum_i \hat{c}^\dagger_i \hat{c}^\dagger_i \hat{c}_i \hat{c}_i} \tag{27}$$

- The trade-off in the kinetic energy terms (which encourage hopping with the negative $-t$ factor) and the Coulomb potential (which encourages repulsion by the positive U factor) gives rise to interesting behaviour.

- $\Longrightarrow$ To Be Continued ...