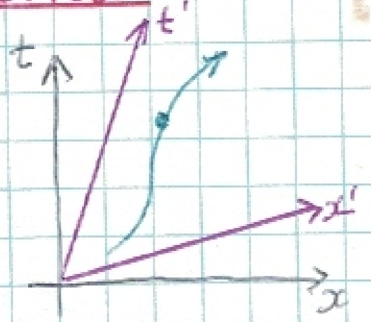
Finally - Relativistic Constant Acceleration

Review:



- a particle moves thru space-time
- we can describe its path as a function $x(t)$ wrt some inertial reference frame O.
- but it's equally correct to describe its coordinates in another frame O' as $x'(t')$.

- (x,t) & (x',t') are related by the Lorentz transformation

$$\boxed{\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \end{aligned}}$$ where $c \equiv 1$ & $\gamma = \frac{1}{\sqrt{1-v^2}}$.

- which frame you use is arbitrary & it would be nice to have a standard, canonical or PROPER way of doing this all inertial frames can agree on, i.e, one $X(T)$ function.

- recall the idea of invariants from before (e.g. bouncing light beam on a moving boxcar).

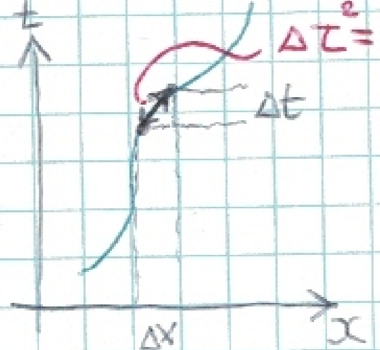
① The Interval $\Delta s^2 = \Delta x^2 - c\Delta t^2$

or

② Proper Time $\Delta \tau^2 = c\Delta t^2 - \Delta x^2$ → note $\Delta \tau^2 = -\Delta s^2$

- use whichever is convenient.

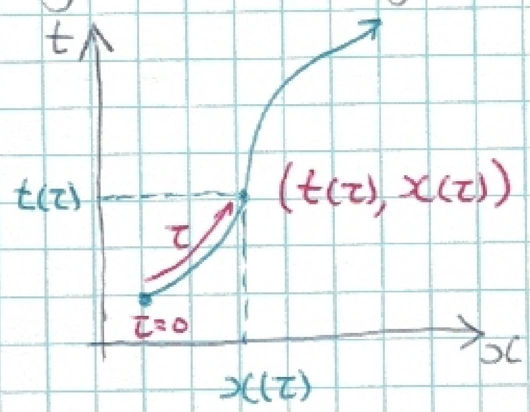
- all inertial frames "see" the same proper time.



(setting $c=1$ from now on).

- $\Delta\tau$ (or $-\Delta s$) acts as an arclength along the path, or world line of the particle.
- it's just the time measured by the traveller's watch as he moves about.

- parameterizing the co-ord's (x,y,z) of points on a curve by the arclength along the curve is standard.



- we can write the curve as

$$\begin{aligned} x^\alpha &= (x^0(\tau), x(\tau)) \\ &= (t(\tau), x(\tau)) \\ &= x^\alpha(\tau). \end{aligned}$$

DEFINITION OF 4-VELOCITY

- the velocity of the particle is just the rate of change of its position wrt time.
- using proper time this define the particle's 4-velocity.

$$\boxed{u^\alpha \equiv \frac{dx^\alpha}{d\tau}}$$

→ this vector is tangent to the curve

- now find the components of the velocity 4-vector in familiar terms.

1 - recall we showed that between inertial frames

that $\Delta t = \gamma \Delta \tau$ $\Rightarrow$ Why? The clock in the moving frame is stationary in the moving frame.

$\therefore \Delta x' = 0$

So $c^2 \Delta \tau^2 - \cancel{\Delta x'^2}^{\,0} = c^2 \Delta t^2 - \Delta x^2$

$c^2 \Delta \tau = \sqrt{\Delta t^2 - \frac{\Delta x^2}{c^2}}$

$\Delta \tau = \sqrt{1 - \frac{\Delta x^2}{\Delta t^2}\frac{1}{c^2}} \cdot \Delta t$

$\Delta \tau = \sqrt{1 - v^2/c^2} \cdot \Delta t$

$\boxed{\Delta \tau = \frac{1}{\gamma} \Delta t}$ ✓

- so $\frac{\Delta t}{\Delta \tau} = \gamma$

or $\frac{dt}{d\tau} = \gamma$

or $\boxed{\frac{dx^0}{d\tau} = \gamma}$ ← time component

2 $\frac{dx^1}{d\tau} = \frac{dx}{dt} \cdot \frac{dt}{d\tau}$

$\boxed{u^x = v_x \cdot \gamma}$ ← x-component

3 Similarly $\boxed{\begin{array}{l} u^y = \gamma v_y \\ u^z = \gamma v_z \end{array}}$

$\Rightarrow$ Putting the pieces together we get the 4-velocity

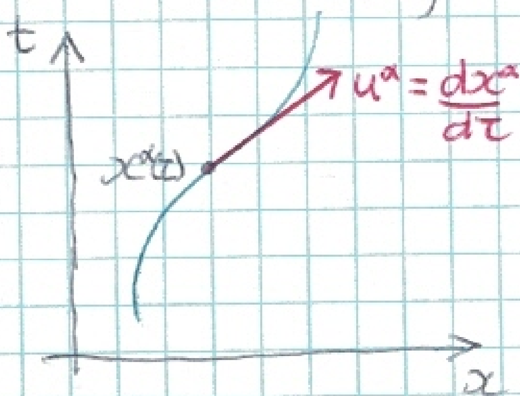
$$\boxed{u^\alpha = (\gamma, \gamma \vec{v}) = \gamma(1, v^x, v^y, v^z) = \gamma(1, \vec{v})}$$

- when stationary in (x,y,z) then $\vec{v} = 0$ & $\gamma = 1$
$\therefore u^\alpha = (1,0,0,0)$

NOTE! This is the rest frame of the moving particle.

$\therefore$ the 4-velocity is the time-axis for a frame (momentarily) co-moving (at rest) with the particle.

- this is a first step in defining a local co-ordinate system for an arbitrarily moving particle.



MAGNITUDE OF 4-VELOCITY

- recall the Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta \quad \text{where} \quad \eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \emptyset \\ & 1 & & \\ & & 1 & \\ \emptyset & & & 1 \end{pmatrix}$$

- the general inner product of two 4-vectors, $\underset{\sim}{a}$, $\underset{\sim}{b}$, in M.S. is given by

$$\boxed{\underset{\sim}{a} \cdot \underset{\sim}{b} = \eta_{\alpha\beta}\, a^\alpha b^\beta} \quad \text{and} \quad \boxed{|\underset{\sim}{a}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a}}$$

- So for the 4-velocity,

$$|\underset{\sim}{u}|^2 = \underset{\sim}{u} \cdot \underset{\sim}{u}$$

$$\boxed{|\underset{\sim}{u}|^2 = \eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}}$$

- but look at the metric,

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$-d\tau^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$d\tau^2 = -\eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$\frac{d\tau^2}{d\tau^2} = -\eta_{\alpha\beta} \frac{dx^\alpha dx^\beta}{d\tau^2}$$

$$1 = -\eta_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

$$\therefore \boxed{\eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = -1}$$

substitute.

$$\therefore \boxed{|\underset{\sim}{u}|^2 = -1}$$

← note the negative makes the vector time-like, as expected

- and it's <u>already</u> normalized to a unit.

<u>Independent Check</u>.

- we showed the 4-velocity has the form:

$$u^\alpha = (\gamma, \gamma v_x) \qquad \leftarrow \text{In Mink. 1-1 space.}$$

- here $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

• so $|\underline{u}|^2 = \eta_{\alpha\beta} u^\alpha u^\beta$

$$= \underset{\eta_{tt}}{\underset{\uparrow}{(-1)}} \cdot \underset{u^t}{\underset{\uparrow}{(\gamma)}}^2 + \underset{\eta_{xx}}{\underset{\uparrow}{(1)}} \cdot \underset{u^x}{\underset{\uparrow}{(\gamma v_x)}}^2$$

$$= -1 \cdot \left(\frac{1}{\sqrt{1-v_x^2}}\right)^2 + \left(\frac{1}{\sqrt{1-v_x^2}} \cdot v_x\right)^2$$

$$= \frac{-1}{1-v_x} + \frac{v_x^2}{1-v_x^2}$$

$$= -\frac{(1-v_x^2)}{1-v_x^2}$$

$\therefore \boxed{|\underline{u}|^2 = -1}$ ✓ ← consistency checks.

4-ACCELERATION

• similarly we can define the 4-acc^n.

$$\boxed{\underline{a}(\tau) = \frac{d\underline{u}}{d\tau}}$$

• but we have $\underline{u} \cdot \underline{u} = 1$

• diff' both sides

$$\frac{d}{d\tau}(\underline{u} \cdot \underline{u}) = \frac{d}{d\tau}(-1).$$

So $\underset{\sim}{u} \cdot \frac{d\underset{\sim}{u}}{d\tau} + \frac{d\underset{\sim}{u}}{d\tau} \cdot \underset{\sim}{u} = 0$

$\rightarrow 2\underset{\sim}{u} \cdot \frac{d\underset{\sim}{u}}{d\tau} = 0$

or $\boxed{\underset{\sim}{u} \cdot \underset{\sim}{a} = 0}$

$\therefore$ the 4-velocity & 4-acc^n are always orthogonal.

PROPER ACC^N

- in the moving & accelerating observer's stationary frame, his velocity is alway

$$\boxed{\underset{\sim}{u}' = (1, \vec{0})}$$

- since his acc^n is alway orthogonal, the observer's frame's acc^n must have the form

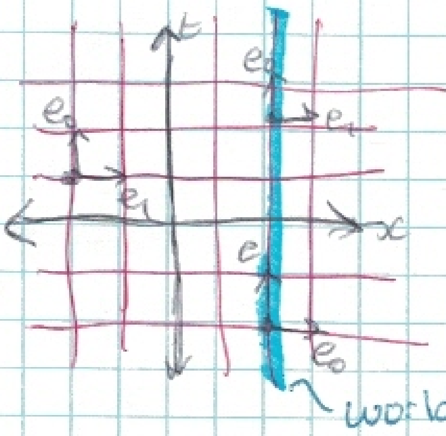
$$\boxed{a' = (0, \vec{a})}$$

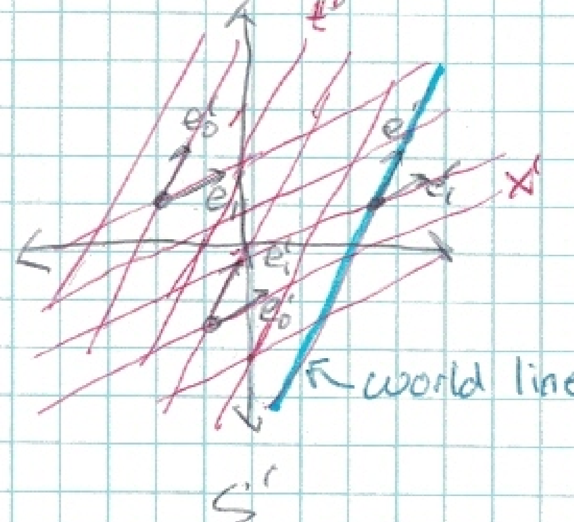
where $\vec{a} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)_{\text{in co-moving frame}}$

$$= \frac{d}{dt}(v_x, v_y, v_z)\Big|_{\text{in cmf.}}$$

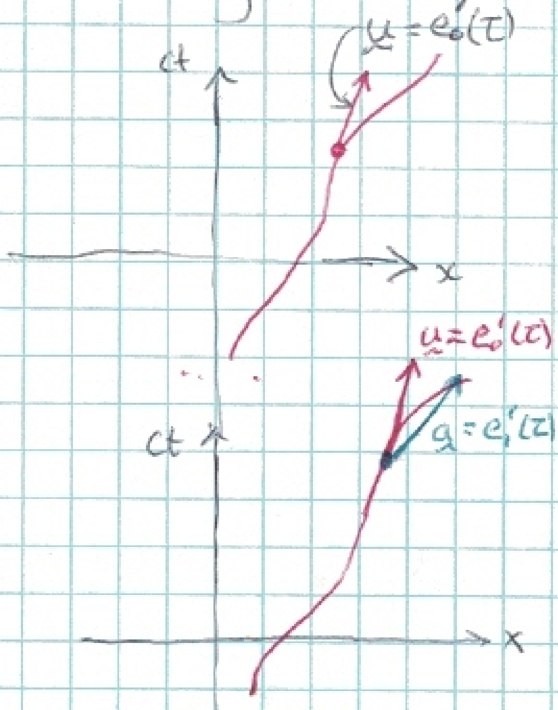
$$\boxed{\vec{a} = \frac{d}{dt}\vec{v}\Big|_{\text{in cmf}}}$$ $\leftarrow$ this is called the PROPER acc^n

Recall the co-ord frames of a stationary & interial frame





* - the co-ord system's basis are fixed for const. veloc.
- when the velocity varies under acc^n define the instantaneous rest frame S' for which the moving observer is momentarily at rest.



- so the velocity vector in general defines a local time axis.
- so $\boxed{e_0'(\tau) = \hat{u}(\tau)}$
- in (t, x) 1-1 space we can the the orthogonal $a(\tau)$ to be the spatial basis vector at that pt

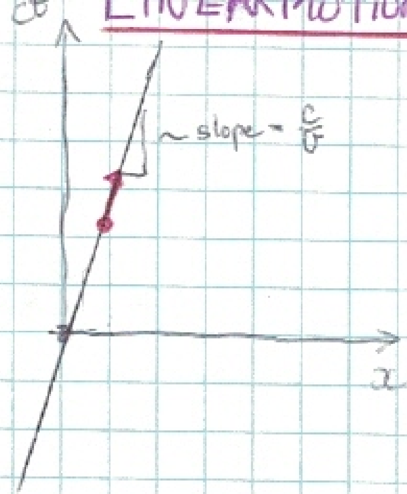
$$\boxed{e_1'(\tau) = \hat{a}(\tau)}$$

* This forms a (momentarily inertial) co-moving frame along the moving observer's world line *

Hartley p5.3

Simple Linear Motion - const. vel 7b

LINEAR MOTION

① 

here $V = \frac{1}{3}c$

- $\gamma = \frac{1}{\sqrt{1-V^2}}$
- $\Delta\tau = \frac{1}{\gamma}\Delta t$
- $\because$ go thru the origin we have

$\tau = \frac{t}{\gamma} \rightarrow \boxed{t = \gamma\tau}$

- set $c=1$ so velocity is the fraction of c (here $V = 1/3$).
- line's equation $y = mx + b$.

$\Longrightarrow \quad t = \frac{1}{v}x + 0$

$t = \frac{x}{v}$

$x = vt$

$\boxed{x = v\gamma\tau}$

- parametrically $\boxed{\begin{aligned} t(\tau) &= \gamma\tau \\ x(\tau) &= v\gamma\tau \end{aligned}}$

$\underset{\sim}{r}(\tau) = (\gamma\tau, v\gamma\tau)$

$\boxed{\underset{\sim}{r}(\tau) = \gamma\tau(1, v)}$

Check:

$v = \frac{3}{5}$

$\gamma = \frac{5}{4}$

$\underset{\sim}{r}(\tau=4) = \frac{5}{\cancel{4}} \cdot \cancel{4}\,(1, \frac{3}{5})$

$\boxed{\underset{\sim}{r}(4) = (5, 3)}$ ✓

② What is the "4"-velocity?

$\frac{d\underset{\sim}{r}}{d\tau} = \frac{d}{d\tau}(\gamma\tau, v\gamma\tau)$.

$\boxed{\underset{\sim}{u} = \gamma(1, v)}$ ✓ $\Longrightarrow$ also the moving observers time axis basis vect.

CONSTANT ACC$^{\underline{N}}$ – Parametric Eq$^{\underline{n}}$'s

- suppose we have constant proper acc$^{\underline{n}}$ as experienced by the moving observer.
- In the observer's frame $\underset{\sim}{a} = (0, a)$ so $|\underset{\sim}{a}|^2 = a^2$

In general then, for any frame,

$$\boxed{\begin{aligned} |\underset{\sim}{a}|^2 = a^\mu a_\mu &= a^2 \\ u^\mu a_\mu &= 0 \\ u^\mu u^\mu &= -1 \end{aligned}}$$

← constant acc$^{\underline{n}}$

← orthogonality of velocity & acc$^{\underline{n}}$

← unit magnitude of time-like veloci

- recalling $x^\mu y_\mu = \eta_{\mu\nu} x^\mu y^\mu$ for inner products of vectors, where $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- we can expand the equations as.

$$\boxed{\begin{aligned} &① \quad -a^0 a^0 + a^1 a^1 = a^2 \\ &② \quad -u^0 a^0 + u^1 a^1 = 0 \\ &③ \quad -u^0 u^0 + u^1 u^1 = -1 \end{aligned}}$$

← set of coupled 2'nd order polynomial eq$^{\underline{n}}$'s.

- solve for the two acc$^{\underline{n}}$'s in terms of the two velocitie
- can solve these to show:

$$\boxed{\begin{aligned} a^0 &= a u^1 \\ a^1 &= a u^0 \end{aligned}}$$

← use Maple, Mathematic to do.

– substituting into eq$^{\underline{n}}$s ①, ②, ③ verifies the sol$^{\underline{n}}$.

- So we now have a pair of coupled first order ordinary differential eq^{n}s to solve:

$$\boxed{\begin{aligned} \frac{du^0}{d\tau} &= a u^1 \\ \frac{du^1}{d\tau} &= a u^0 \end{aligned}}$$

$\Rightarrow$ impose initial conditions:

$t(\tau=0) = 0$

$x(\tau=0) = \frac{1}{a}$

– will be seen to be convenient.

- these have the solutions:

$$\boxed{\begin{aligned} t(\tau) &= \frac{1}{a}\sinh(a\tau) \\ x(\tau) &= \frac{1}{a}\cosh(a\tau) \end{aligned}}$$

← again, use a computer algebra package.

← check with substituting using $\cosh(x)' = \sinh(x)$
$\sinh(x)' = \cosh(x)$.

Some Interesting Points.

(i) It's a hyperbola

- we have $x^2 - t^2 = \frac{1}{a^2}\cosh^2(a\tau) - \frac{1}{a^2}\sinh^2(a\tau)$

$$= \frac{1}{a^2}\underbrace{\left(\cosh^2(a\tau) - \sinh^2(a\tau)\right)}_{1}$$

$$\boxed{x^2 - t^2 = \frac{1}{a^2}}$$

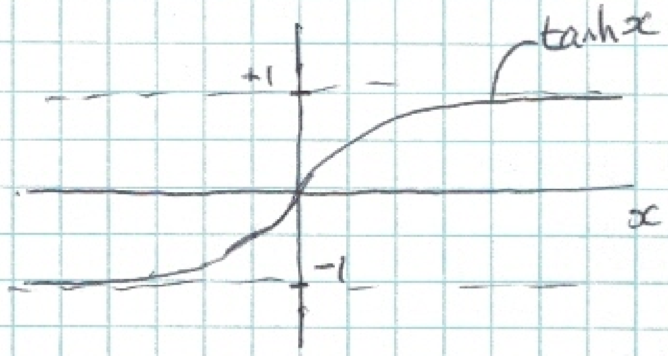
② Co-ordinate velocity only approaches c.

$$v_x = \frac{dx}{dt} = \frac{dx}{d\tau} \cdot \frac{d\tau}{dt}$$

$$= \frac{dx}{d\tau} \Big/ \frac{dt}{d\tau}$$

$$= \frac{\sinh(a\tau)}{\cosh(a\tau)}.$$

$$\boxed{v_x = \tanh(a\tau)}$$



③ Lines of constant τ are radial.

• we have $\dfrac{t(\tau)}{x(\tau)} = \dfrac{1/a \sinh(a\tau)}{1/a \cosh(a\tau)}.$

$$\therefore \frac{t}{x} = \tanh(a\tau).$$

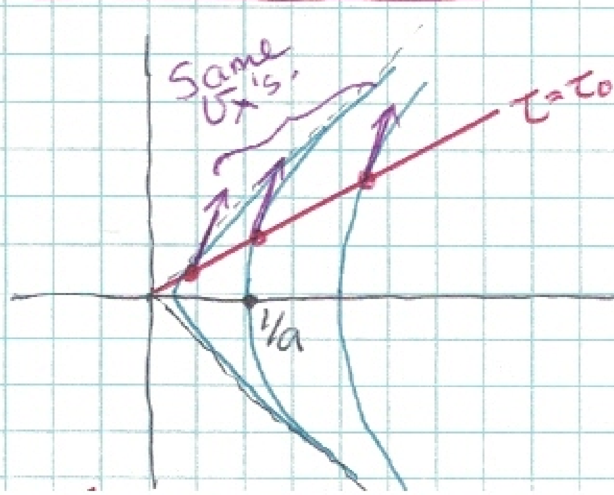
$$\boxed{t = \tanh(a\tau) \cdot x}$$

→ for a given proper time $\tau = \tau_0$:

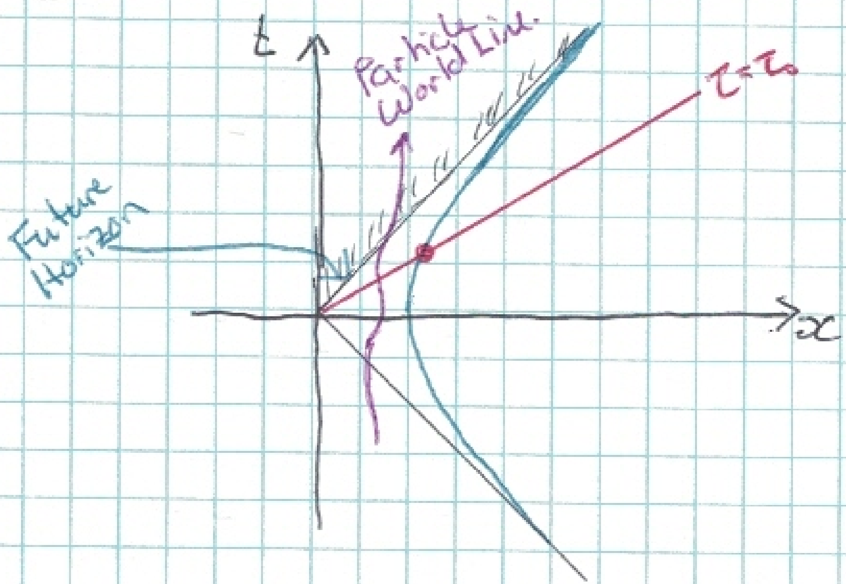
$$t = \tanh(a\tau_0) \cdot x$$

constant for given $\tau = \tau_0$

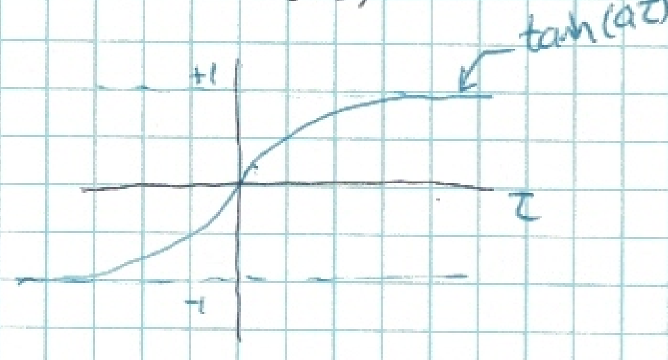
④ Picture.



5 There is an event horizon.



$t = \tanh(a\tau)$



- to the inertial observer the particle just moves along normally.
- the accelerated observer never sees the particle cross the future horizon

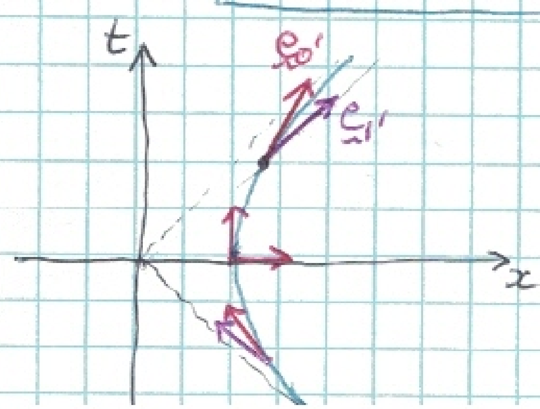
→ $\boxed{\text{it will slow and slow as } \tau \to \infty \text{ and freeze at the point it crosses the horizon.}}$

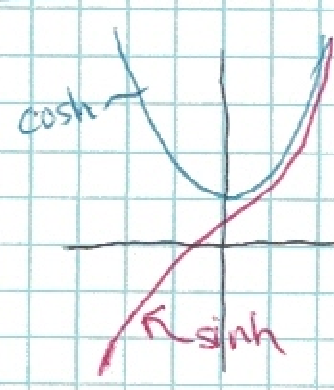
6 Local accelerating observer co-ords.

$$\boxed{\begin{array}{ll} \text{position:} & \underset{\sim}{r}(\tau) = \frac{1}{a}(\sinh(a\tau), \cosh(a\tau)) \\ \text{velocity:} & \underset{\sim}{u}(\tau) = (\cosh(a\tau), \sinh(a\tau)) \\ \text{acc}^{\underline{n}}: & \underset{\sim}{a}(\tau) = a(\sinh(a\tau), \cosh(a\tau)) \end{array}}$$

$\Rightarrow \hat{\underset{\sim}{e}}_{0'}$ ← local time basis

$\Rightarrow \hat{\underset{\sim}{e}}_{1'} = (\mathrm{sh}(a\tau), \mathrm{ch}(a\tau))$ ← local spatial basis.

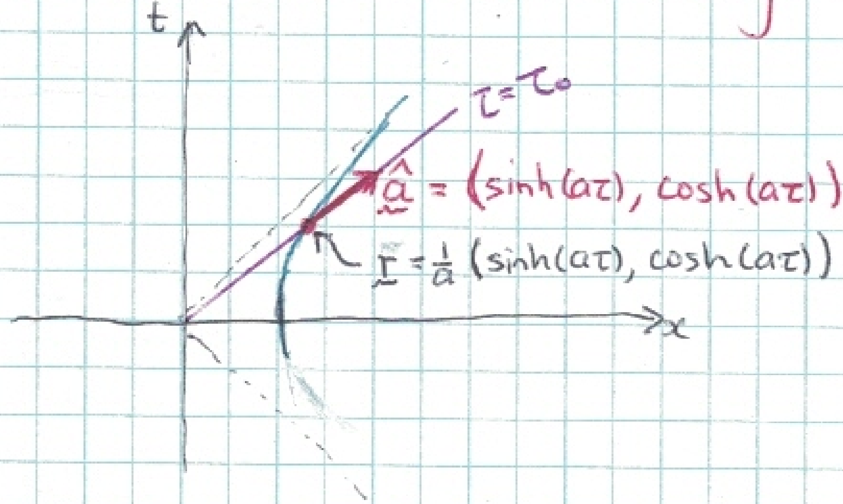




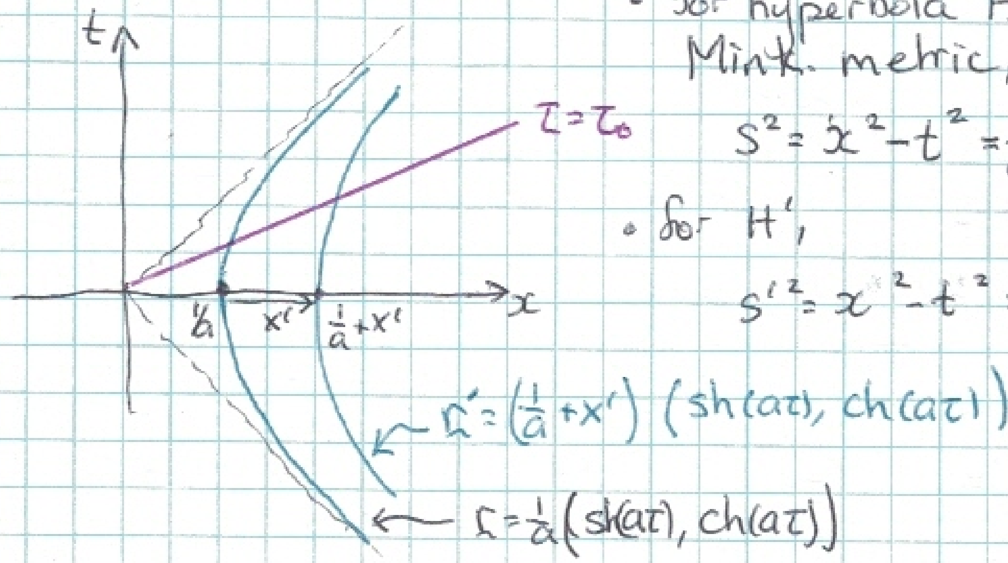
LOCAL Coord' SYSTEM OF AN ACC'd OBSERVER.

POINTS:

① Unit acceleration vectors lie along lines of constant τ.



② In the Minkowski metric, hyperbsalas a separated by a constant distance.



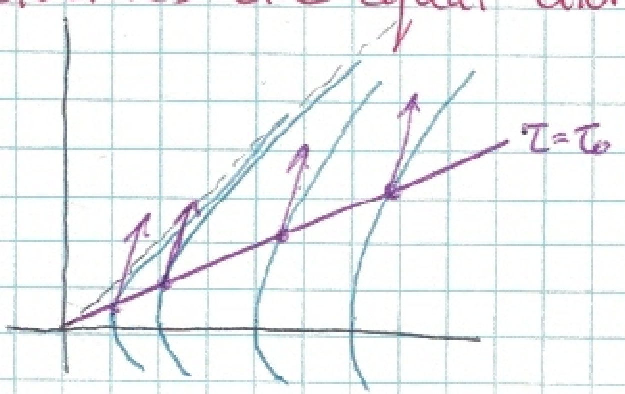
- for hyperbola H, under the Mink. metric,

$$s^2 = x^2 - t^2 = \frac{1}{a^2}$$

- for H',

$$s'^2 = x^2 - t^2 = \frac{1}{a^2} + x'$$

③ The 4-velocities are equal along lines of constant τ.



- as shown on pg. ⑩,

$$v_x = \frac{dx}{dt} = \tanh(a\tau).$$

- all the points move rigidly!

LOCAL COORD'S

- for a rocket acc'ing $\bar{c}$ acc^{n} a, define a spatial co-ord x' as the distance from the floor (the origin).
- define time as the proper time measured by the rocket, $t' \equiv \tau$.
- the relationship btwn the acc'ing local co-ord's and inertial co-ords, (x, t), are:

$$\boxed{\begin{aligned} t &= \left(\frac{1}{a} + x'\right)\sinh(at'). \\ x &= \left(\frac{1}{a} + x'\right)\cosh(at'). \end{aligned}}$$

METRIC ↓

- now calculate the metric $ds^2 = dt^2 - dx^2$ in terms of the local co-ords:

$$\boxed{\begin{aligned} dt &= \sinh(at')\,dx' + (1+ax')\cosh(at')\,dt' \\ dx &= \cosh(at')\,dx' + (1+ax')\sinh(at')\,dt' \end{aligned}}$$

$$\Rightarrow dt^2 - dx^2 = s^2 \cdot dx'^2 + (1+ax')^2 c^2 dt'^2 + \cancelto{0}{2(1+ax')s \cdot c \cdot dx'dt'}$$

$$- c^2 \cdot dx'^2 - (1+ax')^2 s^2 dt'^2 - \cancelto{0}{2(1+ax')\cdot s \cdot c \cdot dx'dt'}$$

$$\boxed{ds^2 = (1+ax')^2 dt'^2 - dx'^2}$$

Recall:

$\cosh^2\theta - \sinh^2\theta = 1$

NOTE: This is still Minkowski space!

Transformation of metric:

- for convenience define the new variables:

$$\boxed{\begin{aligned}\rho &= \frac{1}{a} + x' \\ \varphi &= a t'\end{aligned}} \quad \Longrightarrow \quad \begin{aligned} dx' &= d\rho \\ dt' &= \frac{1}{a} d\varphi.\end{aligned}$$

- now $ds^2 = \cancel{a^2}\rho^2 \frac{1}{\cancel{a^2}} d\varphi^2 - d\rho^2$

$$\boxed{ds^2 = \rho^2 d\varphi^2 - d\rho^2}$$

← The Rindler metric.

- compare $\bar{c}$ Euclidean space metric:

	Cartesian	Polar
Euclidean	$ds^2 = dx^2 + dy^2$	$ds^2 = r^2 d\theta^2 + dr^2$
Minkowsi	$ds^2 = dt^2 - dx^2$	$ds^2 = \rho^2 d\varphi^2 - d\rho^2$

Rindler

Christophel Symbols of Metric

- standard calculations show:

$$\boxed{\begin{aligned}\Gamma^{\varphi}_{\varphi\rho} &= \Gamma^{\varphi}_{\rho\varphi} = \frac{1}{\rho} \\ \Gamma^{\rho}_{\varphi\varphi} &= \rho\end{aligned}}$$

Geodesic Eq$^{\underline{n}}$:

- geodesic eq$^{\underline{n}}$: $\boxed{\dfrac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \dfrac{dx^\beta}{d\tau} \dfrac{dx^\gamma}{d\tau} = 0}$

NOTE: τ is proper time along the geodesic

where $\Gamma^\varphi_{\varphi\rho} = \Gamma^\varphi_{\rho\varphi} = \dfrac{1}{\rho}$

$$\Gamma^\rho_{\varphi\varphi} = \rho$$

- substituting we get

$$\boxed{\begin{array}{ll} a.) & \ddot{\varphi} + \dfrac{2}{\rho}\dot{\varphi}\dot{\rho} = 0 \\ b.) & \ddot{\rho} + \rho(\dot{\varphi})^2 = 0 \end{array}}$$

Solving the Geodesic Equations.

- solve for the functions $\varphi(\tau), \rho(\tau)$ OR $\rho(\varphi)$ - eliminate the τ.
- the metric provides a third relationship

$$d\tau^2 = \rho^2 d\varphi^2 - d\rho^2$$

→ divide by ds^2 on the left & right hand side.

$$1 = \rho^2 \left(\frac{d\varphi}{d\tau}\right)^2 - \left(\frac{d\rho}{d\tau}\right)^2$$

c) $\boxed{1 = \rho^2 (\dot{\varphi})^2 - (\dot{\rho})^2}$

- a.) $\rightarrow$ $$\frac{d^2\varphi}{d\tau^2} + \frac{2}{\rho}\frac{d\varphi}{d\tau}\frac{d\rho}{d\tau} = 0$$

Rewrite as:

$$\frac{1}{\rho^2}\frac{d}{d\tau}\left(\rho^2\frac{d\varphi}{d\tau}\right) = 0$$

$$\therefore \frac{d}{d\tau}\left(\rho^2\frac{d\varphi}{d\tau}\right) = 0$$

d.) $\therefore \boxed{\rho^2\frac{d\varphi}{d\tau} = k}$ - a constant.

- Insert d.) into c.) : $$1 = \overbrace{\rho^2\left(\frac{d\varphi}{d\tau}\right)^2} - \left(\frac{d\rho}{d\tau}\right)^2$$

$$1 = \frac{1}{\rho^2}\left(\rho^2\dot{\varphi}\right)^2 - \dot{\rho}^2$$

$$\therefore 1 = \frac{k^2}{\rho^2} - \dot{\rho}^2$$

$$\Rightarrow \boxed{\frac{d\rho}{d\tau} = \sqrt{\frac{k^2}{\rho^2} - 1}}$$

- from d.) $\boxed{\frac{d\varphi}{d\tau} = \frac{k}{\rho^2}}$

Eliminate the τ-dependence by dividing:

$$\frac{d\varphi}{d\rho} = \frac{\frac{d\varphi}{ds}}{\frac{d\rho}{ds}}$$

$$\frac{d\varphi}{d\rho} = \frac{k}{\rho^2}\Big/\left(\frac{k^2}{\rho^2} - 1\right)^{1/2}.$$

So $$\boxed{\frac{d\psi}{d\rho} = \frac{k}{\rho^2}\left(\frac{k^2}{\rho^2}-1\right)^{-1/2}}$$

- solve by integration.

$$\int_\psi d\psi = \int_\rho \frac{k}{\rho^2}\left(\frac{k^2}{\rho^2}-1\right)^{-1/2} d\rho.$$

$$\Rightarrow \quad \psi - \psi_0 = -\cosh^{-1}\left(\frac{k}{\rho}\right)$$

$$\cosh(\psi_0 - \psi) = \frac{k}{\rho}.$$

So: $$\boxed{\boxed{\rho\cosh(\psi - \psi_0) = k}}$$

← Geodesic Eq^{n} in Rindler co-ords.

Geodesic In Inertial (x,t) Coord's

- expand the cosh fcn:

$$\rho\,(\mathrm{ch}\,\psi \cdot \mathrm{ch}\,\psi_0 - \mathrm{sh}\,\psi\,\mathrm{sh}\,\psi_0) = k$$

$$\mathrm{ch}\,\psi_0 \cdot x - \mathrm{sh}\,\psi_0 \cdot t = k$$

$$x = \left(\frac{\mathrm{sh}\,\psi_0}{\mathrm{ch}\,\psi_0}\right) t + \frac{k}{\mathrm{ch}\,\psi_0}$$

$$\boxed{x = \tanh(\psi_0)\cdot t + \frac{k}{\mathrm{ch}\,\psi_0}}$$

Compare to the eq^{n} of a line:

$$x = mt + b$$

slope y-intercept.

⇒ a line as expected.

Physical Meaning of ψ_0, k?

- We have $x = \tanh(\psi_0) \cdot t + \dfrac{k}{\cosh(\psi_0)}$

$$\therefore \frac{dx}{dt} = \tanh(\psi_0).$$

OR $\boxed{\begin{aligned} \tanh \psi_0 &= v_x \\ \psi_0 &= \tanh^{-1}(v_x) \end{aligned}}$ $\Rightarrow$ the velocity of the moving particle.

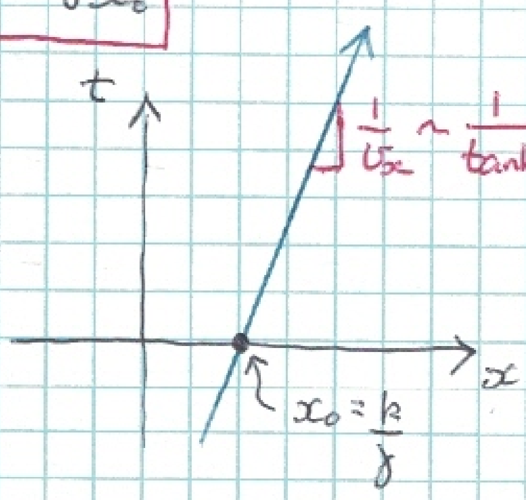
- also $x(t=0) = \boxed{\dfrac{k}{\cosh \psi_0} = x_0}$ - the x-intercept.
- But $\cosh(\psi_0) = \cosh\left(\tanh^{-1}(v_x)\right)$

$$= \frac{1}{\sqrt{1 - v_x^2}}$$

$= \gamma$! The Lorentz factor $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

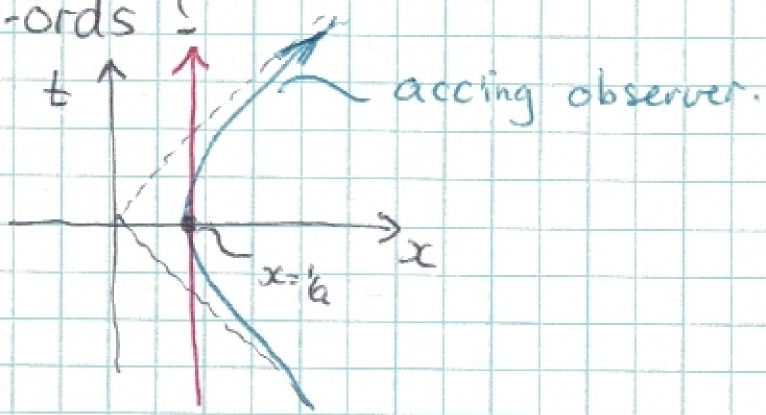
$\therefore \boxed{k = \gamma x_0}$

Picture



What Does The Acc'ng Observer "See"?

- suppose the acc'ing observer is moving $\bar{c}$ acc'n a & a particle is placed at $x = \frac{1}{a}$ & stationary.
- what does the observer measure in his local co-ords?



- the particle's velocity is $v_x = 0 \Rightarrow \gamma = 1$ & $\varphi_0 = 0$

It's x intercept is $x_0 = \frac{1}{a}$

- it's eq^n of motion in Rindler co-ords is

$$\boxed{\rho \cosh \varphi = \frac{1}{a}}$$

where $\rho = \frac{1}{a} + x'$

$\varphi = at'$

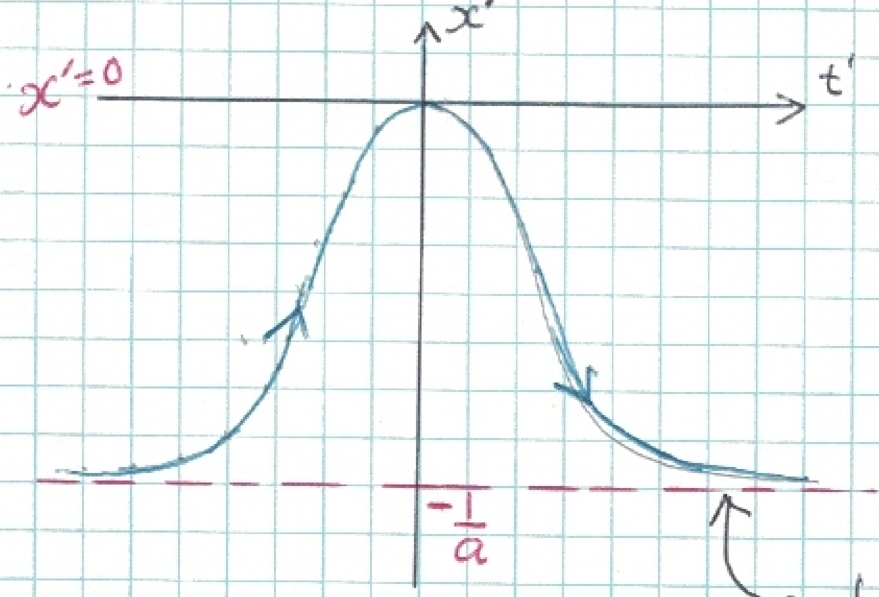
- so $\left(\frac{1}{a} + x'\right) \cosh(at') = \frac{1}{a}$

- or $$\boxed{x' = \frac{1}{a \cosh(at')} - \frac{1}{a}}$$

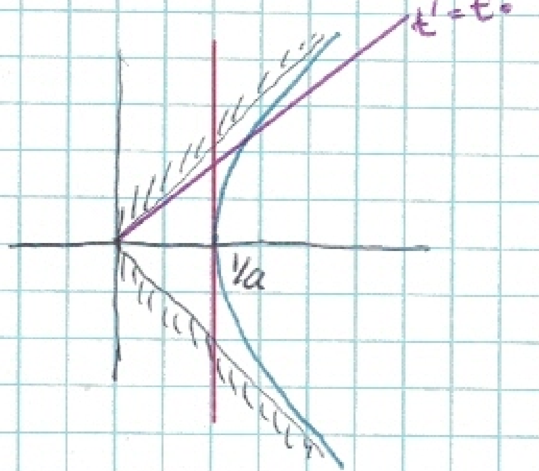
NOTE ① At $t' = 0 \Rightarrow x' = 0$ as expected

② As $t' \to \pm\infty \Rightarrow x' = -\frac{1}{a}$ ⇐ suprise!

Plot of Stationary Particle in Acc'd Coord's



a horizon!



Expansion Near $t' = 0$

- Taylor expanding $x' = \dfrac{1}{a \cdot \cosh(at')} - \dfrac{1}{a}$ about $t' = 0$:

$$\boxed{x' = \underbrace{-\frac{1}{2} a t'^2}_{\text{classical result!}} + \frac{5}{24} a^3 t'^4 + \mathcal{O}(t'^6)}$$