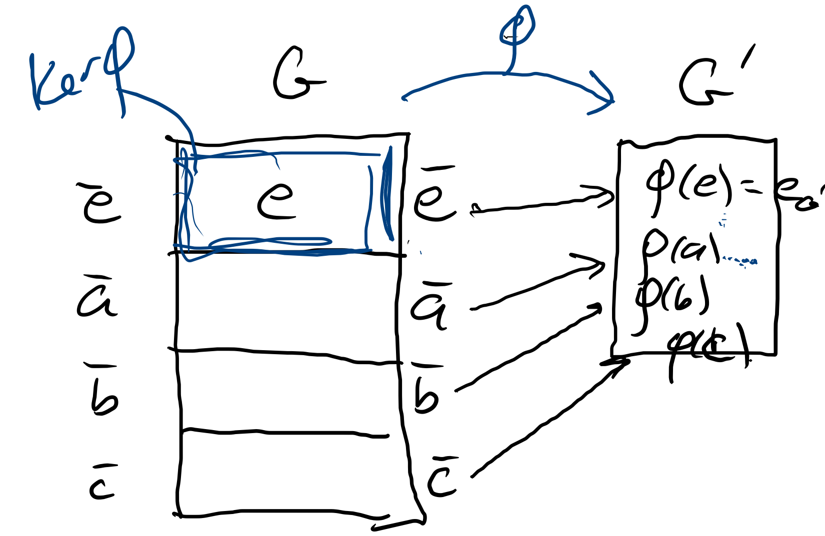
Claim ① All elts in same coset get mapped to same elt in co-domain ✓ ✓

② Elts in different cosets get mapped to different elts in codomain ✓

Claim ③ There is a bijection between the cosets of the kernel & the elts of G'.



i) Injective. – ① & ②

ii) Surjective.

Claim: There is a bijection between $G/\ker\varphi = \{\bar{e}, \bar{a}, \bar{b}, \bar{c}\}$ & elts in G'

$\{\varphi(e)\ \varphi(a)\ \varphi(b)\ \varphi(c)\}$

Define: $f : G/\ker\varphi \longrightarrow G'$

$\bar{a} \longrightarrow \varphi(a)$.

1.) Bijection. ✓

✓ 2.) Compatability: $f(\bar{a} \circ_c \bar{b}) = f(\bar{a}) \circ_{G'} f(\bar{b})$

LHS: $f(\bar{a} \circ \bar{b})$

$= \varphi(a \circ b)$

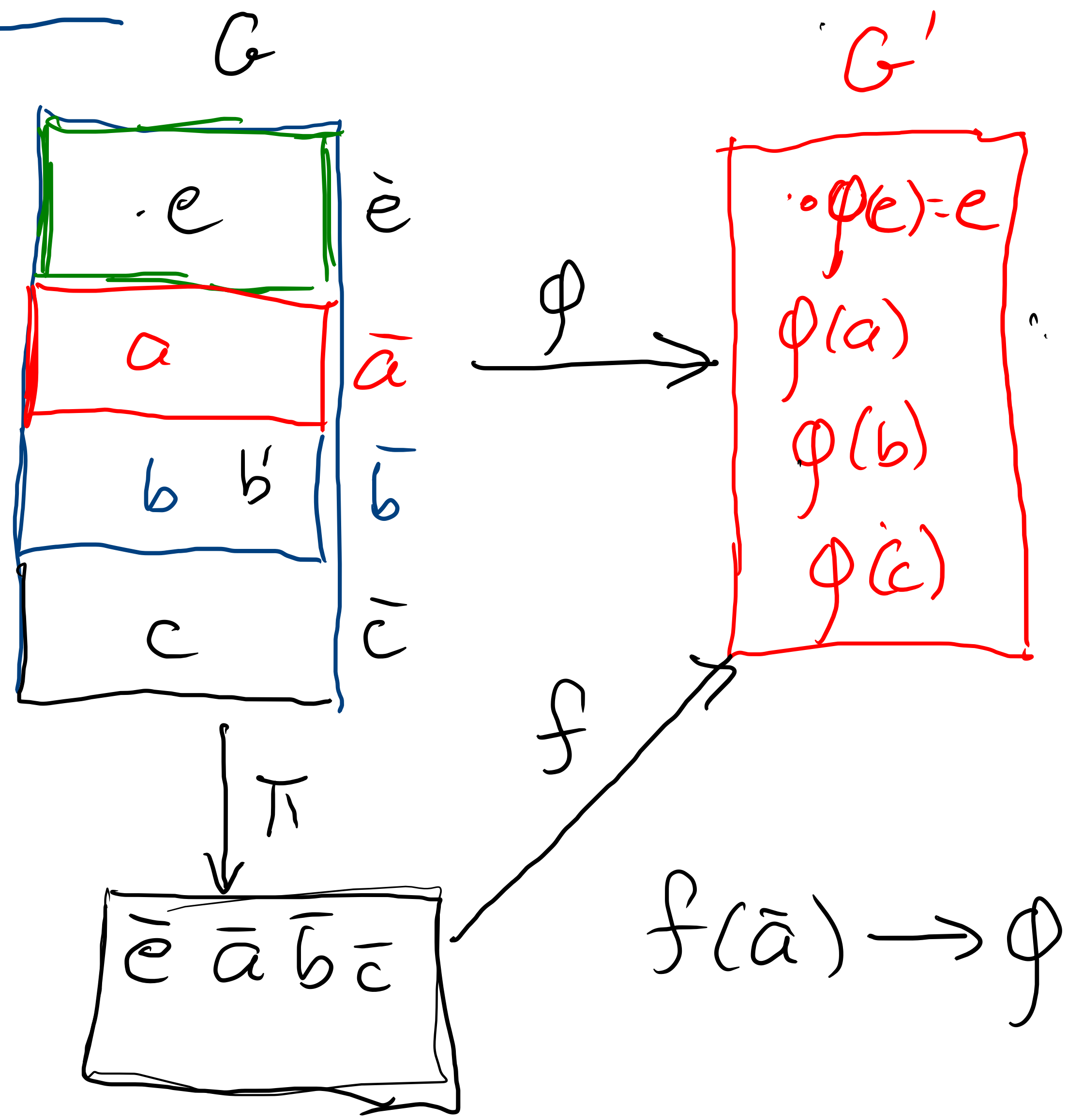
$= \varphi(a) \circ \varphi(b)$

RHS:

$f(\bar{a}) \circ_{G'} f(\bar{b})$

$= \varphi(a) \circ_{G'} \varphi(b)$.

Picture: ✓



$G/\ker\phi$

$f(\bar{a}) \longrightarrow \phi(a).$

$$\boxed{G/\ker\phi \simeq G'}$$

Commutative Diagram:

$$\boxed{\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & G' \\ G/\ker\phi \arrow[ur, "f"'] & \end{tikzcd}}$$

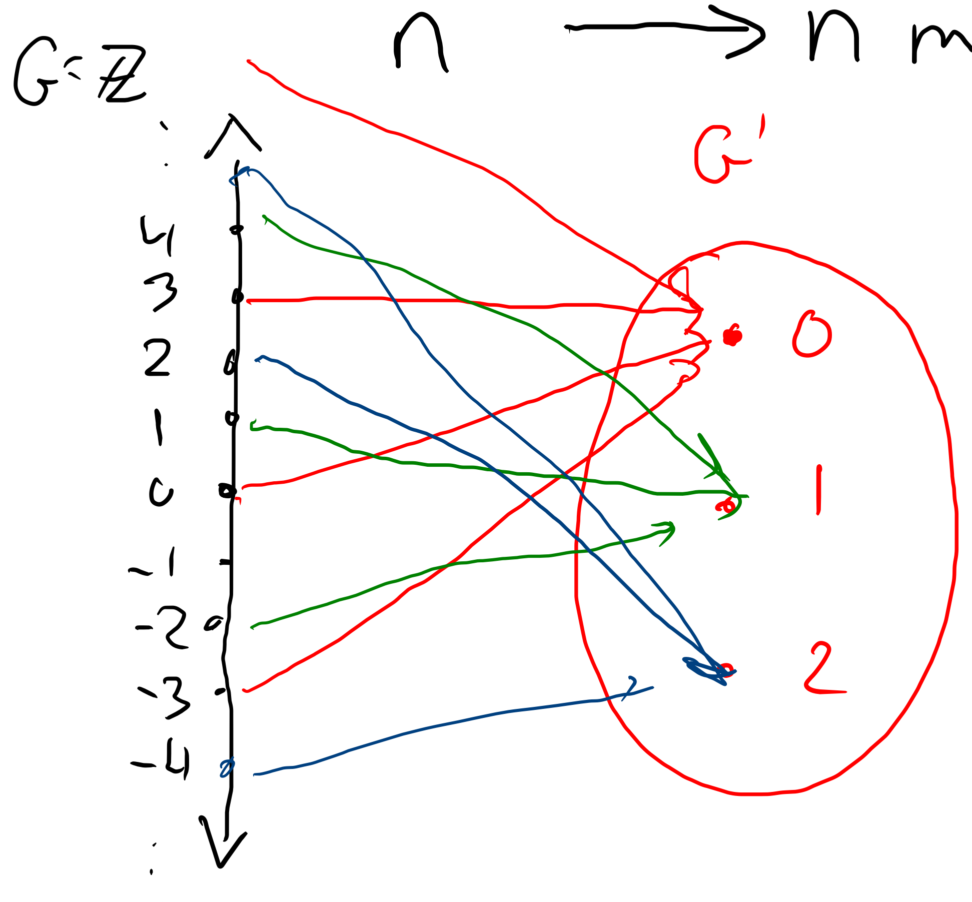
Examples:

1. $\phi : \overset{G}{\mathbb{Z}} \longrightarrow \overset{G'}{\mathbb{Z}_3}$

$$n \longmapsto n \bmod 3 \quad \text{(remainder after dividing by 3)}$$

1'st Isom. Thm.

$$\boxed{G/\ker\phi \simeq G'}$$



$$\ker\phi = \{0, \pm 3, \pm 6, \cdots\} = 3\mathbb{Z}$$

$$\left.\begin{aligned} &G = \mathbb{Z} \\ &G' = \mathbb{Z}_3 \\ &\ker\phi = 3\mathbb{Z} \end{aligned}\right\} \quad \boxed{\mathbb{Z}/3\mathbb{Z} \simeq \mathbb{Z}_3}$$

Example 3 DGT

$AB = C \Rightarrow \cancel{A^{-1}A}B = A^{-1}C$

$B = A^{-1}C$

Let $\boxed{G = (M, \times)}$ be group of 2×2 invertible real matrice under matri mult^n.

Let $\boxed{G' = \{\mathbb{R}^*, \times_{\mathbb{R}}\}}$ where $\mathbb{R}^*$ is the reals except 0.

• already showed the determinat of a matrix is a homomorphism.

$\begin{bmatrix} 3 & \\ & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & \\ & -2 \end{bmatrix}$ $\det M \longrightarrow \mathbb{R}$: + Identity : $\boxed{\det(AB) = \det A \times \det B}$.

$\phi(a \circ b) = \phi(a) \circ \phi(b)$.

$\begin{bmatrix} 3 & \\ & -\frac{2}{3} \end{bmatrix}$

Define: $\phi : \underset{G}{(M, \times)} \longrightarrow \underset{G'}{(\mathbb{R}^*, \times\}}$

$A \longmapsto \det(A)$

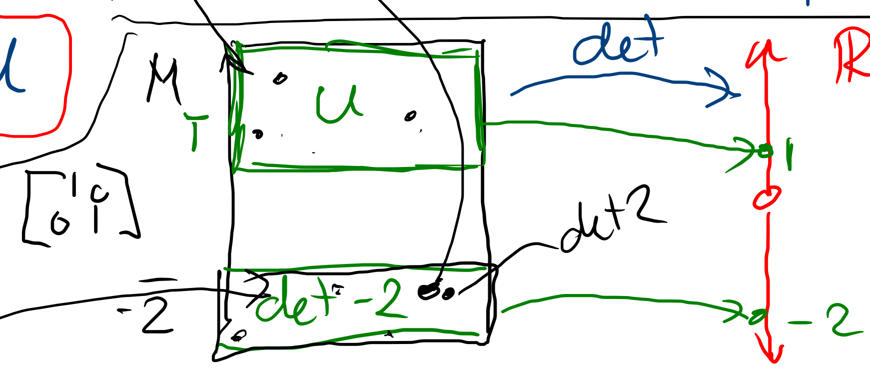
$\begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

$\ker\phi$? $\Rightarrow$ Identity in co-domain (G') ? 1

$\ker\phi = \{M, 2\times 2, \text{real, invertible} \mid \det A = 1\}$

$\equiv U$

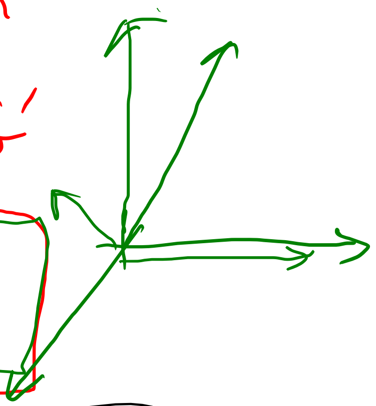
$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \cdot U$



1'st Isom^m Th^m

$G/\ker\phi \cong G'$

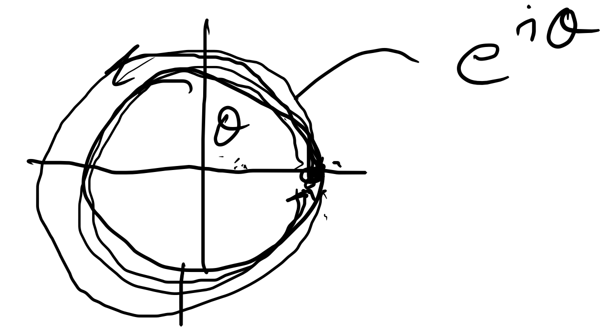
$\boxed{M/U \cong \mathbb{R}^*}$



$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\det} \boxed{ad - bc}$

③ Let $\phi : \underbrace{(\overset{G}{\mathbb{R}}, +)}_{} \longrightarrow (\overset{G'}{\mathbb{C}}, \times)$

$$x \longmapsto e^{2\pi i \cdot x} = \mathbb{T}$$



- It's a homom:

$$\phi(x+y) = e^{2\pi i(x+y)}$$
$$= e^{2\pi i x + 2\pi i y}$$
$$= e^{2\pi i x} \times e^{2\pi i y}$$
$$= \phi(x) \times \phi(y).$$

$$\boxed{G/\ker\phi \simeq G'}$$

$$\boxed{\mathbb{R}/\mathbb{Z} \simeq \mathbb{T}}$$

$$\ker\phi = \{x \in \mathbb{R} \mid e^{2\pi i x} = 1\}$$
$$= \{0, \pm 1, \pm 2 \dots\}$$
$$= \mathbb{Z}$$

- like the real line is wrapped around the circle.