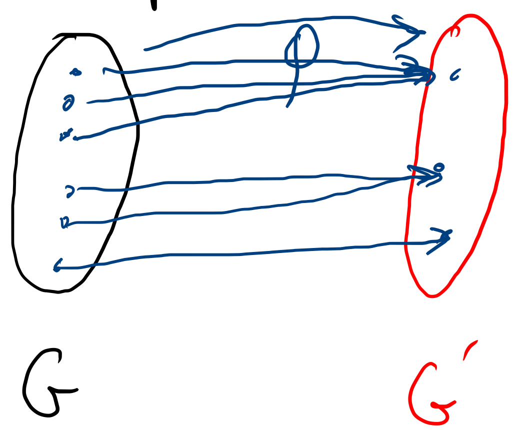
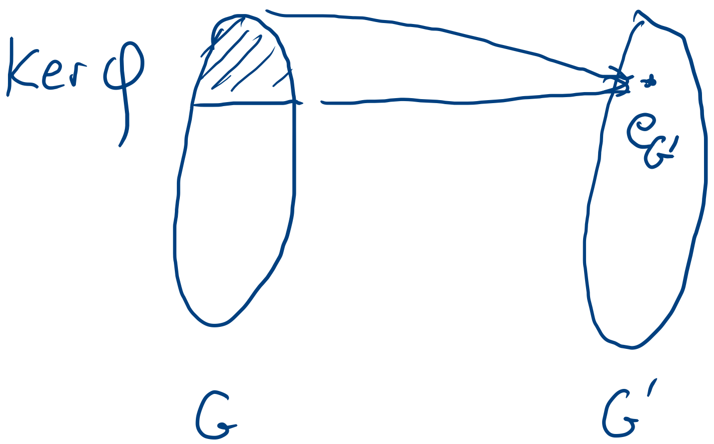
Homomorphism - review



$$\varphi(a \circ_G b) = \varphi(a) \circ_{G'} \varphi(b)$$



$$\operatorname{Ker}\varphi \equiv \{ g \in G \mid \varphi(g) = e_{G'} \}$$

$$\boxed{\operatorname{Ker}\varphi < G}$$

Conjugation of a group elt by another:

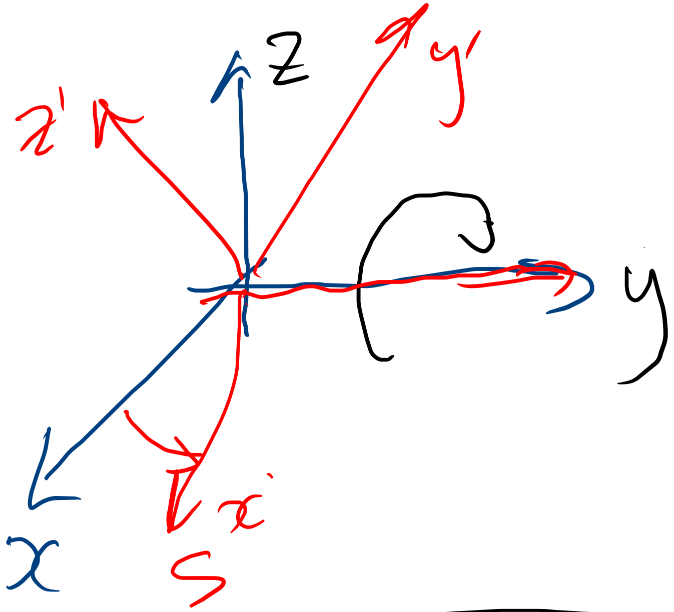
Defⁿ: $\boxed{g \longrightarrow a \circ g \circ a^{-1}}$ · conjⁿ of g by a.

In linear algebra - two matrices are similar
$A \sim B$ if $A = SBS^{-1}$

→ SBS^{-1} is just the linea transfⁿ described by B in another coord system.

If B was a rotⁿ by 90°.

Defⁿ: Normal subgroup H is a subgroup of G $H < G$ such that H is stable under conjⁿ in G : Meaning: take any elt $h \in H$
→ $ghg^{-1} \in H$



$B \longrightarrow SBS^{-1} = B'$

Normal Subgroups: $\boxed{ghg^{-1} \in H}$ $\forall h \in H$ $\forall g \in G$.

Big Deal is left & right cosets are equal:

$ghg^{-1} \in H \longrightarrow gHg^{-1} = H$ as sets

$\boxed{gH = Hg}$

$gH \subset Hg$
$Hg \subset gH$ $\Longrightarrow$

New material:

Claim: The kernel of a homomorphism is normal.

Proof: Need to show for $H = \ker\varphi$, $\forall h \in \ker\varphi$ & $\forall g \in G$, $ghg^{-1} \in \ker\varphi$.

To be in $\ker\varphi$: $\varphi(ghg^{-1}) = e_{G'}$

$$LHS = \varphi(g) \circ_{G'} \cancel{\varphi(h)} \circ \varphi(g^{-1})$$

$\varphi(h) = e_{G'}$

$$= \varphi(g) \circ \varphi(g^{-1})$$

$$= \varphi(g) \circ \varphi(g)^{-1}$$

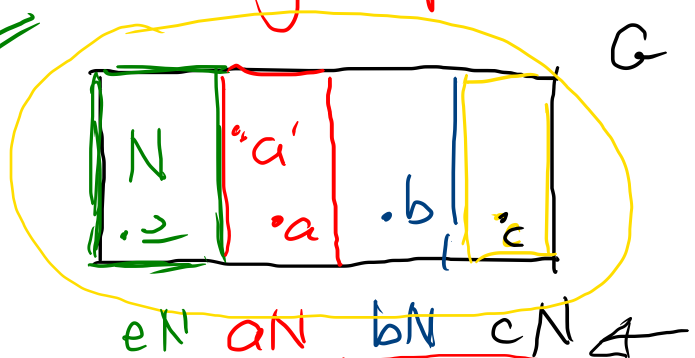
$$= e_{G'}$$

$\therefore \boxed{ghg^{-1} \in \ker\varphi}$ $\therefore \ker\varphi$ is normal.

QUOTIENT GROUPS : G/N

- pick a normal subgroup N of G : $N \trianglelefteq G$
- form a group of the cosest of N in G.

Step 1



Notation : $\boxed{G/N}$ dividing G up into pieces → coset.

$$G/N = \{\{\ldots\}, \{\ldots\}, \{\ldots\}, \{\ldots\}\}$$

$$= \boxed{\{\bar{e}, \bar{a}, \bar{b}, \bar{c}\}}$$

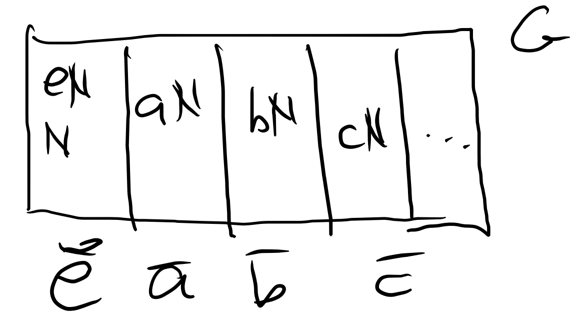
eN aN bN cN

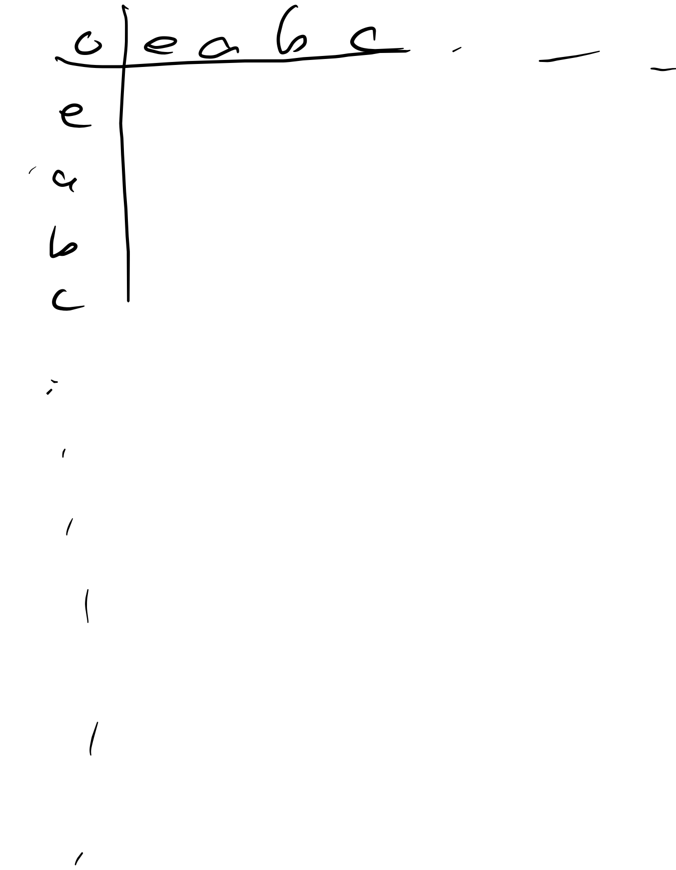
← symbols rep'ing the cosets.

→ $\bar{e}, \bar{a}, \bar{b}$ & $\bar{c}$ are equivalence classes under conj^n.

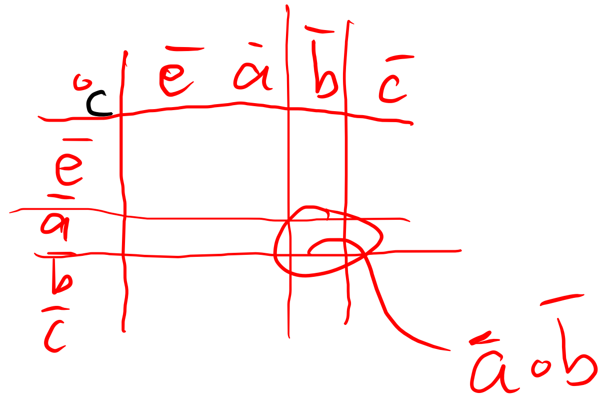
- $aN = \{a \circ n \mid n \in N\}$.
- Because N is normal, $aN = Na$.
- If a & a' are in the same coset, $a'N = aN$.

Step 2 Composition Law/Rule:



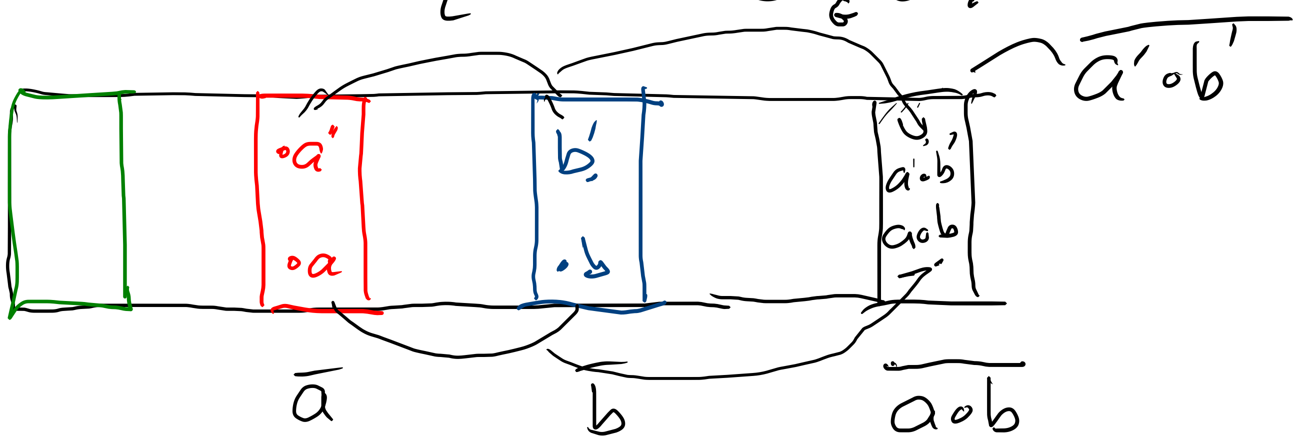


- how do we define $\bar{a} \circ \bar{b}$?



Define: $\bar{a} \circ_C \bar{b} = \overline{a \circ_G b}$. ⇐ Is it well defined?

Claim → YES.



Meaning: $\bar{a} \circ \bar{b}$ has only one

$= \overline{a \circ b}$ one answer.

Proof: Relies on N being N. $N \triangleleft G$.

To show if $\boxed{\overline{a}\circ\overline{b} = \overline{a\circ b}}$, and we choose $a' = a\circ n$ for $n \in N$.

" " " $b' = b\circ n'$,

then $\overline{a'\circ b'} = \overline{a\circ b}$

→ LHS & sub':

$$a'\circ b' = a\circ n\circ b\circ n'$$

RHS:

Coset of $a\circ b$ is $a\circ b\circ n''$.

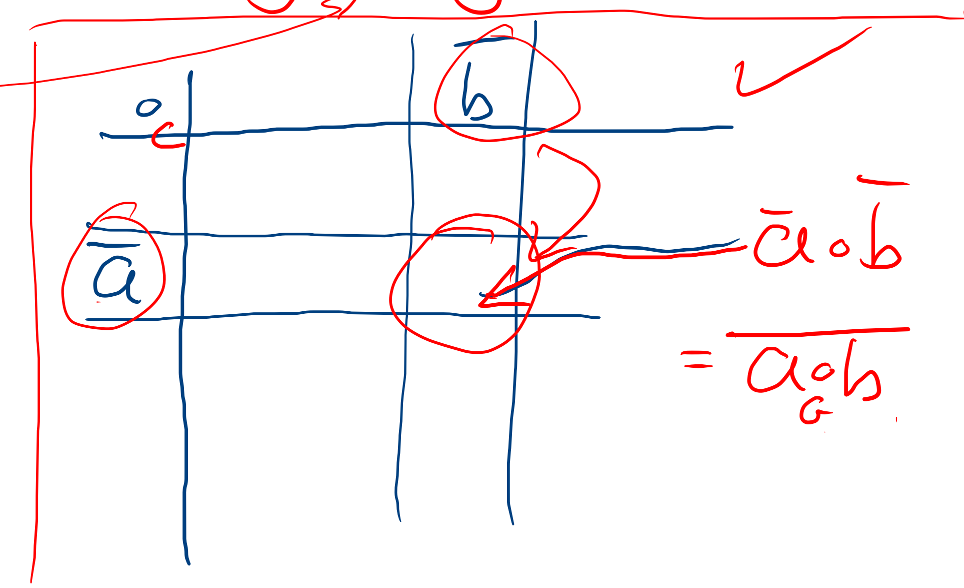
We want to show: $\boxed{a\circ n\circ b\circ n' = \boxed{a\circ b\circ n''}}$? , $n, n', n'' \in N$.

LHS $= a\circ b\circ \underbrace{b^{-1}\circ n\circ b}_{\in N}\circ n'$

$= a\circ b\circ n''\circ n'$

$= \boxed{a\circ b\circ n'''}$

$n \longrightarrow g\circ n\circ g^{-1} \in N$



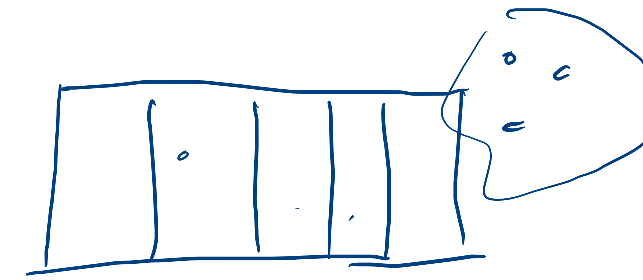
Dfn: Quotient Group: G/N

$$G/N = \{\{\bar e, \bar a, \bar b, \bar c\}, \ \bar a \circ \bar b = \overline{a \circ b}\}$$

Claim: this is a group. → satisfy group criteria.

1.) Identity elt: $\bar e$

$$\bar e \circ \bar a = \overline{e \circ a} = \bar a$$



2.) Assoc'y: Does

	LHS		RHS
	$(\bar a \circ \bar b) \circ \bar c$	$\overset{?}{=}$	$\bar a \circ (\bar b \circ \bar c)$
$=$	$\overline{a \circ b} \circ \bar c$		$\bar a \circ \overline{b \circ c}$
$=$	$\overline{a \circ b \circ c}$		$\overline{a \circ b \circ c}$

3.) Inverse: What's inverse of $\bar a$? → $\overline{a^{-1}}$

Check $\overline{a^{-1}}$: $\bar a \circ \overline{a^{-1}} = \overline{a \circ a^{-1}} = \bar e$

4.) Closure: Does composition of two cosets give another coset?

$\bar a \circ \bar b \longrightarrow \overline{a \circ b}$ ✓

- Compatability cond^n.

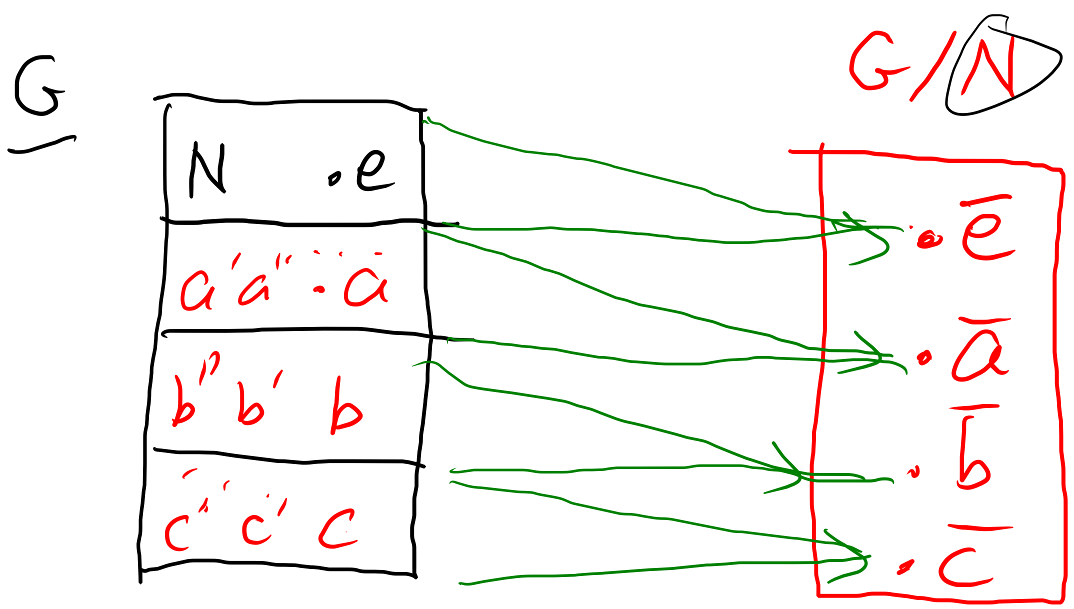
$$\varphi(a\circ b) = \varphi(a)\circ\varphi(b).$$

tr $tr(A+B) = trA + trB$

det $det(A\times B) = detA \times detB$

Cosets comp^n.

$$\overline{a\circ b} = \bar{a}\circ\bar{b}$$



Natural homo^n

Fundamenta homo^n.

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