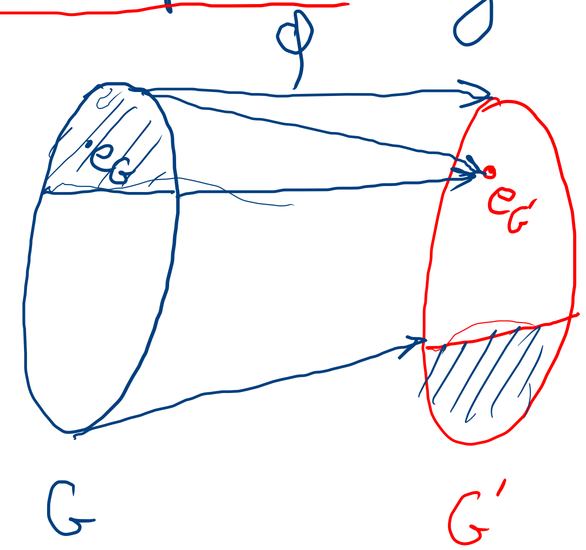
Homomorphisms – generalization of isomorphisms.

Ex1:

$\ker\phi$
$< G$



$\mathrm{Im}\,\phi = \{\phi(g) \mid g \in G\}$
– is a subgroup of G'
$\mathrm{Im}\,\phi < G'$

$\phi(e_G) = e_{G'}$

Example: $\phi: \underset{G}{(\mathbb{Z},+)} \longrightarrow \underset{G'}{S_2} = \{(E,O), \circ\}$

+	0	+1	-1	+2	...
0					
-1					
+1					
2					
⋮					

∘	E	O
E	E	O
O	O	E

$e_{G'} = E$

$$\phi(x) = \begin{cases} E & \text{if } x \text{ is even} \\ O & \text{if } x \text{ is odd} \end{cases}$$

$\ker\phi = \{g \in \mathbb{Z} \mid g \text{ is even number}\}$ = {set of all even ~~Numbers~~ integers}

Ex2: Let $G = (\{M_{n\times n} \text{ matrices } n\times n\}, +_M)$

$G' = (\mathbb{R}, +)$

Recall $\mathrm{tr}A = \mathrm{tr}\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{12} & a_{22} & a_{23} \\ \vdots & & a_{33} \end{pmatrix} = \sum_i a_{ii}$

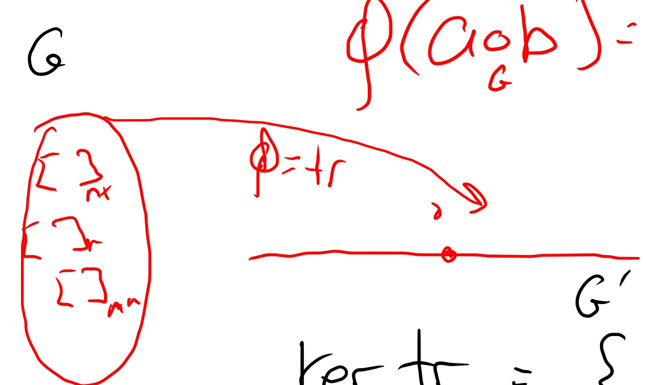
$(A +_M B)_{ii} = a_{ii} + b_{ii}$

$\mathrm{tr}(A+B) = \sum_i a_{ii} + b_{ii} = \sum_i a_{ii} + \sum_j b_{jj}$

$$\boxed{\mathrm{tr}(A +_M B) = \mathrm{tr}A +_R \mathrm{tr}B}$$

Recall homomorphism:

$\phi(a \circ_G b) = \phi(a) \circ_{G'} \phi(b)$.



$\ker \mathrm{tr} = \{M_{n\times n} \mid \mathrm{tr}\, M = 0\}$

su(N) = {Unitary matrices $\bar{c}$ 0 trace}

Ex3: $G = (\{M_{n\times n}, \text{matrices}\}, \times_m)$

$G' = (\mathbb{R}, \times)$

$\phi = \det$

$$\boxed{\det(AB) = \det A \times \det B}$$

Idenity in linear algebra.

$\ker \det = \{M_{n\times n} \mid \det M = 1\}$.

$SU(N) = \{M_{N\times N} \mid \det = 1\}$.

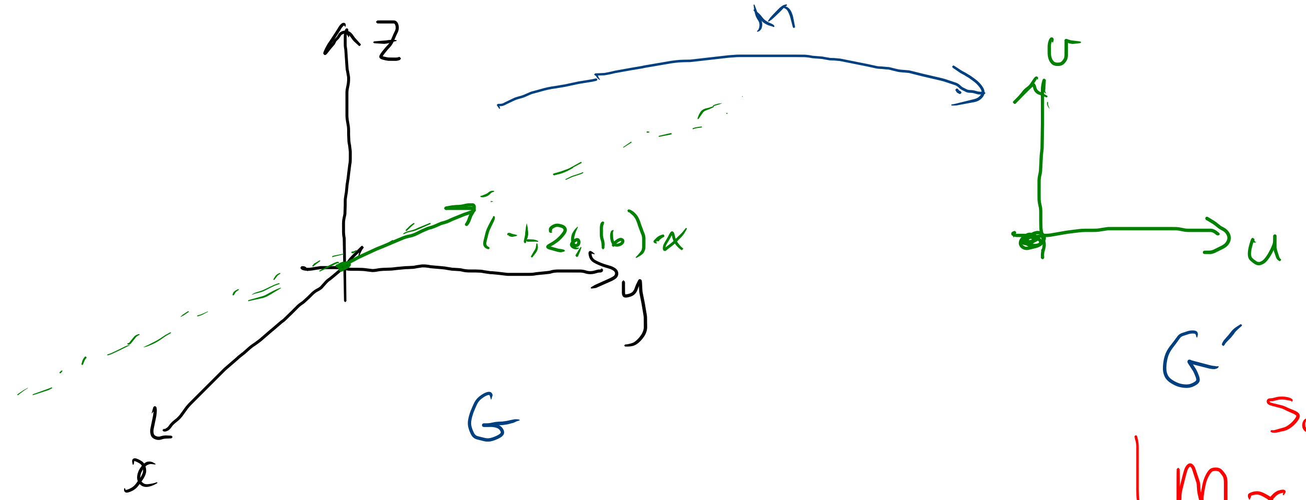
Ex4: Let M be a matrix.

$\text{nullspace}(M) \equiv \{\underset{\sim}{v} \mid M\underset{\sim}{v} = \underset{\sim}{0}\}$ $\quad \times \to +$

e.g. say $M = \begin{pmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{pmatrix}_{2\times 3}$, $\quad M: \mathbb{R}^3 \to \mathbb{R}^2$

$\boxed{\text{null}(M)} = \begin{pmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha\begin{pmatrix} -1 \\ -26 \\ 16 \end{pmatrix}$ where $\alpha \in \mathbb{R}$



Group theory:

$G = \{\mathbb{R}^3, +_{\text{vector}}\}$

$G' = \{\mathbb{R}^2, +_{\text{vector}}\}$.

- what's G' identity?

$\underset{\sim}{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\boxed{\ker M} = \{\underset{\sim}{v} \in \mathbb{R}^3 \mid M\underset{\sim}{v} = \underset{\sim}{0}\}$.

Solve for $\underset{\sim}{x}$:

$M\underset{\sim}{x} = \underset{\sim}{b}$

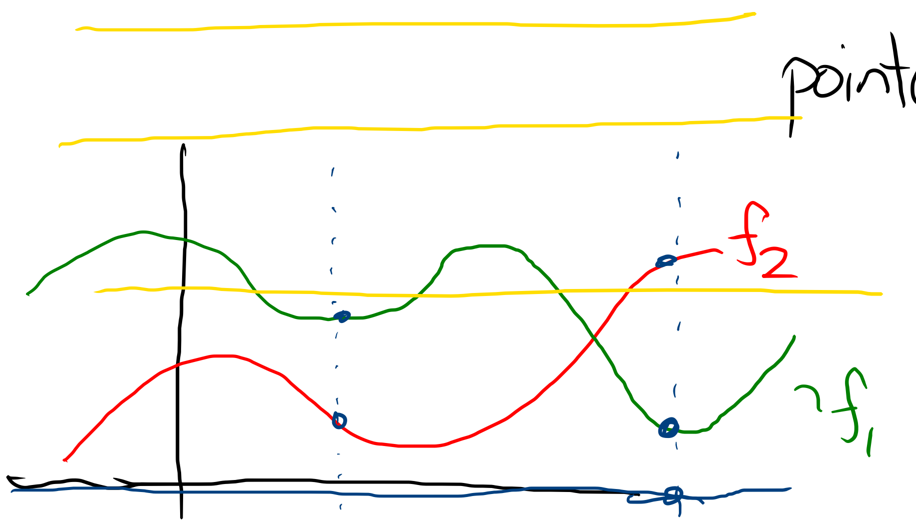
$\underset{\sim}{x} = \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 \\ -26 \\ 16 \end{pmatrix}\alpha$

$M \cdot \left[\begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 \\ -26 \\ 16 \end{pmatrix}\right]$

$\underbrace{M\cdot\begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix}}_{\underset{\sim}{b}} + \underbrace{M\begin{pmatrix} -1 \\ -26 \\ 16 \end{pmatrix}}_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

4) Differential eq^n's.

Let $G = C^\infty(\mathbb{R})$ - set off all ∞-diff'ble functions of $\mathbb{R} \to \mathbb{R}$ under pointwise addition.



$f_1 \circ f_2 = f_1(x) + f_2(x)$

$(f_1 \circ f_2)(x) = f_1(x) + f_2(x)$

identity$(C^\infty(\mathbb{R})) = \boxed{f(x) = 0}$

Let $G' = C^\infty(\mathbb{R})$

• define $\overset{\varphi}{D}(f) : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$

$$D(f) \equiv \frac{df}{dx}$$

$f(x) = x^2$

$Df = 2x$

$$\varphi(f_1 + f_2) \overset{?}{=} \varphi(f_1) + \varphi(f_2).$$

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x).$$

$$\operatorname{Ker} D = \left\{ f(x) \;\middle|\; \frac{d}{dx} f = 0 \right\}$$

$$= \{ f(x) = C \}$$

Normal Subgroups - Kernels of homomorphism are normal subgroups.

First: 1. New concept - to conjugate an elt of a group by another element.

- when we say "take the conjugate of g by a" it means.

$$\boxed{\boxed{g} \longrightarrow a\circ g\circ a^{-1}} = \mathrm{conj}_a(g)$$

- Matrix similarity: $A \sim B$ if $\underbrace{A = SBS^{-1}}$ for some S.

$g\circ x \longrightarrow \pi_g$

$a\circ g\circ a^{-1}\circ x \longrightarrow \pi$

Fact: 1. Suppose the group is Abelian?

$\longrightarrow$ all elts commute.

$$g \longrightarrow a\circ g\circ a^{-1} = aa^{-1}\circ g = g$$

BORING

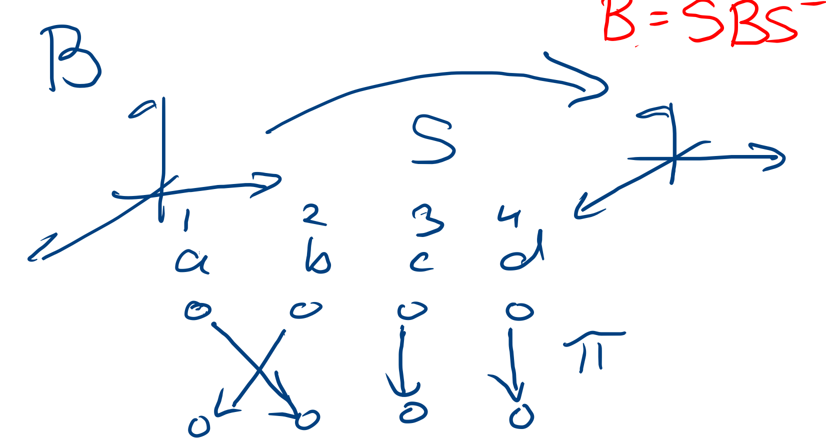
$a \to c$
$b \to d$
$c \to d$
$d \to c$

$\longrightarrow$ INTERESTING!!

- what are subgroups of Abelian groups?

$\longrightarrow$ Abelian.

$B' = SBS^{-1}$



$(ab)(c)(d)$

$\longrightarrow$ one swap of two elts
- the other two stay the same.

$\boxed{(\ \ )(\ )(\ )}$

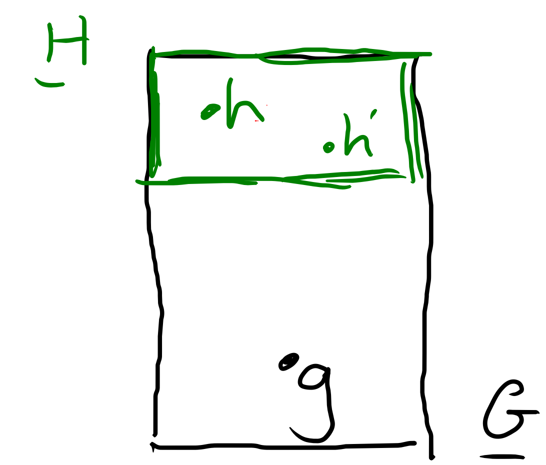
Def^{n}: Normal subgroup.

Given $H < G$ a subgroup. A normal subgroup $H \trianglelefteq G$ means:

The group is stable under conjugation by any elt g in G.

So that $\forall\, h \in H$ and $\forall g \in G$,

$$\boxed{g' \equiv ghg^{-1} \in H,}$$



FACTS

2. If is Abelian $\longrightarrow$ H Abelian.

- pick $h \in H$ $\longrightarrow$ conjugate by some $g \in G$,

then $h \longrightarrow ghg^{-1}$

$$= gg^{-1}h$$

$$= h \in H.$$

$\therefore$ all subgroups of Abelian groups are normal.

In general, subgroups are not normal. → it's special.

2.) If H is normal in G, $H \trianglelefteq G$, then left & richt cosets are equal

- given $H \trianglelefteq G$, take gH as a left coset.

∴ H is normal we can write.

$\boxed{g \circ h \circ g^{-1} \in H}$

$$gHg^{-1} = H$$

$$gH\cancel{g^{-1} \circ g} = H \circ g$$

$$\boxed{gH = Hg}$$ ← left & right cosets are equal.