Robust Control of Wave Energy Converters Using Unstructured Uncertainty

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Abstract— **In the design of ocean wave energy converters, proper control design is essential to the maximization of the power generation performance for the device. However, in realistic applications, this control design must be undertaken in the presence of model uncertainty. This paper considers the use of robust control theory to optimize the nominal performance for a wave energy converter in stochastic waves, subject to the constraint that the controller be stability-robust to unstructured uncertainties. We formulate the problem as a multi-objective optimal control problem, in which the primary objective is the maximization of power generation for the nominal system, and the competing objective is the H_∞ norm of the uncertainty input/output channel. This optimal control problem is nonconvex, and we therefore propose an iterative algorithm that can be used to arrive at a local optimal solution. This iterative approach employs the concept of Iterative Convex Overbounding, in the context of the classical Method of Centers. The methodology is demonstrated on a model of a single, buoy-type wave energy converter.**

I. INTRODUCTION

Ocean waves constitute a promising renewable energy resource, which has received a great deal of attention over the last several decades [1]. Many types of wave energy converters (WECs) have been proposed to harness the resource, such as heaving buoy systems, pitching devices, and submerged flaps [2]. Although different types of WECs differ in size and geometry, many of the basic principles behind their operation are similar. In all cases, incident wave forces induce the WEC to respond dynamically, and this motion is resisted by a power take-off (PTO) device. The PTO consequently extracts power from the WEC system, which is then transmitted to a power bus by a controllable power train.

The PTO must be controlled to regulate the power it extracts from the WEC, by imposing a feedback law between the measured dynamic response of the WEC and the opposing forces it imparts. In early studies, WEC control systems were designed to maximize extracted power, assuming a single frequency of wave excitation [3]. It was recognised that this problem is an application of the impedance matching technique, commonly used in aerospace and mechanical engineering. Via this technique, optimal power extraction is achieved by imposing a linear colocated feedback law between the velocity experienced by the PTO device, and the resultant force it applies. More specifically, the transfer function of this feedback law, when evaluated at the excitation frequency, must be the Hermitian adjoint of the mechanical driving point impedance seen by the PTO device. This technique has been applied extensively in the literature on WEC control [4].

However, realistic wave motions are stochastic, and exhibit significant power across a continuous spectrum of frequencies. As such, the use of the impedance matching technique can be problematic because, to impose the matching condition across a nontrivial frequency band requires that the feedback law be noncausal. Over the last decade, this has led to a significant body of work on the optimal control of WECs to maximise power generation in stochastic waves [5]–[8]. The majority of this work has focused on the use of Model-Predictive Control (MPC) techniques. An advantage of the MPC framework is that nonlinearities in the system dynamics (such as viscous drag forces and nonlinear buoyant force) can be explicitly accommodated, and mechanical constraints (such as PTO force constraints) can be explicitly imposed.

In situations where nonlinearities and hard constraints are not overly restrictive, another approach to the stochastic wave energy harvesting problem is to impose a causality constraint on the linear feedback law for the PTO, and optimize power generation over this constrained domain. It was recently shown in [9] that, under the assumption of linear dynamics and stationary stochastic sea state, the optimal control of the WEC system is a non-standard Linear Quadratic Gaussian (LQG) control problem. The closed-form solution can therefore be found assuming accurate models are available for the WEC dynamics and the wave spectrum.

This observation opens the door to the use of many tools from optimal causal control, in the context of WEC systems. In the case in which nonlinearities in the WEC dynamics are significant, Gaussian Closure (e.g., quasilinearization) techniques can be used [10]. Furthermore, the LQG theory for optimal energy harvesting can be extended to an adaptive control framework [11]. In the case where competing response objectives must be met for the WEC response, multi-objective optimal control can be used [12].

In this paper we consider the application of robust control techniques to the causal WEC control problem. In particular, we are interested here in the use of H_∞ techniques to impose a robust stability condition on the design of the control law for a WEC, for which uncertainty exists in its dynamic model. This problem can be posed as a multi-objective control design problem, in which the primary objective is the maximization of power generation for the nominal model of the WEC system, while the competing objective is the requirement that an H_∞ norm constraint be satisfied for the uncertainty input-output channel. As such, the problem of

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stability-robust optimization of a causal controller for a WEC distills to a LQG problem with an associated H_∞ constraint.

In the literature, there have been many studies on the design of H_2/LQG controllers with an associated H_∞ bound [13]–[16], which is also called mixed H_2/H_∞ control design. In [13], instead of directly minimizing the H_2 norm of the transfer function, it focuses on minimizing an upper bound of this norm, subjected to an H_∞ constraint. The necessary condition for optimality of controllers of a specific order is given in the form of three coupled nonlinear Riccati equations. There are some numerical methods on solving Riccati equation, but there is no effective procedure to solve these cross-coupled Riccati equations. In order to provide a reliable and efficient algorithm to solve this type of control problem, it was proposed that mixed H_2/H_∞ control design can be converted into a convex optimization problem over a bounded set of real matrices [15]. In [16], the Lyapunov-shaping paradigm is used to derive a LMI technique to solve the mixed H_2/H_∞ problem. However, the convex optimization approach and LMI technique are sub-optimal. They guarantee the design specification at the expense of conservatism.

The main contributions of this paper are (i) the novel formulation of the causal robust control problem for WEC systems, and (ii) the use of an iterative convex over-bounding (ICO) to solve the WEC robust control problem. Theoretically, ICO can arrive at local optimum solution and does not introduce any conservatism. The robust control problem can be interpreted as a semidefinite program in which the constraints are nonlinear matrix inequalities. Such inequalities can be solved straight-forwardly by ICO methods, for local optima.

II. Mathematical Modeling

A. Hydrodynamic Model

For a single floating buoy, responding in heave, its dynamic behavior can be modelled by using Newton's Second Law.

$$M\dot{v}(t) = f_r(t) + f_\omega(t) + f_b(t) + u(t) \tag{1}$$

where $v(t)$ is the heaving velocity of buoy, $f_r(t)$ is the radiation force, $f_w(t)$ is the incident wave force, $f_b(t)$ is the buoyancy force, $u(t)$ is the control force acting on the buoy by the PTO system [17]. To model these forces, we assume linear potential theory [17], implying:

1) We assume an ideal fluid; i.e., that it is incompressible, irrotational and inviscid.
2) We assume the wave amplitude is small.
3) We assume a small-body approximation for the buoy.

Incident wave force $f_w(t)$ is related to the wave elevation $a(t)$ by a linear convolution

$$f_w(t) = \int_{-\infty}^{\infty} h_w(t-\tau)a(\tau)d\tau \tag{2}$$

where $h_w(t)$ is the associated convolution kernel. Regarding the wave elevation $a(t)$, we assume it is a stationary stochastic process with known power spectral density $S_a(\omega)$. Consequently, the power spectral density of wave force is

$$S_w(\omega) = \hat{h}_w(j\omega)S_a(\omega)\hat{h}_w^H(j\omega) \tag{3}$$

Because $S_w(\omega)$ is a valid power spectral density, by the spectral factorization theorem [18] there exists an asymptotically-stable, minimum-phase, transfer function $\hat{g}_w(j\omega)$ such that

$$S_w(\omega) = \hat{g}_w(j\omega)\hat{g}_w^H(j\omega) \tag{4}$$

and consequently one can characterize

$$f_w(t) = \int_{-\infty}^{t} g_w(t-\tau)d\nu(\tau) \tag{5}$$

where $g_w(t)$ is the impulse response of $\hat{g}_w(j\omega)$ and $\nu(t)$ is a Wiener process with unit rate.

Buoyancy force f_b is modeled as

$$f_b(t) = -\rho g A \int_{-\infty}^{t} v(\tau)d\tau \tag{6}$$

where ρ is the water density, g is the gravity acceleration, A is the cross-sectional area of the buoy at the free surface.

Radiation force f_r is modeled as

$$f_r(t) = -\int_{-\infty}^{t} z_r(\tau)v(t-\tau)d\tau \tag{7}$$

where $z_r(t)$ is the radiation kernel. The Fourier transform of $z_r(t)$ is the radiation impedance $\hat{z}_r(j\omega)$, which is typically decomposed as:

$$\hat{z}_r(j\omega) = r(\omega) + j\omega m(\omega) \tag{8}$$

where $r(\omega)$ is the radiation resistance and $m(\omega)$ is the added mass.

Substituting (5), (6), and (7) into (1), we can express dynamic equation in frequency domain as

$$\hat{v}(\omega) = G_{uv}(j\omega)\hat{u}(\omega) + G_{wv}(j\omega)\hat{w}(\omega) \tag{9}$$

where $w(t) = \frac{d}{dt}\nu(t)$ is unit-intensity white noise, where

$$G_{uv}(j\omega) = \frac{j\omega}{K_b + r(\omega)j\omega - (m(\omega)+M)\omega^2} \tag{10}$$

$$G_{wv}(j\omega) = \frac{\hat{g}_w(\omega)j\omega}{K_b + r(\omega)j\omega - (m(\omega)+M)\omega^2} \tag{11}$$

and where $K_b \triangleq \rho g A$.

B. Control Objective

We assume the PTO for the WEC imposes a linear feedback law between u and v; i.e.,

$$\hat{u}(j\omega) = K(j\omega)\hat{v}(j\omega) \tag{12}$$

where K is the feedback gain. The objective of the control design is to maximize the expected value of the generated power in stationary response. To formulate this, we first note that the power absorbed from the waves by the WEC is $P_a(t) = -u^T(t)v(t)$. The generated power is the absorbed power minus the transmission losses incurred in the PTO power train. Here, we assume these losses to be quadratic in

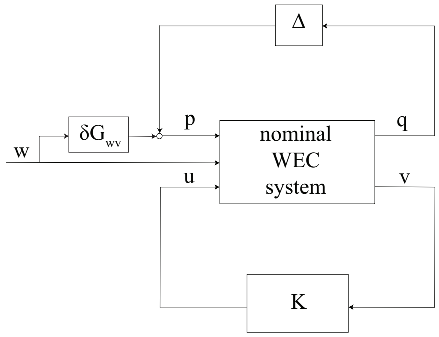

Fig. 1. system diagram

u; i.e., $P_\ell(t) = u^T(t)Ru(t)$ where $R > 0$. As such, we have that the objective of the design for K is to maximize

$$\bar{P} \triangleq -\mathcal{E}\left\{u^T v + u^T R u\right\} \tag{13}$$

where $\mathcal{E}\{\cdot\}$ denotes the expectation taken in stationarity.

C. Model Uncertainty

We assume that transfer functions G_{uv} can be modeled as the sum of a nominal model G_{uv}^n and an additive perturbation $\delta G_{uv}(j\omega)$, i.e.,

$$G_{uv}(j\omega) = G_{uv}^n(j\omega) + \delta G_{uv}(j\omega) \tag{14}$$

Perturbation $\delta G_{uv}(j\omega)$ is used to describe a frequency dependent unstructured uncertainty as

$$\delta G_{uv}(j\omega) = \Delta(j\omega) W_{uv}(j\omega) \tag{15}$$

where the normalized perturbation function $\Delta \in H_\infty$ adheres to

$$\|\Delta\|_\infty \leq 1 \tag{16}$$

and $W_{uv} \in H_\infty$ is a weighting function.

The uncertainty in G_{wv} can be modeled similarly, as

$$G_{wv}(j\omega) = G_{wv}^n(j\omega) + \delta G_{wv}(j\omega) \tag{17}$$

However, it is not necessary to characterize perturbation δG_{wv} in order to accomplish the objectives of the control design problem to be considered in this paper. This is because in this paper, we are only concerned with the implications of model uncertainty for stability robustness, rather than performance robustness, and the presence of perturbation δG_{wv} has no impact on the stability of the closed-loop system. Let

$$\hat{p}(j\omega) = \Delta(j\omega) W_{uv}(j\omega)\hat{u}(j\omega) + \delta G_{wv}(j\omega)\hat{w}(j\omega) \tag{18}$$

Then the dynamics of the closed-loop system, accounting for uncertainty, are characterized by the dynamics of the nominal system model

$$\begin{bmatrix} \hat{v}(j\omega) \\ \hat{q}(j\omega) \end{bmatrix} = \begin{bmatrix} G_{uv}^n(j\omega) & G_{wv}^n(j\omega) & I \\ W_{uv}(j\omega) & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}(j\omega) \\ \hat{w}(j\omega) \\ \hat{p}(j\omega) \end{bmatrix} \tag{19}$$

together with the feedback relationships

$$\hat{p}(j\omega) = \Delta(j\omega)\hat{q}(j\omega) + \delta G_{wv}\hat{w}(j\omega) \tag{20}$$

$$\hat{u}(j\omega) = K(j\omega)\hat{v}(j\omega) \tag{21}$$

The system diagram is shown at Figure 1

D. State-Space Model

As framed in Section II-A, the model of the WEC system exhibits infinite-dimensional dynamics. This infinite-dimensionality arises as a consequence of several aspects of the problem. Firstly, recall that the transfer function from the wave elevation $a(t)$ to the incident wave force $f_w(t)$ is $\hat{h}(j\omega)$. To find this transfer function at a given frequency ω requires that the partial differential equation characterizing the resultant flow around the cylinder be numerically solved. From this solution, the resultant pressure around the wetted area of the buoy is found, and integrated to get the total incident force. As such, the solution to $\hat{h}(j\omega)$ is a nonparametric function of ω. Similarly, to find the radiation impedance $\hat{z}_r(j\omega)$ at a given ω requires the solution to another partial differential equation, leading again to a nonparametric function of ω. Furthermore, typically the spectrum for the wave amplitude, $S_a(\omega)$ is an irrational function of ω, implying that a stochastic signal $a(t)$ producing this spectrum cannot be modeled via a finite-dimensional Gauss-Markov process.

In order to frame the robust control problem in the next section in a way that is tractable, it will be necessary to approximate the dynamics of the system via a finite-dimensional state space. To begin, let

$$M_\infty \triangleq \lim_{\omega\to\infty} m(\omega) \tag{22}$$

and define

$$\Gamma(j\omega) \triangleq r(\omega) + j\omega\left(m(\omega) - M_\infty\right) \tag{23}$$

Then we assume a finite-dimensional approximation

$$\Gamma(j\omega) \sim \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & 0 \end{array}\right] \tag{24}$$

In the above state space model, the values of A_r, B_r, and C_r can be obtained, for example, via subspace identification techniques, as described in [19]. We then have that the buoy dynamics in (1) can be restated as the state space

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c\left(f_w(t) + u(t)\right) \\ v(t) = C_c x_c(t) + p(t) \end{cases} \tag{25}$$

with A_c, B_c, and C_c defined appropriately.

For wave force, f_w we assume a state space model characterized by

$$\begin{cases} dx_w(t) = A_w x_w(t)dt + B_w d\nu(t) \\ f_w(t) = C_w x_w(t) \end{cases} \tag{26}$$

where A_w, B_w, and C_w are chosen such that they approximate the true wave spectrum; i.e.,

$$C_w[j\omega I - A_w]^{-1}B_w \approx \hat{g}_w(j\omega) \tag{27}$$

This can be accomplished, for example, using subspace-based spectral identification techniques, as detailed in [19].

For the uncertainty filter W_{uv}, we assume a finite-dimensional model as well, i.e.,

$$W_{uv} \sim \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array}\right] \tag{28}$$

We consequently have the following augmented state space model for the nominal system:

$$\begin{cases} dx(t) = (Ax(t) + B_{ut}u(t))\,dt + B_{wt}d\nu(t) \\ q(t) = C_q x(t) + D_{uq}u(t) \\ v(t) = C_v x(t) + p(t) \end{cases} \tag{29}$$

where:

$$A \triangleq \begin{bmatrix} A_c & B_c C_w & 0 \\ 0 & A_w & 0 \\ 0 & 0 & A_u \end{bmatrix}, B_{ut} \triangleq \begin{bmatrix} B_c \\ 0 \\ B_u \end{bmatrix}, B_{wt} \triangleq \begin{bmatrix} 0 \\ B_w \\ 0 \end{bmatrix}$$

$$C_q \triangleq \begin{bmatrix} 0 & 0 & C_u \end{bmatrix}, C_v \triangleq \begin{bmatrix} C_c & 0 & 0 \end{bmatrix}, D_{uq} \triangleq D_u$$

III. STABILITY-ROBUST CONTROL DESIGN

Let $\mathcal{K}$ be the set of all casual feedback laws K which are internally stabilizing for all $\Delta_{uv} \in H_\infty$ with $\|\Delta_{uv}\|_\infty \leqslant 1$. Furthermore, let $\bar{P}_n$ be the performance of casual controller K assuming the nominal plant model; i.e.,

$$\bar{P}_n = -\mathcal{E}\{u^T v + u^T R u\}\big|_{\Delta_{uv}=0,\ \delta G_{wv}=0} \tag{30}$$

Then we may precisely state the design problem we consider here as the optimization

$$\text{OP1}: \begin{cases} \text{Maximize:} & \bar{P}_n \\ \text{Domain:} & \text{Causal } K: v \mapsto u \\ \text{Constraint:} & K \in \mathcal{K} \end{cases}$$

Note that, in this problem, the optimization objective is to maximize the performance of the *nominal* system, not the worst-case performance evaluated over the uncertainty domain.

We assume the casual controller K is linear, time-invariant, proper, and has the following state space form:

$$K: \begin{cases} \dot{x}_k(t) = A_k x_k(t) + B_k v(t) \\ u(t) = C_k x_k(t) + D_k v(t) \end{cases} \tag{31}$$

with $\dim A_k = \dim A$. Defining the augmented state $\hat{x} \triangleq \begin{bmatrix} x & x_k \end{bmatrix}^T$, the close-loop system becomes:

$$T: \begin{cases} d\hat{x}(t) = \left(\hat{A}\hat{x}(t) + \hat{B}_p p(t)\right) dt + \hat{B}_{wt} d\nu(t) \\ q(t) = \hat{C}_q x(t) + \hat{D}_{pq} p(t) \\ v(t) = \hat{C}_v x(t) + p(t) \\ u(t) = \hat{C}_u x(t) + \hat{D}_{pu} p(t) \end{cases} \tag{32}$$

where:

$$\hat{A} \triangleq \begin{bmatrix} A + B_{ut} D_k C_v & B_{ut} C_k \\ B_k C_v & A_k \end{bmatrix}, \hat{B}_p \triangleq \begin{bmatrix} B_{ut} D_k \\ B_k \end{bmatrix}$$

$$\hat{B}_{wt} \triangleq \begin{bmatrix} B_{wt} \\ 0 \end{bmatrix}, \hat{C}_q \triangleq \begin{bmatrix} C_q^T + C_v^T D_k^T D_u^T \\ C_k^T D_u^T \end{bmatrix}^T, \hat{D}_{pq} \triangleq D_u D_k$$

$$\hat{C}_v \triangleq \begin{bmatrix} C_v \\ 0 \end{bmatrix}, \hat{C}_u \triangleq \begin{bmatrix} D_k C_v & C_k \end{bmatrix}, \hat{D}_{pu} \triangleq D_k$$

We refer to the various input-output channels of closed-loop system T by subscripts. For example, T_{pq} refers to the closed-loop transfer function from p to q.

Based on small gain theorem [20], the closed loop system is internally stable for all $\Delta \in H_\infty$ with $\|\Delta\|_\infty < 1$ if and only if $\|T_{pq}\|_\infty \leq 1$. Optimization problem OP1 can therefore be restated as

$$\text{OP1}: \begin{cases} \text{Maximize:} & \bar{P}_n \\ \text{Domain:} & A_k, B_k, C_k, D_k \\ \text{Constraint:} & \|T_{pq}\|_\infty \leq 1 \end{cases}$$

We have the following theorem to solve this optimization problem.

Theorem 1: A casual controller K, parametrized as in (31), is internally stabilizing for all $\|\Delta\|_\infty < 1$, and satisfies $\bar{P}_n > \theta$ if and only if $\exists\ P = P^T > 0$, $S = S^T > 0$ and $W = W^T$ such that

$$\begin{bmatrix} \hat{A}^T P + P\hat{A} & P\hat{B}_{wt} \\ \hat{B}_{wt}^T P & -I \end{bmatrix} < 0 \tag{33}$$

$$\begin{bmatrix} W & \hat{C}_u - F \\ \hat{C}_u^T - F^T & P \end{bmatrix} > 0 \tag{34}$$

$$\mathrm{Tr}\{RW\} - \bar{P}_{\max} + \theta < 0 \tag{35}$$

$$\begin{bmatrix} \hat{A}^T S + S\hat{A} & S\hat{B}_p & \hat{C}_q^T \\ \hat{B}_p^T S & -I & \hat{D}_{pq}^T \\ \hat{C}_q & \hat{D}_{pq}^T & -I \end{bmatrix} \leq 0 \tag{36}$$

where $F = \begin{bmatrix} F_1 & 0 \end{bmatrix}$, $\bar{p}_{\max} = -\mathrm{Tr}\{B_{wt}^T Q B_{wt}\}$, $F_1 = -R^{-1}(B_{ut}^T Q + \frac{1}{2}C_v)$, and $Q = Q^T$ is the solution of the Ricatti equation

$$A^T Q + QA - (QB_{ut} + \frac{1}{2}C_v^T)R^{-1}(B_{ut}^T Q + \frac{1}{2}C_v) = 0 \tag{37}$$

Proof: In [9] it is shown that $\bar{P}_n > \theta$ if and only if $\exists\ P > 0$ such that inequalities (33), (34) and (35) hold. Meanwhile, it is a standard result (see, e.g., [21]) that $\|T_{pq}\|_\infty \leq 1$ if and only if $\exists\ S > 0$ such that (36) holds. ■

Optimization problem OP1 can therefore be framed as the maximization of θ, over the domain $\{A_k, B_k, C_k, D_k\}$, subject to matrix inequality constraints (33), (34), (35), and (36). This optimization is nonconvex, due to the bilinearity of matrix inequalities (33) and (36). Beyond the challenges associated with the nonconvexity of the problem, framing OP1 in this manner has another disadvantage, in that for each K, the optimization domain $\{A_k, B_k, C_k, D_k\}$ contains infinite realizations of it. It is consequently the case that for any given design point in the $\{A_k, B_k, C_k, D_k\}$ domain, there will be search directions for which $\bar{P}_n$ has zero sensitivity.

To remedy this, we make use of the coordinate transformation detailed in [16]. Without loss of generality, define

$$P = \begin{bmatrix} Y & N \\ N^T & * \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} X & M \\ M^T & * \end{bmatrix} \tag{38}$$

where $\{X, Y\} \subset R^{n\times n}$ and symmetric. * means the term that is unnecessary to define. Then Theorem 1 can be shown to be equivalent to Theorem 2, below.

Theorem 2: There exists controller K, parametrized as in (31), which is internally stabilizing for all $\|\Delta\|_\infty < 1$, and such that $\bar{P}_n > \theta$, if and only if $\exists$ matrices $X = X^T, Y =$

$Y^T, \hat{S} = \hat{S}^T, W = W^T, \hat{A}, \hat{B}, \hat{C}$ and $\hat{D}$ such that

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \quad (39)$$

$$\mathrm{He}\left\{\begin{bmatrix} AX + B_{ut}\hat{C} & A + B_{ut}\hat{D}C_v & B_{wt} \\ \hat{A} & YA + \hat{B}C_v & YB_{wt} \\ 0 & 0 & -\frac{1}{2}I \end{bmatrix}\right\} < 0 \quad (40)$$

$$\mathrm{He}\left\{\begin{bmatrix} \frac{1}{2}W & \hat{C} - F_1 X & \hat{D}C_v - F_1 \\ 0 & \frac{1}{2}X & I \\ 0 & 0 & \frac{1}{2}Y \end{bmatrix}\right\} > 0 \quad (41)$$

$$\mathrm{Tr}\{RW\} - \bar{P}_{\max} + \theta < 0 \quad (42)$$

$$\mathrm{He}\left\{\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ 0 & -\frac{1}{2}I & 0 \\ \begin{bmatrix} C_q X + D_u\hat{C} & C_q + D_u\hat{D}C_v \end{bmatrix} & D_u\hat{D} & -\frac{1}{2}I \end{bmatrix}\right\} \leq 0 \quad (43)$$

where

$$\Lambda_{11} \triangleq \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \hat{S} \begin{bmatrix} AX + B_{ut}\hat{C} & A + B_{ut}\hat{D}C_v \\ \hat{A} & YA + \hat{B}C_v \end{bmatrix} \quad (44)$$

$$\Lambda_{12} \triangleq \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \hat{S} \begin{bmatrix} B_{ut}\hat{D} \\ \hat{B} \end{bmatrix} \quad (45)$$

and where $\mathrm{He}\{W\} = W + W^H$.

Proof: The proof for this theorem is similar to that in [16], and consequently is omitted here in the interest of brevity. ■

Regarding this theorem, we make the following comments:

1) Assume a solution of the above optimization problem is found. Then it is known from (39) that $I - XY$ is nonsingular. One may consequently find M and N as any matrices satisfying $MN^T = I - XY$.
2) With M and N found, one may find controller parameters $\{A_k, B_k, C_k, D_k\}$ associated with the optimization variables as

$$\begin{cases} D_k &= \hat{D} \\ C_k &= (\hat{C} - D_k C X)M^{-T} \\ B_k &= N^{-1}(\hat{B} - YBD_k) \\ A_k &= N^{-1}[\hat{A} - NB_k CX - YBC_k M^T - \\ & \quad Y(A + BD_k C)X]M^{-T} \end{cases} \quad (46)$$

So, Theorem 2 reframes optimization problem OP1 as the following optimization problem:

$$\mathrm{OP2:} \begin{cases} \text{Given :} & A, B_u, B_w, C_v, C_q, D_u, F_1 \\ \text{Maximize:} & \theta \\ \text{Domain:} & X = X^T, Y = Y^T, \hat{S} = \hat{S}^T \\ & W = W^T, \hat{A}, \hat{B}, \hat{C}, \hat{D}, \theta \\ \text{Constraints:} & (39), (40), (41), (42), (43), \hat{S} > 0 \end{cases}$$

Optimization problem OP2 is still nonconvex, since it involves tri-linear matrix inequality in (43). There is no (known) coordinate transformation that can turn it into linear matrix inequality. Consequently, we develop an iterative convex over-bounding algorithm to give a locally-optimal solution to OP2. This is described next.

IV. Solution to OP2 Via Iterative Convex Overbounding

A. *Convex overbounding of matrix inequality* (43)

In this section, we derive a linear matrix inequality which conservatively approximates tri-linear matrix inequality (43), through the process of convex overbounding. To explain the technique, we first introduce the notation

$$\mathcal{D} \triangleq \left\{X, Y, \hat{S}, W, \hat{A}, \hat{B}, \hat{C}, \hat{D}\right\} \quad (47)$$

to refer to the optimization variables in OP2. Then to apply the convex overbounding technique, we first need a feasible parameter set $\mathcal{D}_0$ satisfying (43), which we call a *design point*. Then, through the process described below, we find a linear matrix inequality in the variables $\mathcal{D}$ which has the following properties:

1) Any $\mathcal{D}$ satisfying the linear matrix inequality also satisfies (43).
2) The linear matrix inequality is feasible at the design point; i.e., with $\mathcal{D} = \mathcal{D}_0$.

For convenience, we introduce the following variables, which are the linear functions of variables $\mathcal{D}$ as

$$\alpha \triangleq \begin{bmatrix} AX + B\hat{C} & A + B\hat{D}C \\ \hat{A} & YA + \hat{B}C \end{bmatrix}, \qquad \beta \triangleq \begin{bmatrix} B_u\hat{D} & 0 \\ \hat{B} & 0 \end{bmatrix}$$

$$\delta \triangleq \begin{bmatrix} -\frac{1}{2}I & 0 \\ D_u\hat{D} & -\frac{1}{2}I \end{bmatrix}, \qquad \zeta \triangleq \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$$

$$\chi \triangleq \begin{bmatrix} 0 \\ \begin{bmatrix} C_q X + D_u\hat{C} & C_q + D_u\hat{D}C_v \end{bmatrix} \end{bmatrix}$$

Equation (43) is equivalent to

$$\mathrm{He}\left\{\begin{bmatrix} \zeta\hat{S} \\ 0 \end{bmatrix}\begin{bmatrix} \alpha & \beta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \chi & \delta \end{bmatrix}\right\} \leq 0 \quad (48)$$

Let $\{\alpha_0, \beta_0, \zeta_0, \hat{S}_0\}$ be the values of $\{\alpha, \beta, \zeta, \hat{S}\}$, when evaluated at the design point $\mathcal{D}_0$. Then (48) can be re-expressed as

$$\begin{aligned} &\mathrm{He}\left\{\begin{bmatrix} \zeta\hat{S} - \zeta_0\hat{S}_0 \\ 0 \end{bmatrix}\begin{bmatrix} \alpha_0 & \beta_0 \end{bmatrix}\right\} + \mathrm{He}\left\{\begin{bmatrix} \zeta_0 \\ 0 \end{bmatrix}\hat{S}_0\begin{bmatrix} \alpha & \beta \end{bmatrix}\right\} \\ &- \mathrm{He}\left\{\begin{bmatrix} \zeta\hat{S} - \zeta_0\hat{S}_0 \\ 0 \end{bmatrix}\begin{bmatrix} \alpha - \alpha_0 & \beta - \beta_0 \end{bmatrix}\right\} + \begin{bmatrix} 0 & \chi^T \\ \chi & \delta + \delta^T \end{bmatrix} \\ &\leq 0 \end{aligned} \quad (49)$$

For the third term in the left side of the inequality (49), we may overbound it as

$$\begin{aligned} &\mathrm{He}\left\{\begin{bmatrix} (\zeta\hat{S} - \zeta_0\hat{S}_0) \\ 0 \end{bmatrix}\begin{bmatrix} \alpha - \alpha_0 & \beta - \beta_0 \end{bmatrix}\right\} \\ &\leq \begin{bmatrix} \zeta\hat{S} - \zeta_0\hat{S}_0 \\ 0 \end{bmatrix} W_1 \begin{bmatrix} \zeta\hat{S} - \zeta_0\hat{S}_0 \\ 0 \end{bmatrix}^T \\ &+ \begin{bmatrix} \alpha^T - \alpha_0^T \\ \beta^T - \beta_0^T \end{bmatrix} W_1^{-1} \begin{bmatrix} \alpha - \alpha_0 & \beta - \beta_0 \end{bmatrix} \end{aligned} \quad (50)$$

where $W_1 = W_1^T > 0$ is any positive-definite matrix. Based on this inequality, (49) is conservatively guaranteed (via a

Schur complement) by

$$\begin{bmatrix} (*) & \begin{bmatrix}\zeta\hat{S}-\zeta_0\hat{S}_0 \\ 0\end{bmatrix} & \begin{bmatrix}\alpha^T-\alpha_0^T \\ \beta^T-\beta_0^T\end{bmatrix} \\ \text{(sym)} & -W_1 & 0 \\ \text{(sym)} & \text{(sym)} & -W_1^{-1} \end{bmatrix} \leq 0 \tag{51}$$

where

$$\begin{aligned}(*) &= \mathrm{He}\left\{ \begin{bmatrix}(\zeta\hat{S}-\zeta_0\hat{S}_0) \\ 0\end{bmatrix} \begin{bmatrix}\alpha_0 & \beta_0\end{bmatrix} \right\} \\ &+ \mathrm{He}\left\{ \begin{bmatrix}\zeta_0 \\ 0\end{bmatrix} \hat{S}_0 \begin{bmatrix}\alpha & \beta\end{bmatrix} \right\} + \begin{bmatrix}0 & \chi^T \\ \chi & \delta+\delta^T\end{bmatrix}\end{aligned}$$

As such, we have converted tri-linear matrix inequality (48) into a bilinear matrix inequality. Furthermore, note that (49) and (51) are equivalent at the design point; i.e., with $\{\alpha, \beta, \zeta, \hat{S}\} = \{\alpha_0, \beta_0, \zeta_0, \hat{S}_0\}$.

The same technique can be applied again, to convert (51) from a bilinear matrix inequality into an LMI. Suppressing the details, we merely state the resultant over-bounded LMI as

$$\begin{bmatrix} (\star) & \begin{bmatrix}\zeta\hat{S}_0+\zeta_0\hat{S}-2\zeta_0\hat{S}_0 \\ 0\end{bmatrix} & \begin{bmatrix}\alpha^T-\alpha_0^T \\ \beta^T-\beta_0^T\end{bmatrix} & \begin{bmatrix}\zeta-\zeta_0 \\ 0\end{bmatrix} & \begin{bmatrix}\alpha_0^T \\ \beta_0^T\end{bmatrix}(\hat{S}-\hat{S}_0)^T \\ \text{(sym)} & -W_1 & 0 & 0 & (\hat{S}-\hat{S}_0)^T \\ \text{(sym)} & \text{(sym)} & -W_1^{-1} & 0 & 0 \\ \text{(sym)} & \text{(sym)} & \text{(sym)} & -W_2 & 0 \\ \text{(sym)} & \text{(sym)} & \text{(sym)} & \text{(sym)} & -W_2^{-1} \end{bmatrix} \leq 0 \tag{52}$$

where $W_2 = W_2^T > 0$ is an arbitrary positive-definite matrix. Now we arrive at a linear matrix inequality which has the desired overbounding features described at the beginning of this section.

It remains to determine how to choose weighting matrices W_1 and W_2. Here, we consider that these parameters can be treated as free variables. To do this, consider that for any fixed $W_{i0} > 0$, and for all $W_i > 0$,

$$W_i^{-1} \geq 2W_{i0}^{-1} - W_{i0}^{-1} W_i W_{i0}^{-1} \tag{53}$$

where W_{i0} is the initial optimized value of W_i. Substituting these overbounds into (52) gives a matrix inequality which is also linear in W_1 and W_2.

As such, (52) becomes the LMI in the augmented variable domain

$$\bar{\mathcal{D}} \triangleq \{X, Y, \hat{S}, W, \hat{A}, \hat{B}, \hat{C}, \hat{D}, W_1, W_2\} \tag{54}$$

centered around the augmented design point $\bar{\mathcal{D}}_0$.

B. Method of centers with iterative convex overbounding

We now combine the convex overbounding approach discussed above, with the classical Method of Centers algorithm [22], to arrive at a technique for solving OP2. To begin, first assume we have a feasible augmented design point $\bar{\mathcal{D}}_0$, and choose an initial value $\theta^0 < \bar{P}_{\max} - \mathrm{Tr}(RW_0^0)$. Then iteration k of the Method of Centers has the following steps:

1) Given feasible point $\bar{\mathcal{D}}_0^k$ with associated θ^k, convert (39), (40), (41), (42), and (52) into an augmented matrix inequality of the form

$$M(x, \theta^k) = M_0 + M_\theta \theta^k + \sum_{i=1}^{n} M_i x_i > 0 \tag{55}$$

where x is a vectorized form of $\bar{\mathcal{D}}$.

2) Define a barrier function $f(x, \theta^k)$

$$f(x, \theta^k) \triangleq \begin{cases} -\log\det M(x,\theta^k) & : \quad M(x,\theta^k) > 0 \\ \infty & : \quad \text{otherwise} \end{cases} \tag{56}$$

and solve for the corresponding analytic center $x^\circ(\theta^k)$ by the following optimization

$$x^\circ(\theta^k) = \mathrm{sol}\begin{cases} \text{Given} & \theta^k, M_0, M_\theta, M_i, i \in \{1...n\} \\ \text{Minimize} & f(x, \theta^k) \\ \text{Domain} & x \in \mathbb{R}^n \end{cases} \tag{57}$$

3) Find the derivative of the analytic center $x^\circ(\theta^k)$ via

$$\left.\frac{\partial x^\circ(\theta)}{\partial \theta}\right|_{\theta=\theta^k} = -f_{xx}^{-1}(\theta^k) f_{x\theta}(\theta^k) \tag{58}$$

where

$$\{f_{x\theta}(\theta)\}_i \triangleq \mathrm{Tr}\{M^{-1}(x,\theta)M_\theta M^{-1}(x,\theta)M_i\}\big|_{x=x^\circ(\theta)}$$
$$\{f_{xx}(\theta)\}_{ij} \triangleq \mathrm{Tr}\{M^{-1}(x,\theta)M_j M^{-1}(x,\theta)M_i\}\big|_{x=x^\circ(\theta)}$$

and define the sensitivity function for the analytic center, with respect to θ, and linearized about θ^k; i.e.,

$$\chi(\theta, \theta^k) \triangleq x^\circ(\theta^k) + \left.\frac{\partial x^\circ(\theta)}{\partial \theta}\right|_{\theta=\theta^k} (\theta - \theta^k) \tag{59}$$

4) Find the next θ value, θ^{k+1}, as the maximum value for which $\chi(\theta, \theta^k)$ is feasible; i.e.,

$$\theta^{k+1} = \mathrm{sol}\begin{cases} \text{Given} & \theta^k, x^\circ(\theta^k), \left.\frac{\partial x^\circ(\theta)}{\partial\theta}\right|_{\theta=\theta^k} \\ & M_0, M_\theta, M_i, i \in \{1...n\} \\ \text{Maximize} & \theta \\ \text{Domain} & \theta \in \mathbb{R} \\ \text{Constraint} & f(\chi(\theta,\theta^k), \theta) < \infty \end{cases} \tag{60}$$

5) Find the next design point $\bar{\mathcal{D}}_0^{k+1}$ from the parameter vector $\bar{x}^\circ(\theta^{k+1}, \theta^k)$, update $\theta^k \leftarrow \theta^{k+1}$, and then return to step 1.

By iterating the above loop, θ^k will converge monotonically to a local maximum.

V. Example

We consider a single-degree-of-freedom floating buoy model in Figure 2 as an example. The floating buoy is connected with a sea-floor-mounted generator through a pre-tensioned tether cable. We assume the buoy oscillates only in heave. The natural period of the free response of the buoy is approximately 5.5s. The equailibrium point for the WEC

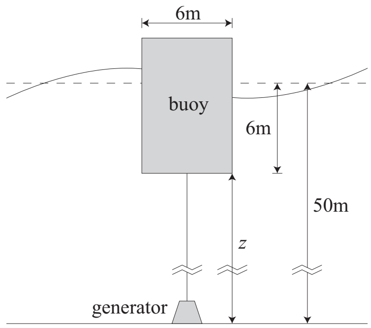

Fig. 2. Diagram of example WEC

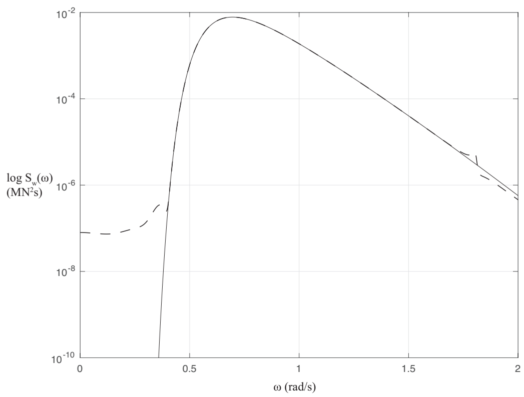

Fig. 3. Wave force spectrum $S_w(\omega)$: infinite-dimension (solid) and finite-dimension approximation (dash)

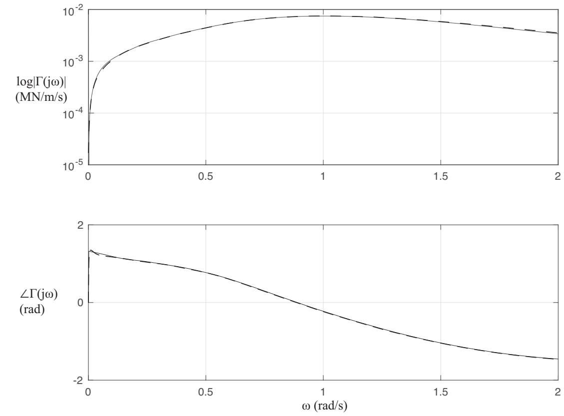

Fig. 4. Transfer function $\Gamma(j\omega)$: infinite-dimension (solid) and finite-dimension approximation (dash)

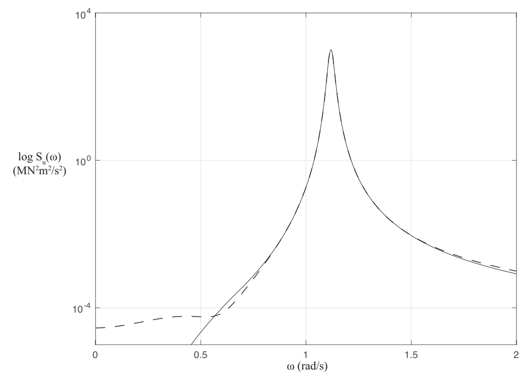

Fig. 5. Power spectrum of uncertainty filter: infinite-dimension (solid) and finite-dimension approximation (dash)

system is taken to be where $z = 0$m. For the translation loss model, we choose $R = 10^{-8}$s/kg. For more modeling details of the example in Fig.2, refer to [23].

We choose $S_a(\omega)$ as a Pierson-Moskowitz spectrum, with a mean wave period of 7s and a significant wave height of 1m. The spectrum of wave force $S_w(\omega)$ is generated based on (3) from $S_a(\omega)$. The spectral factor $\hat{g}_w(j\omega)$ is approximated via a finite-dimensional state space, as in (27), using subspace-based system identification techniques as described in [19]. Figure 3 shows the infinite-dimensional spectrum $S_w(\omega)$ and its finite-dimensional approximation.

The infinite-dimensional transfer function $\Gamma(j\omega)$ is found by the linearized hydrodynamic analysis (4). Using the subspace techniques, finite-dimensionalized state space model of the transfer function $\Gamma(j\omega)$ is obtained as in (24). Figure 4 shows the original infinite-dimensional transfer function for $\Gamma(j\omega)$, and the finite-dimensional approximation.

For the purposes of demonstration in this paper, we assume that the uncertainty in the hydrodynamic model stems from variability in the water depth. This change in depth affects both $\hat{h}_w(j\omega)$, as well as $r(\omega)$ and $m(\omega)$. We assume the water depth varies from 40m to 60m, and we choose 50m depth as the nominal model. To identify W_{uv}, we evaluate $G_{uv}(j\omega)$ for depths in the range from 40 to 60m, at intervals of 2m, resulting in a set of nonparametric transfer functions $G^i_{uv}(j\omega)$, for $i \in \{1...11\}$. We then define the spectrum $S_u(\omega)$ at each frequency ω, as

$$S_u(\omega) = \max_i |G^i_{uv}(j\omega) - G^n_{uv}(j\omega)|^2 \tag{61}$$

According to the spectral factorization theorem, there exists an asymptotically stable, minimum phase function $W_{uv}(j\omega)$, which satisfies

$$S_u(\omega) = W_{uv}(j\omega) W^H_{uv}(j\omega) \tag{62}$$

Applying the same subspace-based spectral approximation technique used to identify $S_w(\omega)$, a state space model for $W_{uv}(j\omega)$ as in (28), is therefore obtained from $S_u(\omega)$.

To implement the algorithm for robust controller design in section IV, the only aspect involving design is the choice of feasible initial point $\mathcal{D}^0_0$. Here, we make the simplest choice, by choosing these parameters such that they correspond to a feedback law $K = 0$, which is guaranteed to satisfy the associated H_∞ constraint.

Figure 6 shows data related to the resultant optimized robust controller. The plot shows two curves for power generation performance, as a function of water depth. The blue solid curve is the causal limit on power generation at each water depth. This is the power generation that would be possible if K were optimized at each depth, in the absence of any system uncertainty. Although not shown in the figure, the controllers achieving the causal limit on performance are extremely sensitive to a mismatch between the modeled and actual depth. Furthermore, a sufficiently-large mismatch results in closed-loop instability.

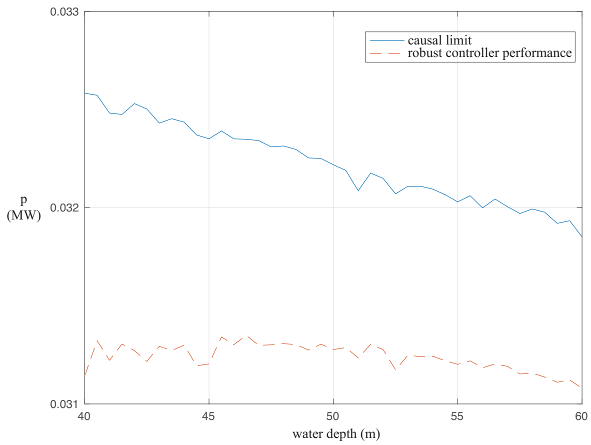

Fig. 6. Average power generation comparison between causal limit and robust controller performance in different water depths

Meanwhile, the red dashed curve is the performance of the robust controller, optimized as described above, and evaluated at the various water depths. At nominal water depth (i.e., 50m), the causal power generation limit is 32.2KW. After implementing the robust controller, the mean power generation goes down to 31.3kW, which is 2.8% decrease in performance. As for other water depths, the robust controller performs reasonably well, with a maximum of 4.5% decrease in power generation.

VI. Conclusions and Future Work

The analysis in this paper demonstrates that techniques from robust control theory can be brought to bear on the control design problem for ocean wave energy conversion systems. In particular, we have demonstrated how robust stability can be assured for the closed-loop system design, through a competing H_∞ constraint, which must be balanced against the primary objective of harvesting maximal power. In the example, we demonstrated how one source of uncertainty in the system dynamics, arising from variability in the water depth, can be accommodated. However, clearly the technique has broader application, to accommodate many other uncertainties which may also arise in this application. For example, such uncertainties could arise as a consequence of unmodeled nonlinearities in the system dynamics.

The problem formulated in this paper considers the optimization of *nominal* performance, subject to the constraint that the feedback law be stability-robust in the presence of model uncertainty. However, it would be preferable to formulate the problem such that the feedback law is optimized to be performance-robust; i.e., so that the worst-case power generation performance is optimized. This problem, which is more challenging than the one considered in the present paper, remains an item for future work.

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