Homework 1 Solutions

Problem 1.

With foci at $(1, 1)$ and $(-1, -1)$, this ellipse must be described by an equation of the form

$$\sqrt{(x-1)^2+(y-1)^2} + \sqrt{(x+1)^2+(y+1)^2} = C.$$

Substituting $(x, y) = (2, 2)$ gives that $C = 4\sqrt{2}$. Thus

$$\sqrt{(x-1)^2+(y-1)^2} + \sqrt{(x+1)^2+(y+1)^2} = 4\sqrt{2}$$

is an equation for the ellipse. To put this equation in the proper form, we move one square root to the right and then square both sides

$$\begin{aligned} \sqrt{(x-1)^2+(y-1)^2} &= 4\sqrt{2} - \sqrt{(x+1)^2+(y+1)^2}, \\ \text{so} \qquad (x-1)^2+(y-1)^2 &= 32 - 8\sqrt{2}\sqrt{(x+1)^2+(y+1)^2} + (x+1)^2 + (y+1)^2. \end{aligned}$$

Next we solve for the remaining square root and then square again:

$$\begin{aligned} \sqrt{(x+1)^2+(y+1)^2} &= \frac{x+y+8}{2\sqrt{2}}, \\ \text{so} \qquad (x+1)^2+(y+1)^2 &= \frac{(x+y+8)^2}{8}. \end{aligned}$$

Rearranging gives the desired equation:

$$\boxed{7x^2 - 2xy + 7y^2 = 48}$$

Problem 2.

The center $\vec{c}(t)$ of the smaller circle moves along a circle of radius 3:

$$\vec{c}(t) = (3\cos t, 3\sin t)$$

The vector $\vec{v}(t)$ from the center to the marked point rotates at a constant speed, making three full clockwise rotations as the smaller circle rolls around once. Thus

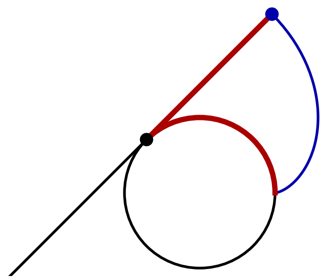
$$\vec{v}(t) = (\cos 3t, -\sin 3t).$$

Adding these together gives the desired parametrization:

$$\vec{x}(t) = \vec{c}(t) + \vec{v}(t) = \boxed{(3\cos t + \cos 3t, 3\sin t - \sin 3t)}$$

Problem 3.

The distance rolled on the circle must equal the distance rolled on the line:



The point $\vec{p}(t)$ at which the line is tangent to the circle is $(\cos t, \sin t)$. The vector $\vec{v}(t)$ from the point of tangency to the marked point is in the direction of $(\sin t, -\cos t)$ and has a length of t (the same as the length of the arc on the circle), so $\vec{v}(t) = t(\sin t, -\cos t) = (t \sin t, -t\cos t)$. Adding these together gives

$$\vec{x}(t) \;=\; \vec{p}(t) + \vec{v}(t) \;=\; \boxed{(\cos t + t\sin t, \sin t - t\cos t)}$$

Problem 4.

Let $\vec{p}(t) = (t, t^2)$ be the point at which the segment touches the parabola. The unit tangent vector to the parabola is $\vec{T}(t) = \left(\frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}\right)$, so the unit normal vector to the parabola is $\vec{U}(t) = \left(-\frac{2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}}\right)$. Adding these together gives

$$\vec{x}(t) \;=\; \vec{p}(t) + \vec{U}(t) \;=\; \boxed{\left(t - \frac{2t}{\sqrt{1+4t^2}},\; t^2 + \frac{1}{\sqrt{1+4t^2}}\right)}$$