Homework 2 Solutions

Problem 1.

The curve $\mathcal{C}$ can be parameterized as follows:

$$\vec{x}(t) \;=\; (3\cos t, 2\sin t) \qquad \text{for } 0 \le t \le \pi/2.$$

Then

$$\begin{aligned}
\int_{\mathcal{C}} xy\,ds \;&=\; \int_0^{\pi/2} x(t)\,y(t)\,\sqrt{x'(t)^2 + y'(t)^2}\,dt \\
&=\; \int_0^{\pi/2} (3\cos t)(2\sin t)\sqrt{(-3\sin t)^2 + (2\cos t)^2}\,dt \\
&=\; \int_0^{\pi/2} 6\cos t \sin t\,\sqrt{4 + 5\sin^2 t}\,dt \\
&=\; \int_4^9 \frac{3}{5}\sqrt{u}\,du \qquad (\text{where } u = 4 + 5\sin^2 t) \\
&=\; \left[\frac{2}{5}u^{3/2}\right]_4^9 \;=\; \boxed{\frac{38}{5}}
\end{aligned}$$

Problem 2.

Let $\mathcal{C}$ be any curve of the form $r = f(\theta)$ for $a \le \theta \le b$, where f is a differentiable function. Then $\mathcal{C}$ can be parameterized by

$$\vec{x}(\theta) \;=\; \big(f(\theta)\cos\theta, f(\theta)\sin\theta\big) \qquad \text{for } a \le \theta \le b$$

where θ is the parameter. Taking the derivative gives

$$\begin{aligned}
\vec{x}\,'(\theta) \;&=\; \big(f'(\theta)\cos\theta - f(\theta)\sin\theta,\; f'(\theta)\sin\theta + f(\theta)\cos\theta\big) \\
&=\; f(\theta)\,(-\sin\theta, \cos\theta) \;+\; f'(\theta)\,(\cos\theta, \sin\theta).
\end{aligned}$$

Since $(-\sin\theta, \cos\theta)$ and $(\cos\theta, \sin\theta)$ are perpendicular unit vectors, it follows that

$$\|\vec{x}\,'(\theta)\| \;=\; \sqrt{f(\theta)^2 + f'(\theta)^2},$$

and therefore

$$\int_{\mathcal{C}} ds \;=\; \int_a^b \|\vec{x}\,'(\theta)\|\,d\theta \;=\; \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta.$$

For the curve $r = e^{2\theta}$ between the points $(1,0)$ and $(0,e^{\pi})$, this gives a length of

$$\int_0^{\pi/2} \sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2}\,d\theta \;=\; \int_0^{\pi/2} \sqrt{5}\,e^{2\theta}\,d\theta \;=\; \left[\frac{\sqrt{5}}{2}e^{2\theta}\right]_0^{\pi/2} \;=\; \boxed{\frac{\sqrt{5}(e^{\pi} - 1)}{2}}$$

Problem 3.

Let $\vec{x}(t) = (t - \tanh t, \operatorname{sech} t)$ for $t > 0$. Then

$$\vec{x}\,'(t) \;=\; (1 - \operatorname{sech}^2 t,\, -\tanh t \operatorname{sech} t) \;=\; \tanh t\,(\tanh t, -\operatorname{sech} t)$$

Since $(\tanh t, -\operatorname{sech} t)$ is a unit vector, it follows that $\|\vec{x}\,'(t)\| = \tanh t$. Then

$$s(t) \;=\; \int \|\vec{x}\,'(t)\|\,dt \;=\; \int \tanh t\,dt \;=\; \log(\cosh t) + C.$$

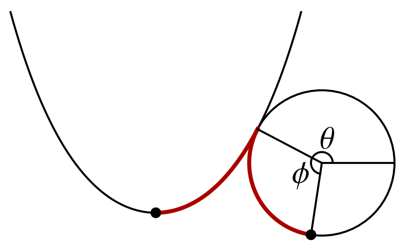
Since the problem didn't specify, we might as well assume that $C = 0$, so $s(t) = \log(\cosh t)$.

To find a unit-speed parametrization, we solve the equation $s = \log(\cosh t)$ for t. This gives $t = \cosh^{-1}(e^s)$, which we plug in for t:

$$\begin{aligned} \vec{x}\,'(t) \;&=\; \big(\cosh^{-1}(e^s) - \tanh(\cosh^{-1} e^s),\, \operatorname{sech}(\cosh^{-1} e^s)\big) \\ &=\; \boxed{\Big(\cosh^{-1}(e^s) - \sqrt{1 - e^{-2s}},\, e^{-s}\Big)} \end{aligned}$$

Problem 4.

Consider the situation at the instant that the point of tangency is $(t, \cosh t)$.



The unit tangent vector to the catenary is $(\operatorname{sech} t, \tanh t)$, so the center of the circle is

$$(t, \cosh t) \;+\; (\tanh t, -\operatorname{sech} t).$$

Then the point P is at

$$(t, \cosh t) \;+\; (\tanh t, -\operatorname{sech} t) \;+\; \big(\cos(\theta + \phi), \sin(\theta + \phi)\big)$$

where θ and ϕ are the angles shown in the picture.

The angle ϕ is the same as the length of the red arc, which is the same as the arc length of the catenary from $(0, 1)$ to $(t, \cosh t)$. It follows that $\phi = \sinh t$. As for the angle θ, observe the the vector from the center of the circle to the point of tangency is $(-\tanh t, \sinh t)$, and thus $\theta = \cos^{-1}(-\tanh t)$. We conclude that

$$\boxed{\vec{x}(t) \;=\; \big(t + \tanh t + \cos(\cos^{-1}(-\tanh t) + \sinh t),\; \cosh t - \operatorname{sech} t + \sin(\cos^{-1}(-\tanh t) + \sinh t)\big)}$$