

**Data 100, Summer 2020**

## Homework #5

*Due Date: Sunday, July 19th at 11:59 PM*

**Total Points: 26**

## Submission Instructions

You must submit this assignment to Gradescope by **Sunday, July 19th, at 11:59 PM**. While Gradescope accepts late submissions, you will not receive **any** credit for a late submission if you do not have prior accommodations (e.g. DSP).

You can work on this assignment in any way you like.

- One way is to download this PDF, print it out, and write directly on these pages (we've provided enough space for you to do so). Alternatively, if you have a tablet, you could save this PDF and write directly on it.
- Another way is to use some form of LaTeX. Overleaf is a great tool.
- You could also write your answers on a blank sheet of paper.

Regardless of what method you choose, the end result needs to end up on Gradescope, as a PDF. If you wrote something on physical paper (like options 1 and 3 above), you will need to use a scanning application (e.g. CamScanner) in order to submit your work.

When submitting on Gradescope, you **must** assign pages to each question correctly (it prompts you to do this after submitting your work). This significantly streamlines the grading process for our tutors. Failure to do this may result in a score of 0 for any questions that you didn't correctly assign pages to. If you have any questions about the submission process, please don't hesitate to ask on Piazza.

## Collaborators

Data science is a collaborative activity. While you may talk with others about the homework, we ask that you write your solutions individually. If you do discuss the assignments with others please include their names at the top of your submission.

## Properties of Simple Linear Regression

1. (7 points) In Lecture 12, we spent a great deal of time talking about simple linear regression, which you also saw in Data 8. To briefly summarize, the simple linear regression model assumes that given a single observation  $x$ , our predicted response for this observation is  $\hat{y} = \hat{\theta}_0 + \hat{\theta}_1 x$ .

In Lecture 12, we saw that the  $\hat{\theta}_0$  and  $\hat{\theta}_1$  that minimize the average  $L_2$  loss for the simple linear regression model are:

$$\begin{aligned}\hat{\theta}_0 &= \bar{y} - \hat{\theta}_1 \bar{x} \\ \hat{\theta}_1 &= r \frac{\sigma_y}{\sigma_x}\end{aligned}$$

Or, rearranging terms, our predictions  $\hat{y}$  are:

$$\hat{y} = \bar{y} + r \sigma_y \frac{x - \bar{x}}{\sigma_x}$$

(a) (3 points) As we saw in lecture, a residual  $e_i$  is defined to be the difference between a true  $y_i$  and predicted  $\hat{y}_i$ . Specifically,  $e_i = y_i - \hat{y}_i$ .

Prove, using the equation for  $\hat{y}$  above, that  $\sum_{i=1}^n e_i = 0$ .

$$\begin{aligned}\sum_{i=1}^n y_i - \hat{y}_i &= \sum_{i=1}^n y_i - (\bar{y} - \hat{\theta}_1 \bar{x} + \hat{\theta}_1 x_i) \\ &= \sum_{i=1}^n y_i - \sum_{i=1}^n \bar{y} + \sum_{i=1}^n \hat{\theta}_1 \bar{x} - \sum_{i=1}^n \hat{\theta}_1 x_i \\ &= \sum_{i=1}^n y_i - n\bar{y} + n\hat{\theta}_1 \bar{x} - \hat{\theta}_1 \sum_{i=1}^n x_i = \sum_{i=1}^n y_i - n(\bar{y}) + n(\hat{\theta}_1 \sum_{i=1}^n x_i) - \hat{\theta}_1 \sum_{i=1}^n x_i \\ &= \sum_{i=1}^n y_i - \sum_{i=1}^n \hat{y}_i + \hat{\theta}_1 \sum_{i=1}^n x_i - \hat{\theta}_1 \sum_{i=1}^n x_i = 0, \therefore \sum_{i=1}^n e_i = \sum_{i=1}^n y_i - \hat{y}_i = 0\end{aligned}$$

(b) (2 points) Using your result from part a, prove that  $\bar{y} = \hat{y}$ .

$$\begin{aligned}\sum_{i=1}^n y_i - \hat{y}_i &= 0 \\ \sum_{i=1}^n y_i - \sum_{i=1}^n \hat{y}_i &= 0, \sum_{i=1}^n y_i = \sum_{i=1}^n \hat{y}_i \\ \frac{1}{n} \left( \sum_{i=1}^n y_i \right) &= \frac{1}{n} \left( \sum_{i=1}^n \hat{y}_i \right) \rightarrow \bar{y} = \hat{y}\end{aligned}$$

(c) (2 points) Prove that  $(\bar{x}, \bar{y})$  is on the simple linear regression line.

Sub  $\bar{x}$  into regression line:  $\hat{y} = \bar{y} + r \sigma_y \frac{x - \bar{x}}{\sigma_x}$

$$\hat{y} = \bar{y} + r \sigma_y \frac{\bar{x} - \bar{x}}{\sigma_x}, \hat{y} = \bar{y}$$

Subbing in  $\bar{x}$  to the regression line results in  $\hat{y} = \bar{y}$ ,

$\therefore (\bar{x}, \bar{y})$  is on the regression line.  $\blacksquare$

## Geometric Perspective of Least Squares

2. (7 points) In Lecture 13, we viewed both the simple linear regression model and the multiple linear regression model through the lens of linear algebra. The key geometric insight was that if we train a model on some design matrix  $\mathbb{X}$  and true response vector  $\mathbb{Y}$ , our predicted response  $\hat{\mathbb{Y}} = \mathbb{X}\hat{\theta}$  is the vector in  $\text{span}(\mathbb{X})$  that is closest to  $\mathbb{Y}$ .

In the simple linear regression case, our optimal vector  $\theta$  is  $\hat{\theta} = [\hat{\theta}_0, \hat{\theta}_1]^T$ , and our design matrix is

$$\mathbb{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} | & | \\ \mathbf{1} & \vec{x} \\ | & | \end{bmatrix}$$

This means we can write our predicted response vector as  $\hat{\mathbb{Y}} = \mathbb{X} \begin{bmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \end{bmatrix}$ , and also as  $\hat{\mathbb{Y}} = \hat{\theta}_0 \mathbf{1} + \hat{\theta}_1 \vec{x}$ .

Note, in this problem,  $\vec{x}$  refers to the  $n$ -length vector  $[x_1, x_2, \dots, x_n]^T$ . In other words, it is a feature, not an observation.

For this problem, assume we are working with the simple linear regression model, though the properties we establish here hold for any linear regression model that contains an intercept term.

- (a) (3 points) Using the geometric properties from lecture, prove that  $\sum_{i=1}^n e_i = 0$ .  
*Hint: Recall, we define the residual vector as  $e = \mathbb{Y} - \hat{\mathbb{Y}}$ , and  $e = [e_1, e_2, \dots, e_n]^T$ .*

The residual vector  $\vec{e}$  is perpendicular to the column space of  $\mathbb{X}$  as seen in lecture. Since the  $\vec{1}$  vector column is in the span of  $\mathbb{X}$ ,  $\vec{e}$  must also be perpendicular to  $\vec{1}$ ,  $\therefore \vec{e} \cdot \vec{1} = 0$ , or

$$[e_1, e_2, \dots, e_n] \cdot [1, 1, \dots, 1] = 0$$

$$= e_1(1) + e_2(1) + \dots + e_n(1) = 0$$

$$= e_1 + e_2 + \dots + e_n = 0$$

$$= \sum_{i=1}^n e_i = 0 \quad \blacksquare$$

- (b) (2 points) Explain why the vector  $\vec{x}$  (as defined in the problem) and the residual vector  $e$  are orthogonal. Hint: Two vectors are orthogonal if their dot product is 0.

Using geometric properties from lecture,  $\vec{e}$  is orthogonal to  $\text{Span}(X)$ . Since  $\vec{x} \in \text{Span}(X)$ ,  $\vec{e}$  must also be orthogonal to  $\vec{x}$ , and therefore  $\vec{e} \cdot \vec{x} = 0$ .  $\square$

- (c) (2 points) Explain why the predicted response vector  $\hat{Y}$  and the residual vector  $e$  are orthogonal.

The response vector  $\hat{Y} = \hat{\theta}_0 \vec{1} + \hat{\theta}_1 \vec{x}$ .

From part (a) and (b), we know that  $\vec{e} \cdot \vec{1} = 0$ , and  $\vec{e} \cdot \vec{x} = 0$ , so

$$\begin{aligned}\vec{e} \cdot \hat{Y} &= \vec{e} \cdot (\hat{\theta}_0 \vec{1}) + \vec{e} \cdot (\hat{\theta}_1 \vec{x}) \\ &= \hat{\theta}_0 (\vec{e} \cdot \vec{1}) + \hat{\theta}_1 (\vec{e} \cdot \vec{x}) = \hat{\theta}_0 (0) + \hat{\theta}_1 (0) = 0 \\ \vec{e} \cdot \hat{Y} &= 0.\end{aligned}$$

$\therefore$  Since  $\vec{e} \cdot \hat{Y} = 0$ ,  $\vec{e}$  must be perpendicular to the response vector  $\hat{Y}$ .  $\square$

## Properties of a Linear Model With No Constant Term

Suppose that we don't include an intercept term in our model. That is, our model is now simply  $\hat{y} = \gamma x$ , where  $\gamma$  is the single parameter for our model that we need to optimize. (In this equation,  $x$  is a scalar, corresponding to a single observation.)

As usual, we are looking to find the value  $\hat{\gamma}$  that minimizes the average squared loss ("empirical risk") across our observed data  $\{(x_i, y_i)\}, i = 1, \dots, n$ .

$$R(\gamma) = \frac{1}{n} \sum_{i=1}^n (y_i - \gamma x_i)^2$$

The normal equations derived in lecture no longer hold. In this problem, we'll derive a solution to this simpler model. We'll see that the least squares estimate of the slope in this model differs from the simple linear regression model, and will also explore whether or not our properties from the previous problem still hold.

3. (4 points) Use calculus to find the minimizing  $\hat{\gamma}$ . That is, prove that

$$\hat{\gamma} = \frac{\sum x_i y_i}{\sum x_i^2}$$

Note: This is the slope of our regression line, analogous to  $\hat{\theta}_1$  from our simple linear regression model.

$$\begin{aligned}
 \frac{d}{d\gamma} R(\gamma) &= \frac{d}{d\gamma} \left( \frac{1}{n} \sum_{i=1}^n (y_i - \gamma x_i)^2 \right) = \frac{1}{n} \sum_{i=1}^n \frac{d}{d\gamma} (y_i - \gamma x_i)^2 \\
 &= \frac{1}{n} \sum_{i=1}^n 2(y_i - \gamma x_i) \cdot (-x_i) \quad (\text{Use power rule}) \\
 R'(\gamma) &= \frac{1}{n} \sum_{i=1}^n -2x_i(y_i - \gamma x_i) \quad R''(\gamma) = \frac{1}{n} \sum_{i=1}^n -2(-x_i) \\
 0 &= \frac{1}{n} \sum_{i=1}^n -2x_i(y_i - \gamma x_i) \quad R''(\gamma) = \frac{1}{n} \sum_{i=1}^n 2x_i^2 \\
 0 &= \sum_{i=1}^n -2x_i y_i + 2x_i^2 \gamma & R''(\gamma) > 0 \quad \forall \gamma, \\
 0 &= \sum_{i=1}^n -2x_i y_i + \sum_{i=1}^n 2x_i^2 \gamma & R''(\hat{\gamma}) > 0 \\
 -\sum_{i=1}^n 2x_i^2 \gamma &= -\sum_{i=1}^n 2x_i y_i & \text{Therefore,} \\
 -2\gamma \sum_{i=1}^n x_i^2 &= -2 \sum_{i=1}^n x_i y_i \\
 \hat{\gamma} &= \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2} & \hat{\gamma} \text{ is the minimizing } \hat{\gamma}.
 \end{aligned}$$

4. (8 points) For our new simplified model, our design matrix is

$$\mathbb{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} | \\ | \\ | \\ | \end{bmatrix}$$

And so our predicted response vector  $\hat{\mathbb{Y}}$  can be expressed as  $\hat{\mathbb{Y}} = \hat{\gamma}\vec{x}$ . ( $\vec{x}$  here is defined the same way it was in Question 2.)

Earlier in this homework, we established several properties that held true for the simple linear regression model that contained an intercept term. For each of the following four properties, state whether or not they still hold true even when there isn't an intercept term. Be sure to justify your answer.

- (a) (2 points)  $\sum_{i=1}^n e_i = 0$ .

No longer holds true when there isn't an intercept term.  
Without an intercept term,  $\vec{1}$  (vector of length  $n$  with all 1s) is no longer in  $\text{Span}(X)$ . Since  $\vec{1} \notin \text{Span}(x)$ ,  $\vec{e} \cdot \vec{1} = \sum_{i=1}^n e_i \neq 0$ .  $\blacksquare$

- (b) (2 points) The column vector  $\vec{x}$  and the residual vector  $e$  are orthogonal.

Still holds true. The residual vector is always perpendicular to the  $\text{Span}(X)$ . Since  $\vec{x} \in \text{Span}(X)$ ,  $\vec{e}$  is also orthogonal to  $\vec{x}$ .  $\blacksquare$

- (c) (2 points) The predicted response vector  $\hat{\mathbb{Y}}$  and the residual vector  $e$  are orthogonal.

The response vector  $\hat{\mathbb{Y}} = \hat{\gamma}\vec{x}$ , so  $\vec{e} \cdot \hat{\mathbb{Y}} = \vec{e} \cdot (\hat{\gamma}\vec{x})$   
 $\vec{e} \cdot \hat{\mathbb{Y}} = \hat{\gamma}(\vec{e} \cdot \vec{x})$ ,  $\vec{e} \cdot \hat{\mathbb{Y}} = \hat{\gamma}(0)$ ,  $\vec{e} \cdot \hat{\mathbb{Y}} = 0$

$\vec{e} \cdot \hat{\mathbb{Y}} = 0$ , therefore, the response vector  $\hat{\mathbb{Y}}$  and the residual vector  $\vec{e}$  are orthogonal.

- (d) (2 points)  $(\bar{x}, \bar{y})$  is on the regression line.

Sub  $\bar{x}$  into  $\hat{y} = \hat{\gamma}\vec{x}$ ,  $\hat{y} = \hat{\gamma}\bar{x} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2} \left( \frac{1}{n} \sum_{i=1}^n x_i \right)$   
 $\hat{y} = \frac{1}{n} \left( \frac{\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2} \right)$ , this summation does not simplify to  $\bar{y}$ .  
 $\therefore$  subbing in  $\bar{x}$  does not result in  $\bar{y}$ , so  $(\bar{x}, \bar{y})$  is not on the regression line.  $\blacksquare$

