

---

## General Topology, Exercises 2

2019 Spring Semester

---

Youngwan Kim

April 15, 2019

**Problem 1.** Let  $A = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 2\}$ . Find  $\text{int}(A)$ ,  $\bar{A}$  and  $\partial A$ .

*Proof.*

1.  $\text{int}(A) = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 2\}$
2.  $\bar{A} = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}$
3.  $\partial A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ or } x^2 + y^2 = 2\}$

□

**Problem 2.** Let  $A = \{0\} \cup \{1/n : n \in \mathbb{N}\}$  be a subspace of  $\mathbb{R}$ . Find the limit points and the isolated points of  $A$ .

*Proof.*

1. limit points :  $\phi$
2. isolated points :  $A$

□

**Problem 3.** Let  $X$  be a finite metric space. Prove that  $X$  has no limit points.

*Proof.* As  $X$  is finite, we can denote its elements as  $X = \{x_i\}$  for some  $i \in I \subset \mathbb{N}$ . Let  $\epsilon > 0$  be the smallest distance among all of the finite points of  $X$ , i.e, we will let  $\epsilon = \min\{d(x_i, x_j)\}$ . Now for  $\forall x_i \in X$ , consider the open ball  $B(x_i, \epsilon)$ . As for  $\forall x_i$ , there exists such  $\epsilon$  that  $B(x_i, \epsilon) \cap X = \{x_i\}$ , every point of such finite metric space  $X$  is an isolated point. As every point in  $X$  is an isolated point, it implies that  $X$  has no limit points. □

**Problem 4.** Let  $p, q \in X$ . Show that there exists open subsets  $U, V$  of  $X$  such that  $p \in U$  and  $q \in V$  and  $U \cap V = \emptyset$ .

*Proof.* Let  $\epsilon = \frac{1}{N}d(p, q) > 0$  for sufficiently big  $N > 0$ . Consider two open balls  $B(p, \epsilon)$  and  $B(q, \epsilon)$ . As open balls are open, let each open balls as  $U, V$ . Then by definition it is obvious that  $p \in U$  and  $q \in V$ . Now we claim that  $U \cap V = \emptyset$ . If there was such element in  $U \cap V$ , let it  $z$ , then by the triangular identity of  $d$ ,  $d(p, q) \leq d(p, z) + d(z, q)$ . But as  $d(p, z)$  and  $d(q, z)$  are smaller than  $\epsilon < d(p, q)$  it leads to a contradiction that  $d(p, q) < d(p, q)$ . Thus such  $z$  should not exist, and thus we conclude that for such  $U, V$ ,  $U \cap V = \emptyset$ . □

**Problem 5.** Consider  $\mathbb{Q}$  as a subspace of  $\mathbb{R}$ . Find the interior, the closure, the limit points and the isolated points of  $\mathbb{Q}$ .

*Proof.*

1.  $\text{int}(\mathbb{Q}) = \emptyset$

For any  $q \in \mathbb{Q}$ , for any  $r > 0$ , as the open ball  $B(q, r)$  itself is consisted with real numbers, it never gets to be  $B(q, r) \subset \mathbb{Q}$ . Thus  $\text{int}(\mathbb{Q})$  is the empty set.

2.  $\overline{\mathbb{Q}} = \mathbb{R}$

For any  $x \in \mathbb{R}$ , there exists some  $r > 0$  such that  $B(x, r) \cap \mathbb{Q} \neq \emptyset$  due to the property of  $\mathbb{Q}$ . To elaborate, for any  $(a, b) \subset \mathbb{R}$  due to the density of rationals, there exists some  $r \in \mathbb{R}$  such that  $r \in (a, b)$ . Also due to the Archimedean property, there exists some  $n \in \mathbb{N}$  such that  $a < a + \frac{b}{n} < b$ .

3. limit points :  $\mathbb{R}$

As the disjoint union of limit points and isolated points should be  $\overline{\mathbb{Q}}$  and as

4. isolated points :  $\emptyset$

For any  $q \in \mathbb{Q}$ ,

□

**Problem 6.** Prove that the set of irrational numbers  $\mathbb{I}$  is dense in  $\mathbb{R}$ .

*Proof.* It suffices to show that  $\bar{\mathbb{I}} = \mathbb{R}$  by the definition of dense subsets, which is same to show that for  $\forall x \in \mathbb{R}$ , there exists some  $r' > 0$  such that  $B(x, r') \cap \mathbb{I} \neq \emptyset$ . Then showing that for any  $(a, b) \subset \mathbb{R}$ , there exists some  $k \in \mathbb{I}$  that is in  $(a, b)$  would suffice it. Due to the density of rationals, there exists some  $r \in \mathbb{R}$  such that  $r \in (a, b)$ . Using the Archimedean property, as  $\frac{b-r}{2} > 0$ , there exists some  $n \in \mathbb{N}$  such that  $\frac{b-r}{2} > \frac{1}{n}$ , which implies that  $r + \frac{2}{n} < b$ . As there exists some  $k \in \mathbb{I}$  such that  $a < k = r + \frac{\sqrt{2}}{n} < r + \frac{2}{n} < b$  for any  $(a, b) \in \mathbb{R}$ , the closure of  $\mathbb{I}$  is  $\mathbb{R}$ .

□

**Problem 7.** Let  $A = \{0, 1, 2, 3\}$  and  $d_1$  be the usual metric on  $A$  as a subspace of  $\mathbb{R}$ , and  $d_2$  be the discrete metric on  $A$ . Are  $d_1$  and  $d_2$  equivalent?

*Proof.* First let us consider the open sets of  $(A, d_2)$ . As  $d_2$  is a discrete metric, any singleton subset of  $A$  is an open ball. That is because for any  $x \in A$ ,  $B(x, 1) = \{x\}$ . And using the fact that open balls are open and any arbitrary union of open sets are open, we can conclude that every subset of  $A$  is actually open in  $A$ . Now let us consider the open sets of  $(A, d_1)$ . As  $d_1$  is given as the usual metric of  $\mathbb{R}$  and  $A$  is a finite subset of such metric space, we know that every singleton subset of  $A$  is closed in  $A$ . Thus using the fact that any finite intersection of closed sets are still closed, again we lead to the conclusion that every subset of  $A$  is closed. Also using the fact that the complement of a closed set is open, we finally conclude that also in  $(A, d_1)$  every subset of  $A$  is also an open set. Thus we can say that  $d_1$  and  $d_2$  are equivalent.

□

**Problem 8.** Let  $A = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$  be a subspace of  $\mathbb{R}^2$ . Show that it is not a complete space by giving an example of a Cauchy sequence in  $A$  that doesn't converge in  $A$ .

*Proof.* Consider a sequence  $\{x_n\}_{n=1}^{\infty}$  where we define  $x_n = (1 - \frac{1}{n}, 0)$ . As  $x_n \rightarrow (1, 0)$  it doesn't converge in  $A$ . Also  $x_n$  is a Cauchy sequence as there exists  $\epsilon > 0$  and  $N > 0$  such that  $d(x_n, x_m) = |\frac{1}{n} - \frac{1}{m}|$  for every  $n, m \geq N$ . Thus as there exists such  $x_n$  is a Cauchy sequence in  $A$  that doesn't converge in  $A$ , we can conclude that  $A$  is not complete.

□