Konkuk University Dept. Of Physics

General Topology, Exercises 2

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Problem 1. *Let* $A = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 2\}$. *Find* $int(A), \overline{A}$ *and* ∂A.

Proof.

1. $int(A) = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 2\}$
2. $\overline{A} = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}$
3. $\partial A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ or } x^2 + y^2 = 2\}$

$\square$

Problem 2. *Let* $A = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ *be a subspace of* $\mathbb{R}$. *Find the limit points and the isolated points of* A.

Proof.

1. limit points : $\{0\}$
2. isolated points : $A \setminus \{0\}$

$\square$

Problem 3. *Let X be a finite metric space. Prove that X has no limit points.*

Proof. As X is finite, we can denote its elements as $X = \{x_i\}$ for some $i \in I \subset \mathbb{N}$. Let $\epsilon > 0$ be the smallest distance among all of the finite points of X, i.e, we will let $\epsilon = min\{d(x_i, x_j)\}$. Now for $\forall x_i \in X$, consider the open ball $B(x_i, \epsilon)$. As for $\forall x_i$, there exists such ϵ that $B(x_i, \epsilon) \cap X = \{x_i\}$, every point of such finite metric space X is an isolated point. As every point in X is an isolated point, it implies that X has no limit points.

□

Problem 4. *Let $p, q \in X$. Show that there exists open subsets U, V of X such that $p \in U$ and $q \in V$ and $U \cap V = \phi$.*

Proof. Let $\epsilon = \frac{1}{N} d(p, q) > 0$ for sufficiently big $N > 0$. Consider two open balls $B(p, \epsilon)$ and $B(q, \epsilon)$. As open balls are open, let each open balls as U, V. Then by definition it is obvious that $p \in U$ and $q \in V$. Now we claim that $U \cap V = \phi$. If there was such element in $U \cap V$, let it z, then by the trinagular identity of d, $d(p, q) \leq d(p, z) + d(z, q)$. But as $d(p, z)$ and $d(q, z)$ are smaller thatn $\epsilon < d(p, q)$ it leads to a contradiction that $d(p, q) < d(p, q)$. Thus such z should not exist, and thus we conclude that for such U, V, $U \cap V = \phi$.

□

Problem 5. *Consdier $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Find the interior, the closure, the limit points and the isolated points of $\mathbb{Q}$.*

Proof.

1. $int(\mathbb{Q}) = \phi$

   For any $q \in \mathbb{Q}$, for any $r > 0$, as the open ball $B(q, r)$ itself is consisted with real numbers, it never gets to be $B(q, r) \subset \mathbb{Q}$. Thus $int(\mathbb{Q})$ is the empty set.

2. $\overline{\mathbb{Q}} = \mathbb{R}$

   For any $x \in \mathbb{R}$, there exists some $r > 0$ such that $B(x, r) \cap \mathbb{Q} \neq \phi$ due to the property of $\mathbb{Q}$. To elaborate, for any $(a, b) \subset \mathbb{R}$ due to the density of rationals, there exists some $r \in \mathbb{R}$ such that $r \in (a, b)$. Also due to the Archimedean property, there exists some $n \in \mathbb{N}$ such that $a < a + \frac{b}{n} < b$.

3. limit points : $\mathbb{R}$

   As the disjoint union of limit points and isolated points should be $\overline{\mathbb{Q}}$ and as

4. isolated points : ϕ

   For any $q \in \mathbb{Q}$,

□

Problem 6. *Prove that the set of irrational numbers $\mathbb{I}$ is dense in $\mathbb{R}$.*

Proof. It suffices to show that $\bar{\mathbb{I}} = \mathbb{R}$ by the definition of dense subsets, which is same to show that for $\forall x \in \mathbb{R}$, there exists some $r' > 0$ such that $B(x, r') \cap \mathbb{I} \neq \phi$. Then showing that for any $(a, b) \subset \mathbb{R}$, there exists some $k \in \mathbb{I}$ that is in (a, b) would suffice it. Due to the density of rationals, there exists some $r \in \mathbb{R}$ such that $r \in (a, b)$. Using the Archimedean property, as $\frac{b-r}{2} > 0$, there exists some $n \in \mathbb{N}$ such that $\frac{b-r}{2} > \frac{1}{n}$, which implies that $r + \frac{2}{n} < b$. As there exists some $k \in \mathbb{I}$ such that $a < k = r + \frac{\sqrt{2}}{n} < r + \frac{2}{n} < b$ for any $(a, b) \in \mathbb{R}$, the closure of $\mathbb{I}$ is $\mathbb{R}$.

□

Proof. Consider the fact that $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$. Then,

$$\bar{\mathbb{I}} = \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R} \setminus int(\mathbb{Q}) - \mathbb{R}$$

□

Problem 7. *Let $A = \{0, 1, 2, 3\}$ and d_1 be the usual metric on A as a subspace of $\mathbb{R}$, and d_2 be the discrete metric on A. Are d_1 and d_2 equivalent?*

Proof. First let us consider the open sets of (A, d_2). As d_2 is a discrete metric, any singleton subset of A is an open ball. That is because for any $x \in A$, $B(x, 1) = \{x\}$. And using the fact that open balls are open and any arbitrary union of open sets are open, we can conclude that every subset of A is actually open in A. Now let us consider the open sets of (A, d_1). As d_1 is given as the usual metric of $\mathbb{R}$ and A is a finite subset of such metric space, we know that every singleton subset of A is closed in A. Thus using the fact that any finite intersection of closed sets are still closed, again we lead to the conclusion that every subset of A is closed. Also using the fact that the complement of a closed set is open, we finally conclude that also in (A, d_1) every subset of A is also an open set. Thus we can say that d_1 and d_2 are equivalent.

□

Problem 8. *Let $A = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ be a subspace of $\mathbb{R}^2$. Show that it is not a complete space by giving an example of a Cauchy sequence in A that doesn't converge in A.*

Proof. Consider a sequence $\{x_n\}_{n=1}^{\infty}$ where we define $x_n = (1 - \frac{1}{n}, 0)$. As $x_n \to (1, 0)$ it doesn't converge in A. Also x_n is a Cauchy sequence as there exists $\epsilon > 0$ and $N > 0$ such that $d(x_n, x_m) = \left|\frac{1}{n} - \frac{1}{m}\right|$ for every $n, m \geq N$. Thus as there exists such x_n is a Cauchy sequence in A that doesn't converge in A, we can conclude that A is not complete.

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