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General Topology

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1 Metric Spaces

Def 1. *A* ***metric*** d *defined on a set* X *is a mapping* $d : X \times X \to \mathbb{R}$ *which has the following properties.*

(m_1) $\forall x, y \in X : d(x,y) \geq 0$, $d(x,y) = 0 \iff x = y$

(m_2) $\forall x, y \in X : d(x,y) = d(y,x)$

(m_3) $\forall x, y, z \in X : d(x,y) + d(y,z) \geq d(x,z)$

If such map is well defined on a certain set X, we can now introduce the notion of metric spaces, which is merely a set equipped with a well defined metric.

Def 2. *A* ***metric space*** (X, d) *is a set* X *equipped with a metric* $d : X \times X \to \mathbb{R}$.

Ex 1. *These are some simple examples of metric spaces.*

1) $(\mathbb{R}^n, d)$ *where* $d(x,y) =$

2) (X, d) *where* $d(x,y) = 0$ *if* $x = y$ *and* $d(x,y) = 1$ *if* $x \neq y$ ***(discrete metric)***

Now it is natural to think of how the metric would be affected under the subsets, and the following theorem states that the restriction of such metric is still a metric on the subsets.

Thm 1. *Let (X, d) be a metric space and let $Y \subset X$. Then $d \restriction_{Y\times Y}: Y \times Y \to \mathbb{R}$ is a metric on Y.*

Proof. left as an exercise $\square$

Def 3. *Let (X, d) be a metric space and $x \in X$, $r \in \mathbb{R}$ where $r > 0$. The* ***open ball*** *$B(x; r)$ is defined as $\{y \in X : d(x, y) < r\}$ where x is called the* ***center*** *of $B(x; r)$, and r is called the* ***radius*** *of $B(x; r)$.*

Ex 2. *Some basic examples of open balls on different metric spaces.*

1) *On $(\mathbb{R}, d)$ where d is the usual metric, $B(x; r)= (x - r, x + r)$.*
2) *On $(\mathbb{R}, d)$ where d is the discrete metric, $B(x; r)= \{x\}$ if $r \in (0, 1]$ and $B(x; r)= \mathbb{R}$ if $r > 1$.*

Lem 1. *Let (X, d) be a metric space, where $x \in X$ and $r > 0$. Then,*

1. $\bigcup\limits_{r>0} B(x; r)=X$
2. $\bigcap\limits_{r>0} B(x; \frac{1}{r})=\{x\}$

Proof.

1. $\bigcup\limits_{r>0} B(x; r)=X$
   - (i) As $B(x; r) \subset X$ for any r by definition, it is obvious that $\bigcup\limits_{r>0} B(x; r) \subset X$.
   - (ii) Let $y \in X$, then $\exists n \in \mathbb{N}$ such that $d(x, y) < n$. Then $y \in B(x; n)$ where $B(x; n) \subset \bigcup\limits_{r>0} B(x; r)$. Thus $y \in B(x; n) \subset \bigcup\limits_{r>0} B(x; r)$, which implies that $\bigcup\limits_{r>0} B(x; r) \supset X$.
2. $\bigcap\limits_{r>0} B(x; \frac{1}{r})=\{x\}$
   - (i) As for $\forall r > 0$, $x \in B(x; r)$ which makes $\{x\} \subset \bigcap\limits_{r>0} B(x; r)$ obvious.

(ii) Suppose $x \neq y$ for $y \in \bigcap_{r>0} B(x;r)$. Then since $d(x,y) > 0$, there exists some $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < d(x,y)$. Then $y \notin B(x; \frac{1}{n})$, which contradicts our assumption that $x \neq y$. Thus $\forall y \in \bigcap_{r>0} B(x;r) : y = x$, which implies that $\{x\} \supset \bigcap_{r>0} B(x;r)$.

$\square$

Def 4. *Let (X,d) be a metric space where $Y \subseteq X$ and $r \in \mathbb{R}$. We define,*

1. *For $x \in Y$, x is an* ***interior point*** *of Y if $\exists r > 0$ such that $B(x;r) \subset Y$.*
2. *The* ***interior of Y****, where we denote it as $int(A) = \{y \in Y | \exists r > 0 : B(y;r) \subset Y\}$.*
3. *If $int(Y) = Y$, we say that Y is* ***open in X****, or an* ***open subset of X****.*

By defining the concept of open balls, we just introduced the concept of open subsets of a certain metric space. Let's take a look at some examples to get close with these concepts.

Ex 3.

1) *Consider the trivial case, $X \subseteq X$*
2) *For $[0,1) \subset \mathbb{R}$*