

Algebraic Topology

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1 INTRODUCTION

If you ever had a course in topology before, you must have had assignments about whether some topological spaces are homeomorphic. For instance, the open interval $(0, 1)$ in \mathbb{R} is homeomorphic with \mathbb{R} . We can solve certain 'exercise problems' as they state us that they are 'homeomorphic' at first and all we have to do is just to find an adequate homeomorphism that maps one to another. But life is way much complicated than that, think of a situation where we have to actually check if two topological spaces are not homeomorphic. For instance, just with basic topological background it is very hard to show that T^2 and S^2 are not homeomorphic.

Algebraic topology gains certain significance regarding these kinds of situations where some topological spaces are given and we have to show that they are not homeomorphic. We implement algebraic methods to show such properties. From now on we will show that a topological space can be 'mapped' into a certain group, and if two topological spaces are homeomorphic those groups are also isomorphic. Algebraic topology can be broadly classified into two main topics : Homotopy theory and Homology theory, and the main goal of this course is to understand the fundamental group of topological spaces and show that $\pi_1(S^1) = \mathbb{Z}$. We will not go through homology and cohomology theory.

2 HOMOTOPIC PATHS

We start our step by defining homotopic paths, which will later help us define fundamental groups. I skipped the documentation of the notes regarding basic group theory.

Def 1 (Paths).

Let X be a topological space. A **path** in X is a continuous map $\gamma : [0, 1] \rightarrow X$.

There is nothing more than that. A path is simply a continuous map with the domain of $[0, 1]$.

Def 2 (Homotopic Paths and Homotopy).

Let X be a topological space and $a, b \in X$. Also let γ_0, γ_1 be paths from a to b , i.e., $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$. Then γ_0 is **homotopic** to γ_1 with end points fixed if there exists a continuous map $F : [0, 1] \times [0, 1] \rightarrow X$ such that,

1. $F(s, 0) = \gamma_0(s)$ for $s \in [0, 1]$
2. $F(s, 1) = \gamma_1(s)$ for $s \in [0, 1]$
3. $F(0, t) = a$ for $t \in [0, 1]$
4. $F(1, t) = b$ for $t \in [0, 1]$

where we call such F a **homotopy**. We denote homotopic paths as $\gamma_0 \simeq \gamma_1$, *rel* $\{0, 1\}$.

The third and fourth condition stated for such map being a homotopy is to emphasize that the end points are fixed and we even denote it as *rel* $\{0, 1\}$. Homotopy can be easily thought as some kind of continuous map deforming, or a 'movie' showing one path smoothly becoming the other one.

Lem 1. Homotopic relations are equivalence relations.

Proof. We will just show that homotopic relations are transitive. Let $\gamma_0 \simeq \gamma_1$ by a homotopy F and $\gamma_1 \simeq \gamma_2$ by a homotopy G where all homotopy are *rel* $\{0, 1\}$. We will show that there exists a homotopy $H : [0, 1] \times [0, 1] \rightarrow X$ such that $\gamma_0 \simeq \gamma_2$, *rel* $\{0, 1\}$. We define such H by,

$$H(s, t) = \begin{cases} F(s, 2t) & t \in [0, \frac{1}{2}) \\ G(s, 2t - 1) & t \in [\frac{1}{2}, 1] \end{cases}$$

then we can show that,

$$\begin{aligned} H(s, 0) &= F(s, 0) = \gamma_0(s) \\ H(s, 1) &= G(s, 1) = \gamma_2(s) \end{aligned}$$

and,

$$\begin{aligned} H(0, t) &= a \\ H(1, t) &= b \end{aligned}$$

which suffices all of the four conditions to be a homotopy between γ_0 and γ_2 . □

Recall. By the gluing lemma, the continuity of H is guaranteed, but we won't elaborate it here.

As we've seen that homotopy relation is actually an equivalence relation, it is natural to think of the equivalence classes that such relation makes.

Def 3. For a topological space X , where γ is a path defined in X , we define the **homotopy class** of γ as

$$[\gamma] = \{\gamma' : [0, 1] \rightarrow X \text{ s.t. } \gamma \simeq \gamma'\}$$

Homotopy classes will be the basic notion for structuring fundamental groups.

Remark. By definition, $\gamma_0 \simeq \gamma_1 \iff [\gamma_0] = [\gamma_1]$

Def 4. For a topological space X , where $a \in X$, we denote the constant path $\gamma : [0, 1] \rightarrow X$ which maps $\gamma(s) = a$ for $\forall s \in [0, 1]$, by just a .

Lem 2 (Independence of Reparametrization).

Let $\gamma : [0, 1] \rightarrow X$ be a path such that $\gamma(0) = a$ and $\gamma(1) = b$. Also let $\rho : [0, 1] \rightarrow [0, 1]$ be a continuous function such that $\rho(0) = 0$ and $\rho(1) = 1$. Then $[\gamma] = [\gamma \circ \rho]$.

Proof. Define $F : [0, 1] \times [0, 1] \rightarrow X$ by $F(s, t) = \gamma((1-t) \cdot s + t \cdot \rho(s))$. Then $F(s, 0) = \gamma(s)$, $F(s, 1) = \gamma(\rho(s))$, $F(0, t) = a$, and $F(1, t) = b$. Thus for any ρ , F suffices to be a homotopy between γ and $\gamma \circ \rho$ thus $[\gamma] = [\gamma \circ \rho]$. □

Def 5. We define the collection of paths with fixed points and denote it as :

$$P_X(a, b) = \{\gamma : [0, 1] \rightarrow X : \gamma(0) = a, \gamma(1) = b\}$$

That collection looks lonely without a proper operation between paths, so let's define one.

Def 6 (Concatenation).

Let $\alpha \in P_X(a, b)$ and $\beta \in P_X(b, c)$. Then we define the **concatenation** of α and β and denote it as $\alpha\beta : [0, 1] \rightarrow X$ by,

$$\alpha\beta(s) = \begin{cases} \alpha(2s) & s \in [0, \frac{1}{2}) \\ \beta(2s - 1) & s \in [\frac{1}{2}, 1] \end{cases}$$

where $\alpha\beta \in P_X(a, c)$.

Lem 3 (Homotopy Invariance). Let $\alpha_0, \alpha_1 \in P_X(a, b)$ such that $[\alpha_0] = [\alpha_1]$. Let $\beta_0, \beta_1 \in P_X(b, c)$ such that $[\beta_0] = [\beta_1]$. Then $[\alpha_0\beta_0] = [\alpha_1\beta_1]$.

Proof. We need to show that there exists $H : [0, 1] \times [0, 1] \rightarrow X$ such that $\alpha_0\beta_0 \simeq \alpha_1\beta_1$, *rel* $\{0, 1\}$. Let F, G each be the homotopy such that $\alpha_0 \simeq \alpha_1$ and $\beta_0 \simeq \beta_1$ all *rel* $\{0, 1\}$. If we define $H(s, t)$ by,

$$H(s, t) = \begin{cases} F(2s, t) & s \in [0, \frac{1}{2}) \\ G(2s - 1, t) & s \in [\frac{1}{2}, 1] \end{cases}$$

we can easily check that this H suffices to be a homotopy between $\alpha_0\beta_0$ and $\alpha_1\beta_1$. □

Remark. Due to homotopy invariance, we can define $[\alpha][\beta] = [\alpha\beta]$.

Lem 4. Let X be a topological space. For $\alpha \in P_X(a, b)$, $\beta \in P_X(b, c)$, and $\gamma \in P_X(c, d)$,

$$([\alpha][\beta])[\gamma] = [\alpha]([\beta][\gamma]).$$

This lemma states that such operation over homotopy classes is associative, and we also wish there is an identity like object for this class. We will later show that the homotopy class of constant paths actually acts as an identity element for such operation.

Remark. $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ are different paths, but homotopic thanks to **Lem 2**. Do not regard them as identical 'paths'.

Lem 5. Let $\alpha \in P_X(a, b)$. Then $[\alpha][\alpha] = [\alpha]$ and $[\alpha][b] = [\alpha]$ where,

$$a\alpha(s) = \begin{cases} \alpha(2s) & s \in [0, \frac{1}{2}) \\ \alpha(2s - 1) & s \in [\frac{1}{2}, 1] \end{cases}$$

Def 7. Let $\alpha \in P_X(a, b)$. Then we define the inverse path of α as $\alpha^{-1} : [0, 1] \rightarrow X$ which maps $\alpha^{-1}(s) = \alpha(1 - s)$.

Remark. We can see that for $\alpha \in P_X(a, b)$ the inverse of it $\alpha^{-1} \in P_X(b, a)$.

Lem 6. Let $\alpha \in P_X(a, b)$. Then $[\alpha][\alpha^{-1}] = [a]$ and $[\alpha^{-1}][\alpha] = [b]$.

Proof. We will just show the case that $[\alpha][\alpha^{-1}] = [a]$ by taking a homotopy F as,

$$F(s, t) = \begin{cases} \alpha(2s) & s \in [0, \frac{t}{2}) \\ \alpha(t) & s \in [\frac{t}{2}, 1 - \frac{t}{2}) \\ \alpha(2 - 2s) & s \in [1 - \frac{t}{2}, 1] \end{cases}$$

We can then check that F becomes a homotopy between a and $\alpha\alpha^{-1}$. □

Lem 7. Let $\alpha_0, \alpha_1 \in P_X(a, b)$. If $\alpha_0 \simeq \alpha_1$, rel $\{0, 1\}$ then $\alpha_0^{-1} \simeq \alpha_1^{-1}$, rel $\{0, 1\}$.

Proof. Let F be the homotopy between α_0 and α_1 . We define $G(s, t) = F(1 - s, t)$ and then this becomes a homotopy between α_0^{-1} and α_1^{-1} . □

Def 8. Let $\alpha \in P_X(a, b)$, then we define the inverse homotopy class of α by $[\alpha]^{-1} = [\alpha^{-1}]$.

3 THE FUNDAMENTAL GROUP

Throughout the previous section, we defined many concepts that will be essential for us to define the fundamental group of a topological space. Now using the concepts from the previous section, now we define the concept of fundamental group.

Def 9. For a topological space X with $b \in X$, we define a set named $\pi_1(X, b)$ of X by

$$\begin{aligned} \pi_1(x, b) &= L(X, b) / \simeq \\ &= \{[\alpha] \mid \alpha : [0, 1] \rightarrow X, \text{ s.t } \alpha(0) = \alpha(1) = b\} \end{aligned}$$

where $L(X, b)$ is the set of loops in X having $b \in X$ as a base point.

Thm 1. If we equip the set $\pi_1(x, b)$ with the operation $[\alpha][\beta] = [\alpha\beta]$, $\pi_1(X, b)$ becomes a group.

Recall. A topological space X is **path connected** if for $\forall a, b \in X$, there exists a path $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

Thm 2. Let X be a path connected topological space. Then $\pi_1(X, b) \simeq \pi_1(X, c)$ for any $b, c \in X$.

Proof. Define $\gamma : [0, 1] \rightarrow X$ be a path such that $\gamma(0) = c$ and $\gamma(1) = b$. Existence of such path is guaranteed due to the fact that X is path connected. Then define a mapping $\phi : \pi_1(X, b) \rightarrow \pi_1(X, c)$, such that

$$\phi([\alpha]) = [\gamma][\alpha][\gamma]^{-1} = [\gamma\alpha\gamma^{-1}]$$

We will then show that such ϕ is an isomorphism between $\pi_1(X, b)$ and $\pi_1(X, c)$.

1) ϕ is injective.

Suppose $\phi([\alpha]) = \phi([\beta]) \implies [\gamma][\alpha][\gamma]^{-1} = [\gamma][\beta][\gamma]^{-1}$ for any $[\alpha]$ and $[\beta]$. Then,

$$\begin{aligned} [\gamma][\alpha][\gamma]^{-1} &= [\gamma][\beta][\gamma]^{-1} \\ [\gamma]^{-1}[\gamma][\alpha][\gamma]^{-1}[\gamma] &= [\gamma]^{-1}[\gamma][\beta][\gamma]^{-1}[\gamma] \\ [b][\alpha][c] &= [b][\beta][c] \\ [\alpha] &= [\beta] \end{aligned}$$

2) ϕ is surjective.

Let arbitrary $[\alpha'] \in \pi_1(X, c)$. Consider $\phi([\gamma^{-1}\alpha'\gamma])$,

$$\begin{aligned} \phi([\gamma^{-1}\alpha'\gamma]) &= [\gamma][\gamma^{-1}\alpha'\gamma][\gamma]^{-1} \\ &= [c][\alpha'][c] = [\alpha'] \end{aligned}$$

3) ϕ is a homomorphism.

For any $[\alpha], [\beta] \in \pi_1(X, b)$,

$$\begin{aligned} \phi([\alpha][\beta]) &= [\gamma][\alpha][\beta][\gamma]^{-1} \\ &= [\gamma][\alpha][b][\beta][\gamma]^{-1} \\ &= [\gamma][\alpha][\gamma]^{-1}[\gamma][\beta][\gamma]^{-1} \\ &= \phi([\alpha])\phi([\beta]) \end{aligned}$$

As ϕ is a bijective homomorphism between $\pi_1(X, b)$ and $\pi_1(X, c)$, it is an isomorphism. \square

Def 10. As *Thm 2* states that for path connected topological spaces π_1 is independent of the choice of base point, we define the **fundamental group** of a path connected topological space as the isomorphic type of $\pi_1(X, b)$ and denote it as $\pi_1(X)$.

Def 11. A topological space X is **simply connected** if X is path connected and the fundamental group is trivial, i.e., $\pi_1(X) = 0$.

Ex 1. Here are some examples of simply connected and not simply connected spaces.

- Simply connected : $\pi_1(\mathbb{R}^n) = 0$, $\pi_1(D^n) = 0$, $\pi_1(S^n) = 0$ (for $n > 1$) ...
- Not simply connected : $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(\mathbb{R} \setminus \{0\}) = \mathbb{Z}$...

Ex 2. We will explicitly show that \mathbb{R}^2 is simply connected.

Let the base point $b = (0, 0) \in \mathbb{R}^2$. Also let $\alpha : (0, 1] \rightarrow \mathbb{R}^2$ be a loop based at b . Define $F : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^2$ by

$$F(s, t) = t \cdot b + (1 - t) \cdot \alpha$$

which eventually becomes a homotopy such that $\alpha \simeq b$ for any loop based at b . This implies that $\pi_1(X, b) = \{[b]\}$, which is the trivial group.

From the above example, we took advantage of the convex property of \mathbb{R}^2 when we constructed the homotopy F . This means that we could apply the same method of proof to any convex spaces.

Def 12. A subset S of \mathbb{R}^n is **convex** if for $\forall a, b \in S$, the line segment from a to b is contained in S , i.e., $\forall t \in [0, 1]$ the line segment $t \cdot a + (1 - t) \cdot b \in S$.

Ex 3. Some basic examples of convex and non convex subsets of \mathbb{R}^n .

- \mathbb{R}^n itself is convex.
- $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is convex.
- $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is not convex.

Then as we expect, does any convex subset becomes a simply connected space? The next theorem states that it is true.

Thm 3. *A convex subset S of \mathbb{R}^n is simply connected.*

Proof. Let $b \in S$, and $\alpha : [0, 1] \rightarrow S$ be a loop based at b . Define $F : [0, 1] \times [0, 1] \rightarrow S$ as $F(s, t) = t \cdot b + (1 - t) \cdot \alpha$ which becomes a homotopy such that $\alpha \simeq b$. The existence of such homotopy is guaranteed as S is a convex subset. As for any loop based at b becomes homotopic with b , $\pi_1(S) = 0$ thus S is simply connected. \square

4 INDUCED HOMOMORPHISMS

5 COVERING SPACES