

General Topology, Exercises 1

2019 Spring Semester

Youngwan Kim

April 11, 2019

Problem 1. Let $A = \{(x, y) \in \mathbb{R}^2 | 1 < x^2 + y^2 \leq 2\}$. Find $\text{int}(A)$, \bar{A} and ∂A .

Proof.

□

Problem 2. Let $A = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ be a subspace of \mathbb{R} . Find the limit points and the isolated points of A .

Proof.

1. limit points : ϕ
2. isolated points : A

□

Problem 3. Let X be a finite metric space. Prove that X has no limit points.

Proof. As X is finite, we can denote its elements as $X = \{x_i\}$ for some $i \in I \subset \mathbb{N}$. Let $\epsilon > 0$ be the smallest distance among all of the finite points of X , i.e, we will let $\epsilon = \min\{d(x_i, x_j)\}$. Now for $\forall x_i \in X$, consider the open ball $B(x_i, \epsilon)$. As for $\forall x_i$, there exists such ϵ that $B(x_i, \epsilon) \cap X = \{x_i\}$, every point of such finite metric space X is an isolated point. As every point in X is an isolated point, it implies that X has no limit

points.

□

Problem 4. Let $p, q \in X$. Show that there exists open subsets U, V of X such that $p \in U$ and $q \in V$ and $U \cap V = \emptyset$.

Proof. Let $\epsilon = \frac{1}{N}d(p, q) > 0$ for sufficiently big $N > 0$. Consider two open balls $B(p, \epsilon)$ and $B(q, \epsilon)$. As open balls are open, let each open balls as U, V . Then by definition it is obvious that $p \in U$ and $q \in V$. Now we claim that $U \cap V = \emptyset$. If there was such element in $U \cap V$, let it z , then by the triangular identity of d , $d(p, q) \leq d(p, z) + d(z, q)$. But as $d(p, z)$ and $d(q, z)$ are smaller than $\epsilon < d(p, q)$ it leads to a contradiction that $d(p, q) < d(p, q)$. Thus such z should not exist, and thus we conclude that for such U, V , $U \cap V = \emptyset$.

□

Problem 5. Consider \mathbb{Q} as a subspace of \mathbb{R} . Find the interior, the closure, the limit points and the isolated points of \mathbb{Q} .

Proof.

1. $\text{int}(\mathbb{Q}) = \emptyset$

For any $q \in \mathbb{Q}$, for any $r > 0$, as the open ball $B(q, r)$ itself is consisted with real numbers, it never gets to be $B(q, r) \subset \mathbb{Q}$. Thus $\text{int}(\mathbb{Q})$ is the empty set.

2. $\overline{\mathbb{Q}} = \mathbb{R}$

For any $x \in \mathbb{R}$, there exists some $r > 0$ such that $B(x, r) \cap \mathbb{Q} \neq \emptyset$ due to the property of \mathbb{Q} . To elaborate, for any $(a, b) \subset \mathbb{R}$ due to the density of rationals, there exists some $r \in \mathbb{R}$ such that $r \in (a, b)$. Also due to the Archimedean property, there exists some $n \in \mathbb{N}$ such that $a < r + \frac{1}{n} < b$.

3. limit points : \mathbb{R}

4. isolated points : \emptyset

For any $q \in \mathbb{Q}$,

□

Problem 6. Prove that the set of irrational numbers \mathbb{I} is dense in \mathbb{R} .

Proof. It suffices to show that $\overline{\mathbb{I}} = \mathbb{R}$ by the definition of dense subsets, which is same to show that for $\forall x \in \mathbb{R}$, there exists some $r' > 0$ such that $B(x, r') \cap \mathbb{I} \neq \emptyset$. Then showing that for any $(a, b) \subset \mathbb{R}$, there exists some $k \in \mathbb{I}$ that is in (a, b) would suffice it. Due to the density of rationals, there exists some $r \in \mathbb{R}$ such that $r \in (a, b)$. Using the

Archimedean property, as $\frac{b-r}{2} > 0$, there exists some $n \in \mathbb{N}$ such that $\frac{b-r}{2} > \frac{1}{n}$, which implies that $r + \frac{2}{n} < b$. As there exists some $k \in \mathbb{I}$ such that $a < k = r + \frac{\sqrt{2}}{n} < r + \frac{2}{n} < b$ for any $(a, b) \in \mathbb{R}$, the closure of \mathbb{I} is \mathbb{R} . □

Problem 7. Let $A = \{0, 1, 2, 3\}$ and d_1 be the usual metric on A as a subspace of \mathbb{R} , and d_2 be the discrete metric on A . Are d_1 and d_2 equivalent?

Proof. First let us consider the open sets of (A, d_2) . As d_2 is a discrete metric, any singleton subset of A is an open ball. That is because for any $x \in A$, $B(x, 1) = \{x\}$. And using the fact that open balls are open and any arbitrary union of open sets are open, we can conclude that every subset of A is actually open in A . Now let us consider the open sets of (A, d_1) . As □