Konkuk University Dept. Of Physics

General Topology, Exercises 3

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Problem 1. *Is $\mathbb{Z}$ compact? Are $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ compact?*

Proof. As all of them are subsets of $\mathbb{R}$ we could apply Heine-Borel to check if such spaces are compact or not.

1. $\mathbb{Z}$ is not compact.

   $\mathbb{Z}$ is not bounded.

2. $\mathbb{Q}$ is not compact.

   As we know that $\overline{\mathbb{Q}} = \mathbb{R}$, it is not closed. Thus it is not compact.

3. $\mathbb{R} \setminus \mathbb{Q}$ is not compact.

   Again as $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$, it is not compact.

□

Problem 2. *Show that a finite metric space is compact.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a finite metric space $X = \{a_1, \dots, a_N\}$. Then for at least one element a_i of X, the sequence would take it as a value finitely many times. If not, it implies that for every finite element of X it only appears finite times in the sequence, which contradicts the assumption that the sequence is infinite. Thus for any sequences in a finite metric space, we can find a converging subsequence, thus X is compact. □

Problem 3. *Is $S^1 \subset \mathbb{R}^2$ compact?*

Proof. Yes S^1 is compact. As it is a subspace of $\mathbb{R}^2$ we can apply Heine-Borel. S^1 is bounded as we can see that for any $x, y \in S^1$, there exists some $\epsilon > 0$ such that $d(x, y) \leq 2$. Also consider a continuous function $F(x, y) = x^2 + y^2 - 1$. As $\{0\}$ is a closed set in $\mathbb{R}^2$, $F^{-1}(0)$ should also be a closed set too. Here $F^{-1}(0) = (x, y) \in \mathbb{R}^2 : a^2 + y^2 = 1 = S^1$, thus we can also see that S^1 is closed too. Thus S^1 is compact. $\square$

Problem 4. *Is $B^2 \subset \mathbb{R}^2$ compact? Also is $int(B^2)$ compact?*

Proof. Again we can apply Heine-Borel.

1. B^2 is compact.

   First B^2 is closed as $\overline{B^2} = B^2$. B^2 is also bounded as for any $x, y \in B^2$, there exists $\epsilon > 0$ such that $d(x, y) < 2 + \epsilon$.

2. $int(B^2)$ is not compact.

   $int(B^2) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is bounded as for any $x, y \in int(B^2)$, $d(x, y) < 2$ but it is not closed as $\overline{int(B^2)} = \phi \neq B^2$.

$\square$

Problem 5. *Let $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$. Is A compact?*

Proof. A is closed but not bounded, thus A is not compact. $\square$

Problem 6. *Let $A = \{(x, y) \in \mathbb{R}^2 : x \neq 0$ and $y = 1/x\}$. Is A compact?*

Proof. A is not compact. $\square$

Problem 7. *Show that TFAE. Recall that a metric space X is bounded if there exists $b > 0$ such that $d(x, y) < b$ for all $x, y \in X$.*

1) X is bounded.

2) There exists $b > 0$ such that $d(x, y) \leq b$ for all $x, y \in X$.

3) There exists $p \in X$ and $r > 0$ such that $X \subset B(p, r)$.

4) There exists $p \in X$ and $r > 0$ such that $d(p, x) \leq r$ for all $x \in X$.

Proof.

1. 1) $\implies$ 2)

   $d(x, y) < b \implies d(x, y) \leq b$

2. 2) $\implies$ 3)

   For such b of assumption, we can see that $X \in B(p, b + \epsilon)$.

3. 3) $\implies$ 4)

   $d(p,x) < r \implies d(p,x) \leq r$

4. 4) $\implies$ 1)

   $d(x,y) \leq d(x,p) + d(p,y) \leq 2r$ then $^\exists \epsilon > 0$ such that $d(x,y) < 2r + \epsilon$.

$\square$

Problem 8. *Show that if A and B are bounded subsets of X then $A \cup B$ and $A \cap B$ are also bounded.*

Proof. Let $r_A, r_B > 0$ such that each for any $x, y \in A$, $d(x,y) < r_A$ and for any $x, y \in B$, $d(x,y) < r_B$.

1. $A \cap B$

   As $A \cap B \subset A$ and $A \cap B \subset B$ and A, B are bounded $A \cap B$ is also bounded.

2. $A \cup B$ For arbitrary $x, y \in A \cup B$, where $a \in A$ and $b \in B$ is fixed,

$$d(x,y) \leq d(x,a) + d(a,b) + d(b,y) \leq r_1 + d(a,b) + r_2$$

$\square$

Problem 9. *Let X be a metric space with the discrete metric. Show that X is compact iff X is finite.*

Proof. Suppose X is infinite. Then $\{\{x\}\}_{x \in X}$ is an open cover of X since discrete metric is given. But this open cover couldn't have a finite subcover as if we suppose $\{\{x_1\}, \ldots, \{x_n\}\}$ as a finite subcover of $\{\{x\}\}_{x \in X}$ we can easily check that such finite subcover doesn't actually covers X. Thus our initial assumption of X being infinite is wrong. $\square$

Problem 10. *Prove or disprove : If A_1 and A_2 are compact subspaces of a metric space X, then $A_1 \cup A_2$ is also compact.*

Proof. Consider a open cover $\{U_\alpha\}_{\alpha \in A}$ that covers $A_1 \cup A_2$. As A_1 and A_2 are each a subset of $A_1 \cup A_2$, $\{U_\alpha\}_{\alpha \in A}$ is also a cover of $A_1 \cup A_2$. Also as both are compact, there exists a finite subcover of $\{U_\alpha\}_{\alpha \in A}$, which implies that $\exists \alpha_1, \alpha_2, \ldots \alpha_N \in A$ such that $A_1 = \bigcup\limits_{i=1}^{N} U_{\alpha_i}$ and $\exists \beta_1, \ldots, \beta_M \in A$ such that $A_2 = \bigcup\limits_{j=1}^{M} U_{\beta_j}$. Then we can say that there exists a finite subcover of $\{U_{\alpha_i}\} \cup \{U_{\beta_j}\}$ of $\{U_\alpha\}_{\alpha \in A}$, such that $A_1 \cup A_2 = \bigcup\limits_{i=1}^{N} U_{\alpha_i} \cup \bigcup\limits_{j=1}^{M} U_{\beta_j}$. Thus $A_1 \cup A_2$ is also compact. $\square$

Problem 11. *Prove or disprove : If A_1 and A_2 are compact subspaces of a metric space X, then $A_1 \cap A_2$ is also compact.*

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be a open cover of $A_1 \cap A_2$. Consider $X \setminus A_2$ as a subset of X. As A_2 is a compact subset of X, it is complete, which implies it is closed in X thus the complement $X \setminus A_2$ is open in X. Then we can now consider another open cover $\{U_\alpha\}_{\alpha\in A} \cup \{X \setminus A_2\}$ which covers A_1. Since A_1 is compact there exists a finite subcover $\{U_{\alpha_i}\}_{i=1}^N \cup \{X \setminus A_2\}$ for $^\exists\alpha_1, \dots, \alpha_N \in A$ such that $A_1 \subset \bigcup\limits_{i=1}^{N} U_{\alpha_i} \cup (X \setminus A_2)$. As $A_1 \cap A_2 \subset A_1$, we can also conclude that :

$$A_1 \cap A_2 \subset \bigcup_{i=1}^{N} U_{\alpha_i} \cup (X \setminus A_2)$$

But notice that $(A_1 \cap A_2) \cap (X \setminus A_2) = \phi$, which implies that :

$$A_1 \cap A_2 \subset \bigcup_{i=1}^{N} U_{\alpha_i}$$

which directly shows us that there exists a finite subcover $\{U_{\alpha_i}\}_{i=1}^N$ for $A_1 \cap A_2$. ☐

Problem 12. *Let $\mathcal{B} = \{(a, \infty) : a \in \mathbb{R}\}$. Is $\mathcal{B}$ a base of open sets in $\mathbb{R}$ with the usual metric?*

Proof. No. Consider $0 \in \mathbb{R}$ and a neighborhood of it, $(-\epsilon, \epsilon)$. Then $\nexists V \in \mathcal{B}$ such that $0 \in V \subset (-\epsilon, \epsilon)$, thus $\mathcal{B}$ fails to be a basis of $\mathbb{R}$. ☐

Problem 13. *Let X be a metric space with the discrete metric. Find all possible bases of open sets in X.*

Proof. First we claim that for such circumstances $\mathcal{B}$ is a basis iff for $\forall x \in X : \{x\} \in \mathcal{B}$. The only if statement is pretty obvious as the discrete metric is given, any subset of X is an open set in X. Now for the other direction let us assume that for some $x_0 \in X$, $\{x_0\} \notin \mathcal{B}$. Then for $\{x_0\}$ which is an open set in X, $\nexists V \in \mathcal{B}$ such that $x_0 \in V \subset \{x_0\}$, which implies that such $\mathcal{B}$ fails to be a basis. Thus as our assumption faced a contradiction, it is shown that $\forall x \in X : \{x\} \in \mathcal{B}$ for $\mathcal{B}$ in order to be a base. ☐