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Quantum Field Theory And Condensed Matter

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1. Thermodynamics and Statistical Mechanics Review

1.1.

For an ideal gas, show that

$$T(S,V) = \frac{\partial U}{\partial S} = \frac{2}{3nR}U$$

to obtain a function of T. Next construct $F(T,V) = U(T) - S(T,V)T$ and show that,

$$F(T,V) = \frac{3nRT}{2}\Big[(1+\ln C) - \ln\frac{3nRT}{2} - \frac{2}{3}\ln V\Big].$$

Verify that the partial derivatives with respect to T and V give the expected results for the entropy and pressure for an ideal gas.

Sol)

Starting from the definition of U,

$$
\begin{aligned}
U &= C\left(\frac{e^{S/nR}}{V}\right)^{2/3} \\
T = \frac{\partial U}{\partial S} &= \frac{C}{V^{2/3}}\frac{\partial}{\partial S}e^{2S/3nR} \\
&= \frac{C}{V^{2/3}}\frac{2}{3nR}e^{2S/3nR} = \frac{2}{3nR}U.
\end{aligned}
\tag{1.1}
$$

1.2.

Evaluate $Z(\beta)$ for the classical oscillator.

Sol)

$$
\begin{aligned}
Z(\beta) &= \int_{+\infty}^{-\infty}\int_{+\infty}^{-\infty} e^{-\beta\left(\frac{p^2}{2m}+\frac{1}{2}m\omega_0^2x^2\right)}\,\mathrm{d}p\,\mathrm{d}x \\
&= \int_{+\infty}^{-\infty} e^{-\frac{\beta}{2m}p^2}\,\mathrm{d}p \int_{+\infty}^{-\infty} e^{-\frac{\beta}{2}m\omega_0^2x^2}\,\mathrm{d}x \\
&= \sqrt{\frac{2m\pi}{\beta}}\sqrt{\frac{2\pi}{m\omega_0^2\beta}} \\
&= \frac{2\pi}{\omega_0\beta}
\end{aligned}
\tag{1.2}
$$

Thus for a classical oscillator,

$$
Z(\beta)_c = \frac{2\pi}{\omega_0\beta} \tag{1.3}
$$

1.3.

Evaluate $Z(\beta)$ for the quantum oscillator.

Sol)

For the quantum case, consider the Hamiltonian of the osciallator.

$$
H(X,P) = \frac{P^2}{2m} + \frac{1}{2}m\omega_0^2X^2 \tag{1.4}
$$

The eigenvalues E_n of $H(X,P)$ can be derived as

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0 \tag{1.5}$$

The partition function $Z(\beta)$ is,

$$\begin{aligned} Z(\beta) &= \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega_0} \\ &= e^{-\beta\frac{1}{2}\hbar\omega_0}\sum_{n=0}^{\infty} e^{-\beta n\hbar\omega_0} \\ &= e^{-\beta\frac{1}{2}\hbar\omega_0}\left(\frac{1}{1-e^{-\beta\hbar\omega_0}}\right) \\ &= \left[\sinh\left(\frac{1}{2}\beta\hbar\omega_0\right)\right]^{-1} \end{aligned} \tag{1.6}$$

Thus for the quantum oscillator,

$$Z(\beta)_q = \left[\sinh\left(\frac{1}{2}\beta\hbar\omega_0\right)\right]^{-1} \tag{1.7}$$

1.4.

Evaluate $\langle E\rangle$ for the classical and quantum oscillators. Show that they become equal in the appropriate limit.

Sol)

For the classical oscillator,

$$\begin{aligned} \langle E\rangle_c &= -\frac{\partial \ln Z_c}{\partial\beta} \\ &= -\frac{\partial}{\partial\beta}[\ln 2\pi - \ln(\omega_0\beta)] \\ &= \frac{\partial}{\partial\beta}\ln(\omega_0\beta) = \beta^{-1} \end{aligned} \tag{1.8}$$

For the quantum oscillator,

$$
\begin{aligned}
\langle E\rangle_q &= -\frac{\partial \ln Z_q}{\partial\beta} \\
&= \frac{\partial}{\partial\beta}\ln\left(\sinh\left(\frac{1}{2}\beta\omega_0\hbar\right)\right) \\
&= \frac{1}{2}\omega_0\hbar\left[\tanh\left(\frac{1}{2}\beta\omega_0\hbar\right)\right]^{-1}
\end{aligned} \tag{1.9}
$$

Consider $\hbar\sim 0$ limit,

$$
\langle E\rangle_q \sim \frac{1}{2}\omega_0\hbar\left(\frac{1}{2}\beta\omega_0\hbar\right)^{-1} = \beta^{-1} \tag{1.10}
$$

Thus under the limit $\hbar\sim 0$, $\langle E\rangle_q \sim \langle E\rangle_c$.

1.5.

Apply the grand canonical ideas to a system which is a quantum state of energy ϵ that may be occupied by fermions(bosons). Show that in the two cases,

$$
\langle N\rangle = n_{F/B} = \frac{1}{e^{\beta(\epsilon-\mu)}\pm 1}.
$$

These averages are commonly referred to with lower case symbols as n_F and n_B respectively.

Sol)

First, let us consider the grand partition function of both fermionic and bosonic systems. Let it $\mathcal{Z}_F$ and $\mathcal{Z}_B$ respectively. As bosons are indistinguishable and any number can occupy a certain energy state, we could write $\mathcal{Z}_B$ as

$$
\mathcal{Z}_B = \sum_{N=0}^{\infty} e^{N\beta(\mu-\epsilon)} = \frac{1}{1-e^{\beta(\mu-\epsilon)}} \tag{1.11}
$$

A fermionic system will only have 2 states due to the Pauli exclusion theorem, so Z_F would be expressed as,

$$
\mathcal{Z}_F = e^{0\cdot\beta(\mu-\epsilon)} + e^{1\cdot\beta(\mu-\epsilon)} = e^{\beta(\mu-\epsilon)} + 1 \tag{1.12}
$$

Using the relation between $\langle N \rangle$ and $\mathcal{Z}$,

$$\langle N \rangle = \frac{1}{\beta \mathcal{Z}} \left(\frac{\partial \mathcal{Z}}{\partial \mu} \right) = \frac{1}{\beta} \frac{\partial \ln \mathcal{Z}}{\partial \mu} \tag{1.13}$$

For the fermionic system $\mathcal{Z}_F$,

$$n_F = \frac{1}{\beta} \left(\frac{\partial \ln \mathcal{Z}_F}{\partial \mu} \right) = \frac{1}{e^{\beta(\epsilon - \mu)} + 1} \tag{1.14}$$

For the bosonic sysytem $\mathcal{Z}_B$,

$$n_B = \frac{1}{\beta} \left(\frac{\partial \ln \mathcal{Z}_B}{\partial \mu} \right) = \frac{1}{e^{\beta(\epsilon - \mu)} - 1} \tag{1.15}$$

Thus for fermionic(bosonic) quantum systems, $\langle N \rangle$ can be expressed as

$$n_{F/B} = \frac{1}{e^{\beta(\epsilon - \mu)} \pm 1} \tag{1.16}$$

2. The Ising Model in $d = 0$ and $d = 1$

2.1.

Show that

$$\frac{\partial^2 [-\beta F(K, h_1, h_2)]}{\partial h_1 \partial h_2} = \langle s_1 s_2 \rangle - \langle s_1 \rangle \langle s_2 \rangle$$

Sol)

As $Z = e^{-\beta F}$,

$$\begin{aligned} \frac{\partial^2 [-\beta F(K, h_1, h_2)]}{\partial h_1 \partial h_2} &= \frac{\partial^2 \ln Z}{\partial h_1 \partial h_2} \\ &= \frac{1}{Z} \frac{\partial^2 Z}{\partial h_1 \partial h_2} - \frac{1}{Z^2} \frac{\partial Z}{\partial h_1} \frac{\partial Z}{\partial h_2} \end{aligned} \tag{2.1}$$

Also as $Z = \sum e^{Ks_1s_2+h_1s_1+h_2s_2}$,

$$\begin{aligned}
\frac{1}{Z}\frac{\partial^2 Z}{\partial h_1 \partial h_2} &= \frac{1}{Z}\frac{\partial}{\partial h_1}\left(\frac{\partial Z}{\partial h_2}\right) \\
&= \frac{1}{Z}\frac{\partial}{\partial h_1}\left(\sum_{s_1,s_2} s_2 e^{Ks_1s_2+h_1s_1+h_2s_2}\right) \\
&= \frac{1}{Z}\left(\sum_{s_1,s_2} s_1 s_2 e^{Ks_1s_2+h_1s_1+h_2s_2}\right) = \langle s_1 s_2\rangle
\end{aligned} \tag{2.2}$$

The second term,

$$\begin{aligned}
\frac{1}{Z^2}\frac{\partial Z}{\partial h_1}\frac{\partial Z}{\partial h_2} &= \left(\frac{1}{Z}\frac{\partial Z}{\partial h_1}\right)\left(\frac{1}{Z}\frac{\partial Z}{\partial h_2}\right) \\
&= \frac{1}{Z}\left(\sum_{s_1,s_2} s_1 e^{Ks_1s_2+h_1s_1+h_2s_2}\right)\frac{1}{Z}\left(\sum_{s_1,s_2} s_2 e^{Ks_1s_2+h_1s_1+h_2s_2}\right) \\
&= \langle s_1\rangle\langle s_2\rangle
\end{aligned} \tag{2.3}$$

Thus ,

$$\langle s_1 s_2\rangle_c \equiv \frac{\partial^2[-\beta F(K,h_1,h_2)]}{\partial h_1 \partial h_2} = \langle s_1 s_2\rangle - \langle s_1\rangle\langle s_2\rangle \tag{2.4}$$

2.2.

Show that

$$\begin{aligned}
\langle s_1 s_2 s_3 s_4\rangle_c &\equiv \frac{\partial^4(-\beta F)}{\partial h_1 \partial h_2 \partial h_3 \partial h_4}\Big|_{h=0} \\
&= \langle s_1 s_2 s_3 s_4\rangle - \langle s_1 s_2\rangle\langle s_3 s_4\rangle - \langle s_1 s_3\rangle\langle s_2 s_4\rangle - \langle s_1 s_4\rangle\langle s_2 s_3\rangle
\end{aligned}$$

Sol)

The partition function for four spins with their source terms can be written as

$$Z = \sum_{s_1,s_2,s_3,s_4} e^{K(s_1s_2s_3s_4)+h_1s_1+h_2s_2+h_3s_3+h_4s_4} \tag{2.5}$$

We know that $s_i \in \{+1,-1\}$ and the source terms were originally

$$h(s_1+s_2+s_3+s_4) = h_1s_1+h_2s_2+h_3s_3+h_4s_4 \tag{2.6}$$

so for the $h = 0$ condition, there shouldn't be

2.3.

Derive $\langle s_1 s_2 \rangle$ and discuss its K dependence at $h = 0$.

Sol)

Recall that $Z = \sum e^{K(s_1 s_2)+h(s_1+s_2)} = 2e^K \cosh(2h) + 2e^{-K}$.

$$\begin{aligned}
\langle s_1 s_2 \rangle &= \frac{1}{Z} \sum_{s_1, s_2} s_1 s_2 e^{K(s_1 s_2)+h(s_1+s_2)} \\
&= \frac{1}{Z} \frac{\partial Z}{\partial K} = \frac{\partial \ln Z}{\partial K} \\
&= \frac{\partial}{\partial K} \left[\ln(2e^K \cosh(2h) + 2e^{-K}) \right] \\
&= \frac{e^K \cosh(2h) - e^{-K}}{e^K \cosh(2h) + e^{-K}}
\end{aligned} \tag{2.7}$$

Considering the $h = 0$ case,

$$\begin{aligned}
\langle s_1 s_2 \rangle \big|_{h=0} &= \frac{e^K - e^{-K}}{e^K + e^{-K}} \\
&= \frac{\sinh(K)}{\cosh(K)} = \tanh(K)
\end{aligned} \tag{2.8}$$

Thus at $h = 0$, K dependence of $\langle s_1 s_2 \rangle$ shows up to be $\tanh(K)$.

2.4.

Show that if you use the rates specified by the Metropolis algorithm, you get

$$\frac{p(i)}{p(j)} = e^{-\beta(E_i - E_j)}$$

Given a computer that can generate a random number between 0 and 1, how will you accept a jump with probability $e^{-\beta(E_i - E_j)}$?

Sol)

Let us first assume that the energy levels of two states i and j is given as $E_j > E_i$. At equilibrium, we will have $\frac{dp(i)}{dt} = 0$.

3. Statistical to Quantum Mechanics

3.1.

(i). Consider $U(x, x'; \tau)$ for the oscillator as $\tau \to \infty$ and read off the ground-state wave-function and energy. Compare to what you learned as a child. Given this, try to pull out the next state from the subdominant terms.

(ii). Set $x = x' = 0$ and extract the energies from $U(\tau)$ in Eq. (3.24). Why are some energies missing?

Sol)

For an arbitrary initial state, $|\psi(0)\rangle$

$$\lim_{\tau\to\infty} \langle x|U(\tau)|x'\rangle|\psi(0)\rangle = \lim_{\tau\to\infty} U(x, x'; \tau)|\psi(0)\rangle = \psi_0(x)\psi_0(x')^*|\psi(0)\rangle e^{-\frac{1}{\hbar}E_0\tau} \tag{3.1}$$

Also for a quantum harmonic oscillator we know that $U(x, x'; \tau)$ is given as

$$\begin{aligned} U(x, x'; \tau) &= \sqrt{\frac{m\omega}{2\pi\hbar\sinh\omega\tau}} \exp\left[-\frac{m\omega}{2\pi\hbar\sinh\omega\tau}((x^2 + x'^2)\cosh\omega\tau - 2xx')\right] \\ &= \sqrt{\frac{m\omega}{2\pi\hbar}}\sqrt{\frac{1}{\sinh\omega\tau}} \exp\left[-\frac{m\omega}{2\pi\hbar}((x^2 + x'^2)\frac{1}{\tanh\omega\tau} - 2\frac{xx'}{\sinh\omega\tau})\right] \end{aligned} \tag{3.2}$$

Considering the limit of $\tau \to \infty$, where $\sinh\omega\tau \to \frac{1}{2}e^{\omega\tau}$ and $\tanh\omega\tau \to 1$

$$\begin{aligned} \lim_{\tau\to\infty} U(x, x'; \tau) &= \lim_{\tau\to\infty} \sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{1}{e^{\omega\tau}}} \exp\left[-\frac{m\omega}{2\pi\hbar}((x^2 + x'^2) - \frac{4xx'}{e^{\omega\tau}})\right] \\ &= \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left[-\frac{m\omega}{2\pi\hbar}((x^2 + x'^2) - \frac{1}{2}\omega\tau)\right] \end{aligned} \tag{3.3}$$

Comparing the result with (3.1), terms with τ should be identical

$$e^{-\frac{1}{\hbar}E_0\tau} = e^{-\frac{1}{2}\omega\tau} \tag{3.4}$$

resulting the ground state energy to be $E_0 = \frac{1}{2}\hbar\omega$. For the ground state wave function,

$$\begin{aligned}\psi_0(x)\psi_0(x')^* &= \sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{2\pi\hbar}(x^2+x'^2)\right] \\ &= \left[\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}\right]\left[\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x'^2}{2\hbar}}\right]\end{aligned} \tag{3.5}$$

Thus we can conclude that the ground state wave function $\psi_0(x)$ is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} \tag{3.6}$$

Setting $x = x' = 0$

3.2.

Check that the free energy per site is the same as above for periodic boundary conditions starting with

$$Z = \mathrm{Tr}\ T^N. \tag{3.7}$$

Sol)

Let's assume λ_0, λ_1 as the eigenvalues of T and their respective eigenvectors as $|0\rangle, |1\rangle$. We can expand T and express T^N as,

$$T = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1|\ ,\ T^N = \lambda_0^N |0\rangle\langle 0| + \lambda_1^N |1\rangle\langle 1| \tag{3.8}$$

and it tells us that the eigenvalues for T^N are λ_0^N, λ_1^N. Using the periodic boundary condition $s_0 = s_N$ for the $d = 1$ Ising model,

$$Z = \mathrm{Tr}\ T^N = \lambda_0^N + \lambda_1^N \tag{3.9}$$

Let us consider $\lambda_0 > \lambda_1$, and calculate the free energy per site under thermodynamic limits.

$$
\begin{aligned}
-f &\sim \lim_{N\to\infty} \frac{1}{N} \ln Z \\
&= \lim_{N\to\infty} \frac{1}{N} \ln \left[\lambda_0^N \left(1 + \mathcal{O}\big(\lambda_1/\lambda_0\big)^N \right) \right] \\
&= \lim_{N\to\infty} \left[\ln \lambda_0 + \frac{1}{N} \mathcal{O}\big(\lambda_1/\lambda_0\big)^N \right] = \ln \lambda_0
\end{aligned}
\tag{3.10}
$$

It shows up to be that the free energy per site f is the same whether we choose fixed or periodic boundary conditions for the $d = 1$ Ising model.

3.3.

Show that if you invert

$$
\tanh K^*(K) = e^{-2K} \tag{3.11}
$$

you find $\tanh K = e^{-2K^*}$.

Sol)

Starting from $\tanh K^* = e^{-2K}$

$$
\begin{aligned}
\tanh K^* = \frac{e^{2K^*} - 1}{e^{2K^*} + 1} &= e^{-2K} \\
1 - \frac{2}{e^{2K^*} + 1} &= e^{-2K} \\
\frac{2}{e^{2K^*} + 1} &= 1 - e^{-2K} \\
e^{2K^*} + 1 &= \frac{2}{1 - e^{-2K}} \\
e^{2K^*} &= \frac{2}{1 - e^{-2K}} - 1 = \frac{1 + e^{-2K}}{1 - e^{-2K}} \\
e^{-2K^*} &= \frac{e^K - e^{-K}}{e^K + e^{-K}} = \tanh K
\end{aligned}
\tag{3.12}
$$

3.4.

Verify that when the correlation function of Eq.(3.61) is inserted into Eq.(3.62), the power law prefactor drops out in the determination of ξ.

Sol)

Eq.(3.61) states that

$$\lim_{|j-i|\to\infty} \langle s_i s_j \rangle_c \sim \frac{e^{-|j-i|/\xi}}{|j-i|^{d-2+\eta}} \tag{3.13}$$

Inserting this into Eq.(3.62),

$$\begin{aligned}
\lim_{|j-i|\to\infty} \left[-\frac{\ln\langle s_i s_j\rangle_c}{|j-i|} \right] &= \lim_{|j-i|\to\infty} \left[-\frac{1}{|j-i|} \ln\left(\frac{e^{-|j-i|/\xi}}{|j-i|^{d-2+\eta}} \right) \right] \\
&= \lim_{|j-i|\to\infty} \frac{1}{|j-i|} \left[\frac{|j-i|}{\xi} + (d-2+\eta)\ln|j-i| \right] \\
&= \lim_{|j-i|\to\infty} \left[\frac{1}{\xi} + (d-2+\eta)\frac{\ln|j-i|}{|j-i|} \right] \\
&= \xi^{-1}
\end{aligned} \tag{3.14}$$

3.5.

Show that f in *Eq.(3.74)* agrees with *Eq.(2.31)* upon subtracting $\ln\cosh K^*$ and using the definition of K^*.

Sol)

To avoid confusion let f in Eq.(3.74) as $f' = -K^*$. By the definition of K^*, we know that $\tanh K = e^{-2K^*}$.

$$\begin{aligned}
f' - \ln\cosh K^* &= -K^* - \ln\cosh K^* \\
&=
\end{aligned} \tag{3.15}$$

3.6.

(Very Important) Find the eigenvalues of T in *Eq. (3.78)* and show that there is degeneracy only for $h = K^* = 0$. Why does this degeneracy not violate the Perron-Frobenius theorem? Show that the magnetization is

$$\langle s\rangle = \sinh h / \left(\sqrt{\sinh^2 h + e^{-4K}} \right)$$

in the thermodynamic limit.

3.7.

Consider the correlation function for the $h = 0$ problem with periodic boundary conditions and write it as a ratio of two traces. Saturate the denominator with the largest eigenket, but keep both eigenvectors in the numerator and show that the answer is invariant under $j - i \leftrightarrow N - (j - i)$. Using the fact that σ_3 exchanes $|0\rangle$ and $|1\rangle$ should speed things up. Argue that as long as $|j - i|$ is much smaller than N, only one term is needed.

3.8.

Recall the remarkable fact that the correlation function $\langle s_j s_i \rangle$

4. Quantum to Statistical Mechanics

4.1.

Write $T_{ss'} = e^{R(s,s')}$, and expand the function R in a series, allowing all possible powers of s and s'. Show that $R(s, s') = Ass' + B(s + s') + C$ follows, given that $s^2 = s'^2 = 1$ and both $T_{ss'}$ and R are symmetric under $s \leftrightarrow s'$.

4.2.

(Important)

(i). Solve *Eq.(4.9)* for K, h and c terms of B_1, B_3 and ε by choosing $s = s' = \pm 1$ and $s = -s'$, and show that

$$
\begin{aligned}
\tanh h &= \frac{B_3}{B} \tanh \varepsilon B \\
e^{-2K+c} &= \frac{B_1}{B} \sinh \varepsilon B \\
e^2 c \left(1 - e^{-4K}\right) &= 1
\end{aligned}
$$

4.3.

Verify that the exact results in *Eqs. (4.10) - (4.12)* from *Exercise 4.2* reduce to *Eqs. (4.20) and (4.21)* and $c = 0$ to $\mathcal{O}(\varepsilon)$.

Sol)

content...

5. The Feynman Path Integral

5.1.

Derive *Eq. (5.8)* by introducing a resolution of the identity in terms of momentum states between the exponential operator and the position eigenket on the left-hand side of *Eq. (5.8)*.

Sol)

content...

5.2.

To show that the higher derivatives can be ignored in the limit $\alpha \to \infty$, first shift the origin to $x = x_0$ and estimate the average value of the new x^2 in terms of α. Argue that a factor $e^{\alpha x^3}$ will be essentially equal to 1 in the region where the Gaussian integral has any real support.

Sol)

content...

5.3.

(Important) Consider the oscillator. First solve for a trajectory that connects the spacetime points $(x, 0)$ to (x', t) by choosing the two free parameters in its solution. (Normally you choose them given the initial position and velocity; now you want the solution to go through the two end points in spacetime.) Then find its action and show that

$$U(x', x; t) = A(t)\left[\frac{im\omega}{2\hbar\sin\omega t}\left[(x^2 + x'^2)\cos\omega t - 2x'x\right]\right]$$

If you want $A(t)$ you need to modify the trick used for the free particle,since the exponential is not a Gaussian in $(x - x')$. Note, however, that if you choose $x = 0$, it becomes a Gaussian in x' which allows you to show that $A(t) = (m\omega/2\pi i\hbar\sin\omega t)^{1/2}$. That the answer for the fluctuation integral A does not depend on the classical path, especially the end ponts x or x', is a propert of the quadratic action. It amounts in our toy model to the fact that if $F(x)$ is quadrati, that is $F = ax^2 + bx + c$, then $F'' = 2a$ for all x, including the minimum.

Sol)

content...

5.4.

Include the kinetic energy $p^2/2m$ of the atoms in the Boltzmann wight and do the p integrals as part of Z. Obtain an overall factor $(\sqrt{2\pi mkT})^{N-1}$ and show that the statistical properties of the x's are still given by the Z in *Eq.(5.44)*.

Sol)

content...

5.5.

(Very Important) Assume that

$$H = \sum_{1}^{N} E_0 \left|n\right\rangle \left\langle n\right| - t \left(\left\langle n\right| \left|n+1\right\rangle + \left\langle n+1\right| \left|n\right\rangle \right)$$

describes the low-energy Hamiltonian of a particle in a periodic potential with minima at integers n. The integers n go from 1 to N since it is assumed that the world is a ring of length N, so that the $(N+1)$th point is the ïňĄrst. Thus, the problem has symmetry under translation by one site despite the ïňĄnite length of the world. The ïňĄrst term in H represents the energy of the Gaussian state centered at $x = n$

Sol)

content...