Konkuk University Dept. Of Physics

Algebraic Topology

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1 Introduction

If you ever had a course in topology before, you must have had assignments about whether some topological spaces are homeomorphic. For instance, the open interval $(0, 1)$ in $\mathbb{R}$ is homeomorphic with $\mathbb{R}$. We can solve certain 'exercise problems' as they state us that they are 'homeomorphic' at first and all we have to do is just to find an adequate homeomorphism that maps one to another. But life is way much complicated than that, think of a situation where we have to actually check if two topological spaces are not homeomorphic. For instance, just with basic topological background it is very hard to show that T^2 and S^2 are not homeomorphic.

Algebraic topology gains certain significance regarding these kinds of situations where some topological spaces are given and we have to show that they are not homeomorphic. We implement algebraic methods to show such properties. From now on we will show that a topological space can be 'mapped' into a certain group, and if two topological spaces are homeomorphic those groups are also isomorphic. Algebraic topology can be broadly classified into two main topics : Homotopy theory and Homology theory, and the main goal of this course is to understand the fundamental group of topological spaces and show that $\pi_1(S^1) = \mathbb{Z}$. We will not go through homology and cohomology theory.

2 Homotopic Paths

We start our step by defining homotopic paths, which will later help us define fundamental groups. I skipped the documentation of the notes regarding basic group theory.

Def 1 (Paths)**.**
Let X be a topological space. A ***path*** *in X is a continuous map $\gamma : [0,1] \to X$.*

There is nothing more than that. A path is simply a continuous map with the domain of $[0,1]$.

Def 2 (Homotopic Paths and Path Homotopy)**.**
Let X be a topological space and $a, b \in X$. Also let γ_0, γ_1 be paths from a to b, i.e, $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$. Then γ_0 is ***path homotopic*** *to γ_1 with end points fixed if there exists a continuous map $F : [0,1] \times [0,1] \to X$ such that,*

1. *$F(s,0) = \gamma_0(s)$ for $s \in [0,1]$*
2. *$F(s,1) = \gamma_1(s)$ for $s \in [0,1]$*
3. *$F(0,t) = a$ for $t \in [0,1]$*
4. *$F(1,t) = b$ for $t \in [0,1]$*

where we call such F a ***homotopy****. We denote homotopic paths as $\gamma_0 \simeq \gamma_1$, rel $\{0,1\}$.*

The third and fourth condition stated for such map being a homotopy is to emphasize that the end points are fixed and we even denote it as *rel* $\{0,1\}$. Homotopy can be easily thought as some kind of continuous map deforming, or a 'movie' showing one path smoothly becoming the other one.

Lem 1. *Homotopic relations are equivalence relations.*

Proof. We will just show that homotopic relations are transitive. Let $\gamma_0 \simeq \gamma_1$ by a homotopy F and $\gamma_1 \simeq \gamma_2$ by a homotopy G where all homotopy are *rel* $\{0,1\}$. We will show that there exists a homotopy $H : [0,1] \times [0,1] \to X$ such that $\gamma_0 \simeq \gamma_2$, *rel* $\{0,1\}$. We define such H by,

$$H(s,t) = \begin{cases} F(s,2t) & t \in [0,\frac{1}{2}) \\ G(s,2t-1) & t \in [\frac{1}{2},1] \end{cases}$$

then we can show that,

$$H(s,0) = F(s,0) = \gamma_0(s)$$
$$H(s,1) = G(s,1) = \gamma_2(s)$$

and,

$$H(0,t) = a$$
$$H(1,t) = b$$

which suffices all of the four conditions to be a homotopy between γ_0 and γ_2.

□

Recall. *By the gluing lemma, the continuity of H is guranteed, but we won't elaborate it here.*

As we've seen that homotopy relation is actually an equivalence relation, it is natural to think of the equivalence classes that such relation makes.

Def 3. *For a topological space X, where γ is a path defined in X, we define the* ***homotopy class*** *of γ as*

$$[\gamma] = \{\gamma' : [0,1] \to X \ \ s.t \ \ \gamma \simeq \gamma'\}$$

Homotopy classes will be the basic notion for structuring fundamental groups.

Remark. *By definition,* $\gamma_0 \simeq \gamma_1 \iff [\gamma_0] = [\gamma_1]$

Def 4. *For a topological space X, where $a \in X$, we denote the constant path $\gamma : [0,1] \to X$ which maps $\gamma(s) = a$ for $\forall s \in [0,1]$, by just a.*

Lem 2 (Independence of Reparametrization)**.**
Let $\gamma : [0,1] \to X$ be a path such that $\gamma(0) = a$ and $\gamma(1) = b$. Also let $\rho : [0,1] \to [0,1]$ be a continuous function such that $\rho(0) = 0$ and $\rho(1) = 1$. Then $[\gamma] = [\gamma \circ \rho]$.

Proof. Define $F : [0,1]\times[0,1] \to X$ by $F(s,t) = \gamma((1-t)\cdot s + t\cdot\rho(s))$. Then $F(s,0) = \gamma(s)$, $F(s,1) = \gamma(\rho(s))$, $F(0,t) = a$, and $F(1,t) = b$. Thus for any ρ, F suffices to be a homotopy between γ and $\gamma \circ \rho$ thus $[\gamma] = [\gamma \circ \rho]$.

□

Def 5. *We define the collection of paths with fixed points and denote it as :*

$$P_X(a,b) = \{\gamma : [0,1] \to X : \gamma(0) = a, \gamma(1) = b\}$$

That collection looks lonely without a proper operation between paths, so let's define one.

Def 6 (Concatenation)**.**
Let $\alpha \in P_X(a,b)$ and $\beta \in P_X(b,c)$. Then we define the ***concatenation*** *of α and β and denote it as $\alpha\beta : [0,1] \to X$ by,*

$$\alpha\beta(s) = \begin{cases} \alpha(2s) & s \in [0, \frac{1}{2}) \\ \beta(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

where $\alpha\beta \in P_X(a,c)$.

Lem 3 (Homotopy Invariance)**.** *Let $\alpha_0, \alpha_1 \in P_X(a,b)$ such that $[\alpha_0] = [\alpha_1]$. Let $\beta_0, \beta_1 \in P_X(b,c)$ such that $[\beta_0] = [\beta_1]$. Then $[\alpha_0\beta_0] = [\alpha_1\beta_1]$.*

Proof. We need to show that there exists $H : [0,1] \times [0,1] \to X$ such that $\alpha_0\beta_0 \simeq \alpha_1\beta_1$, *rel* $\{0,1\}$. Let F, G each be the homotopy such that $\alpha_0 \simeq \alpha_1$ and $\beta_0 \simeq \beta_1$ all *rel* $\{0,1\}$. If we define $H(s,t)$ by,

$$H(s,t) = \begin{cases} F(2s,t) & s \in [0, \frac{1}{2}) \\ G(2s-1,t) & s \in [\frac{1}{2}, 1] \end{cases}$$

we can easily check that this H suffices to be a homotopy between $\alpha_0\beta_0$ and $\alpha_1\beta_1$.

$\square$

Remark. *Due to homotopy invariance, we can define $[\alpha][\beta] = [\alpha\beta]$.*

Lem 4. *Let X be a topological space. For $\alpha \in P_X(a,b)$, $\beta \in P_X(b,c)$, and $\gamma \in P_X(c,d)$,*

$$([\alpha][\beta])[\gamma] = [\alpha]([\beta][\gamma]).$$

This lemma states that such operation over homotopy classes is associative, and we also wish there is an identity like object for this class. We will later show that the homotopy class of constant paths actually acts as an identity element for such operation.

Remark. *$(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ are different paths, but homotopic thanks to* ***Lem 2****. Do not regard them as identical 'paths'.*

Lem 5. *Let $\alpha \in P_X(a,b)$. Then $[a][\alpha] = [\alpha]$ and $[\alpha][b] = [\alpha]$ where,*

$$a\alpha(s) = \begin{cases} \alpha(2s) \\ \alpha(2s-1) \end{cases}$$

Def 7. *Let $\alpha \in P_X(a,b)$. Then we define the inverse path of α as $\alpha^{-1} : [0,1] \to X$ which maps $\alpha^{-1}(s) = \alpha(1-s)$.*

Remark. *We can see that for $\alpha \in P_X(a,b)$ the inverse of it $\alpha^{-1} \in P_X(b,a)$.*

Lem 6. *Let $\alpha \in P_X(a,b)$. Then $[\alpha][\alpha^{-1}] = [a]$ and $[\alpha^{-1}][\alpha] = [b]$.*

Proof. We will just show the case that $[\alpha][\alpha^{-1}] = [a]$ by taking a homotopy F as,

$$F(s,t) = \begin{cases} \alpha(2s) & s \in [0, \frac{t}{2}) \\ \alpha(t) & s \in [\frac{t}{2}, 1 - \frac{t}{2}) \\ \alpha(2-2s) & s \in [1 - \frac{t}{2}, 1] \end{cases}$$

We can then check that F becomes a homotopy between a and $\alpha\alpha^{-1}$.

□

Lem 7. *Let $\alpha_0, \alpha_1 \in P_X(a,b)$. If $\alpha_0 \simeq \alpha_1$, rel $\{0,1\}$ then $\alpha_0^{-1} \simeq \alpha_1^{-1}$, rel $\{0,1\}$.*

Proof. Let F be the homotopy between α_0 and α_1. We define $G(s,t) = F(1-s,t)$ and then this becomes a homotopy bewtween α_0^{-1} and α_1^{-1}.

□

Def 8. *Let $\alpha \in P_X(a,b)$, then we define the inverse homotopy class of α by $[\alpha]^{-1} = [\alpha^{-1}]$.*

3 The Fundamental Group

Throughout the previous section, we defined many concepts that will be essential for us to define the fundamental group of a topological space. Now using the concepts from the previous section, now we define the concept of fundamental group.

Def 9. *For a topological space X with $b \in X$, we define a set named $\pi_1(X,b)$ of X by*

$$\begin{aligned}\pi_1(x,b) &= L(X,b)/\simeq \\ &= \{[\alpha] \mid \alpha : [0,1] \to X,\ s.t\ \alpha(0) = \alpha(1) = b\}\end{aligned}$$

where $L(X,b)$ is the set of loops in X having $b \in X$ as a base point.

Thm 1. *If we equip the set $\pi_1(x,b)$ with the operation $[\alpha][\beta] = [\alpha\beta]$, $\pi_1(X,b)$ becomes a group.*

Recall. *A topological space X is* ***path connected*** *if for $\forall a, b \in X$, there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$.*

Thm 2. *Let X be a path connected topological space. Then $\pi_1(X,b) \simeq \pi_1(X,c)$ for any $b, c \in X$.*

Proof. Define $\gamma : [0,1] \to X$ be a path such that $\gamma(0) = c$ and $\gamma(1) = b$. Existence of such path is guranteed due to the fact that X is path connected. Then define a mapping $\phi : \pi_1(X,b) \to \pi_1(X,c)$, such that

$$\phi([\alpha]) = [\gamma][\alpha][\gamma]^{-1} = [\gamma\alpha\gamma^{-1}]$$

We will then show that such ϕ is an isomorphism between $\pi_1(X,b)$ and $\pi_1(X,c)$.

1) ϕ is injective.

   Suppose $\phi([\alpha])$=$\phi([\beta]) \implies [\gamma][\alpha][\gamma]^{-1} = [\gamma][\beta][\gamma]^{-1}$ for any $[a]$ and $[\beta]$. Then,

$$\begin{aligned}[\gamma][\alpha][\gamma]^{-1} &= [\gamma][\beta][\gamma]^{-1} \\ [\gamma]^{-1}[\gamma][\alpha][\gamma]^{-1}[\gamma] &= [\gamma]^{-1}[\gamma][\beta][\gamma]^{-1}[\gamma] \\ [b][\alpha][c] &= [b][\beta][c] \\ [\alpha] &= [\beta]\end{aligned}$$

2) ϕ is surjective.

   Let arbitrary $[\alpha'] \in \pi_1(X,c)$. Consider $\phi([\gamma^{-1}\alpha'\gamma])$,

$$\begin{aligned}\phi([\gamma^{-1}\alpha'\gamma]) &= [\gamma][\gamma^{-1}\alpha'\gamma][\gamma^{-1}] \\ &= [c][\alpha'][c] = [\alpha']\end{aligned}$$

3) ϕ is a homomorphism.

For any $[\alpha], [\beta] \in \pi_1(X, b)$,

$$\begin{aligned}\phi([\alpha][\beta]) &= [\gamma][\alpha][\beta][\gamma]^{-1} \\ &= [\gamma][\alpha][b][\beta][\gamma]^{-1} \\ &= [\gamma][\alpha][\gamma]^{-1}[\gamma][\beta][\gamma]^{-1} \\ &= \phi([\alpha])\phi([\beta])\end{aligned}$$

As ϕ is a bijective homomorphism between $\pi_1(X, b)$ and $\pi_1(X, c)$, it is an isomorphism. □

Def 10. *As **Thm 2** states that for path connected topological spaces π_1 is independent of the choice of base point, we define the **fundamental group** of a path connected topological space as the isomorphic type of $\pi_1(X, b)$ and denote it as $\pi_1(X)$.*

Def 11. *A topolgical space X is **simply connected** if X is path connected and the fundamental group is trivial, i.e, $\pi_1(X) = 0$.*

Ex 1. *Here are some examples of simply connected and not simply connected spaces.*

- *Simply connected : $\pi_1(\mathbb{R}^n) = 0$, $\pi_1(D^n) = 0$, $\pi_1(S^n) = 0$ (for $n > 1$) ...*
- *Not simply connected : $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(\mathbb{R} \setminus \{0\}) = \mathbb{Z}$ ...*

Ex 2. *We will explicitly show that $\mathbb{R}^2$ is simply connected.*

Let the base point $b = (0, 0) \in \mathbb{R}^2$. Also let $\alpha : (0, 1] \to \mathbb{R}^2$ be a loop beased at b. Define $F : [0, 1] \times [0, 1] \to \mathbb{R}^2$ by

$$F(s, t) = t \cdot b + (1 - t) \cdot \alpha$$

which eventually becomes a homotopy such that $\alpha \simeq b$ for any loop based at b. This implies that $\pi_1(X, b) = \{[b]\}$, which is the trivial group.

From the above example, we took advantage of the convex property of $\mathbb{R}^2$ when we constructed the homotopy F. This means that we could apply the same method of proof to any convex spaces.

Def 12. *A subset S of $\mathbb{R}^n$ is **convex** if for $\forall a, b \in S$, the line segment from a to b is contained in S, i.e, $\forall t \in [0, 1]$ the line segment $t \cdot a + (1 - t) \cdot b \in S$.*

Ex 3. *Some basic examples of convex and non convex subsets of* $\mathbb{R}^n$.

- $\mathbb{R}^n$ *itself is convex.*
- $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ *is convex.*
- $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ *is not convex.*

Then as we expect, does any convex subset becomes a simply connected space? The next theorem states that it is true.

Thm 3. *A convex subset* S *of* $\mathbb{R}^n$ *is simply connected.*

Proof. Let $b \in S$, and $\alpha : [0,1] \to S$ be a loop based at b. Define $F : [0,1] \times [0,1] \to S$ as $F(s,t) = t \cdot b + (1-t) \cdot \alpha$ which becomes a homotopy such that $\alpha \simeq b$. The existence of such homotopy is guranteed as S is a convex subset. As for any loop based at b becomes homotopic with b, $\pi_1(S) = 0$ thus S is simply connected.

□

Remark. *If a subset of* $\mathbb{R}^n$ *is convex, it is also path connected.*

Def 13. *A subset* S *of* $\mathbb{R}^n$ *is* ***star shaped*** *if there exists some* $b \in S$ *such that for each* $x \in S$ *the line between* b *and* x *is contained in* S, *i.e, for* $\forall t \in [0,1]$ *the line segment* $t \cdot a + (1-t) \cdot b \in S$.

Thm 4. *A star shaped subset is simply connected.*

Proof. The proof is skipped.

□

Remark. *convex* $\implies$ *star shaped* $\implies$ *simply connected*

The homotopy concept can be actually extended to a more general one, as our definition of homotopy can be regarded a continuous function mapping one path which maps $[0,1] \to X$ to another path. Instead of $[0,1]$ we can similarly define homotopy with any topological space.

Def 14. *Let* X, Y *be topological spaces. Let* $A \subset Y$ *and* $f_0, f_1 : Y \to X$ *be maps such that* $f_0(a) = f_1(a)$ *for* $\forall a \in A$. *Then* f_0 *is* ***homotopic*** *to* f_1 ***relative to*** A *if there exists* $F : Y \times [0,1] \to X$ *such that* $F(y,0) = f_0(y)$ *and* $F(y,1) = f_1(y)$ *for* $y \in Y$ *and* $F(a,t) = f_0(a) = f_1(a)$ *and* $F(a,t) = f_0(a) = f_1(a)$ *for* $\forall a \in A$. *We denote such relation as* $f_0 \simeq f_1$, *rel* A.

Def 15. *Let X be a topological space where $x_0 \in X$. X is* ***contractible to*** *$x_0 \in X$ with x_0 held fixed if there exists a map $F : X \times [0,1] \to X$ such that $F(x,0) = x$ and $F(x,1) = x_0$ for $\forall x \in X$ and $F(x_0,t) = x_0$ for $\forall t \in [0,1]$.*

Thm 5. *If X is star shaped, then X is contractible.*

Thm 6. *If X is contractible, then $\pi_1(X) = 0$.*

Proof. Let $\alpha : [0,1] \to X$ be a path such that $\alpha(0) = \alpha(1) = x_0$. Let $F : X \times [0,1] \to X$ be a map such that $F(x,0) = x$, $F(x,1) = x_0$, $F(x_0,t) = x_0$ as X is contractible. Now let us define $G : [0,1] \times [0,1] \to X$ such that $G(s,t) = F(\alpha(s),t)$. Then,

$$\begin{cases} G(s,0) = F(\alpha(s),0) = \alpha(s) \\ G(s,1) = F(\alpha(s),1) = x_0 \\ G(0,t) = F(x_0,t) = x_0 \\ G(1,t) = F(x_0,t) = x_0 \end{cases}$$

where we can check that G becomes a homotopy between α and x_0, *rel* $\{0,1\}$.

□

Consider S^n, we know that when $n = 1$ it isn't contractible, and we can show this both implicitly by taking arbitrary points in S^1 and showing that not all of the line segments are in S^1.

X	contractible?	$\pi_1(X)$
S^1	No	$\mathbb{Z}$
S^n	??	0

Only with homotopy theory we can't show whether S^n are contractible or not for $n > 1$. This question can be solved with homology theory but that is currently out of our scope.

4 Induced Homomorphisms

We can think of obtaining a fundamental group of a certain topological space as some kind of correspondence, or a map sending a topological space to a group. The main question we deal with in this section is about whether there would also be a homomorphism between the fundamental groups induced by homeomorphisms between the topological spaces.

Def 16. *Let X, Y be topological spaces where $f : X \to Y$ is a homeomorphism between them such that for $b \in X$ and $c \in Y$, $f(b) = c$. Then we define the* ***induced homomorphism*** *of f as $f_* : \pi_1(X, b) \to \pi_1(X, c)$ which maps $f_*([\alpha]) = [f \circ \alpha]$.*

We can first define such induced homomorphisms, and the next lemma states that such homomorphisms are well defined.

Lem 8. *Let X, Y be a topological space with $f : X \to Y$. For $[\alpha_0], [\alpha_1] \in \pi_1(X, b)$ such that $[\alpha_0] = [\alpha_1]$, $[f \circ \alpha_0] = [f \circ \alpha_1]$ in $\pi_1(Y, c)$.*

Proof. Since $[\alpha_0] = [\alpha_1]$, there exists a homotopy between them and let it F. Define $G : [0, 1] \times [0, 1] \to Y$ by $G(s, t) = (f \circ F)(s, t)$. We can check that this becomes a homotopy between $f \circ \alpha_0$ and $f \circ \alpha_1$.

□

Thm 7. *Let X, Y, Z be a topological space.*

1. *If $id : X \to X$ is the identity map, then $(id)_* : \pi_1(X, b) \to \pi_1(x, b)$ is the identity homomorphism of $\pi_1(X, b)$.*
2. *If $f : (X, b) \to (Y, c)$ and $g : (Y, c) \to (Z, d)$, then $(g \circ f)_* = g_* \circ f_*$.*

Proof.

1) For $\forall [\alpha] \in \pi_1(X, b)$, by definition $(id_*)([\alpha]) = [id \circ \alpha] = [\alpha]$. Thus $(id)_*$ is the identity homomorphism of $\pi_1(X, b)$.
2) For $\forall [\alpha] \in \pi_1(X, b)$, the

$$\begin{aligned}(g \circ f)_*([\alpha]) &= [(g \circ f) \circ \alpha] \\ &= [g \circ (f \circ \alpha)] \\ &= g_*([f \circ \alpha]) \\ &= g_*(f_*([\alpha])) = g_* \circ f_*([\alpha])\end{aligned}$$

□

Thm 8. *If $f : X \to Y$ is a homeomorphism, then $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.*

Proof. Since $f : X \to Y$ is a homeomorphism there exists a continuous map $f^{-1} : Y \to X$ such that $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$. Then,

$$\begin{cases} (f^{-1} \circ f)_* = (id_X)_* & : \pi_1(X) \to \pi_1(X) \\ (f \circ f^{-1})_* = (id_Y)_* & : \pi_1(Y) \to \pi_1(Y) \end{cases}$$

Also,

$$\begin{cases} (f^{-1} \circ f)_* = (f^{-1})_* \circ f_* \\ (f \circ f^{-1})_* = f_* \circ (f^{-1})_* \end{cases}$$

and $(id_X)_* = id_{\pi_1(X)}$ and $(id_Y)_* = id_{\pi_1(Y)}$,

$$\begin{cases} (f^{-1})_* \circ f_* = id_{\pi_1(X)} \\ f_* \circ (f^{-1})_* = id_{\pi_1(Y)} \end{cases}$$

As there exists such $(f^{-1})_*$, we conclude that such f_* is a bijective map. Also, as f_* is an induced homomorphism, f_* is a bijective homomorphism, which is an isomorphism. Thus f_* is an isomorphism.

□

Cor 1. $X \cong Y \implies \pi_1(X) \simeq \pi_1(Y)$ *which is,* $\pi_1(X) \not\simeq \pi_1(Y) \implies X \ncong Y$

This is a very useful corollary, because it directly tells us how to check if two topological spaces are non homeomorphic by comparing if their fundamental groups are isomorphic. This is how algebraic topology converts a topological problem into a purely algebraic situation.

Cor 2. *If $X \cong Y$ and X is simply connected, then Y is also simply connected.*

Def 17. *Let $A \subseteq X$ for a topological space X.*

1. *A map $r : X \to A$ is a **retraction** of X into A if $r(a) = a$ for $\forall a \in A$*
2. *For a retraction $r : X \to A$, we call A the **retract** of X.*

Ex 4. *Let $X = \mathbb{R}^2 \setminus \{0\}$, and $A = \{x \in \mathbb{R}^2 : |x| = 1\}$. Define $r : X \to A$ as $r(x) = \frac{x}{|x|}$. Then as for $\forall a \in A$, $r(a) = a$ so r becomes a retraction of X into A and A becomes a retract of X. However, if we consider this map in $\mathbb{R}^2$ instead of X it doesn't become a retraction and it is not easy to show directly.*

Thm 9. *If A is a retract of X and X is simply connected, then A is also simply connected.*

Proof. left as an exercise (check assignment sheet)

□

Until now we defined and stated what are fundamental groups are, and the relation between them. But we haven't yet introduced how to calculate fundamental groups of a given topological space. There are two methods of calculating the fundamental group : one is to use the Seifert Van Kampen theorem, and the other is to deform the given topological space. The above theorem can be related to the second method, which was kind of a brief introduction. Now we also give a brief introduction about the first method of calculating the fundamental group by stating a simple case of the Seifert Van Kampen theorem.

Thm 10. *Let $U, V \subseteq X$ for a topological space X. Suppose that,*

1. *U, V are open in X and both are simply connected.*
2. *$U \cap V = \phi$ and $U \cup V = X$.*
3. *$U \cap V$ is path connected.*

Then $\pi_1(X) = 0$, i.e, X is also simply connected.

In order to prove this theorem, we need to introduce a lemma named Lebesguq number lemma, which we might have learned (or not) from general topology.

Lem 9. *Let X be a compact metric space. Suppose $\{U_\alpha\}$ is a collection of open sets of X such that $\bigcup U_\alpha = X$ ($\alpha \in I$), i.e, is an open cover of X. Then there exists some $\delta > 0$ such that if $diam(A) < \delta$ for some $A \subseteq X$, then there exists some $\alpha_0 \in I$ such that $A \subseteq U_{\alpha_0}$.*

Proof. Let $b \in U \cap V$, and $\alpha : [0,1] \to X$ be a loop based at b. Since U, V are open and $U \cup V = X$, $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ are open in $[0,1]$ and $\alpha^{-1}(U) \cup \alpha^{-1}(V) = [0,1]$, which implies that $\{\alpha^{-1}(V), \alpha^{-1}(U)\}$ becomes an open cover of $[0,1]$. Also we know that $[0,1]$ is a compact metric space, thus we can apply the Lebesgue number lemma we stated above. Thus, there exists $\delta > 0$ such that $A \subseteq [0,1]$ and $diam(A) \leq \delta$ such that $A \subseteq \alpha^{-1}(U)$ or $A \subseteq \alpha^{-1}(V)$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$, and also choose $0 = s_0 < s_1 < \cdots < s_N = 1$ such that $s_{i+1} - s_i = 1/N$ for every i. Then for each i, $[s_i, s_{i+1}]$ becomes subsets of U or V, i.e, $\alpha([s_i, s_{i+1}]) \subset U$ or $\subset V$. For each i, define $\alpha_i : [0,1] \to X$ by $\alpha_i(t) = \alpha((s_{i+1} - s_i)t + s_i)$. Since $U \cap V$ is path connected for each $i = 0, \cdot, N-1$ we can take a path $\beta_i : [0,1] \to U \cap V$ such that $\beta_i(0) = \alpha_i(1)$ and $\beta_i(1) = b$ Then,

$$\alpha \simeq (\alpha_0\beta_0)(\beta_0^{-1}\alpha_1\beta_1)\dots(\beta_{N-3}^{-1}\alpha_{N-2}\beta_{N-2})(\beta_{N-2}^{-1}\alpha_{N-1})$$

where each of the paths within a paranthesis is a loop in either U or V. Since $\pi_1(U) = \pi_1(V) = 0$,

$$\alpha_0\beta_0 \simeq \beta_i^{-1}\alpha_{i+1}\beta_{i+1} \simeq \beta_{N-2}^{-1}\alpha_{N-1} \simeq b \qquad \text{rel } \{0,1\}$$

Thus, $\alpha \simeq bb \dots bb \simeq b \; rel \; \{0,1\}$.

□

5 Covering Spaces

Our goal in this section is to understand how $\pi_1(S^1) = \mathbb{Z}$ using a mathematical notion known as the covering spaces. Before getting into the general idea of covering spaces and liftings, let us first deal with the example of S^1. We will then introduce the general definitions of them.

Def 18. *Let $p : \mathbb{R} \to S^1$ be a map defined as $p(t) = e^{2\pi i t}$. An open subset of U of S^1 is* ***evenly covered*** *by p if $p^{-1}(U) = \bigsqcup_{i \in \mathbb{Z}} V_i$ such that $p|_{V_i} : V_i \to U$ is a homeomorphism for each $i \in \mathbb{Z}$.*

Lem 10. *Every $x \in S^1$ has an open neighborhood which is evenly covered by $p : \mathbb{R} \to S^1$ such that $p(t) = e^{2\pi i t}$.*

Proof.

1. For $x \neq -1$ let $U = S^1 \backslash \{-1\}$. Then $p^{-1}(U) = \bigsqcup_{i \in \mathbb{Z}} \left(-\frac{1}{2} + i, \frac{1}{2} + i\right)$ and $p|_{\left(-\frac{1}{2}+i, \frac{1}{2}+i\right)} : \left(-\frac{1}{2} + i, \frac{1}{2} + i\right) \to U$ is a homeomorphism.
2. For $x \neq 1$, let $U = S^1 \setminus \{1\}$. Then $p^{-1}(U) = \bigsqcup_{i \in \mathbb{Z}} (i, i+1)$ and also $p|_{(i,i+1)}$ is also a homeomorphism.

□

Thm 11. *Let $\alpha : [0,1] \to S^1$ be a loop such that $\alpha(0) = \alpha(1) = 1 \in S^1$. Let $N \in \mathbb{Z}$, and $p : \mathbb{R} \to S^1$ be a map such that $p(t) = e^{2\pi i t}$. Then there exists a map $\tilde{\alpha}_N : [0.1] \to \mathbb{R}$ such that*

$$p \circ \tilde{\alpha}_N = \alpha \qquad \tilde{\alpha}_N(0) = N$$

We call this lifting a path, and it could be also understood by the following commutative diagram :

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\alpha"] \arrow[ur, dashed, "\tilde{\alpha}_N"] & S^1 \end{tikzcd}$$

Thm 12. *Let $F : [0,1] \times [0,1] \to S^1$ be a map such that $F(0,t) = F(1,t) = 1$ for $\forall t \in [0,1]$ and $F(s,0) = \alpha$ and $F(s,1) = \beta$. Let $N \in \mathbb{Z}$, then there exists a map $\tilde{F} : [0,1] \times [0,1] \to \mathbb{R}$ such that*

$$p \circ \tilde{F} = F \qquad \tilde{F}(0,1) = N \qquad \tilde{F}(1,t) = M$$

for some $M \in \mathbb{Z}$ and for every $t \in [0,1]$.

As paths can be lifted, the above theorem states that also path homotopy maps can be lifted too. This can also be represented as a commutative diagram :

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1] \times [0,1]} \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & S^1 \end{tikzcd}$$

Cor 3. *Let α, β be loops in S^1 based at $1 \in S^1$ and $N \in \mathbb{Z}$. If $\alpha \simeq \beta$ with $rel\{0,1\}$ then $\tilde{\alpha}_N \simeq \tilde{\beta}_N$ with $rel\{0,1\}$ and hence $\tilde{\alpha}_N(1) = \tilde{\beta}_N(1)$.*

Thm 13. $\pi_1(S^1) = \mathbb{Z}$

Proof. Let us first define $\Phi : \pi_1(S^1) \to \mathbb{Z}$ by $\Phi([\alpha]) = \tilde{\alpha}_0(1)$. We want to show that such Φ is an isomorphism.

1. Φ is well defined

   If $[\alpha] = [\beta]$ then it implies that $\Phi([\alpha]) = \Phi([\beta])$ which directly follows from **Cor 3**.

2. Φ is surjective.

   Let arbitrary $N \in \mathbb{Z}$. Define $\alpha : [0,1] \to S^1$ by $\alpha(s) = e^{2\pi i N s}$. Then $\tilde{\alpha}_0 : [0,1] \to \mathbb{R}$ is obtained by $\tilde{\alpha}_0(s) = Ns$. So as $\tilde{\alpha}_0(1) = N$, for any $N \in \mathbb{Z}$ there exists $[\alpha] \in \pi_1(S^1)$ such that $\Phi([\alpha]) = \tilde{\alpha}_0(1) = N$.

3. Φ is injective

   Suppose $\Phi([\alpha]) = \Phi([\beta]) \implies \tilde{\alpha}_0(1) = \tilde{\beta}_0(1)$. Then $\tilde{\alpha}_0\tilde{\beta}_0^{-1}$ is a loop in $\mathbb{R}$, and since $\pi_1(\mathbb{R}) = 0$, it implies that $\alpha_0\tilde{\beta}_0^{-1} \simeq 0$. So $p \circ (\tilde{\alpha}_0\tilde{\beta}_0^{-1}) \simeq p(0)\ rel\{0,1\}$. This again implies that $\alpha\beta^{-1} \simeq 1\ rel\{0,1\}$,

   $$[\alpha][\beta]^{-1} = [1] \implies [\alpha] = [\beta]$$

   thus for any $[\alpha], [\beta]$ such that $\Phi([\alpha]) = \Phi([\beta])$, we've shown that $[\alpha] = [\beta]$ which implies that Φ is injective.

4. Φ is a homomorphism

   Suppose $\Phi([\alpha]) = N$ and $\Phi([\beta]) = M$. In order to show that Φ is a homomorphism, it suffices to show that

   $$\Phi([\alpha][\beta]) = \Phi([\alpha]) + \Phi([\beta]) = N + M$$

   And as $[\alpha][\beta] = [\alpha\beta]$ it suffices to show that $\tilde{\alpha\beta}_0(1) = N + M$. Let us define $\tilde{h} : [0,1] \to \mathbb{R}$ by

   $$\tilde{h}(s) = \begin{cases} \tilde{\alpha}_0(2s) & (0 \le s \le \frac{1}{2}) \\ \tilde{\beta}_N(2s-1) = \tilde{\beta}_0(2s-1) + N & (\frac{1}{2} \le s \le 1) \end{cases}$$

   Then $\tilde{h} = \tilde{\alpha\beta}_0$ since,

   $$p \circ \tilde{h} = \begin{cases} \alpha(2s) \\ \beta(2s-1) \end{cases} = \alpha\beta \qquad \tilde{h}(0) = 0$$

Thus $\widetilde{\alpha\beta_0}(1) = \tilde{\beta}_N(1) = N + \tilde{\beta}_0(1) = N + M$, which proves that Φ is a homomorphism.

Thus as we've shown that such $\Phi : \pi_1(S^1) \to \mathbb{Z}$ is an isomorphism and it exists, we can conclude that $\pi_1(S^1) \simeq \mathbb{Z}$.

□

Now we will attempt to extend such idea we've used to show the fundamental group of a circle is $\mathbb{Z}$, to general topological spaces.

Let E, X be topological spaces, where $p : E \to X$ is a well defined map.

Def 19. *An open subset U of X is **evenly covered** if $p^{-1}(U) = \bigsqcup_{\alpha \in A} U_\alpha$ such that $p|_{U_\alpha}$ is a homeomorphism and each U_α are open in E.*

Def 20. *A map $p : E \to X$ is called a **covering map** and E a **covering space** over X if*

1. *p is a surjection.*
2. *For each $x \in X$, x has an open neighborhood which is evenly covered by p.*

Def 21. *Let $p : E \to X$ be a covering map.*

1. *The set $p^{-1}(x)$ for some $x \in X$ is called the **fibre** over x.*
2. *For an open subset U of X which is evenly covered by p such that $p^{-1}(U) = \bigsqcup_{\alpha \in A} U_\alpha$, each U_α is called a **sheet** of $p^{-1}(U)$.*

Thm 14. *Let $p : (E, e_0) \to (X, x_0)$ be a covering map. Then the induced homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ is injective.*

Proof. Let $\tilde{\sigma} : [0,1] \to E$ such that $\tilde{\sigma}(0) = \tilde{\sigma}(1) = e_0$. Also let $\sigma = p \circ \tilde{\sigma} : [0,1] \to X$. Then $\tilde{\sigma}$ is a lift of σ by definition. Also then it implies that $[\sigma] = p_*([\tilde{\sigma}])$. To show that p_* is injective, it suffices to show that $p_*([\tilde{\sigma}]) = [x_0] \implies [\tilde{\sigma}] = [e_0]$. As

$$p_*([\tilde{\sigma}]) = [x_0] \implies p \circ \tilde{\sigma} \simeq x_0 = p(e_0)$$

and applying the homotopy lift theorem, we can show that $\tilde{\sigma} \simeq e_0$ which implies that $[\tilde{\sigma}] = [e_0]$. □

This theorem states us that while covering spaces get *bigger* as they cover more and more, the fundamental group of them get *smaller and smaller*. If it gets small until the biggest covering space has a trivial fundamental group recall that we called such cover an **universal cover**.

Thm 15. *Let $p : E \to X$ be a covering map. Then,*

1) For $x_0, x_1 \in X$, $|p^{-1}(x_0)| = |p^{-1}(x_1)|$

2) For $x \in X$, $|p^{-1}(x)| = [\pi_1(X) : p_*(\pi_1(E))]$

Ex 5. *Consider $p : \mathbb{R} \to S^1$ where $t \mapsto e^{2\pi i t}$. Then $p^{-1}(1) = \mathbb{Z}$ and $p^{-1}(-1) = \frac{1}{2} + \mathbb{Z}$.*

(1) $|p^{-1}(-1)| = |p^{-1}(1)|$ as $|p^{-1}(\mathbb{Z})| = |p^{-1}(\frac{1}{2} + \mathbb{Z})|$

(2) $\pi(S^1) \simeq \mathbb{Z}$ and $\pi(\mathbb{R}) = 0$ which implies that $p_*(\pi(\mathbb{R})) = 0$ then :

$$[\pi_1(S^1) : p_*(\pi(\mathbb{R}))] = [\mathbb{Z} : 0] = |\mathbb{Z}|$$

whcih is equal to $|p^{-1}(1)|$.

Ex 6. *Consider $p : S^1 \to S^1$ where $z \mapsto z^2$. Then $p^{-1}(1) = \{+1, -1\}$ and $p^{-1}(-1) = \{-i, +i\}$.*

(1) $|p^{-1}(-1)| = |p^{-1}(1)| = 2$

(2) Let $p_* : \pi_1(S^1) \to \pi_1(S^1)$ such that $n \mapsto 2n$. Then as $\pi_1(S^1) \simeq \mathbb{Z}$, $p_*(\pi_1(S^1)) \simeq 2\mathbb{Z}$. Then :

$$[\pi_1(S^1) : p_*(\pi(S^1))] = [\mathbb{Z} : 2\mathbb{Z}] = 2$$

which is equal to $|p^{-1}(1)|$.

Def 22. *Let $p : E \to X$ be a covering map. E is called a* ***n-fold covering space*** *if $|p^{-1}(x)| = n \in \mathbb{N}$ for $x \in X$.*

Such n is well defined due to **Thm 15**. For covering spaces such that $p^{-1}(x) = \mathbb{Z}$ we call it an **infinite (cyclic) covering space** or a $\mathbb{Z}$ **covering space**.

Recall. *The 2 dimensional real projective space $\mathbb{R}P^2$ can be defined as*

1) $\left(\mathbb{R}^3 \setminus \{0\}\right) /_{x \sim \lambda x}$ where $\lambda \in \mathbb{R} \setminus \{0\}$.

2) $S^2 /_{x \sim -x}$

3) $S^2_+/_{x\sim -x}$ *where* $x \in S^1$

Thm 16. $\pi_1(\mathbb{R}P^2) \simeq \mathbb{Z}_2$.

Proof. Let $p : S^2 \to \mathbb{R}P^2$ as $x \mapsto [x]$ which is given by the equivalence of $x \sim -x$. Then such p becomes a 2-fold covering map as $p^{-1}(x) = \{+x, -x\}$ for any $x \in S^2$. Then applying **Thm 15**, we can see that

$$[\pi_1(\mathbb{R}P^2) : p_*(\pi_1(S^2))] = [\pi_1(\mathbb{R}P^2) : 0] = 2$$

which implies that $|\pi_1(\mathbb{R})P^2| = 2$. As for any finite groups with order 2 are isomorphic to $\mathbb{Z}_2$ we can conclude that $\mathbb{R}P^2 \simeq \mathbb{Z}_2$. $\square$

Thm 17. *If* $p_1 : E_1 \to X_1$ *and* $p_2 : E_2 \to X_2$ *are covering maps, then*

$$p : E_1 \times E_2 \to X_1 \times X_2$$

is also a covering map.

Ex 7. *Consider two covering maps* $p_1 : \mathbb{R} \to S^1$ *and* $p_2 : \mathbb{R} \to S^1$ *which both maps* $t \mapsto e^{2\pi it}$. *Then the following product* $p_1 \times p_2$ *is also a covering map :*

$$p_1 \times p_2 : \mathbb{R}^2 \to S^1 \times S^1 = T^2$$

is also a covering map.

6 Homotopic Maps

Recall. *We dealt with a special map* $r : A \subset X \to X$, *which we called it asa retraction map, which makes the following diagram commutative :*

$$\begin{tikzcd} X \arrow[r, "r"] & A \\ A \arrow[u, hook, "\iota_A"] \arrow[ur, dashed, "id_A"'] & \end{tikzcd}$$

where ι_A *is the trivial inclusion map.*

Such retraction was somehow useful but still it doesn't give us a lot of information. So we define such maps :

Def 23. *Let $A \subset X$, and $\iota_A : A \hookrightarrow X$ be the inclusion map.*

1) *A map $r : X \to A$ is a **deformation retract** if $r \circ \iota_A = id_A$ and $\iota_A \circ r \simeq id_X$. That is, there exists $F : X \times [0.1] \to X$ such that $F(x, 0) = (\iota_A \circ r)(x)$ and $F(x, 1) = x$ for any $x \in X$.*

2) *A map $r : X \to A$ is a **strong deformation retract** if $r \circ \iota_A = id_A$ and $\iota_A \circ r \simeq id_X$ with rel A condition. That is, there exists $F : X \times [0, 1] \to X$ such that $F(x, 0) = (\iota_A \circ r)(x)$, $F(x, 1) = x$ and $F(a, t) = a$ for $\forall t \in [0, 1]$ and $a \in A$.*

Thm 18. *If $r : X \to A \subset X$ is a deformation retract, then $r_* : \pi_1(X) \to \pi_1(A)$ is an isomorphism. Hence $\pi_1(X) \simeq \pi_1(A)$.*

Proof. Let $r : X \to A$ be a deformation retraction. Then,

$$\begin{cases} r \circ \iota_A = id_A \\ \iota_A \circ r \simeq id_X \end{cases}$$

holds which implies,

$$\begin{cases} r_* \circ (\iota_A)_* = id_{\pi_1(A)} \\ (\iota_A)_* \circ r_* \text{ is an isomorphism} \end{cases}$$

The first and second implication tells us that r_* is surjective, and r_* is injective, which implies that r_* is bijective. As r_* is a bijective homomorphism, it is an isomorphism. □

Ex 8. *Consider $X = \mathbb{R}^2 \setminus \{(0, 0)\}$ and S^1 as a subset of X. If we let $r : X \to A$ which maps $x \mapsto \frac{x}{|x|}$. Then such r is a deformation retract as :*

i) $r \circ \iota_A = id_A$

For $\forall a \in S^1 : (r \circ \iota_A)(a) = a/|a| = a$. Thus it holds.

ii) $\iota_A \circ r \simeq id_X$

Define $F : X \times [0, 1] \to X$ which maps $(x, t) \mapsto \frac{x}{|x|^t}$. Such F lets ι_A homotopic to r. Thus the above holds.

*Then due to **Thm 18**, as r is a deformation retract $\pi_1(\mathbb{R} \setminus \{0\}) \simeq \pi_1(S^1) \simeq \mathbb{Z}$.*

Ex 9. *Let M be a Möbius strip. Then S^1 is a deformation retract of X, thus $\pi_1(M) \simeq \pi_1(S^1) \simeq \mathbb{Z}$.*

Ex 10. *Let X be a puncutred 2-torus, i.e $X = T^2 \setminus \{p\}$ for some $p \in T^2$. Then $S^1 \vee S^1$ is a deformation retract of X. Thus $\pi_1\left(T^2 \setminus \{p\}\right) \simeq \pi_1\left(S^1 \vee S^1\right) \simeq \mathbb{Z} * \mathbb{Z}$.*

Def 24. *Let X, Y be topological spaces.*

(1) Then a map $f : X \to Y$ is a ***homotopy equivalence*** *if*

i)

ii)

(2) We say that X is ***homotopy equivalent*** *to Y if there exists a homotopy equivalence between X and Y.*

Thm 19. *Again let X, Y be topological spaces. Then,*

1) $X \cong Y \implies X \simeq Y$

2) $f : X \to Y$ is a homeomorphism $\implies$ f is a homotopy equivalence.

3) Homotopy equivalence is an equivalence relation.

Thm 20. *If $f : X \to Y$ is a homotopy equivalence then $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.*

Proof. As f is a homotopy equivalence $\exists g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. This implies that $g_* \circ f_*$ and $f_* \circ g_*$ are both an isomorphism, which again implies that f_* is an isomorphism. □

Cor 4. $X \simeq Y \implies \pi_1(X) \simeq \pi_1(Y)$

Lem 11. *Let $A \subset X$ and $r : X \to A$ be a deformation retraction. Then r is a homotopy equivalence.*

Proof. content... □

Remark. *To sum up every situation :*

$$\begin{array}{ccccc} & & SDR & \Longrightarrow DR & \Longrightarrow\ R \\ & & & \Downarrow & \\ X \cong Y & \Longrightarrow & & X \simeq Y & \\ & & & \Downarrow & \\ & & & \pi_1(X) \simeq \pi_1(Y) & \end{array}$$

7 The Seifert-van Kampen Theorem