Konkuk University Dept. Of Physics

Algebraic Topology, Exercises 1

2019 Spring Semester

Youngwan Kim

April 12, 2019

Recall. *Recall that a map $f : X \to Y$ is **homotopic** to a map $g : X \to Y$ if there is a map $F : X \times [0,1] \to Y$ such that $F(x,0) = f$ and $F(x,1) = g$ for $\forall x \in X$. (no relative conditions)*

Problem 1. *Show that if the topological spaces X and Y are homeomorphic and X is simply connected, then so is Y.*

Proof. If $X \cong Y$ then $\pi_1(X) \simeq \pi_1(Y)$. As $\pi_1(X) = 0$ and $\pi_1(X) \simeq \pi_1(Y)$ it implies that $\pi_1(Y) = 0$.

$\square$

Problem 2. *Let n be a positive integer. Let $f : X \to S^n$ and $g : X \to S^n$ be maps. Suppose that $f(x) \neq -g(x)$ for any $x \in X$. Show that f is homotopic to g.*

Proof. Define a map $F : X \times [0,1] \to S^n$ such that

$$F(x,t) = \frac{(1-t) \cdot f(x) + t \cdot g(x)}{|(1-t) \cdot f(x) + t \cdot g(x)|}$$

then we can check that

$$\begin{cases} F(x,0) = \frac{f(x)}{|f(x)|} = f(x) \\ F(x,1) = \frac{g(x)}{|g(x)|} = g(x) \end{cases}$$

as $f(x), g(x) \in S^n$. Also as $f(x) \neq -g(x)$ for all $x \in X$, such $F(x,t)$ is well defined as $|(1-t) \cdot f(x) + t \cdot g(x)| \neq 0$ for all $(x,t) \in X \times [0,1]$.

□

Remark. *The trick of normalizing line segments are well used as a method of constructing a homotopy between maps which maps to S^n, due to its property.*

Problem 3. *Let $X = \{x \in \mathbb{R}^n : 1 \leq |x| \leq 2\}$. Let $f : X \to X$ be a map defined by $f(x) = x/|x|$. Show that f is homotopic to the identity map $id : X \to X$.*

Proof. Let $F : X \times [0,1] \to X$ as

$$F(x,t) = (1-t) \cdot \frac{x}{|x|} + t \cdot x$$

Then $F(x,0) = f$ and $F(x,1) = id$, thus as there exists such homotopy F, $f \simeq id$.

□

Remark. *For the above problem, $F(x,t) = \frac{x}{|x|^t}$ is also a homotopy.*

Problem 4. *Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to the direct product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be projections, i.e, for $\forall (x,y) \in X \times Y$, the maps are defined as $p_X(x,y) = x$ and $p_Y(x,y) = y$. Now let $\phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ for all $[\alpha] \in \pi_1(X \times Y, (x_0, y_0))$ which maps as,

$$\phi([\alpha]) = ([p_X \circ \alpha], [p_Y \circ \alpha])$$

ϕ is well defined. Such ϕ is obviously a bijection, as we can define $\phi^{-1} : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$. We just have to show that it is a homomorphism. For any $[\beta], [\gamma] \in \pi_1(X \times Y, (x_0, y_0))$

$$\begin{aligned}
\phi([\alpha][\beta]) = \phi([\alpha\beta]) &= ([p_X \circ \alpha\beta], [p_Y \circ \alpha\beta]) \\
&= ([p_X \circ \alpha][p_X \circ \beta], [p_Y \circ \alpha][p_Y \circ \beta]) \\
&= ([p_X \circ \alpha], [p_Y \circ \alpha])([p_X \circ \beta], [p_Y \circ \beta]) \\
&= \phi([\alpha])\phi([\beta])
\end{aligned}$$

□

Problem 5. *Prove that the product of simply connected spaces is simply connected.*

Proof. Using the results of **Problem 4**, if $\pi_1(X) = 0$ and $\pi_1(Y) = 0$ then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y) = 0$. Also as both X, Y are path connected, $X \times Y$ is also path connected, as there exists any path $\gamma(s) = (\alpha(s), \beta(s))$ for all $(x_0, y_0), (x_1, y_1) \in X$ such that $\gamma(0) = (x_0, y_0)$ and $\gamma(1) = (x_1, y_1)$, where the existence of α and β is guranteed as X, Y are path connected. Thus as $\pi_1(X \times Y) = 0$ and $X \times Y$ is path connected, it is simply connected.

$\square$

Problem 6. *Prove that if $n \geq 3$, then $\mathbb{R}^n \setminus \{0\}$ is simply connected. (Hint : use the fact that S^{n-1} is simply connected.)*

Proof. For any $[\alpha] \in \mathbb{R}^n \setminus \{0\}$, by some homotopy, let it F, $\alpha \simeq \frac{\alpha}{|\alpha|} \in S^{n-1}$. In S^{n-1}, $\alpha/|\alpha| \simeq 1$ by a homotopy, let it G as S^{n-1} is simply connected for $n \geq 3$. As $S^{n-1} \subset \mathbb{R}^n$, we can say that $\alpha \simeq \alpha/|\alpha| \simeq 1$, thus $\mathbb{R}^n \setminus \{0\}$ is also simply connected.

$\square$

Remark. *This problem can be easily solved using retract deformations, or considering homotopy equivalence of S^{n-1} and $\mathbb{R}^n \setminus \{0\}$.*

Problem 7. *Let A be a subspace of X and $j : A \hookrightarrow X$ be the inclusion map. Let a map $r : X \to A$ be a retraction of X onto A, that is $r \circ j = id_A$. Prove the following,*

(a) $j_* : \pi_1(A, b) \to \pi_1(X, b)$ is one-to-one.

(b) $r_* : \pi_1(X, b) \to \pi_1(A, b)$ is onto.

(c) If X is simply connected, then so is A.

Proof. As $r \circ j = id_A$, $(r \circ j)_* = (id_A)_* \implies r_* \circ j_* = id_{\pi_1(A)}$. This implies (a) and (b). Also assuming X is simply connected, i.e, $\pi_1(X, b) = 0$ then due to (a) and (b), so does $\pi_1(A) = 0$. Path connectedness of A is also guranteed as $r : X \to A$ is a continuous map and X is simply connected.

$\square$