

General Topology, Exercises 1

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Problem 1. *Show that $[0, 1)$ is neither closed nor open in \mathbb{R} .*

Proof. As $\text{int}([0, 1)) = (0, 1)$ and $\overline{[0, 1)} = [0, 1]$ thus it is neither closed or open in \mathbb{R} . □

Problem 2. *Let $A = \{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}$. Consider A as a subspace of \mathbb{R} . Is it closed or open?*

Proof. Consider $\mathbb{R} \setminus A$, which becomes

$$\mathbb{R} \setminus A = (-\infty, 0) \cup \left(\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n} \right) \right) \cup (1, \infty)$$

which is an union of open sets, implying that $\mathbb{R} \setminus A$ is also open. As $\mathbb{R} \setminus A$ is open, A is closed. □

Problem 3. *Show that for $p \in X$, the singleton $\{p\}$ is closed in X . Also show that a finite subset of X is closed in X .*

Proof.

1. The singleton is closed in X .

Consider $q \in X \setminus \{p\}$, then there exists $B(q, r)$ where $r = d(p, q) > 0$. Then $p \notin B(q, r)$ which implies that $B(q, r) \subset X \setminus \{p\}$. Thus $X \setminus \{p\}$ is open, and $\{p\}$ is thus closed.

2. A finite subset of X is closed in X .

For any finite subset, let it P , it could be expressed as $P = \bigcup_{p \in P} \{p\}$, which is a finite union of closed sets. As any finite subset is such a finite union of closed sets, it is closed.

□

Problem 4. Let us consider $A = \{0, 1, 2, 3\}$ and $B = \{0, 1\}$ both as subspaces of \mathbb{R} .

- (a) Prove that A is closed in \mathbb{R} .
- (b) Is B open in \mathbb{R} ?
- (c) Is B open in A ?
- (d) Find all open subsets of A .

Proof. content...

□

Problem 5. A set of the form $\{y \in X | d(x, y) \leq r\}$ is called a closed ball. Show that a closed ball is a closed set. Is the closed ball $\{y \in X | d(x, y) \leq r\}$ always the closure of the open ball $B(x, r)$?

Proof.

1. Closed ball is closed.
2. Is it always the closure of the open ball? (No)

Consider \mathbb{R} which is equipped with a discrete metric. The open ball $B(0, 1)$ under this metric space is the singleton $\{0\}$. So the closure is, $\overline{B(0, 1)} = \{0\} = \{0\}$. Meanwhile the closed ball with the same center and radius gives us, $B'(0, 1) = \mathbb{R}$. Thus as this counterexample exists we can say that not always the closed ball becomes the closure of the open ball.

□

Problem 6. Let $A \subset B \subset X$. Show that $\text{int}(A) \subset \text{int}(B)$ and $\overline{A} \subset \overline{B}$.

Proof.

1. $\text{int}(A) \subset \text{int}(B)$

Consider $x \in \text{int}(A)$ which implies that there exists $r > 0$ such that $B(x, r) \subset A$. But as $A \subset B$, it is also true that $B(x, r) \subset B$. Thus $x \in \text{int}(B)$ holds.

2. $\overline{A} \subset \overline{B}$

Consider $x \in \overline{A}$, which implies that for any $r > 0$, $B(x, r) \cap A \neq \emptyset$. Again as $A \subset B$, the previous statement implies that $B(x, r) \cap A \subset B(x, r) \cap B$. Thus $x \in \overline{B}$ also holds.

□

Problem 7. Let $A \subset Y \subset X$. Show the following:

(a)

(b)

(c)

(d)