KONKUK UNIVERSITY DEPT. OF PHYSICS

General Topology, Exercises 1

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Problem 1. *Show that* $[0, 1)$ *is neither close dnor open in* $\mathbb{R}$*.*

Proof. As $int([0, 1)) = (0, 1)$ and $\overline{[0, 1)} = [0, 1]$ thus it is neither close or open in $\mathbb{R}$. □

Problem 2. *Let* $A = \{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}$*. Consider* A *as a subspace of* $\mathbb{R}$*. Is it closed or open?*

Proof. Consider $\mathbb{R} \setminus A$, which becomes

$$\mathbb{R} \setminus A = (-\infty, 0) \cup \left(\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n} \right) \right) \cup (1.\infty)$$

which is an union of open sets, implying that $\mathbb{R} \setminus A$ is also open. As $\mathbb{R} \setminus A$ is open, A is closed. □

Problem 3. *Show that for* $p \in X$*, the singleton* $\{p\}$ *is closed in* X*. Also show that a finite subset of* X *is closed in* X*.*

Proof.

1. The singleton is closed in X.

   Consider $q \in X \setminus \{p\}$, then there exists $B(q, r)$ where $r = d(p, q) > 0$. Then $p \notin B(q, r)$ which implies that $B(q, r) \subset X \setminus \{p\}$. Thus $X \setminus \{p\}$ is open, and $\{p\}$ is thus closed.

2. A finite subset of X is closed in X.

   For any finite subset, let it P, it could be expressed as $P = \bigcup_{p \in P} \{p\}$, which is a finite union of closed sets. As any finite subset is such a finite union of closed sets, it is closed.

□

Problem 4. *Let us consider $A = \{0, 1, 2, 3\}$ and $B = \{0, 1\}$ both as subspaces of $\mathbb{R}$.*

(a) *Prove that A is closed in $\mathbb{R}$.*

(b) *Is B open in $\mathbb{R}$?*

(c) *Is B open in A?*

(d) *Find all open subsets of A.*

Proof. content... □

Problem 5. *A set of the form $\{y \in X | d(x, y) \leq r\}$ is called a closed ball. Show that a closed ball is a closed set. Is the closed ball $\{y \in X | d(x, y) \leq r\}$ always the closure of the open ball $B(x, r)$?*

Proof.

1. Closed ball is closed.

2. Is it always the closure of the open ball? (No)

   Consider $\mathbb{R}$ which is equipped with a discrete metric. The open ball $B(0, 1)$ under this metric space is the singleton $\{0\}$. So the closure is, $\overline{B(0, 1)} = \overline{\{0\}} = \{0\}$. Meanwhile the closed ball with the same center and radius gives us, $B'(0, 1) = \mathbb{R}$. Thus as this counterexample exists we can say that not always the closed ball becomes the closure of the open ball.

□

Problem 6. *Let $A \subset B \subset X$. Show that $int(A) \subset int(B)$ and $\overline{A} \subset \overline{B}$.*

Proof.

1. $int(A) \subset int(B)$

   Consider $x \in int(A)$ which implies that there exists $r > 0$ such that $B(x,r) \subset A$. But as $A \subset B$, it is also true that $B(x,r) \subset B$. Thus $x \in int(B)$ holds.

2. $\overline{A} \subset \overline{B}$

   Consider $x \in \overline{A}$, which implies that for any $r > 0$, $B(x,r) \cap A \neq \phi$. Again as $A \subset B$, the previous statement implies that $B(x,r) \cap A \subset B(x,r) \cap B$. Thus $x \in \overline{B}$ also holds.

$\square$

Problem 7. *Let $A \subset Y \subset X$. Show the following:*

(a)

(b)

(c)

(d)