Konkuk University Dept. Of Physics

Algebraic Topology, Exercises 1

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Recall. *Recall that a map $f : X \to Y$ is* ***homotopic*** *to a map $g : X \to Y$ if there is a map $F : X \times [0,1] \to Y$ such that $F(x,0) = f$ and $F(x,1) = g$ for $\forall x \in X$.*

Problem 1. *Show that if the topological spaces X and Y are homeomorphic and X is simply connected, then so is Y.*

Proof. If $X \cong Y$ then $\pi_1(X) \simeq \pi_i(Y)$. As $\pi_1(X) \simeq 0$ and $\pi_1(X) \simeq \pi_i(Y)$ it implies that $\pi_1(Y) \simeq 0$.

□

Problem 2. *Let n be a positive integer. Let $f : X \to S^n$ and $g : X \to S^n$ be maps. Suppose that $f(x) \neq -g(x)$ for any $x \in X$. Show that f is homotopic to g.*

Problem 3. *Let $X = \{x \in \mathbb{R}^n : 1 \leq |x| \leq 2\}$. Let $f : X \to X$ be a map defined by $f(x) = x/|x|$. Show that f is homotopic to the identity map $id : X \to X$.*

Proof. Let $F : X \times [0,1] \to X$ as

$$F(x,t) = (1-t) \cdot \frac{x}{|x|} + t \cdot x$$

Then $F(x,0) = f$ and $F(x,1) = id$, thus as there exists a homotopy F, $f \simeq id$.

□

Problem 4. *Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to the direct product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be projections, i.e, for $\forall (x, y) \in X \times Y$, the maps are defined as $p_X(x, y) = x$ and $p_Y(x, y) = y$. Now let $\phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ for all $[\alpha] \in \pi_1(X \times Y, (x_0, y_0))$ which maps as,

$$\phi([\alpha]) = (p_{X*}([\alpha]), p_{Y*}([\alpha]))$$

Such ϕ is obviously a bijection, as it is a product of two projections, which are both bijections. We just have to show that it is a homomorphism. For any $[\beta], [\gamma] \in \pi_1(X \times Y, (x_0, y_0))$

$$\begin{aligned} \phi([\alpha][\beta]) = \phi([\alpha\beta]) &= (p_{X*}([\alpha\beta]), p_{Y*}([\alpha\beta])) \\ &= d \end{aligned}$$

□

Problem 5. *Prove that the product of simply connected spaces is simply connected.*

Proof. Using the results of **Problem 4**, if $\pi_1(X) \cong 0$ and $\pi_1(Y) \cong 0$ then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y) \cong 0$. Thus as $\pi_1(X \times Y) \cong 0$, it is simply connected.

□

Problem 6. *Prove that if $n \geq 3$, then $\mathbb{R}^n \setminus \{0\}$ is simply connected.*

Proof. Consider the stereographic map $\sigma : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ such that

□

Problem 7. *Let A be a subspace of X and $j : A \hookrightarrow X$ be the inclusion map. Let a map $r : X \to A$ be a retraction of X onto A, that is $r \circ j = id_A$. Prove the following,*

(a) $j_* : \pi_1(A, b) \to \pi_1(X, b)$ is one-to-one.

(b) $r_* : \pi_1(X, b) \to \pi_1(A, b)$ is onto.

(c) If X is simply connected, then so is A.

Proof. As $r \circ j = id_A$, $(r \circ j)_* = (id_A)_* \implies r_* \circ j_* = id_{\pi_1(A)}$. This implies (a) and (b). Also assuming X is simply connected, i.e, $\pi_1(X, b) \cong 0$ then due to (a) and (b), so does $\pi_1(A) \cong 0$.

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