1. (a) Write down the cartesian coordinates, corresponding to the following homogeneous coordinates : $(6, 4, 2)$ and $(12, 8, 4)$. [2]

The following questions refer to the matrix M below:

$$M = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) Show that the homogeneous transformation matrix M is a special orthogonal matrix. [3]

(ii) Find the inverse of the above matrix and explain why this is not difficult in this case. [3]

(iii) Show that one of the row vectors can be transformed into the Y axis. [3]

(b) The perspective transformation matrix M_p is given by:

$$M_p = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where n is the distance to the near plane and f to the far.

(i) A point in the viewing system, (x_v, y_v, z_v, w) is found by applying M_p to a point; $(x, y, z, 1)$. Find an expression for z_v (the transformed z), in terms of z, n, f. [3]

(ii) What are the limits of z and z_v? [1]

(iii) If the near plane is placed at $n = 10$ and the far plane at $f = 100$, sketch a graph of z_v against z. Using the graph explain any problems that might arise in the z-buffer algorithm when rendering large triangles. [5]

(Note to examiner: "Bookwork" is used to denote problems that are common in course texts or were presented in the lecture notes. "Advanced" problems require significant creative thinking beyond the ideas presented in the course or textbooks)

(a) This part is mostly bookwork.

Write down the cartesian coordinates, corresponding to the following homogeneous coordinates : $(6, 4, 2)$ *and* $(12, 8, 4)$

(x, y, W) *homogeneous*
To change to Cartesian coordinate in $2D$ divide by W. $(x/W, y/W)$
Note that there is a dimension reduction.
$(6, 4, 2)$ goes to $(6/2, 4/2) = (3, 2)$ [1]
$(12, 8, 4)$ goes to $(12/4, 8/4) = (3, 2)$ [1]

(i) *Show that the homogeneous transformation matrix* M*, below is a special orthogonal matrix.*

Three properties that show special orthogonal.
In the upper $2x2$ matrix.

1. magnitude of row vectors (and column vectors) are 1 [1]

$$\sqrt{\cos(\theta) * cos(\theta) + -sin(\theta) * -sin(\theta)} = 1$$

2. determinant is 1: $cos(\theta) * cos(\theta) + -sin(\theta) * -sin(\theta) = 1$ [1]
3. orthogonal vectors (dot product is zero)
$(cos(\theta), -sin(\theta)).(sin(\theta), cos(\theta)) = 1$ [1]

(ii) *Find the inverse of the above matrix and explain why this is not difficult in this case.*
Since the matrix is special orthogonal the inverse is the transpose.

$$\begin{bmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ [2]

(iii) *Show that one of the row vectors can be transformed into the* Y *axis.* [2]

$$\begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sin(\theta) \\ cos(\theta) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(b) This part is mainly advanced especially part iii

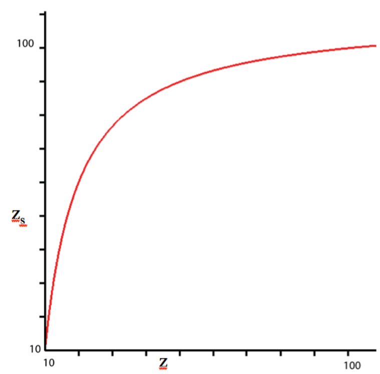
A point in the viewing system, (x_v, y_v, z_v, w) *is found by applying* M_p *to a point;* $(x, y, z, 1)$*. Find an expression for* z_v *(the transformed* z*), in terms of* z, n, f*.*

$$M_p \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z\frac{n+f}{n} - f \\ \frac{z}{n} \end{bmatrix} \text{ Homogenize gives: } \begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n + f - \frac{fn}{z} \\ 1 \end{bmatrix}$$

In other words the new $z_s = n + f - \frac{fn}{z}$ [2]

(ii) *What are the limits of* z and z_v?
$n \leq z \leq f$ and $n \leq z_v \leq f$ [1]

(iii) *If the near plane is placed at* $n = 10$ *and the far plane at* $f = 100$*, sketch a graph of* z_v *against* z*. Using the graph explain any problems that might arise in the z-buffer algorithm when rendering large triangles.*



[3]

It can be seen from the graph that values of z in the camera system that lie close to the $z = n$ plane, bear a nearly linear relationship to z_v. Since z_v values in a triangle face in the $z - buffer$ algorithm are calculated by linear interpolation between edge values, the algorithm can correct distinguish which objects are closer to the eye with respect to a particular pixel. In contrast, z_v values far from the near plane do not vary linearly with z and errors can result. Essentially there are fewer bits of resolution available to distinguish z values far from the eye.
(i.e. the wrong object appears to be in front at a closer z-value). [5]

2. (a) Distinguish between parameter space and modelling (object) space for parametric curves. [3]

(b) Given the four basis functions ($b_i(t)$ $0 \leq i \leq 3$) and ($0 \leq t \leq 1$) and four points, (P_i $0 \leq i \leq 3$). Find an expression for the desired curve in the modelling space. [2]

(c) A generalized cylinder is built around a cubic curve given by the parametric equations:

$$f_x(t) = 4t^3 - 6t^2 + 4t + 1$$
$$f_y(t) = 6t^3 - 9t^2 + 2t + 2$$
$$f_z(t) = 0$$

(i) Derive expressions for the Frenet frame at the point where $t = 0$. (In other words don't evaluate the expressions). [10]

(ii) What problem is encountered if the frame is calculated at $t = 0.5$? Describe briefly how the problem can be overcome in an algorithm for building the cylinder from circular cross sections. [6]

(a) *Distinguish between parameter space and modelling (object) space for parametric curves.* Basis functions are defined in parametric space. For a particular value of t a point on the object space curve is defined by weighting each control point (in object space) by the appropriate basis function. The curve,in modelling space, is drawn by finding a series of points along the curve as t varies along the unit interval (for uniform parametric curves). [5]

(b) $\sum P_i * b_i(t)$

(c) (i) *Derive expressions for the Frenet frame at the point where $t = 0$.*
Find the tangent vector and acceleration vector at the point $(t = 0.0)$ and at $(t = 0.5)$ [6]

function	$t = 0$	$t = 0.5$
$f_x(t) = 4t^3 - 6t^2 + 4t + 1$	1	2
$f_y(t) = 6t^3 - 9t^2 + 2t + 2$	2	0
$1st$ *derivative*	$t = 0$	$t = 0.5$
$f'_x(t) = 12t^2 - 12t + 4$	4	-2
$f'_y(t) = 18t^2 - 18t + 2$	2	-2.5
$2nd$ *derivative*	$t = 0$	$t = 0.5$
$f''_x(t) = 24t - 12$	-12	0
$f''_y(t) = 36t - 18$	18	0

(ii) *What problem is encountered if the frame is calculated at $t = 0.5$? Describe briefly how the problem can be overcome in an algorithm for building the cylinder from circular cross sections.*
At $t = 0$ we have: [4]

$$\mathbf{v} = (4, 2)$$
$$\mathbf{q} = (-12, -36)$$
$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$
$$\mathbf{B} = \frac{\mathbf{v} \times \mathbf{q}}{\|\mathbf{v} \times \mathbf{q}\|}$$
$$\mathbf{N} = \frac{\mathbf{v} \times \mathbf{q} \times \mathbf{v}}{\|\mathbf{v} \times \mathbf{q} \times \mathbf{v}\|}$$

It can be seen from the above that the vector formed from the 2nd derivative is zero. This means that there is a point at which the Frenet frame cannot be calculated analytically as indicated above. [2]

A rotation minimising frame is used. After finding an initial frame successive frames are found by rotating the previous frame around an axis. If the frame at point p_i is known then the frame at point p_{i+1} is found by rotating around an axis. [2]

The axis of rotation is derived from the following cross product: [2]

$$\mathbf{T_i} \times \mathbf{T_{i+1}}$$

If this is zero the frame is simply translated to p_{i+1}. The angle of rotation is found from the dot product:

$$\mathbf{T_i} \cdot \mathbf{T_{i+1}} = \|\mathbf{T_i}\|\|\mathbf{T_{i+1}}\|cos(\alpha)$$

[2]

$\mathbf{B_{i+1}}$ and $\mathbf{N_{i+1}}$ are computed by rotating $\mathbf{B_i}$ and $\mathbf{N_i}$ [2]

3. (a) In what order are polygons fed to a z-buffer algorithm? What is the limit on the number of polygons a Z-Buffer program could handle? [5]

(b) What is a scene graph? [4]

(c) Describe why and how jittering is used in ray tracing. [4]

(d) In a viewing system the vector from the eye into the scene is given by $\mathbf{g}$, and the up vector by $(0\ 1\ 0)$. Derive a coordinate system $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ for viewing and write down the rotation matrix that will align the viewing system with the $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ world system. [4]

(e) Describe Phong shading and Phong illumination, and distinguish between them. [4]

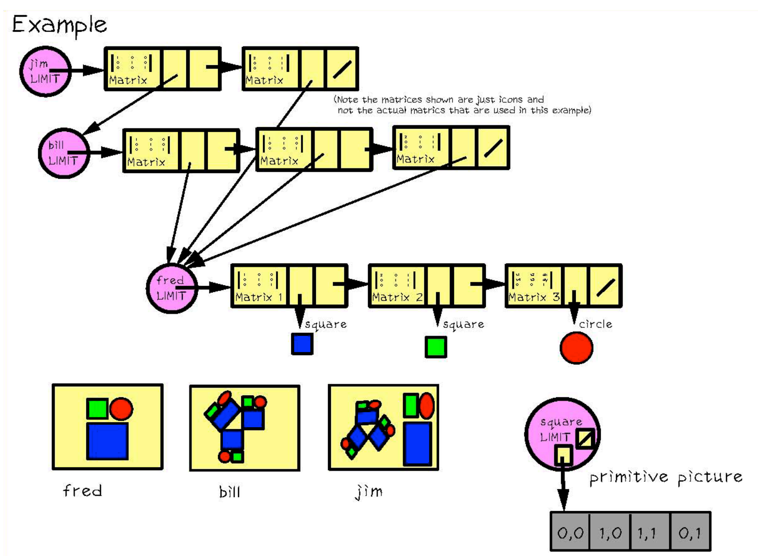
(f) What is twice the answer to Life the Universe and everything? [1]

In what order are polygons fed to a z-buffer algorithm?
Since z-buffer deals with a fixed sized storage, only storing the pixel colour corresponding to the point on the nearest triangle, the order of the triangles is immaterial. [2.5]

What is the limit on the number of polygons a Z-Buffer program could handle?
Since the triangles are not stored, the only limitation is the time allowed for one frame. This depends on how fast the triangles can be read into the appropriate memory, scan converted and z values compared. [2.5]

(b) *What is a scene graph?* [5]

A scene graph describes models in terms of a directed acyclic graph of geometric transformations of other models. Each node contains a reference to an instance of a model and a transformation matrix that will be applied to the model referenced when the data structure is traversed. A model may be thought of as the head of a linked list of nodes. When the data structure is traversed the current matrix is concatenated with the matrix stored in each node. The resulting matrix is then applied to the model referenced from the node.



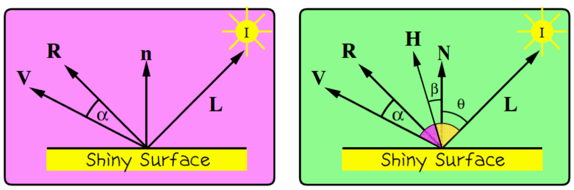
(c) *Describe why and how jittering is used in ray tracing.* Jittering is a means of anti-aliasing that introduces noise in exchange for aliasing artefacts based on the fact that humans are less susceptible to noise than aliasing. Each pixel is subdivided (for example 9x9) and a ray sent through the centre of each subdivision, offset by some small randomized amount to simulate noise. The pixel value is calculated as an average of the ray results. [5]

(d) *In a viewing system the vector from the eye into the scene is given by* $\mathbf{g}$, *and the up vector by* $(0\ 1\ 0)$. *Derive a coordinate system* $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ *for viewing and write down the rotation* *matrix that will align the viewing system with the* $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *world system.*

Constructing a basis from a single vector. let the up vector $\mathbf{t} = (0\ 1\ 0)$ [1]
$\mathbf{w} = \frac{\mathbf{g}}{||\mathbf{g}||}$ [1]
$\mathbf{u} = \frac{\mathbf{t} \times \mathbf{w}}{||\mathbf{t} \times \mathbf{w}||}$ [1]
$\mathbf{v} = \mathbf{w} \times \mathbf{u}$ [1]

The eye coordinate system views along the negative $\mathbf{w}$ axis. [1]

(e) *Describe Phong shading and Phong illumination, and distinguish between them.* The Phong Light model is an empirical model that incorporates specular reflection using $(\mathbf{R}.\mathbf{V}$ or $(\mathbf{H}.\mathbf{N})$, where: the vectors are shown in the figure.



Vectors used in the Phong specular calculation

Phong Shading averages surface normals. In a triangular mesh where no normals are given the vertex normals are calculated from the face normals. Normals are averaged along edges and across faces and the new normal is used for each illumination calculation. Thus specular highlights are seen when using the Phong Light Model.

Specular component= $K_s \mathbf{I_p} (\mathbf{R}.\mathbf{V})^n$
or using the halfway vector where $\mathbf{H} = \frac{\mathbf{L}+\mathbf{V}}{||\mathbf{L}+\mathbf{V}||}$ Specular component= $K_s \mathbf{I_p} (\mathbf{N}.\mathbf{H})^n$ [5]

(f) *What is twice the answer to Life the Universe and everything?*
$2 * 42 = 84$ [1]