Facts and Understanding about Morse Function and Reeb Graphs

Tao Hou

(All the statements should be true, however more rigorous algebraic proof or more easy-to-use topological tools are needed to fully understand why.)

Let $\mathbb{M}$ be a connected and orientable (compact?) 2-manifold and let a height function $f : \mathbb{M} \to \mathbb{R}$ be a Morse function, then all critical points (derivative be 0) of f can be classified into 3 categories: *minimum*, *maximum*, and *saddle point*.

For a minimum c, a new connected component of the sublevel set (as well as level set) is created.

For a saddle point c, we can classifiy the saddle point based on their local behaviors (it also seems that the level set variation corresponds to the local behavior): either one component forks into two components in the level set (up-forking) or two connected components of the level set get merged into one (down-forking).

- Up-forking: A new non-trivial loop (1st homology class) in the sublevel set is created on the saddle point, because a new hole is created on the connected component of c. (Let n be the number of holes on $\mathbb{S}^2$, then the space is homotopy equivalent to a wedge sum of $n-1$ $\mathbb{S}^1$'s.) Also, a new connected component in the level set is created.
- Down-forking: If the two level set components are from different connected components in the sublevel set, then the two components of the sublevel set merge into one.

  Otherwise, both the two connected components of the level set cannot be directly from a minimum because otherwise the level set components will not be connected in the sublevel set. This means that both level set components are from a previous up-forking, then a new 1st homology class is created (represented by the "longitude" of the newly attached holed torus).

For a maximum c, it always closes a hole in the sublevel set.

- If c is not the last critical point (the last critical point is always a maximum), then the hole that it closes cannot be the only hole of the connected component in the sublevel set, because otherwise the component of c forms a closed connected component and the last critical point must belong to another connected component, contradicting to the fact that $\mathbb{M}$ is connected. Then c must cause a non-trivial hole of the connected component (which is a holed $\mathbb{S}^1$ or g-torus) to become trivial.
- If c is the last critical point, then since the hole it closes is already a trivial hole being the boundary of the sublevel set, then no topological change is made at this critical point.