An Understanding of the Constructible Cosheaf

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[1] gives an description of the constructible cosheaf, which is a beautiful structure revealing some applications of the cosheaf theory. However, given the sets defined on the "basic intervals", it gives no explicit construction of the corresponding cosheaf nor proving it. Here I will construct the full functor and proves that it is a cosheaf.

First recall the definition of constructible cosheaf:

The building blocks for the constructible cosheaf are: given a set of critical values $S = \{a_0, ..., a_n\}$ with $a_0 = -\infty$ and $a_n = \infty$, assign a set to each vertex intervals (a_{i-1}, a_{i+1}) for $i = 1, \dots, n-1$ and to each edge intervals (a_i, a_{i+1}) for $i = 0, \dots, n-1$. Given these building blocks, we can define a functor $\mathcal{F} : \mathbf{Int} \to \mathbf{Set}$ by first assigning the edge and vertex intervals their corresponding sets. Then for each "whole" intervals $J = (a_i, a_j)$, with $i \geq 0$, $j \leq n$, $j - i \geq 3$, we can define

$$\mathcal{F}(J) = \coprod_{I \subseteq J} \mathcal{F}(I) / \sim$$

where I is an edge or vertex interval and "$\sim$" is the equivalence relation by identifying all $x \in \mathcal{F}(I)$ with $[\mathcal{F}(I \subseteq I')](x)$. For two "whole" intervals $J \subseteq J'$, the morphism $\mathcal{F}(J \subseteq J')$ is defined by mapping a equivalence class $[x]$ of $\mathcal{F}(J)$ to the corresponding equivalent class $[x]$ of $\mathcal{F}(J')$, by the observation that the "basic" interval I containg x must be a subset of J' and that each two elements identified in $\mathcal{F}(J)$ must also be identified in $\mathcal{F}(J')$.

Then for an arbitrary interval K, we can find the unique "whole" interval J containing the same set of critical values and let $\mathcal{F}(K) = \mathcal{F}(J)$. A morphism between two arbitrary intervals $K \subseteq K'$ can be formed by first finding the their corresponding "whole" intervals J and J', then compose the following maps:

$$\mathcal{F}(K) \xrightarrow{\simeq} \mathcal{F}(J) \to \mathcal{F}(J') \xrightarrow{\simeq} \mathcal{F}(K')$$

It is easy to check the assigning of the morphism preserves the composition of morphisms so this defines a functor $\mathcal{F}$.

Next we are gonna show that this is actually a cosheaf. In order for $\mathcal{F}$ to be a cosheaf, $\mathcal{F}$ must satisfy that for each open interval J and each open interval cover $\{J_i | i \in A\}$, $\mathcal{F}(J) = \varinjlim_{J' \in \mathcal{N}(J)} \mathcal{F}(J')$. Since for a functor on the category $\mathbf{Set}$, $\varinjlim_{J' \in \mathcal{N}(J)} \mathcal{F}(J') = \coprod_{J' \subseteq \mathcal{N}(J)} \mathcal{F}(J') / \sim$, where the relation "$\sim$" is defined similarly as above, we will show that for each "whole" interval nerve $\mathcal{N}(J)$ of a "whole" interval J:

$$\coprod_{J' \subseteq \mathcal{N}(J)} \mathcal{F}(J') / \sim = \mathcal{F}(J)$$

Let $C = \coprod_{J' \subseteq \mathcal{N}(J)} \mathcal{F}(J') / \sim$, we define a bijective map $f : \mathcal{F}(J) \to C$ by $f([x]) = [x]$, that is, sending the equivalence class of x in $\mathcal{F}(J)$ to the equivalence class of x in C. This is well defined because for each $J_i \in \mathcal{N}(J)$ containing an equivalence class of x, it must contain the basic interval X_0 of x. The intersection of all such J_i's in the nerve also contains $[x]$ and will identify all $[x]$'s, so there will be only one element in C containing x. Surjectiviy of f is easy to see and suppose $[x] \neq [x']$ in $\mathcal{F}(U)$ but

$f([x]) = f([x'])$. Then there must be intervals $J \subseteq J' \in \mathcal{N}(J)$ such that $[x] \mapsto [x']$ or $[x'] \mapsto [x]$ by the morphism from $\mathcal{F}(J)$ to $\mathcal{F}(J')$. However, this is not possible because otherwise x will be identified with x' in $\mathcal{F}(J)$.

References

[1] De Silva, Vin, Elizabeth Munch, and Amit Patel. "Categorified reeb graphs." Discrete & Computational Geometry 55.4 (2016): 854-906.