Chapter 6
Heath–Jarrow–Morton (HJM) Methodology

As we have seen in Chap. 5, short-rate models are not always flexible enough to calibrating them to the observed initial term-structure. In the late eighties, Heath, Jarrow and Morton (henceforth HJM) [90] proposed a new framework for modeling the entire forward curve directly. This chapter provides the essentials of the HJM framework.

6.1 Forward Curve Movements

The stochastic setup is as in Sect. 4.1. We consider $\mathbb{P}$ as objective probability measure, and let W be a d-dimensional Brownian motion.

We assume that we are given an $\mathbb{R}$-valued and $\mathbb{R}^d$-valued stochastic process $\alpha = \alpha(\omega, t, T)$ and $\sigma = (\sigma_1(\omega, t, T), \dots, \sigma_d(\omega, t, T))$, respectively, with two indices, t, T, such that

(HJM.1) α and σ are $\mathrm{Prog} \otimes \mathcal{B}$-measurable;
(HJM.2) $\int_0^T \int_0^T |\alpha(s,t)|\, ds\, dt < \infty$ for all T;
(HJM.3) $\sup_{s,t \le T} \|\sigma(s,t)\| < \infty$ for all T.[1]

For a given integrable initial forward curve $T \mapsto f(0,T)$ it is then assumed that, for every T, the forward rate process $f(\cdot, T)$ follows the Itô dynamics

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\, ds + \int_0^t \sigma(s,T)\, dW(s), \quad t \le T. \tag{6.1}$$

This is a very general setup. The only substantive economic restrictions are the continuous sample paths assumption for the forward rate process, and the finite number, d, of random drivers $W_1, \dots, W_d$.

The integrals in (6.1) are well defined by **(HJM.1)**–**(HJM.3)**. Note that $\alpha(t,T)$ and $\sigma(t,T)$ enter the dynamic equation (6.1) and the sequel only for $t \le T$; we can and will set them equal to zero for all $t > T$ without loss of generality. Moreover, it follows from Corollary 6.3 below that the short-rate process

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)\, ds + \int_0^t \sigma(s,t)\, dW(s)$$

[1] Note that this is a ω-wise boundedness assumption.

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has a progressive modification—again denoted by $r(t)$—satisfying $\int_0^t |r(s)|\,ds < \infty$ a.s. for all t. Hence the money-market account $B(t) = \mathrm{e}^{\int_0^t r(s)\,ds}$ is well defined. More can be said about the zero-coupon bond prices $P(t,T) = \mathrm{e}^{-\int_t^T f(t,u)\,du}$:

Lemma 6.1 *For every maturity T, the zero-coupon bond price follows an Itô process of the form*

$$P(t,T) = P(0,T) + \int_0^t P(s,T)\,(r(s) + b(s,T))\,ds + \int_0^t P(s,T)v(s,T)\,dW(s), \tag{6.2}$$

for $t \le T$, where

$$v(s,T) = -\int_s^T \sigma(s,u)\,du, \tag{6.3}$$

is the T-bond volatility and

$$b(s,T) = -\int_s^T \alpha(s,u)\,du + \frac{1}{2}\|v(s,T)\|^2.$$

Proof Using the classical Fubini Theorem and Theorem 6.2 below for stochastic integrals twice, we calculate

$$\begin{aligned}
&\log P(t,T)\\
&\quad = -\int_t^T f(t,u)\,du\\
&\quad = -\int_t^T f(0,u)\,du - \int_t^T\int_0^t \alpha(s,u)\,ds\,du - \int_t^T\int_0^t \sigma(s,u)\,dW(s)\,du\\
&\quad = -\int_t^T f(0,u)\,du - \int_0^t\int_t^T \alpha(s,u)\,du\,ds - \int_0^t\int_t^T \sigma(s,u)\,du\,dW(s)\\
&\quad = -\int_0^T f(0,u)\,du - \int_0^t\int_s^T \alpha(s,u)\,du\,ds - \int_0^t\int_s^T \sigma(s,u)\,du\,dW(s)\\
&\qquad + \int_0^t f(0,u)\,du + \int_0^t\int_s^t \alpha(s,u)\,du\,ds + \int_0^t\int_s^t \sigma(s,u)\,du\,dW(s)\\
&\quad = -\int_0^T f(0,u)\,du + \int_0^t \left(b(s,T) - \frac{1}{2}\|v(s,T)\|^2\right) ds + \int_0^t v(s,T)\,dW(s)\\
&\qquad + \int_0^t \underbrace{\left(f(0,u) + \int_0^u \alpha(s,u)\,ds + \int_0^u \sigma(s,u)\,dW(s)\right)}_{=r(u)} du\\
&\quad = \log P(0,T) + \int_0^t \left(r(s) + b(s,T) - \frac{1}{2}\|v(s,T)\|^2\right) ds + \int_0^t v(s,T)\,dW(s).
\end{aligned}$$

Itô's formula now implies (6.2) (→ Exercise 6.2). □

As a corollary, we derive the dynamic equation of the discounted bond price process as follows:

Corollary 6.1 *We have, for $t \le T$,*

$$\frac{P(t,T)}{B(t)} = P(0,T) + \int_0^t \frac{P(s,T)}{B(s)} b(s,T)\,ds + \int_0^t \frac{P(s,T)}{B(s)} v(s,T)\,dW(s).$$

Proof Itô's formula (→ Exercise 6.2). □

6.2 Absence of Arbitrage

In this section we investigate the restrictions on the dynamics (6.1) under the assumption of no arbitrage. In what follows we let $\mathbb{Q} \sim \mathbb{P}$ be an equivalent probability measure of the form (4.8) for some $\gamma \in \mathcal{L}$. With $dW^* = dW - \gamma^\top dt$ we denote the Girsanov transformed $\mathbb{Q}$-Brownian motion, see Theorem 4.6. According to Definition 4.1, we call $\mathbb{Q}$ an ELMM for the bond market if the discounted bond price process $\frac{P(t,T)}{B(t)}$ is a $\mathbb{Q}$-local martingales for $t \le T$, for all T.

Theorem 6.1 (HJM Drift Condition) *$\mathbb{Q}$ is an ELMM if and only if*

$$b(t,T) = -v(t,T)\,\gamma(t)^\top \quad \textit{for all } T,\ d\mathbb{P} \otimes dt\textit{-a.s.} \tag{6.4}$$

In this case, the $\mathbb{Q}$-dynamics of the forward rates $f(t,T)$ are of the form

$$f(t,T) = f(0,T) + \int_0^t \underbrace{\left(\sigma(s,T) \int_s^T \sigma(s,u)^\top du\right)}_{\textit{HJM drift}} ds + \int_0^t \sigma(s,T)\,dW^*(s), \tag{6.5}$$

and the discounted T-bond price satisfies

$$\frac{P(t,T)}{B(t)} = P(0,T)\mathcal{E}_t(v(\cdot,T) \bullet W^*) \tag{6.6}$$

for $t \le T$.

Proof In view of Corollary 6.1 we find that

$$d\frac{P(t,T)}{B(t)} = \frac{P(t,T)}{B(t)} \left(b(t,T) + v(t,T)\,\gamma(t)^\top\right) dt + \frac{P(t,T)}{B(t)} v(t,T)\,dW^*(t).$$

Hence $\frac{P(t,T)}{B(t)}$, $t \le T$, is a $\mathbb{Q}$-local martingale if and only if $b(t,T) = -v(t,T)\,\gamma(t)^\top$ $d\mathbb{P} \otimes dt$-a.s. Since $v(t,T)$ and $b(t,T)$ are both continuous in T, we deduce that $\mathbb{Q}$ is an ELMM if and only if (6.4) holds.

Differentiating both sides of (6.4) in T yields

$$-\alpha(t,T)+\sigma(t,T)\int_t^T \sigma(t,u)^\top du = \sigma(t,T)\,\gamma(t)^\top \quad \text{for all } T,\, d\mathbb{P}\otimes dt\text{-a.s.}$$

Inserting this in (6.1) gives (6.5). Equation (6.6) now follows from Lemma 4.2. □

Remark 6.1 It follows from (6.2) and (6.4) that

$$dP(t,T) = P(t,T)\left(r(t) - v(t,T)\,\gamma(t)^\top\right)dt + P(t,T)v(t,T)\,dW(t).$$

Whence the interpretation of $-\gamma$ as the market price of risk for the bond market.

The striking feature of the HJM framework is that the distribution of $f(t,T)$ and $P(t,T)$ under $\mathbb{Q}$ only depends on the volatility process $\sigma(t,T)$, and not on the $\mathbb{P}$-drift $\alpha(t,T)$. Hence option pricing only depends on σ. This situation is similar to the Black–Scholes stock price model (→ Exercise 4.7).

We can give sufficient conditions for $\frac{P(t,T)}{B(t)}$ to be a true $\mathbb{Q}$-martingale.

Corollary 6.2 *Suppose that* (6.4) *holds. Then* $\mathbb{Q}$ *is an EMM if either*

(a) *the Novikov condition*

$$\mathbb{E}_{\mathbb{Q}}\left[e^{\frac{1}{2}\int_0^T \|v(t,T)\|^2\,dt}\right] < \infty \quad \textit{for all } T \tag{6.7}$$

holds; *or*

(b) *the forward rates are nonnegative*: $f(t,T)\geq 0$ *for all* $t\leq T$.

Proof By Theorem 4.7, the Novikov condition (6.7) is sufficient for $\frac{P(t,T)}{B(t)}$ in (6.6) to be a $\mathbb{Q}$-martingale.

If $f(t,T)\geq 0$, then $0\leq P(t,T)\leq 1$ and $B(t)\geq 1$. Hence $0\leq \frac{P(t,T)}{B(t)}\leq 1$. Since a uniformly bounded local martingale is a true martingale, the corollary is proved. □

6.3 Short-Rate Dynamics

What is the interplay between the short-rate models in Chap. 5 and the present HJM framework? Let us consider the simplest HJM model: a constant $\sigma(t,T)\equiv\sigma>0$. Suppose that $\mathbb{Q}$ is an ELMM. Then (6.5) implies

$$f(t,T) = f(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma W^*(t).$$

Hence for the short rates we obtain

$$r(t) = f(t,t) = f(0,t) + \frac{\sigma^2 t^2}{2} + \sigma W^*(t).$$

This is just the Ho–Lee model of Sect. 5.4.4.

In general, we have the following:

Proposition 6.1 *Suppose that* $f(0,T)$, $\alpha(t,T)$ *and* $\sigma(t,T)$ *are differentiable in* T *with* $\int_0^T |\partial_u f(0,u)|\,du < \infty$ *and such that* (**HJM.1**)–(**HJM.3**) *are satisfied when* $\alpha(t,T)$ *and* $\sigma(t,T)$ *are replaced by* $\partial_T\alpha(t,T)$ *and* $\partial_T\sigma(t,T)$, *respectively.*

Then the short-rate process is an Itô process of the form

$$r(t) = r(0) + \int_0^t \zeta(u)\,du + \int_0^t \sigma(u,u)\,dW(u), \tag{6.8}$$

where

$$\zeta(u) = \alpha(u,u) + \partial_u f(0,u) + \int_0^u \partial_u\alpha(s,u)\,ds + \int_0^u \partial_u\sigma(s,u)\,dW(s).$$

Proof Recall first that

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)\,ds + \int_0^t \sigma(s,t)\,dW(s).$$

Applying the Fubini Theorem 6.2 below to the stochastic integral gives

$$\begin{aligned}\int_0^t \sigma(s,t)\,dW(s) &= \int_0^t \sigma(s,s)\,dW(s) + \int_0^t (\sigma(s,t) - \sigma(s,s))\,dW(s)\\ &= \int_0^t \sigma(s,s)\,dW(s) + \int_0^t\int_s^t \partial_u\sigma(s,u)\,du\,dW(s)\\ &= \int_0^t \sigma(s,s)\,dW(s) + \int_0^t\int_0^u \partial_u\sigma(s,u)\,dW(s)\,du.\end{aligned}$$

Moreover, from the classical Fubini Theorem we deduce in a similar way that

$$\int_0^t \alpha(s,t)\,ds = \int_0^t \alpha(s,s)\,ds + \int_0^t\int_0^u \partial_u\alpha(s,u)\,ds\,du,$$

and finally

$$f(0,t) = r(0) + \int_0^t \partial_u f(0,u)\,du.$$

Combining these formulas, we obtain (6.8). □

6.4 HJM Models

In the preceding sections we have studied the stochastic behavior of the forward rate process $f(t,T)$ for some generic drift and volatility processes $\alpha(\omega,t,T)$

and $\sigma(\omega, t, T)$. For modeling purposes we would prefer a forward rate dependent volatility coefficient

$$\sigma(\omega, t, T) = \sigma(t, T, f(\omega, t, T))$$

for some appropriate function σ. The simplest choice is a deterministic function $\sigma(t, T)$ which does not depend on ω. This results in Gaussian distributed forward rates $f(t, T)$ and leads to simple bond option price formulas, as we will see in Sect. 7.2 below. A particular case is the constant $\sigma(t, T) \equiv \sigma$, which corresponds to the Ho–Lee model as we have seen in Sect. 6.3 above.

It is shown in [90] and [125] that, for any continuous initial forward curve $f(0, T)$, there exists a unique jointly continuous solution $f(t, T)$ of

$$df(t, T) = \left(\sigma(t, T, f(t, T)) \int_t^T \sigma(t, u, f(t, u))\, du\right) dt + \sigma(t, T, f(t, T))\, dW(t) \tag{6.9}$$

if $\sigma(t, T, f)$ is uniformly bounded, jointly continuous, and Lipschitz continuous in the last argument. It is remarkable that the boundedness condition on σ cannot be substantially weakened as the following example shows.

6.4.1 Proportional Volatility

We consider the special case of a single Brownian motion ($d = 1$) and where $\sigma(t, T, f(t, T)) = \sigma f(t, T)$ for some constant $\sigma > 0$. This volatility function is positive and Lipschitz continuous but not bounded. The solution of (6.9), if it existed, must satisfy ($\rightarrow$ Exercise 6.3)

$$f(t, T) = f(0, T) \mathrm{e}^{\sigma^2 \int_0^t \int_s^T f(s,u)\, du\, ds} \mathrm{e}^{\sigma W(t) - \frac{\sigma^2}{2} t}. \tag{6.10}$$

Following the arguments in Avellaneda and Laurence [5, Sect. 13.6], we now sketch that there is no finite-valued solution to expression (6.10).

Indeed, assume for simplicity that the initial forward curve is flat, i.e. $f(0, T) \equiv 1$, and $\sigma = 1$. Differentiating both sides of (6.10) with respect to T, we obtain

$$\partial_T f(t, T) = f(t, T) \int_0^t f(s, T)\, ds = \frac{1}{2} \partial_t \left(\int_0^t f(s, T)\, ds \right)^2.$$

Integrating this equation with respect to t from $t = 0$ to 1, and interchanging the order of differentiation and integration,[2] yields

$$\partial_T \int_0^1 f(s, T)\, ds = \frac{1}{2} \left(\int_0^1 f(s, T)\, ds \right)^2.$$

[2] This argumentation is somehow sketchy. A full rigorous proof that (6.10) is not finite valued can be found in Morton [125].

Solving this differential equation path-wise for $X(T) = \int_0^1 f(s,T)\,ds$, $T \geq 1$, we obtain as unique solution

$$X(T) = \frac{X(1)}{1 - \frac{X(1)}{2}(T-1)}.$$

In view of (6.10), we have $X(1) > 0$. Hence $X(T) \uparrow \infty$ for $T \uparrow \tau$ where $\tau = 1 + \frac{2}{X(1)}$ is a finite random time. We conclude that $f(\omega, t, \tau(\omega))$ must become $+\infty$ for some $t \leq 1$, for almost all ω.

The nonexistence of HJM models with proportional volatility encouraged the development of the so-called LIBOR market models, which will be further discussed in Chap. 11 below.

6.5 Fubini's Theorem

In this section we prove Fubini's theorem for stochastic integrals. For the classical version of Fubini's theorem, we refer to the standard textbooks in integration theory.

Theorem 6.2 (Fubini's theorem for Stochastic Integrals) *Consider the $\mathbb{R}^d$-valued stochastic process $\phi = \phi(\omega, t, s)$ with two indices, $0 \leq t, s \leq T$, satisfying the following properties:*[3]

(a) *ϕ is $\mathrm{Prog}_T \otimes \mathcal{B}[0,T]$-measurable;*
(b) $\sup_{t,s} \|\phi(t,s)\| < \infty$.[4]

Then $\lambda(t) = \int_0^T \phi(t,s)\,ds \in \mathcal{L}$, and there exists a $\mathcal{F}_T \otimes \mathcal{B}[0,T]$-measurable modification $\psi(s)$ of $\int_0^T \phi(t,s)\,dW(t)$ with $\int_0^T \psi^2(s)\,ds < \infty$ a.s.

Moreover, $\int_0^T \psi(s)\,ds = \int_0^T \lambda(t)\,dW(t)$, that is,

$$\int_0^T \left(\int_0^T \phi(t,s)\,dW(t) \right) ds = \int_0^T \left(\int_0^T \phi(t,s)\,ds \right) dW(t). \tag{6.11}$$

Proof Without loss of generality, we can put $d = 1$, as we just have to prove (6.11) componentwise.

We assume first that (b) is replaced by

(b′) $|\phi| \leq C$ for some finite constant C.

Then clearly $\lambda \in \mathcal{L}$. Denote by $\mathcal{H}$ the set of all ϕ satisfying (a) and (b′) and for which the theorem holds. We will show that $\mathcal{H}$ contains all ϕ satisfying (a) and (b′).

[3] Prog_T denotes the progressive σ-algebra Prog restricted to $\Omega \times [0,T]$.

[4] Note that this is a ω-wise boundedness assumption.

Let K be some bounded progressive process and f some bounded $\mathcal{B}[0,T]$-measurable function. Then $\phi(\omega,t,s) = K(\omega,t) f(s)$ satisfies

$$\int_0^T \phi(t,s)\,ds = K(t)\int_0^T f(s)\,ds, \qquad \int_0^T \phi(t,s)\,dW(t) = f(s)\int_0^T K(t)\,dW(t)$$

and thus $\phi \in \mathcal{H}$. It follows from elementary measure theory that processes of the form Kf generate the σ-algebra $\mathrm{Prog}_T \otimes \mathcal{B}[0,T]$.

Next, we let $\phi_n \in \mathcal{H}$ and suppose that $\phi_n \uparrow \phi$ for some bounded $\mathrm{Prog}_T \otimes \mathcal{B}[0,T]$-measurable process ϕ. We can assume that $\sup_{t,s} |\phi_n| \le N$, for some finite constant N that does not depend on n. Define

$$\psi_n(s) = \int_0^T \phi_n(t,s)\,dW(t).$$

From the Itô isometry and dominated convergence it follows that

$$\mathbb{E}\left[\left(\psi_n(s) - \int_0^T \phi(t,s)\,dW(t)\right)^2\right] = \mathbb{E}\left[\int_0^T |\phi_n(t,s) - \phi(t,s)|^2\,dt\right] \to 0 \tag{6.12}$$

for $n \to \infty$, for all $s \le T$. Define $A = \{(\omega,s) \mid \lim_n \psi_n(\omega,s) \text{ exists}\}$. Then A is $\mathcal{F}_T \otimes \mathcal{B}[0,T]$-measurable and so is the process

$$\psi(\omega,s) = \begin{cases} \lim_n \psi_n(\omega,s), & \text{if } (\omega,s) \in A, \\ 0, & \text{otherwise.} \end{cases} \tag{6.13}$$

In view of (6.12) we have $\psi(s) = \int_0^T \phi(t,s)\,dW(t)$ a.s. for all $s \le T$. Thus, $\psi(s)$ has the desired properties. From Jensen's integral inequality, the Itô isometry and dominated convergence we then have, on one hand,

$$\begin{aligned} &\mathbb{E}\left[\left(\int_0^T \psi_n(s)\,ds - \int_0^T \psi(s)\,ds\right)^2\right] \\ &\le T\int_0^T \mathbb{E}\left[(\psi_n(s) - \psi(s))^2\right] ds \\ &= T\int_0^T \mathbb{E}\left[\int_0^T |\phi_n(t,s) - \phi(t,s)|^2\,dt\right] ds \to 0 \quad \text{for } n \to \infty. \end{aligned} \tag{6.14}$$

On the other hand,

$$\begin{aligned} &\mathbb{E}\left[\left(\int_0^T \left(\int_0^T \phi_n(t,s)\,ds\right) dW(t) - \int_0^T \lambda(t)\,dW(t)\right)^2\right] \\ &= \mathbb{E}\left[\int_0^T \left|\int_0^T \phi_n(t,s)\,ds - \int_0^T \phi(t,s)\,ds\right|^2 dt\right] \\ &\le T\mathbb{E}\left[\int_0^T \int_0^T |\phi_n(t,s) - \phi(t,s)|^2\,ds\,dt\right] \to 0 \quad \text{for } n \to \infty. \end{aligned} \tag{6.15}$$

Combining (6.14) and (6.15) shows that (6.11) also holds for ϕ, and thus $\phi \in \mathcal{H}$.

Since $\mathcal{H}$ is also a vector space, it follows from the monotone class theorem 6.3 below that $\mathcal{H}$ contains all bounded $\mathrm{Prog}_T \otimes \mathcal{B}[0, T]$-measurable processes, which proves the theorem under the assumption (b′).

For the general case, we define the nondecreasing sequence of stopping times

$$\tau_n = \inf\left\{ t \mid \sup_s |\phi(t, s)| > n \right\} \wedge T.$$

Then $\phi_n(t, s) = \phi(t, s) 1_{\{t \le \tau_n\}}$ satisfies (b′). From the above step, we thus obtain $\lambda_n \in \mathcal{L}$ and some $\mathcal{F}_T \otimes \mathcal{B}[0, T]$-measurable $\psi_n(s)$ with $\psi_n(s) = \int_0^{T \wedge \tau_n} \phi(t, s)\, dW(t)$ a.s. for all $s \le T$. Since $\tau_n \uparrow T$, the process ψ is well defined by setting $\psi(s) = \psi_n(s)$ for $s \le \tau_n$ and has the desired properties. Moreover, $\lambda_n(t) = \lambda(t) 1_{\{t \le \tau_n\}}$, and we infer that $\lambda \in \mathcal{L}$ and (6.11) holds on $\{\tau_n = T\}$ for all $n \ge 1$. Since $\mathbb{P}[\tau_n < T] \to 0$, letting $n \to \infty$, the theorem is proved. □

Corollary 6.3 *Let ϕ be as in Theorem 6.2. Then the process*

$$\int_0^s \phi(t, s)\, dW(t), \quad s \in [0, T],$$

has a progressive modification $\pi(s)$ with $\int_0^T \pi^2(s)\, ds < \infty$ a.s.

Proof For $\phi(\omega, t, s) = K(\omega, t) f(s)$, with bounded progressive process K and bounded measurable function f, the process

$$\int_0^s \phi(t, s)\, dW(t) = f(s) \int_0^s K(t)\, dW(t)$$

is clearly progressive and path-wise square integrable. Now use a similar monotone class and localization argument as in the proof of Theorem 6.2 (→ Exercise 6.4). □

Here we recall the monotone class theorem, which is proved in e.g. [154, Sect. 12.6].

Theorem 6.3 (Monotone Class Theorem) *Suppose the set $\mathcal{H}$ consists of real-valued bounded functions defined on a set Ω with the following properties:*

(a) *$\mathcal{H}$ is a vector space;*
(b) *$\mathcal{H}$ contains the constant function 1_Ω;*
(c) *if $f_n \in \mathcal{H}$ and $f_n \uparrow f$ monotone, for some bounded function f on Ω, then $f \in \mathcal{H}$.*

If $\mathcal{H}$ contains a collection $\mathcal{M}$ of real-valued functions, which is closed under multiplication (that is, $f, g \in \mathcal{M}$ implies $fg \in \mathcal{M}$). Then $\mathcal{H}$ contains all real-valued bounded functions that are measurable with respect to the σ-algebra which is generated by $\mathcal{M}$ (that is, $\sigma\{f^{-1}(A) \mid A \in \mathcal{B},\ f \in \mathcal{M}\}$).

6.6 Exercises

Exercise 6.1 Using the monotone class theorem 6.3, show that a process X is progressive if and only if X is Prog-measurable.

Exercise 6.2 Complete the proofs of Lemma 6.1 and Corollary 6.1.

Exercise 6.3 Show that the solution to the proportional volatility HJM model would equal (6.10) if it existed.

Exercise 6.4 Complete the proof of Corollary 6.3.

Exercise 6.5 The goal of this exercise is to show that parallel shifts of the forward curve creates arbitrage. Consider first the one-period model for the forward curve

$$f(0,t) = 0.04, \quad t \geq 0,$$
$$f(\omega, 1, t) = \begin{cases} 0.06, & t \geq 1,\ \omega = \omega_1, \\ 0.02, & t \geq 1,\ \omega = \omega_2, \end{cases}$$

where $\Omega = \{\omega_1, \omega_2\}$ with $\mathbb{P}[\omega_i] > 0$, $i = 1, 2$.

(a) Show that the matrix

$$\begin{pmatrix} P(0,1) & P(0,2) & P(0,3) \\ P(\omega_1,1,1) & P(\omega_1,1,2) & P(\omega_1,1,3) \\ P(\omega_2,1,1) & P(\omega_2,1,2) & P(\omega_2,1,3) \end{pmatrix}$$

is invertible.

(b) Use (a) to find an arbitrage strategy with value process $V(0) = 0$ and $V(\omega_i, 1) = 1$ for both ω_i.

Next, we extend the one-period finding to the continuous time HJM framework with a one-dimensional driving Brownian motion W. An HJM forward curve evolution by parallel shifts is then of the form

$$f(t,T) = h(T-t) + Z(t)$$

for some deterministic initial curve $f(0,T) = h(T)$ and some Itô process $dZ(t) = b(t)\,dt + \rho(t)\,dW(t)$ with $Z(0) = 0$.

(c) Show that the HJM drift condition implies $b(t) \equiv b$, $\rho^2(t) \equiv a$, and

$$h(x) = -\frac{a}{2}x^2 + bx + c$$

for some constants $a \geq 0$, and b, $c \in \mathbb{R}$.

(d) How is this model related to the Ho–Lee model from Sect. 5.4.4?

(e) Argue that, for generic initial curves $f(0,T)$, non-trivial forward curve evolutions by parallel shifts are excluded by the HJM drift condition.

Exercise 6.6 Consider the Hull–White extended Vasiček short-rate dynamics under the EMM $\mathbb{Q} \sim \mathbb{P}$

$$dr(t) = (b(t) + \beta r(t))\,dt + \sigma\,dW^*(t),$$

where W^* is a standard real-valued $\mathbb{Q}$-Brownian motion, β and $\sigma > 0$ are constants, and $b(t)$ is a deterministic continuous function. Using the results from Sect. 5.4.5, find the corresponding HJM forward rate dynamics

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \sigma(s,T)\,dW^*(s).$$

(a) What are $f(0,T)$, $\alpha(s,T)$, $\sigma(s,T)$?
(b) Verify your findings in (a), by checking whether $\alpha(s,T)$ satisfies the HJM drift condition.
(c) Discuss the role of $b(s)$. Do $\alpha(s,T)$ and $\sigma(s,T)$ depend on $b(s)$?
(d) What does this imply for the Vasiček model ($b(s) \equiv b$)?
(e) Verify Proposition 6.1 by showing that $dr(t) = \zeta(t)\,dt + \sigma(t,t)\,dW^*(t)$, where $\zeta(t)$ is given by f, α, σ as in Proposition 6.1.

6.7 Notes

The approach in Sect. 6.4 has been carried out by Heath, Jarrow and Morton [90], and in more depth and generality by Morton [125], and also in [68] and [35]. The proof of Fubini's Theorem 6.2 for stochastic integrals follows along the line of arguments in Protter [132, Sect. IV.6], however cannot be immediately deduced from [132, Theorem 64], as it requires a localization step carried out above.