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Application of Sampling Variance Smoothing Methods for Small Area Proportion Estimation

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Sampling variance smoothing is an important topic in small area estimation. In this article, we propose sampling variance smoothing methods for small area proportion estimation. In particular, we consider the generalized variance function and design effect methods for sampling variance smoothing. We evaluate and compare the smoothed sampling variances and small area estimates based on the smoothed variance estimates through analysis of survey data from Statistics Canada. The results from real data analysis and simulation study indicate that the proposed sampling variance smoothing methods perform very well for small area estimation.

Key words: Coefficient of variation; design effect; generalized variance function; log-linear model; relative error.

1. Introduction

Small area estimation has become very popular and important in both public and private agencies due to the growing demand for reliable small domain estimates. Small area estimation is based on models that borrow strength across areas and combine different sources of information in order to obtain reliable estimates. In this article, we focus on area level models that are based on direct survey estimates aggregated from the unit level data and area level auxiliary variables. Various area level models have been proposed in the literature to improve the precision of the direct survey estimates: a good summary of these methods is discussed in Rao and Molina (2015). The Fay-Herriot model (Fay and Herriot 1979) is a basic area level model that is widely used in practice. The Fay-Herriot model has two components, namely, a sampling model for the direct survey estimates and a linking model for the small area parameters of interest. The sampling model assumes that there exists a direct survey estimator y_i, which is usually design unbiased, for the small area parameter θ_i such that

$$y_i = \theta_i + e_i, \quad i = 1, \ldots, m, \tag{1}$$

where e_i is the sampling error associated with the direct estimator y_i and m is the number of small areas. It is customary in practice to assume that the e_i's are independently distributed normal random variables with mean $E(e_i) = 0$ and sampling variance $Var(e_i) = \sigma_i^2$. The linking model assumes that the small area parameter of interest θ_i is related to area level auxiliary variables $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})'$ through a linear regression model

$$\theta_i = \boldsymbol{x}_i{}'\boldsymbol{\delta} + v_i, \quad i = 1, \ldots, m, \tag{2}$$

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where $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_p)'$ is a $p \times 1$ vector of regression coefficients, and v_i's are area-specific random effects assumed to be independent and identically distributed with $E(v_i) = 0$ and $Var(v_i) = \sigma_v^2$.

The assumption of normality for v_i is generally also included. The model variance σ_v^2 is unknown and needs to be estimated from the data. For the Fay-Herriot model, the sampling variance σ_i^2 is assumed to be known in model (1). As this is a very strong assumption, a smoothing or modeling approach is usually used to estimate σ_i^2. The sampling variance can be smoothed or can be modeled directly as in Wang and Fuller (2003), You and Chapman (2006), Maples et al. (2009), Dass et al. (2012), Sugasawa and Kubokawa (2017), Sugasawa et al. (2017), Ghosh et al. (2018), and so on. It is also shown in You (2021) that the smoothing approach can provide more efficient and accurate model-based estimates than the modeling approach for small areas under the hierarchical Bayes framework. Lesage et al. (2021) also have some discussions on the sampling variance smoothing for the Fay-Herriot model.

The objective of this article is to compare different methods that smooth the direct estimates of the sampling variances for proportions in small area estimation using the Fay-Herriot model. Let $\hat{p}_{iw}$ be the direct design-based estimator for the proportion p_i for a given characteristic in the i-th area. Applying the Fay-Herriot model to $\hat{p}_{iw}$, we have

$$\hat{p}_{iw} = p_i + e_i, \tag{3}$$

where the sampling variance $Var(e_i) = \sigma_i^2$ is unknown. Now let $\hat{V}_i$ be a direct sampling variance estimator for σ_i^2 obtained from the survey data. Usually some of the $\hat{V}_i$'s are very unstable due to small sample sizes. We, therefore, need to smooth the sampling variance estimate, $\tilde{V}_i$, and then treat the resulting smoothed variance estimate $\tilde{V}_i$ in the sampling model (3) as known.

In this article, we compare two smoothing methods. One method is based on the generalized variance function (GVF, see, e.g., Wolter 2007), and the other one is based on design effects (DEFF). We then propose an average smoothed (ASM) variance estimator that combines the GVF and DEFF smoothed estimators. The main purpose of the article is to promote the proposed GVF and DEFF methods. The ASM is used as an additional choice as it pools the GVF and DEFF estimates by taking their average. Recently Hirose et al. (2023) proposed a variance stabilizing transformation for small area estimation of proportions. They used the arc-sin transformation approach to construct a Fay-Herriot model with known sampling variance. Note that their transformation approach is applied to data assumed to have a binomial distribution, and this implies simple random sampling (SRS). However, many surveys are based on various complex sampling designs. The sampling variance smoothing approach can be applied to any estimator of proportions that is based on complex designs. We re-estimate the variance via a smoothing process that involves all small area estimates of the sampling variances.

There are many applications of the GVF in small area estimation, see, for example, the early work of Dick (1995) and the recent application in Hidiroglou et al. (2019). The DEFF can also be used in variance modeling and smoothing for small area estimation. For example, You (2008) used the smoothed design effects over time to obtain the smoothed variance and covariance matrices. Liu et al. (2014) also applied area level models to proportions using design effects for the sampling variance smoothing and modeling.

In this article, we provide a general method to compute the design effect and propose a smoothed variance estimator based on the average design effects over areas. We will also show that the DEFF-smoothed variance estimator and the GVF-smoothed variance estimator are roughly equivalent under certain conditions. We illustrate the smoothing methods via various survey data sets and a simulation study.

The article is organized as follows. In section 2, we propose several sampling variance smoothing procedures that include the GVF and DEFF methods. In section 3, we apply the proposed smoothing methods to two Statistics Canada survey data and compare the smoothing variances. In section 4, we compare the model-based estimates based on different smoothed sampling variance estimates using the Canadian Labor Force Survey (LFS) survey data. Section 5 is a simulation study that compares proposed variance smoothing estimators. In section 6, we offer some concluding remarks and suggestions.

2. Sampling Variance Smoothing Methods

2.1. Smoothing Using Log-Linear Models

In this section, we will construct a GVF model to obtain smoothed sampling variances. This procedure is widely used in practice to model the variance. We apply a log-linear regression model to the direct sampling variance $\tilde{V}_i$ using the sample size n_i as the auxiliary variable in the model as follows:

$$log(\hat{V}_i) = \beta_0 + \beta_1 log(n_i) + \varepsilon_i, \quad i = 1, ..., m, \tag{4}$$

where the model error term is $\varepsilon_i \sim N(0, \tau^2)$, and the model error variance τ^2 is unknown. Note that the proposed regression model (4) is equivalent to the following model:

$$log(\hat{V}_i) = \beta_0 + \beta_1 log(n_i^{-1}) + \varepsilon_i, \quad i = 1, ..., m, \tag{5}$$

where $log(n_i^{-1})$ is used as the auxiliary variable. The proposed GVF models (4) or (5) are the same models used in You (2021) for the hierarchical Bayes (HB) modeling of sampling variance. This GVF model also extends the model proposed by Souza et al. (2009) for sampling variances by using $log(n_i^{-1})$ and adding a normal random effect (ε_i) to the regression part in the model.

Let $\hat{\beta}_0$ and $\hat{\beta}_1$ denote the ordinary least square estimators of the regression coefficients β_0 and β_1. A naïve GVF-smoothed estimator of the sampling variance is obtained by taking the exponential of the fitted value:

$$\tilde{V}_i^{naive} = exp(\hat{\beta}_0 + \hat{\beta}_1 log(n_i)). \tag{6}$$

Dick (1995) used the naïve smoothed estimator $\tilde{V}_i^{naive}$ in the application of census undercoverage small area estimation. As noted by Rivest and Belmonte (2000), the naïve smoothed estimator $\tilde{V}_i^{naive}$ underestimates the sampling variance. This can be seen as follows. If Y is a log-normal random variable with mean μ and variance τ^2, the mean of Y is $E(Y) = exp(\mu)exp(\tau^2/2)$. It follows that the smoothed estimator $\tilde{V}_i^{naive}$ underestimates the true values by ignoring the second term $exp(\tau^2/2)$ in the mean of the log-normal random variable. Denote as $\hat{\omega}_{RB} = exp(\hat{\tau}^2/2)$ the Rivest and Belmonte (2000) correction, where $\hat{\tau}^2$ is the estimated residual variance of the proposed log-linear regression model (4). Then a GVF-smoothed estimator, denoted as $\tilde{V}_i^{GVF.RB}$, is given by

$$\tilde{V}_i^{GVF.RB} = \tilde{V}_i^{naive} \cdot \hat{\omega}_{RB} = \tilde{V}_i^{naive} \cdot exp(\hat{\tau}^2/2). \tag{7}$$

The naïve GVF estimator $\tilde{V}_i^{naive}$ in Equation (6) underestimates the sampling variance by $exp(\hat{\tau}^2/2)$. This term is always greater than 1, and sometimes it can be large, depending on the value of $\hat{\tau}^2$.

Hidiroglou et al (2019) proposed another correction term for the naïve estimator $\tilde{V}_i^{naive}$. Let $\tilde{V}^{naive}$ be the sum of the naïve smoothed variance estimators, that is, $\tilde{V}_i^{naive} = \Sigma_{i=1}^m \tilde{V}_i^{naive}$, and $\tilde{V}^{total}$ be the sum of the direct sampling variances, that is, $\tilde{V}^{total} = \Sigma_{i=1}^m \tilde{V}_i$. Following Hidiroglou et al. (2019), we define a correction term, $\hat{\omega}_{HBY} = \tilde{V}^{total}/\tilde{V}^{naive}$, named as the Hidiroglou, Beaumont, and Yung (HBY) correction term. This leads to a second GVF-smoothed variance estimator, denoted as $\tilde{V}_i^{GVF.HBY}$. It is given by

$$\tilde{V}_i^{GVF.HBY} = \tilde{V}_i^{naive} \cdot \hat{\omega}_{HBY} = \tilde{V}_i^{naive} \cdot \frac{\tilde{V}^{total}}{\tilde{V}^{naive}} \tag{8}$$

Note that $\hat{\omega}_{HBY}$ is obtained as an alternative estimator to $exp(\tau^2/2)$ using the method of moments (Beaumont and Bocci 2016). This avoids the sensitivity of the GVF model to deviations from the normality assumption of ε_i in model (4). The HBY correction term is also equivalent to the so-called smearing estimator, see Duan (1983). A nice property of $\tilde{V}_i^{GVF.HBY}$ is that the average of the smooth variance estimates is equal to the average of the direct sampling variance estimates, that is,

$$\frac{1}{m}\sum_{i=1}^{m} \tilde{V}_i^{GVF.HBY} = \frac{1}{m}\sum_{i=1}^{m} \hat{V}_i.$$

This property may ensure that the smoothing procedure does not systematically overestimate or underestimate the sampling variances.

2.2. *Smoothing Using Design Effects*

Let $\hat{p}_{iw}$ be the direct design-based estimate for a proportion p_i and $\hat{V}_i$ the corresponding direct sampling variance under complex design for the i-th small area. Then the estimated design effect can be approximately computed as

$$def\, f_i = \frac{\hat{V}_i}{\hat{p}_{iw}(1-\hat{p}_{iw})/n_i + \hat{V}_i/n_i}, \tag{9}$$

where n_i is the sample size of the i-th small area; see Gambino (2009, 143), Remark iii for a more detailed discussion of the special case 0-1 variables. Noting that the $def\, f_i$ in Equation (9) is not equal to 1 under simple random sampling design, we modify the $def\, f_i$ by multiplying it by a correction term $(n_i+1)/n_i$:

$$def\, f_i = \frac{\hat{V}_i}{\hat{p}_{iw}(1-\hat{p}_{iw})/n_i + \hat{V}_i/n_i} \cdot \frac{n_i+1}{n_i}. \tag{10}$$

Using Equation (10), we can re-write the design-based sampling variance $\hat{V}_i$ as

$$\hat{V}_i = def\,f_i.\frac{\hat{p}_{iw}(1-\hat{p}_{iw})}{n_i}\cdot\left(1+\frac{1-deff_i}{n_i}\right)^{-1}. \tag{11}$$

If the sample size n_i is large, the term $(1-def\,f_i)/n_i$ may be negligible in Equation (11) so that the Equation (11) reduces to

$$\hat{V}_i = def\,f_i.\frac{\hat{p}_{iw}(1-\hat{p}_{iw})}{n_i}. \tag{12}$$

Equation (12) is used, for example, in Liu et al. (2014) for sampling variance smoothing and modeling. However, in small area estimation, n_i can be very small, and the term $(1-def\,f_i)/n_i$ may not be negligible.

We can compute all the design effects $def\,f_i$'s using Equation (10) for all areas, and the average value over all areas, thereby obtaining a smoothed design effect $\overline{def\,f} = \frac{1}{m}\Sigma_{i=1}^{m} def\,f_i$. The average proportion estimate over all areas is given by $\bar{p}_w = \frac{1}{m}\Sigma_{i=1}^{m}\hat{p}_{iw}$. Replacing the $deff_i$ by $\overline{def\,f}$ and $\hat{p}_{iw}$ by $\bar{p}_w$ in Equation (11), a DEFF-smoothed estimator of the sampling variance for proportion estimate $\hat{p}_{iw}$ is:

$$\tilde{V}_i^{DEFF} = \overline{def\,f}.\frac{\bar{p}_w(1-\bar{p}_w)}{n_i}.\left(1+\frac{1-\overline{def\,f}}{n_i}\right)^{-1}. \tag{13}$$

If the sample size n_i is large, then the term $\left(1-\overline{def\,f}\right)/n_i$ in $\tilde{V}_i^{DEFF}$ can be negligible. The smoothed variance $\tilde{V}_i^{deff}$ can then be simplified to

$$\tilde{V}_i^{def\,f} = \overline{deff}.\frac{\bar{p}_w(1-\bar{p}_w)}{n_i} \tag{14}$$

2.3. Comparing the GVF and DEFF Smoothing

We now show the similarity between the GVF-estimators and the DEFF-estimator $\tilde{V}_i^{DEFF}$ under certain conditions. Using $\tilde{V}_i^{GVF.RB}$ as an illustration, we can express this term as:

$$\tilde{V}_i^{GVF.RB} = exp\left(\hat{\beta}_0+\hat{\beta}_1\cdot log(n_i)\right)\cdot exp(\frac{\hat{\tau}^2}{2})$$

$$= exp(\hat{\beta}_0+\frac{\hat{\tau}^2}{2})\cdot exp\left(\hat{\beta}_1\cdot log(n_i)\right) = C_0\cdot exp(log(n_i)^{\hat{\beta}_1})$$

$$= C_0\cdot n_i^{\hat{\beta}_1}$$

where $C_0 = exp(\hat{\beta}_0+\frac{\hat{\tau}^2}{2})$ is a constant. If the value of the regression coefficient $\hat{\beta}_1$ is close to -1, then the GVF-estimator $\tilde{V}_i^{GVF.RB}$ can be approximately expressed as $\tilde{V}_i^{GVF.RB}\approx C_0/n_i$.

The DEFF-estimator $\tilde{V}_i^{DEFF}$ can be rewritten as follows:

$$\tilde{V}_i^{DEFF} = \overline{def\,f}\cdot\frac{\bar{p}_w(1-\bar{p}_w)}{n_i}\cdot\left(1+\frac{1-\overline{def\,f}}{n_i}\right)^{-1}$$

$$= \frac{C_1}{n_i} \cdot \left(\frac{n_i + 1 - \overline{def\,f}}{n_i} \right)^{-1} = \frac{C_1}{n_i + 1 - \overline{def\,f}}$$

$$\approx \frac{C_1}{n_i},$$

where $C_1 = \overline{def\,f} \cdot \bar{p}_w (1 - \bar{p}_w)$ is a constant. Both the GVF-estimator $\tilde{V}_i^{GVF.RB}$ and the DEFF-estimator $\tilde{V}_i^{DEFF}$ are proportional to n_i^{-1} if the regression coefficient $\hat{\beta}_1$ is close to -1 in the GVF regression model. Given this condition, both the GVF and DEFF smoothed variances should perform similarly. Hirose et al. (2023) used the arc-sin transformation for binomial samples to construct the Fay-Herriot model with a fixed known variance of $1/4n_i$, which on the other hand, shows that the sampling variance is proportional to $1/n_i$. Their result for variance estimation via a binomial data transformation is consistent with our proposed GVF and DEFF smoothing variance.

In practical applications, we can combine the GVF and DEFF smoothed variance estimators by averaging them. We denote this averaged estimator as $\tilde{V}_i^{ASM} = (\tilde{V}_i^{GVF.RB} + \tilde{V}_i^{GVF.HBY} + \tilde{V}_i^{DEFF})/3$, where ASM stands for average smoothed. This simple data pooling method can provide additional smoothing to obtain the final smoothed variance estimate. As we will see in the LFS small area application (Section 4) and a simulation study (Section 5), the average smoothed estimator $\tilde{V}_i^{ASM}$ can perform very well and lead to large bias and CV reductions for small area estimates.

3. Application of Sampling Variance Smoothing

In this section, we will compare the GVF-estimators and DEFF-estimator by analysing two survey data sets. These data sets have information about the disease rate estimates of the Canadian Community Health Survey (CCHS) and adult disability rate estimates from the Participation and Activity Limitation Survey (PALS). Estimated variances for these two surveys are computed via the Rao-Wu bootstrap procedure. This procedure constructs bootstrap weights that reflect the sample details: see Rao and Wu (1988) or Rao et al. (1992) for details on how the bootstrap weights are computed.

3.1. *CCHS Application*

The CCHS is a federal survey conducted by Statistics Canada. The primary objective of CCHS is to provide timely and reliable estimates of health determinants, health status, and health system utilization across Canada. It is a cross-sectional survey that is carried out on a two-year collection cycle. The first year of the survey cycle "x.1" targets individuals aged 12 or older who are living in private dwellings, and it is a general population health survey with a large sample (130,000 persons) designed to provide reliable estimates at the health region, provincial and national levels. The second year of the survey cycle "x.2" has a smaller sample (30,000 persons) allocated based on provincial sample buy-ins and is designed to provide provincial and national level results on specifically focused health topics. Cycle "x.1" of the CCHS collected data corresponds to 136 health regions in the ten provinces and three territories. It primarily used two sampling frames. The first one, used as the primary frame, was based on the area frame designed for the Canadian Labour Force Survey, and within the area frame, a multistage stratified cluster design was used to sample dwellings. The second frame consists of a list of telephone numbers. Random digit dialing methodology is used in some of the health regions for cost reasons. Following You and Zhou (2011), we use a small data set from Cycle 1.1 containing the estimates of asthma rates for 20 health regions in the province of British Columbia (BC) to demonstrate the sampling variance smoothing. In our data analysis, we use direct point estimates and direct sampling variance estimates to obtain the smoothed sampling variance estimates. Details of the methodology for the CCHS are given in Béland (2002).

For the CCHS data set, Figure 1 shows the plot of the log sampling variance $log(\tilde{V}_i)$ vs log sample size $log(n_i)$ with the fitted regression line. The GVF model fitting is very good as shown in Figure 1. The estimated regression parameters with standard errors (in parentheses) in the log-linear regression model (4) are obtained as $\hat{\beta}_0 = -2.861$ (1.321) and $\hat{\beta}_1 = -0.926$ (0.208). The residual correction term $exp(\hat{\tau}^2/2)$ is equal to 1.029. The HBY correction term in (8) is obtained as $\hat{\omega}_{HBY} = \hat{V}^{total}/\tilde{V}^{native} = 1.031$. Since these two correction terms are almost identical for this data set, so the two GVF estimators $\tilde{V}_i^{GVF.RB}$ and $\tilde{V}_i^{GVF.HBY}$ are almost the same. Since the correction term is close to 1, the naïve estimator $\tilde{V}_i^{native}$ just slightly underestimates the sampling variances.

Recall that in section 2, we claimed that the DEFF-estimator and the GVF-estimator should be approximately equivalent if the regression coefficient β_1 was close to -1. In this application, the estimated coefficient is $\hat{\beta}_1 = -0.926$. We would therefore expect $\tilde{V}_i^{DEFF}$, $\tilde{V}_i^{GVF.RB}$ and $\tilde{V}_i^{GVF.HBY}$ to be similar as well. Figure 2 compares the direct and smoothed sampling variance estimates sorted by the corresponding sample size (from small to large). For a more detailed comparison, we use the standard deviation as the squared root of the variance and plot the smoothed deviation with the direct deviation. In this application, all three smoothed variance estimators $\tilde{V}_i^{GVF.RB}$, $\tilde{V}_i^{GVF.HBY}$ and $\tilde{V}_i^{DEFF}$ and are almost identical and perform almost the same. The three smoothed estimators perform as expected and lead to smoothed sampling variances. For areas with large sample sizes, as expected, the smoothed variances and the direct variances are close to one another, and the smoothing method hardly modifies the direct estimate of sampling variance for large sample sizes.

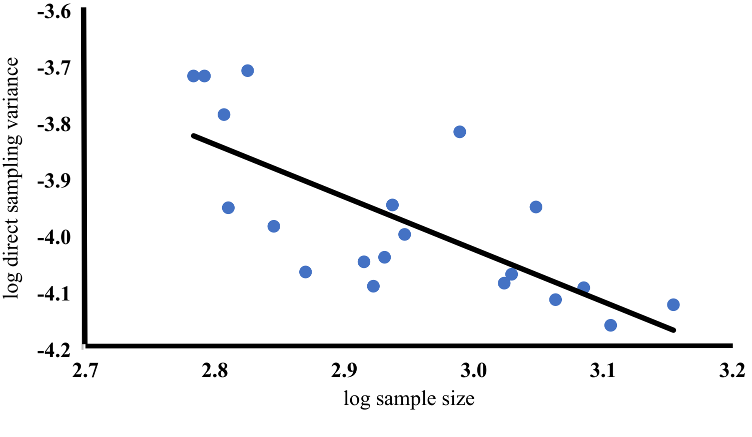

Fig. 1. Log direct sampling variance vs log sample size (BC health data).

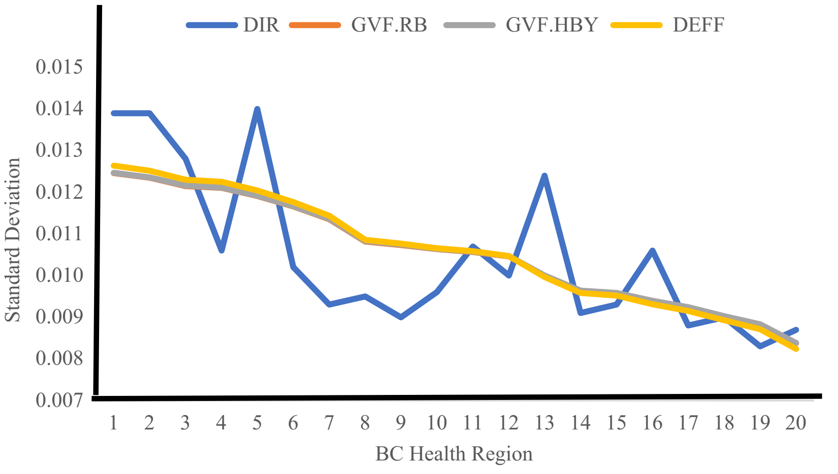

Fig. 2. Comparison of direct and smoothed deviation (BC health data).

3.2. PALS Application

The Participation and Activity Limitation Survey (PALS) is a post census survey that collects information about persons with disabilities whose everyday activities are limited because of a health-related condition or problem. This nationwide survey provides key information on the prevalence of different types of disabilities, on support provided to people with disabilities, on their labour force profile, their income, and their participation in society. The PALS sample was 48,000, consisting of approximately 39,000 adults and 9,000 children. The sample was selected using a two-phase stratified design where in the first phase, a Census questionnaire was distributed to approximately one out of five persons, and in the second phase, a stratified sample was selected based on characteristics from the first phase. However, the number of respondents to the survey does not allow for accurate direct estimates at the sub-provincial level. Following the demands to that effect which were expressed by many provincial governments as well as municipalities, Statistics Canada has put in place a model-based approach to small area estimation for the disability count and rate. Following Bizier et al. (2009), we consider the data of adult disability rate estimates for 116 small areas across Canada. These 116 areas include metropolitan areas, census agglomerations, and other sub-provincial areas. The survey took place between November 2006 and February 2007. Details of the methodology for PALS are given in Langlet et al. (2003).

For the PALS data set, Figure 3 shows the plot of log direct sampling variance vs log sample size. It is very clear from Figure 3 that the linear regression GVF model (4) is suitable for the PALS data. The estimated regression parameters with standard errors in the log-linear regression model (4) are obtained as $\hat{\beta}_0 = -3.033$ (0.263) and $\hat{\beta}_1 \; -1.029$ (0.056). The residual correction term $exp(\hat{\tau}^2/2)$ is equal to 1.334. The HBY correction term is $\hat{\omega}_{HBY} = \hat{V}^{total}/\tilde{V}^{navie} = 1.163$. Since the estimated coefficient is $\hat{\beta}_1 = -1.029$, we would expect the DEFF and GVF estimators to perform similarly. The naïve estimator $\tilde{V}_i^{navie}$ will underestimate the sampling variances, and estimates $\tilde{V}_i^{GVF.RB}$ will be slightly larger than $\tilde{V}_i^{GVF.HBY}$ in this example.

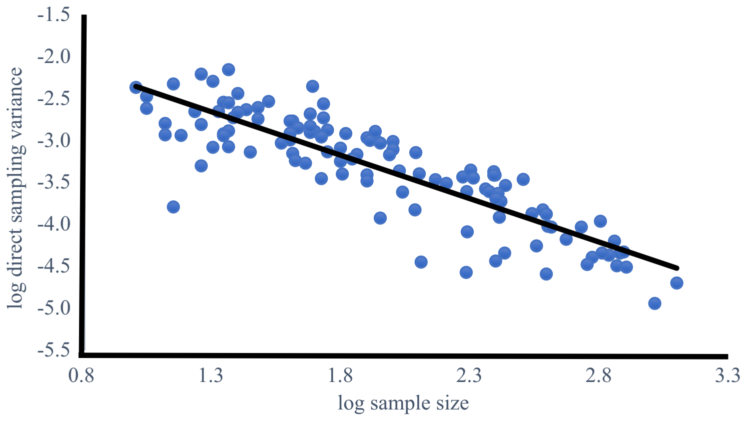

Fig. 3. Log direct sampling variance vs log sample size (PALS data).

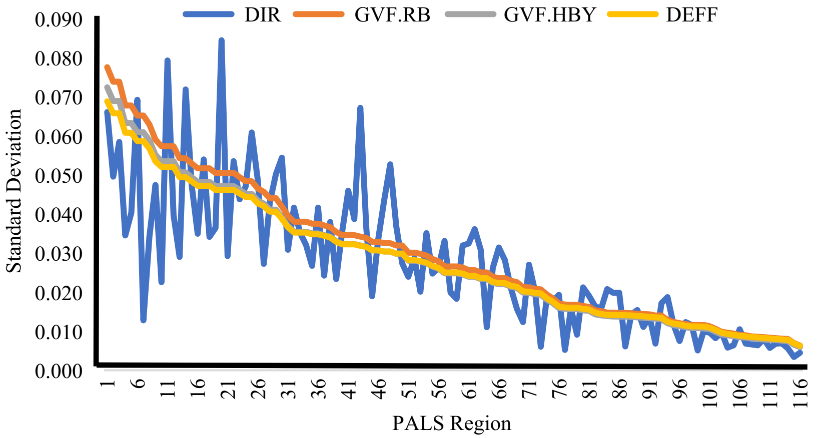

Fig. 4. Comparison of direct and smoothed deviation (PALS data).

Figure 4 compares the direct and smoothed sampling variances for the PALS data. The sampling variance estimates are sorted by the corresponding sample size from small to large. It is clear from Figure 4 that the direct sampling variance estimates have large variations when the sample size is small. The GVF and DEFF smoothed estimators perform similarly and lead to smoothed variance estimates. When the sample size is large, the direct and the three smoothed estimates are about the same as expected. When the sample size is small, the smoothed variances could be different. In this case, we could use the average ASM estimator $\tilde{V}_i^{ASM} = (\tilde{V}_i^{GVF.RB} + \tilde{V}_i^{GVF.HBY} + \tilde{V}_i^{DEFF})/3$ as a simple data pooling method to obtain the final smoothed variance estimate.

4. LFS Small Area Estimation Using Smoothed Sampling Variances

In this section, we apply the variance smoothing methods to the Canadian Labour Force Survey (LFS) data and compare the small area estimates based on the smoothed sampling variances. In the previous section, we displayed the performance of variance smoothing using GVF and DEFF as applied to the CCHS and PALS surveys. For the LFS, we will exhibit the performance of GVF and DEFF, and the improvement of model-based small area estimates that use the smoothed sampling variances. The small area estimates of the LFS will be compared to the corresponding census values for the same reference month (May 2016).

The LFS produces monthly estimates of the unemployment rate at the national and provincial levels. The LFS also releases unemployment estimates for sub-provincial areas such as Census Metropolitan Areas (CMAs) and Census Agglomerations (CAs) across Canada. Details of the methodology of the LFS are given in Methodology of the Canadian Labour Force Survey (2017). The estimated variances for the LFS are also computed through the Rao-Wu bootstrap procedure. For some sub-provincial areas, the direct estimates are not reliable because the sample sizes in some areas are quite small. Small area estimation, as applied to the LFS, usually estimates unemployment rates for local sub-provincial areas such as CMA/CAs using small area models. These models are discussed in Hidiroglou et al. (2019), Lesage et al. (2021), You et al. (2003), and You (2008, 2021).

We apply the Fay-Herriot model given by (1) and (2) to the May 2016 unemployment rate estimates at the CMA/CA level. There are 128 CMA/CAs (areas) in our study: three of these areas have a sample size smaller than or equal to 10, 10 have a sample size smaller than 30, 33 have a sample size smaller than 60, and 59 have a sample size smaller than 120, representing almost 50% of the areas in the study. In contrast, there are also 13 of the 128 areas with a sample size larger than 1,000, including some large cities such as Toronto, Montreal, and Vancouver. The Median sample size of all 128 areas is 129. We used four smoothed variance estimators in the LFS application, namely, $\tilde{V}_i^{GVF.RB}$, $\tilde{V}_i^{GVF.HBY}$, $\tilde{V}_i^{DEFF}$ and the average smoothed estimator $\tilde{V}_i^{ASM} = (\tilde{V}_i^{GVF.RB} + \tilde{V}_i^{GVF.HBY} + \tilde{V}_i^{DEFF})/3$. Figure 5 shows the plot of log direct sampling variance vs log sample size for the LFS data. The GVF model fitting is very good as shown in this figure, except for one or two outliers.

We first obtain the smoothed sampling variances for all the 128 CMA/CAs using the proposed $\tilde{V}_i^{GVF.RB}$, $\tilde{V}_i^{GVF.HBY}$, $\tilde{V}_i^{DEFF}$ and $\tilde{V}_i^{ASM}$. For the GVF model (4), the regression estimates with standard errors are $\hat{\beta}_0 = -3.194$ (0.306) and $\hat{\beta}_1 = -0.901$ (0.058). The RB residual correction term $exp(\hat{\tau}^2/2)$ is equal to 1.467 and the HBY correction term is

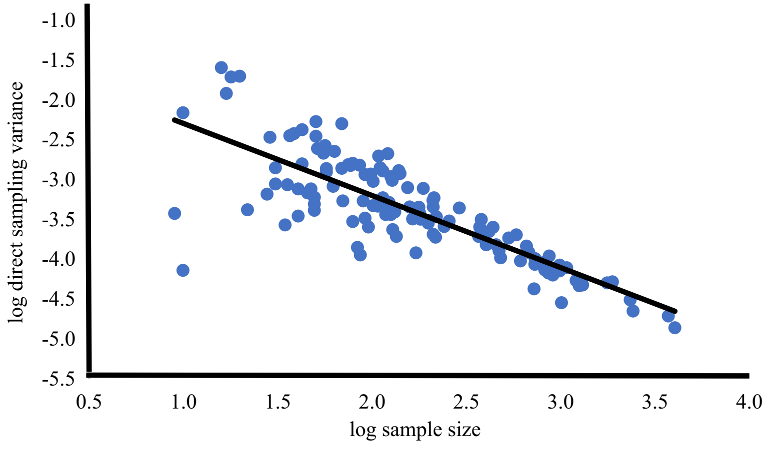

Fig. 5. Log direct sampling variance vs log sample size (LFS data).

$\hat{\omega}_{HBY} = \hat{V}^{total}/\tilde{V}^{naive} = 1.786$. Since the regression coefficient $\hat{\beta}_1 = -0.901$ is close to -1, and the difference between the two correction terms is not large, we could expect the GVF and DEFF lead to similarly smoothed sampling variances for the LFS data. Figure 6 shows the GVF and DEFF smoothed estimators perform very similarly and all lead to smoothed variance estimates.

We applied the empirical best linear unbiased prediction (EBLUP) approach in the LFS application to obtain the model-based estimates. The details of the EBLUP estimator and related mean squared error (MSE) estimation based on the Fay-Herriot model with REML method to estimate the model variance can be found, for example, in Rao and Molina (2015) and You (2021). Local area employment insurance monthly beneficiary rate is used as an auxiliary variable x_i in the linking model (2) as in Hidiroglou et al. (2019) and You (2008, 2021), The resulting linking model (2) is specified as $\theta_i = \delta_1 + x_i\delta_2 + v_i$, and $v_i \sim (0, \sigma_v^2)$. The model-based estimates and the direct estimates are compared with the census estimates to evaluate the effects of sampling variance smoothing. We applied the Fay- Herriot model to the 128 CMA/CA LFS unemployment rate data with the four different smoothed sampling variances and obtained the corresponding EBLUP estimates. The small area EBLUP estimates are compared via the absolute relative error (ARE) of the direct and EBLUP estimates with respect to the census estimates for each CMA/CA as follows:

$$\mathrm{ARE}_i = \left| \frac{\theta_i^{Census} - \theta_i^{Est}}{\theta_i^{Census}} \right|,$$

where θ_i^{Est} is the direct or the EBLUP estimate and θ_i^{Census} is the corresponding census value of the LFS unemployment rate. It is a common practice to evaluate the model-based estimates with the census values, for example, as in Hidiroglou et al. (2019) and You (2021). We then take the average of AREs over CMA/CAs by different subgroups with respect to the sample size, as in Hidiroglou et al. (2019). Table 1 presents the estimates of

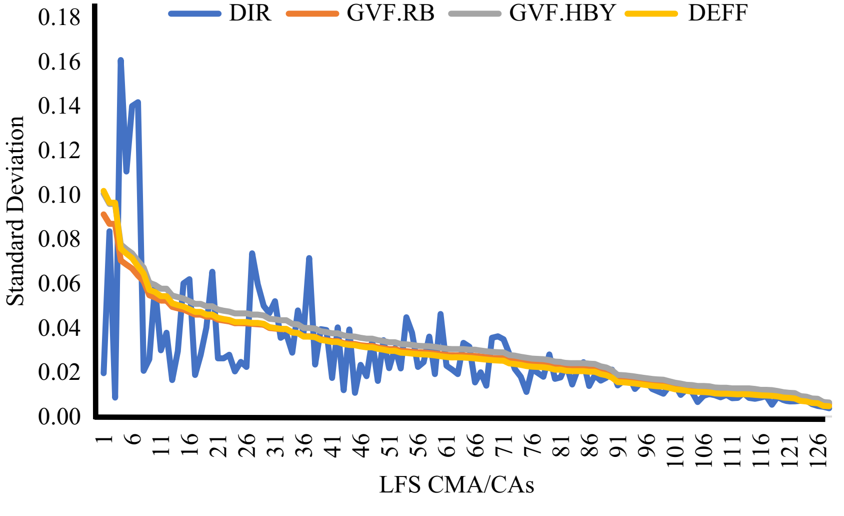

Fig. 6. Comparison of direct and smoothed deviation (LFS data).

the regression parameters and model variance in the linking model for the LFS application with different smoothed input sampling variances.

From Table 1, the estimates of the regression coefficients δ_1 and δ_2 are very similar for the different input smoothed sampling variances. The model variance estimate is slightly smaller using the GVF.HBY sampling variance in our application. Using the ASM sampling variance, the estimates of δ_1, δ_2 and σ_v^2 are quite good and reasonable as compared to the estimates that use the GVF or DEFF sampling variances.

Table 2 presents the average ARE for the direct LFS and EBLUP estimators based on different input sampling variance estimates. For comparison, we also used the direct sampling variance as input sampling variance in the Fay-Herriot model. For example, for EBLUP(DIR) the direct (DIR) sampling variance estimate is used in the Fay-Herriot model. EBLUP(GVF.RB) means that the smoothed sampling variance estimate $\tilde{V}_i^{GVF.RB}$ (GVF.RB) is used, etc. It is clear from Table 2 that the EBLUP estimates substantially improve the direct estimates by reducing the ARE. Even with the use of the direct sampling variance estimates, EBLUP(DIR) results in much smaller ARE than the direct survey estimator. However, by using the smoothed sampling variance estimates, EBLUP performs substantially much better than the direct estimator. The AREs are reduced for each area group, and consequently over all the areas. In general, all the EBLUPs with the four smoothed sampling variances perform very similarly. Amongst the EBLUP estimators using the smoothed sampling variances, EBLUP(GVF.HBY) has a slightly larger ARE than the others, and the EBLUP(DEFF) has a slightly smaller ARE. The respective AREs of EBLUP(GVF.RB), EBLUP(GVF.HBY) and EBLUP(DEFF) over all the 128 CMA/CAs are 0.138, 0.144, and 0.135. EBLUP(DEFF) performs the best in terms

Table 1. Estimates of regression parameter and model variance in the Fay-Herriot model.

Parameters	GVF.RB	GVF.HBY	DEFF	ASM
δ_1	4.884	4.916	4.875	4.892
δ_2	0.796	0.788	0.796	0.793
σ_v^2	0.551	0.269	0.836	0.532

Table 2. Comparison of ARE for EBLUP estimates based on the different input sampling variances.

CMA/CAs	Direct LFS	EBLUP (DIR)	EBLUP (GVF. RB)	EBLUP (GVF.HBY)	EBLUP (DEFF)	EBLUP (ASM)
25 smallest areas (sample size less than 50)	0.489	0.279	0.181	0.184	0.180	0.182
Next 25 smallest areas (sample size 50 to 100)	0.338	0.214	0.146	0.147	0.146	0.146
Next 25 smallest areas (sample size 100 to 180)	0.276	0.198	0.138	0.143	0.134	0.138
Next 25 smallest areas (sample size 180 to 550)	0.198	0.161	0.134	0.141	0.130	0.135
28 largest areas (sample size 550 and over)	0.132	0.125	0.099	0.108	0.091	0.099
Overall areas	0.283	0.194	0.138	0.144	0.135	0.139

of relative error. For the average smoothed sampling variance $\tilde{V}_i^{ASM}$ used in the Fay-Herriot model, the EBLUP(ASM) has an overall ARE value of 0.139, which is between the ARE values of EBLUPs using GVF and DEFF. The EBLUP(ASM) performs very well.

In terms of the overall average CV, EBLUP also reduces the CV substantially over the direct estimator. The direct LFS estimator has an average CV of 39.4%, EBLUP(DIR) has an average CV of 24.5%.The EBLUP(GVF.RB) has an average CV of 10.3%, EBLUP(GVF.HBY) has a slightly smaller average CV of 8.2%, and EBLUP(DEFF) has the average CV value 11.8%. The EBLUP(ASM) has an average CV of 10.2%. This means that using smoothed sampling variances substantially reduces the CV for EBLUPs, and once more the CV for EBLUP(ASM) is between the CV values of EBLUPs that use the GVF and DEFF variances.

EBLUP(ASM) has a smaller ARE value than either EBLUP(GVF.RB) or EBLUP(GVF.HBY). It also has a smaller CV than EBLUP(GVF.RB) and EBLUP(DEFF). The use of the averaged smoothed sampling variances $\tilde{V}_i^{ASM}$ in the model allows us to achieve a balanced reduction for both ARE and CV in our application. By comparing the ARE and CV for the EBLUP estimates, it is clear that the average smoothed estimator $\tilde{V}_i^{ASM}$ performs very well.

Lesage et al. (2021) considered the following smoothing model, denoted as the LBB model, for sampling variance smoothing:

$$log(\hat{V}_i) = \beta_0 + \beta_1 log(z_i) + \beta_2 log(1 - z_i) + \beta_3 log(n_i) + \varepsilon_i, i = 1, ..., m, \tag{15}$$

where z_i is the employment insurance beneficiary rate used in the Fay-Herriot model as an auxiliary variable to obtain the EBLUP estimators. By applying the LBB smoothing model (15) to the 128 area sampling variance data, we have the following regression estimates $\hat{\beta}_0 = -4.443$, $\hat{\beta}_1 = -0.486$, $\hat{\beta}_2 = -29.139$ and $\hat{\beta}_3 = -0.886$. The residual correction term $\hat{\omega}_{RB} = exp(\hat{\tau}^2/2)$ is equal to 1.461 and the HBY correction term is $\hat{\omega}_{HBY} = \hat{V}^{total}/\tilde{V}^{naive} = 1.782$. We denote $\tilde{V}_i^{LBB.RB}$ as the smoothed variance estimator based on the LBB model (15) and formula (7) with a correction term $\hat{\omega}_{RB} = 1.461$. Similarly, let $\tilde{V}_i^{LBB.HBY}$ be the smoothed variance estimator based on the LBB model (15) using formula (8) with a correction term $\hat{\omega}_{HBY} = 1.782$. We now compare the EBLUP estimates based on the LBB smoothing model and the proposed smoothing method. In particular, we compare the proposed EBLUP(ASM) to EBLUP estimates using $\tilde{V}_i^{LBB.RB}$ and $\tilde{V}_i^{LBB.HBY}$, e.g., EBLUP(LBB.RB) and EBLUP(LBB.HBY).

Table 3 presents the average ARE to compare the effects of variance smoothing using the ASM and the LBB procedures. It is clear from Table 3 that all EBLUP estimates perform very well and improve the direct survey estimates by substantially reducing the ARE with respect to the census values. EBLUP(ASM) and EBLUP(LBB.RB) perform almost the same, and EBLUP(LBB.HBY) has slightly larger ARE, same as the performance of EBLUP(GVF.HBY) in Table 2. EBLUP(LBB.HBY) and EBLUP(GVF.HBY) perform almost identically by comparing the results in Table 2 and Table 3. In terms of CV, EBLUP(LBB.RB) and EBLUP(ASM) have the same average CV 10.2%, and EBLUP(LBB.HBY) has the same average CV 8.2% as EBLUP(GVF.HBY).

Table 3. Comparison of ARE for EBLUP estimates based on the different GVF models and smoothed sampling variances.

CMA/CAs	Direct LFS	EBLUP (ASM)	EBLUP (LBB.RB)	EBLUP (LBB.HBY)
25 smallest areas (sample size less than 50)	0.489	0.182	0.181	0.183
Next 25 smallest areas (sample size 50 to 100)	0.338	0.146	0.144	0.145
Next 25 smallest areas (sample size 100 to 180)	0.276	0.138	0.137	0.142
Next 25 smallest areas (sample size 180 to 550)	0.198	0.135	0.135	0.141
28 largest areas (sample size 550 and over)	0.132	0.099	0.099	0.108
Overall areas	0.283	0.139	0.138	0.143

The LFS small area application shows that the proposed GVF model (4) and the proposed sampling variance smoothing methods GVF, DEFF and ASM perform very well by comparing the EBLUP estimates with the census values and other GVF smoothing model for LFS application, for example, Lesage et al. (2021).

5. Simulation Study

In this section, we conduct a simulation study to verify and evaluate the proposed GVF, DEFF, and ASM smoothing estimators for small area estimation with data generated from different mechanisms. Following Lesage et al. (2021), we used the LFS data studied in Section 4 to generate the simulated data. We considered m = 128 areas in the simulation study. Let θ_i be the simulated true parameter of interest. The θ_i's were generated as $\theta_i = \gamma_0 + \gamma_1 z_i + \nu_i$, where z_i is the LFS beneficiary rate used in Section 4, $\gamma_0 = 0.05$, $\gamma_1 = 0.88$, and ν_i was generated from $N(0, \sigma_v^2)$, and $\sigma_v^2 = 4.78653e - 05$. The values of γ_0, γ_1 and σ_v^2 were obtained from the EBLUP estimation of LFS application in Section 4 with the ASM smoothed sampling variances as input data. To generate the direct estimate $\hat{\theta}_i$ for the parameter θ_i and the corresponding direct sampling variance, we considered two approaches. The first one generated $\hat{\theta}_i$ from a binomial distribution, that is, $\hat{\theta}_i = n_i^{-1}$ Binomial(n_i, θ_i): Lesage et al. (2021) used the same simulation setup. The direct variance estimator was then computed as $\hat{V}_i = (n_i - 1)^{-1} \hat{\theta}_i(1 - \hat{\theta}_i)$. We denote this simulation setup as LBB setup. The second method generated the data directly from the Fay-Herriot model using the sampling variance modeling given by You and Chapman (2006) and You et al. (2013). The direct estimate $\hat{\theta}_i$ is generated as $\hat{\theta}_i = n_i + e_i$, where $e_i = \mathrm{N}(0, \sigma_i^2)$ and the sampling variance σ_i^2 is obtained from Section 4 using the average smoothed sampling variance ASM. The sampling variance σ_i^2 is treated as the true sampling variance in the simulation. The direct sampling variance estimate $\hat{V}_i$ is generated using $\hat{V}_i = (d_i)^{-1}\sigma_1^2\chi_{d_i}^2$, where $d_i = n_i - 1$, as of Rivest and Vandal (2002), Wang and Fuller (2003) and You (2021). We denote this simulation setup as FHM (Fay-Herriot modeling) setup.

We generated 5,000 samples for the LBB and FHM simulation setup respectively. Under the LBB simulation setup, the average estimated design effects $\overline{def\,f}$ is 1.0131 and

Table 4. Comparison of ARE, CV and coverage rate (CR) for simulation study.

	LBB simulation setup			FHM simulation setup		
	ARE	CV	CR	ARE	CV	CR
Direct estimator	0.2553	36.57%		0.3112	78.95%	
EBLUP(DIR)	0.0829	10.12%	94.06%	0.0872	10.26%	93.92%
EBLUP(GVF.RB)	0.0713	6.38%	94.05%	0.0776	8.54%	93.71%
EBLUP(GVF.HBY)	0.0716	6.13%	93.11%	0.0776	8.61%	93.85%
EBLUP(DEFF)	0.0714	6.28%	93.73%	0.0765	8.37%	93.21%
EBLUP(ASM)	0.0715	6.22%	93.68%	0.0769	8.27%	93.36%

the estimated $\hat{\beta}_1$ = -1.0081. For simulation under the FHM setup, the average estimated design effects $\overline{deff}$ is 2.6679 and the estimated $\hat{\beta}_1$ = -0.7302. Table 4 presents the ARE comparison results to the average CV for the direct estimator and EBLUPs that use different smoothed sampling variances. Following Hidiroglou and You (2016), we also compute and compare the confidence interval coverage rate (CR) of the EBLUPs. The 95% confidence interval of the EBLUP estimator is obtained as EBLUP $\pm 1.96\sqrt{mse(\text{EBLUP})}$. Table 4 also reports the confidence interval coverage rate for EBLUPs over the 5,000 simulated samples.

Under the LBB simulation setup, the ARE of the direct estimator is 0.2553 with an average CV of 36.57%. As expected, the EBLUP has a much smaller ARE and CV. EBLUP(DIR) has an ARE of 0.0829 with an average CV of 10.12%, whereas the EBLUPs using smoothed sampling variances have even smaller AREs and CVs. The GVF, DEFF and ASM lead to almost the same ARE which is around 0.0715, and GVF.HBY has slightly smaller CV compared with GVF.RB or DEFF as in the real LFS application.

Under the FHM simulation setup, again, the proposed GVF, DEFF and ASM lead to smaller AREs and CVs than either the direct estimator or the EBLUP that uses direct sampling variances. Using the ASM variance estimates leads to a smaller CV than using the GVF and DEFF as shown in Table 4 under the FHM simulation setup. In general the ASM leads to a balanced ARE and CV in the simulation study, which is the same as in the LFS application, which indicates that using ASM as a data pooling to average the GVF and DEFF is useful in practice for the sampling variance smoothing. Note that using the ASM is suggested, but it is also a matter of choice. For confidence intervals, the EBLUPs all provide similar and reasonable coverage of around 93–94%. In summary, as shown in the simulation results reported in Table 4, the proposed GVF, DEFF and ASM smoothing models and methods all perform similarly and very well.

6. Conclusions and Suggestions

In this article, we have proposed sampling variance smoothing estimators using the generalized variance function method and smoothed design effect method for small area estimation. The proposed smoothing models and methods only require the use of the sample size in the model and the computation of design effects. The proposed estimators $\tilde{V}_i^{GVF.RB}$, $\tilde{V}_i^{GVF.HBY}$ and $\tilde{V}_i^{DEFF}$ usually result in similar smoothed variance estimates.

In practical applications, we may use the average smoothed estimator $\tilde{V}_i^{ASM}$ as a data pooling procedure to obtain the final smoothed variance estimate. The use of ASM may be particularly useful and may have more advantages when the GVF and DEFF lead to different smoothing estimates. The proposed smoothing model and method can be easily implemented in practice. The proposed smoothing methods simplify the smoothing procedure for practical users as they do not need other complicated GVF models or auxiliary variables for the sampling variance modeling. The small area estimation results in our LFS application in Section 4 and the simulation study in Section 5 both indicate that the proposed GVF, DEFF and ASM perform very well.

It may also be possible in practice to consider a weighted average of the three variance estimators estimators $\tilde{V}_i^{GVF.RB}$, $\tilde{V}_i^{GVF.HBY}$ and $\tilde{V}_i^{DEFF}$. For example, if DEFF leads to a larger CV for the final small area EBLUP estimates, and GVF leads to a smaller CV for the final EBLUP estimates, then we may want to reduce the weights for DEFF and put more weights on GVF. The weights could be proportional to the inverse of the average CV of the small area estimates corresponding to the smoothed sampling variances, as the idea of small area estimation is to achieve the CV reduction and to obtain reliable small area estimates as shown in our data analysis and simulation study. From Section 4, EBLUP(GVF.RB) has an average CV of 10.3%, EBLUP(GVF.HBY) has a slightly smaller average CV of 8.2%, and EBLUP(DEFF) has an average CV value of 11.8%. Then we may construct a weighted ASM (WASM) smoothed sampling variance as follows:

$$\tilde{V}_i^{WASM} = (\tilde{V}_i^{GVF.RB} + 1.2 * \tilde{V}_i^{GVF.HBY} + 0.8 * \tilde{V}_i^{DEFF})/3.$$

This weighted smoothed sampling variance $\tilde{V}_i^{WASM}$ puts more weight on $\tilde{V}_i^{GVF.HBY}$ and less weight on $\tilde{V}_i^{DEFF}$ according to the inverse of the CVs of the corresponding EBLUPs. We then obtained the EBLUP(WASM) using the weighted $\tilde{V}_i^{WASM}$, and compared the results with the EBLUP(ASM). The ARE for EBLUP(ASM) is 0.1392, whereas the ARE for EBLUP(WASM) is 0.1398. The average CV for EBLUP(ASM) is 10.2%, whereas the average CV for EBLUP(WASM) is 9.96%. Thus by putting more weight on GVF.HBY and less weight on DEFF, the EBLUP(WASM) has a slightly smaller CV than EBLUP(ASM) as expected, and using the WASM leads to a very tiny increase in ARE. The difference between using ASM and WASM in the LFS application is very small and could be ignored. In general, the simple average ASM should provide adequate additional smoothing by combining the GVF and DEFF smoothed variances.

For future work, we may consider more auxiliary variables in the GVF modeling for both the real data analysis and simulation study. It would be interesting to study the relationship between the GVF and DEFF estimators if other auxiliary variables including design variables and/or proxies are available and used in the GVF models. Note that the GVF models need the sample size as an auxiliary variable to make it compatible with the smoothed DEFF procedure. Bertarelli et al. (2018) used a log CV modelling for proportions instead of the variance. It would also be interesting to compare the log CV modelling with the GVF modelling on log variance through real data analysis and simulation study. For the averaged smoothing using both GVF and DEFF, more combinations or choices of weights could be considered and evaluated via additional simulation studies.

In practice, for areas with larger sample sizes, it may also be fine to use the direct variances by assuming that the direct variance estimates are stable enough when the sample size is large. This is a reasonable procedure to follow, because as shown in Figures 2, 4, and 6 in our applications, the direct estimate and smoothed estimate are usually very similar for areas with larger sample sizes. It is possible to set a minima threshold in terms of a minimum sample size, say n_{min}, to decide on the choice between smoothed or direct variance estimates. If the sample size of an area is smaller than n_{min}, then the preferred choice would be to use smoothed variance estimates. If on the other hand, the sample size of an area is greater than n_{min}, then the preferred choice would be the direct variance estimates. For example, Hidiroglou et al. (2019) used direct variance estimates in the LFS small area estimation for the areas with large sample sizes. As a rule of thumb, they set $n_{min} = 400$ in their application and used the direct sampling variance estimate when the sample size was greater than 400. However, care should be taken when choosing the value for n_{min}, as more comparisons and studies may be needed in practice to evaluate the results. In general, we recommend using the smoothed variance estimates for all areas, as it is quite simple to apply, and can avoid the additional problem of settling a value for n_{min}.

Finally, we make some recommendations for total variance smoothing using the proposed methods. For the estimation of totals, we can modify the proposed method as follows: If $\hat{y}_{iw}$ is an estimator of a total, and if we know the corresponding population size N_i, we can transform the total estimator $\hat{y}_{iw}$ to a rate (proportion) estimator by dividing it by N_i, that is, $\hat{p}_{iw} = \hat{y}_{iw}/N_i$, and the corresponding sampling variance is $\hat{V}_i(\hat{p}_{iw}) = \hat{V}_i(\hat{y}_{iw})/N_i^2$. We can apply the proposed smoothing methods for proportions to get smoothed variance estimates $\tilde{V}_i(\hat{p}_{iw})$. Then the smoothed sampling variance for the total can be simply obtained as $\tilde{V}_i(\hat{y}_{iw}) = \tilde{V}_i(\hat{p}_{iw}) \cdot N_i^2$. It is a good idea in practice to transform the estimates of totals into estimates of proportions. The reason for this is that the transformed proportion estimates are more easily modeled than estimates of totals. The total estimates can be obtained by transforming back the model-based rate estimates.

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