Investigation of Replication Error in Black Scholes Model: What happens when one cannot hedge continuously?

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Executive Summary:

In Black Scholes World, delta hedging in a continuous time gives solution to Black Scholes Equation; we hold a portfolio of some stocks and bonds that replicates an option payoff. If the continuous hedging is relaxed to a discrete time, we see a replication error with the portfolio. Namely, theta and gamma, which are changes of option value respect to time and second derivative of the derivative product, generate this error. Through analyses, we understand how varying number of rebalancing given a time period produce a varying distribution of replication error. Also, we try to understand what happens to the replication error when the volatility is no longer constant.

Discussion of Issues:

In the inception of this project, we did not fully understand the mathematics behind the sources of replication error. However, we specified in the proposal that we will find the bid-ask spread such that market maker would not lose money 80% of the time while holding a re-balancing once-a-day portfolio. In retrospect, proposing too specific of a task before understanding the detail was a mistake.

Originally, we suspected gamma and theta as sources of the error, and consulted Hannes Byun[1] and some texts by Emanuel Derman. Once we understood the mathematics, we built a data structure in Python to test ideas. Building a data structure[2] in object-oriented program from scratch was a daunting task that involved 40+ hours. We regret that the inability to implement time varying volatility surface, because having it can generate more realistic back testing for multiple options and insights into the effect of volatility smiles.

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[1] I thank Hannes at Goldman Sachs for his insights into understanding replication portfolios

[2] All codes can be found on Github: https://github.com/wbaik/Finance/tree/master/WIP

Analysis:

Black Scholes Model holds as one of its assumptions that continuous hedging is possible to solve for option prices. It assumes a known future volatility of a stock, thus replicates an option payoff by continuous re-balancing of stocks and bonds.

However, suppose we cannot hedge continuously. Instead, we will aim to replicate the option payoff by assuming daily re-balancing. So the question is, how much of a difference arises from a single rebalance per day compared to continuous hedging assumed by the Black Sholes model? This difference is what we will refer to as *Replication Error* through out this paper.

Theoretically, Black Scholes equation gives some clues of replication error:

$$\left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)\Delta t = r\left(-V + S\frac{\partial V}{\partial S}\right)\Delta t$$

Notice that the first term on the LHS is equivalent to Theta, and the second term is a function of Gamma. Now, if we were to replicate this equation by a daily hedge instead of continuous hedging, one sees the Replication Error simplifies to just the terms on the left hand side.[3] In equation[4]:

$$dP\&L = d(C - \Delta S) = \tfrac{1}{2}\Gamma(\Delta S^2) + \Theta\Delta t \qquad (2)$$

Since $\Delta S \equiv \sigma S N(0,1)\sqrt{\Delta t}$, where $N(0,1)$ is normal, we can rewrite (2) as

$$dP\&L = d(C - \Delta S) = \frac{1}{2}\Gamma(\Delta S^2) + \Theta\Delta t = \frac{1}{2}\Gamma(\sigma^2 S^2 N(0,1)^2 \Delta t) + \Theta\Delta t$$

The total Replication Error is therefore the sum of all of the generated Theta and Gamma P&L.

<u>In the world of Black Scholes:</u>

To put this error into a perspective, we create a hypothetical situation:

Stock BSM: Generate a price path using a geometric Brownian motion $dS = \mu S dt + \sigma S dW$

Spot = \$100, Strike = \$100, $\sigma_{implied} = \sigma_{realized}$ *= 20% per year,* $\mu = 0\%$*, dividend = 0%,*

time = 1 year = 252 days

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[3] E. Derman, N. Taleb: The Illusions of Dynamic Replication
[4] E. Derman, Dynamic Replication: Realities and Myths of Options Pricing

Assume we initially take the position of short a call option, and long the stock accordingly to hedge the delta. We rebalance to be delta neutral by selling/buying the underlying using daily close prices. Note that we borrow US treasuries and rebalance this as well to keep a replication portfolio.

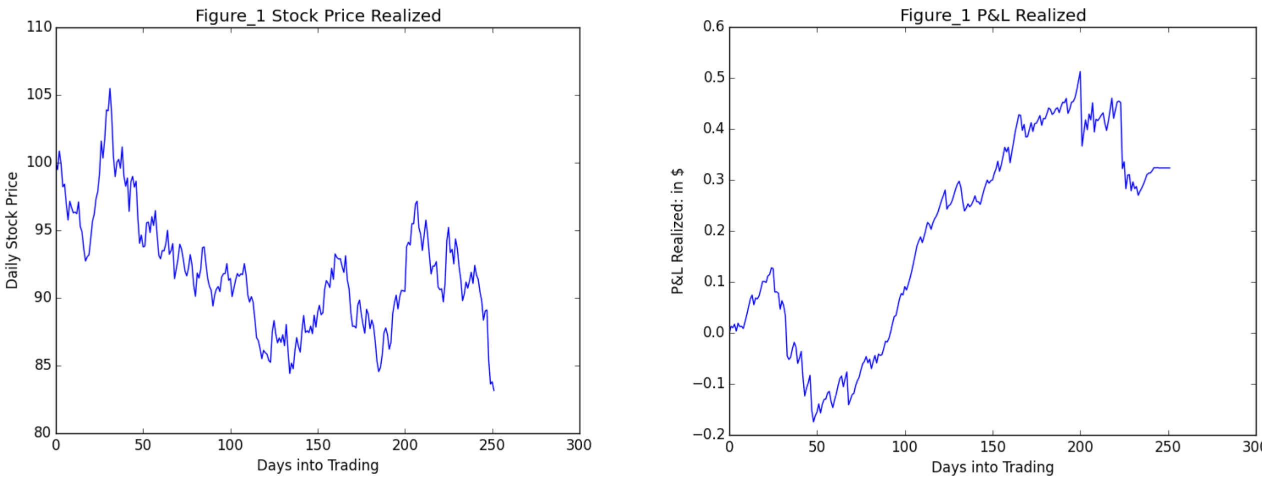

Figure 1: Left: Stock Price Path, Right: Replication Error = Aggregate P&L Realized

The Replication Error, or in other words the P&L realized, can be broken down to the daily gain/loss along the path of the underlying. The errors are explained *mostly*[5] by the gamma and theta: the aggregate of the Daily Gamma and Theta P&L yields net Daily P&L below.

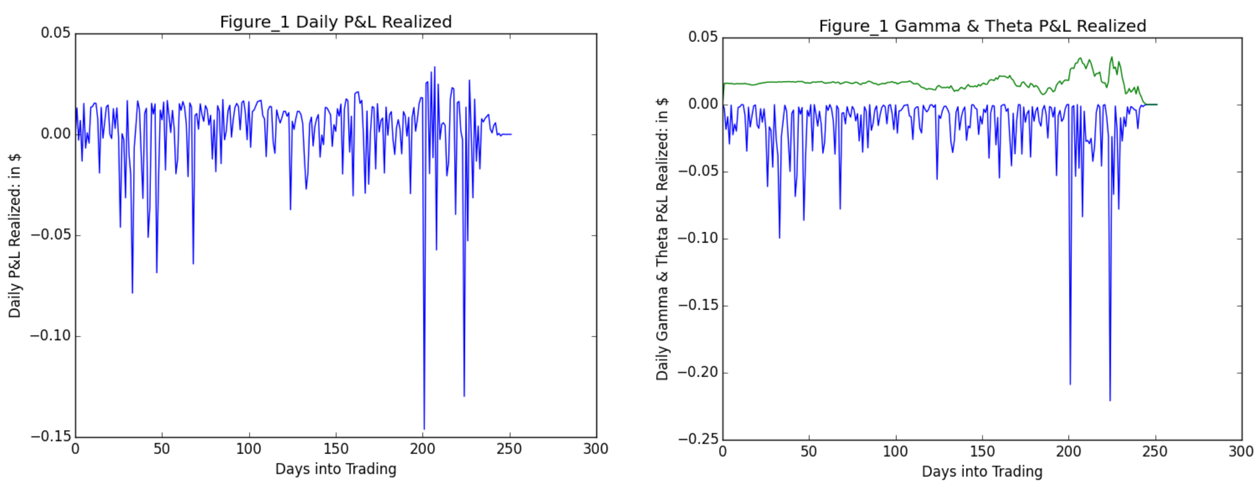

Figure 1: Left: Daily P&L Realized, Right: Gamma vs Theta P&L: Theta is positive (in green), and Gamma is negative (in blue), as we are short the option and long the stock.

Now, if we generate a similar simulation for 200 different price paths, and calculate the daily P&L, we get the following:

[5] '*Mostly'* because the error on equation (2) exist, unless we integrate the equation respect to 'dt'; Gamma and Theta are not necessarily constant over dt

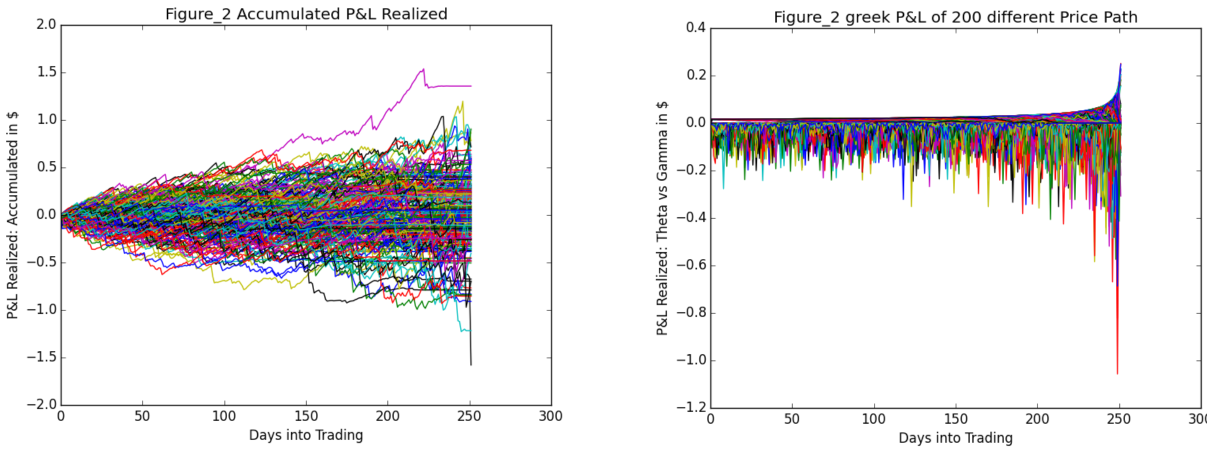

Figure 2: 200 different P&L realization
Left: P&L realized from replication, Right: Daily Distribution of Gamma & Theta P&L

Each of the realized P&L path is a sum of Theta gains and Gamma losses. Theta and Gamma P&L behaves the way we expect intuitively; While it does depend on the price path, Gamma and Theta tends to increase in magnitude as the time to expiry decreases, spiking at the very end. Because this path was generated under a 20% implied volatility and an actual volatility of 20%, net P&L on average tends to be about zero.

By now, it should be obvious what this trade is about. As long as realized volatility (throughout the time till maturity) is lower than the market implied volatility, the gain from Theta should be, on average, greater than the losses from Gamma, thus produce a positive returns. Alternatively, if realized volatility is greater than the market's implied volatility, the realized P&L should be below the x-axis, showing negative profits.

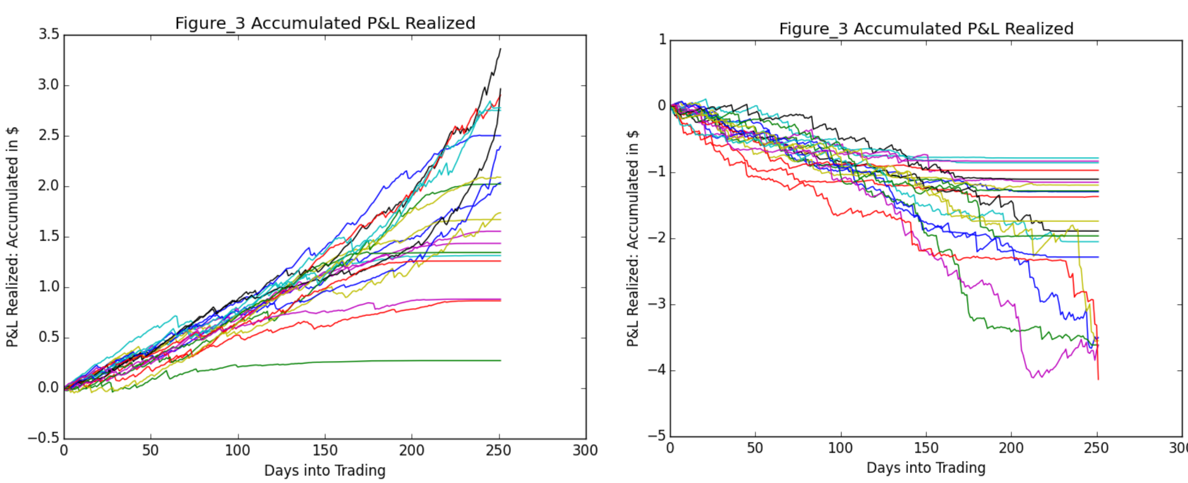

Figure 3: Left: Implied Vol = 20% vs Realized Vol = 15%, Right: Implied Vol = 20% vs Realized Vol = 25%

In Our World:

Instead of relying on the hypothetical paths of Brownian motion, we run a backtesting with the real world data. Take the Dow Johns Industrial Average data from year 2009. From January 2, 2009, we sell one option on one share of stock (ETF index, to be more precise), with maturity slightly greater than 1 year (expiring on 1/15/2010) and rebalance the delta and bond portfolio daily, just as we have been doing in the previous case.

DIA (ETF on Dow Johns Industria Average): Call option Expiry of Jan 2010, Strike price at \$100
Started Trading on 01/02/2009, Finished Replication on 01/15/2010

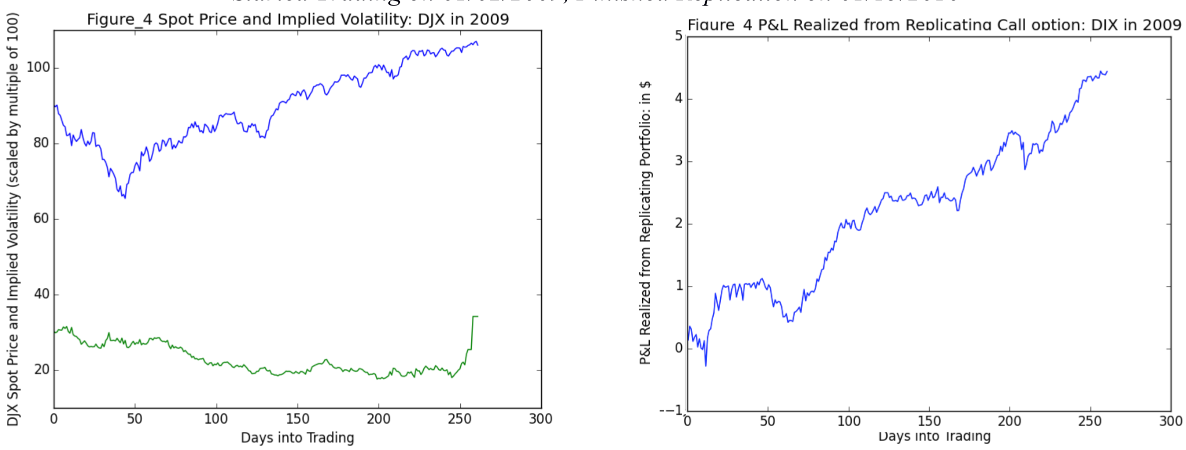

Figure 4: Left:2009 DIA Spot Price and Implied Volatility Path, Right: Replication's P&L

Here, we see that selling one option on one share of stock and holding a replication portfolio generated a positive Replication Error, shown by the P&L realized being positive of roughly \$4.40.

It is natural to ask what are the components of those profits. What is our total Theta gain, and Gamma loss? If we sum the daily P&L from Gamma and Theta throughout the year, we get Theta gain of \$5.61, Gamma loss of \$3.80, which makes total net profit of only \$1.81; this is significantly below the \$4.40 stated above. The difference is coming from Vega, $v = \frac{dV}{d\sigma} = \frac{dC}{d\sigma}$. When Implied Volatility changes, P&L will be generated by Vega difference. Here is the daily realized Vega P&L in the red line, and a comparison to Gamma and Theta.

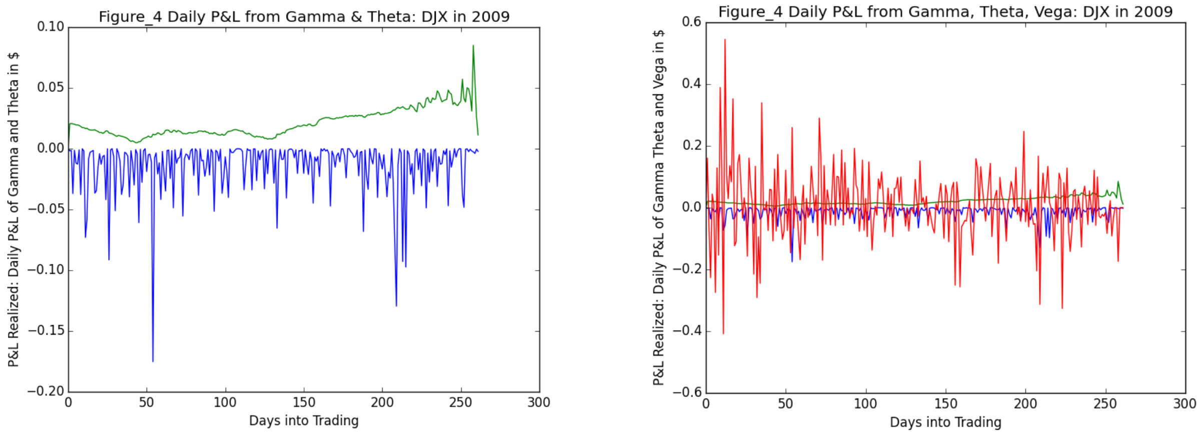

Figure 4: Left: 2009's Gamma & Theta P&L, Right: Vega P&L (in red) on top of the original Gamma & Theta P&L

Notice Vega P&L is so much greater in magnitude in many instances than P&L from Gamma or Theta. Why was this Vega analysis left out in the first analysis in the world of Black Scholes? Because the world in our previous analysis assumed a constant volatility. Since volatility was constant, Vega generated no P&L.

On to the World of Stochastic Volatility:

Observing the result of real world data, we must rewrite the equation (2). We add the Vega component, and the equation of Replication Error is now:

$$dP\&L = d(C - \Delta S) = \frac{1}{2}\Gamma(\Delta S^2) + \Theta\Delta t - v\Delta\sigma$$

And notice that in this world, we have escaped from the Black Scholes because we no longer assume constant volatility. So we must rewrite a stochastic differential equation such that the volatility is time varying. In this study, Heston Model[6] is introduced to test the idea:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^1 \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2 \\ dW_t^1 dW_t^2 &= \rho dt \end{aligned} \qquad (3)$$

S_t and V_t are price and volatility processes, and W_t^1 and W_t^2 are correlated Brownian motion processes with correlation parameter ρ. V_t is a square root mean reverting process, with long-run mean

[6] Steven L. Heston, A Closed-Form Solution for Options with Stochastic Volatility

θ, and rate of reversal κ. σ is volatility of volatility. All parameters of $\mu, \kappa, \theta, \sigma, \rho$ are time and state homogenous.[7]

Using these equations, we generate a new price path and run the replicating portfolio to generate the Replication Error. Let $\rho = -0.5, \sigma = 0.1, \kappa = 2, \theta = 0.04, \mathrm{V} = \sigma_i^2 = 0.04$; these numbers are chosen arbitrarily for the test. Let spot price = \$100, strike = \$100, $\sigma_{implied} = \sigma_{realized}$ = 20% per year, $\mu = 0\%$, dividend = 0%, time = 1 year = 252 days. Running a randomly generated price path and volatility path results the following Replication Error, which is equivalent to realized P&L.

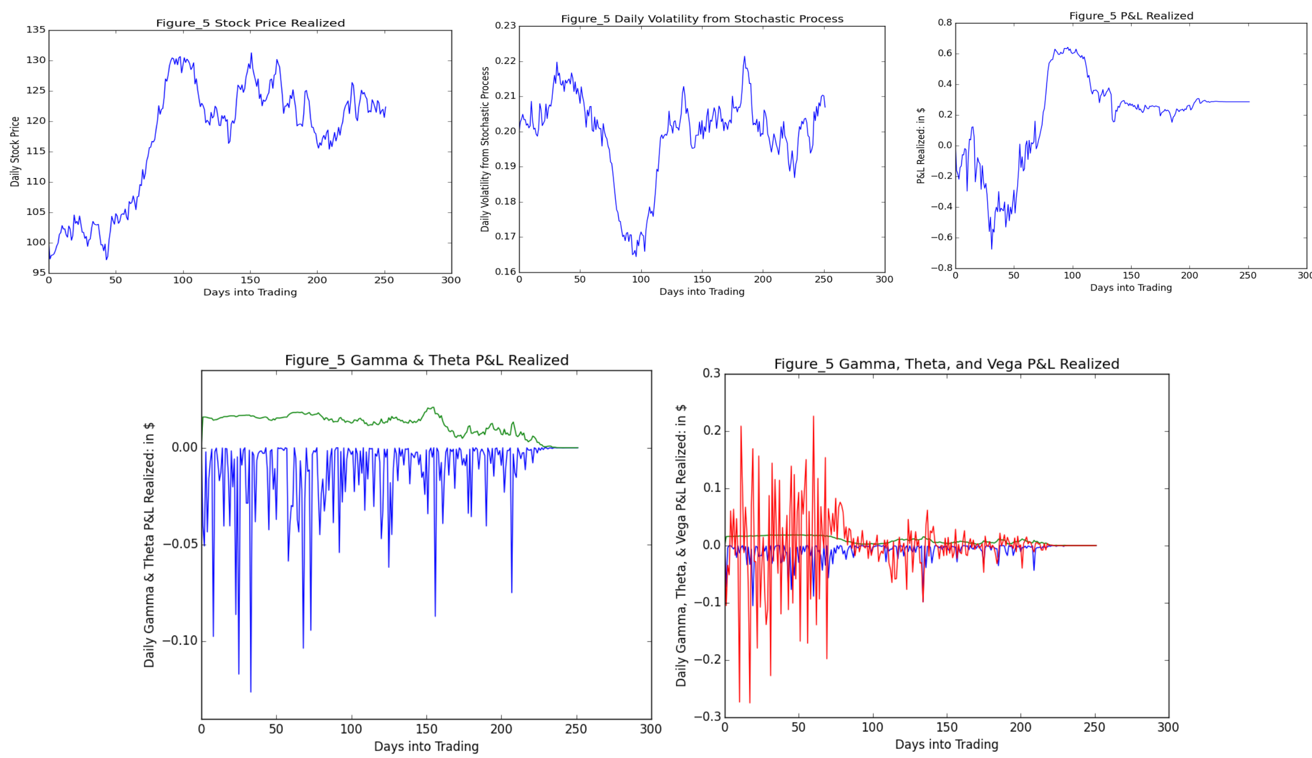

Figure 5: Top row: Left: Stock Path, Mid: Volatility Path, Right: P&L Realized
Bottom row: Left: Gamma & Theta P&L, Right: Vega P&L (in red) on top of Gamma & Theta

Above charts give insights into our replication portfolio. Under the traditional model of Black Scholes, Vega risk did not exist. When volatility is stochastic, Vega P&L may dominate other risks. To prove this point, we run tests with varying volatility of volatility, σ in the equation (3).

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[7] Nimalin Moodley, The Heston Model: A Practical Approach

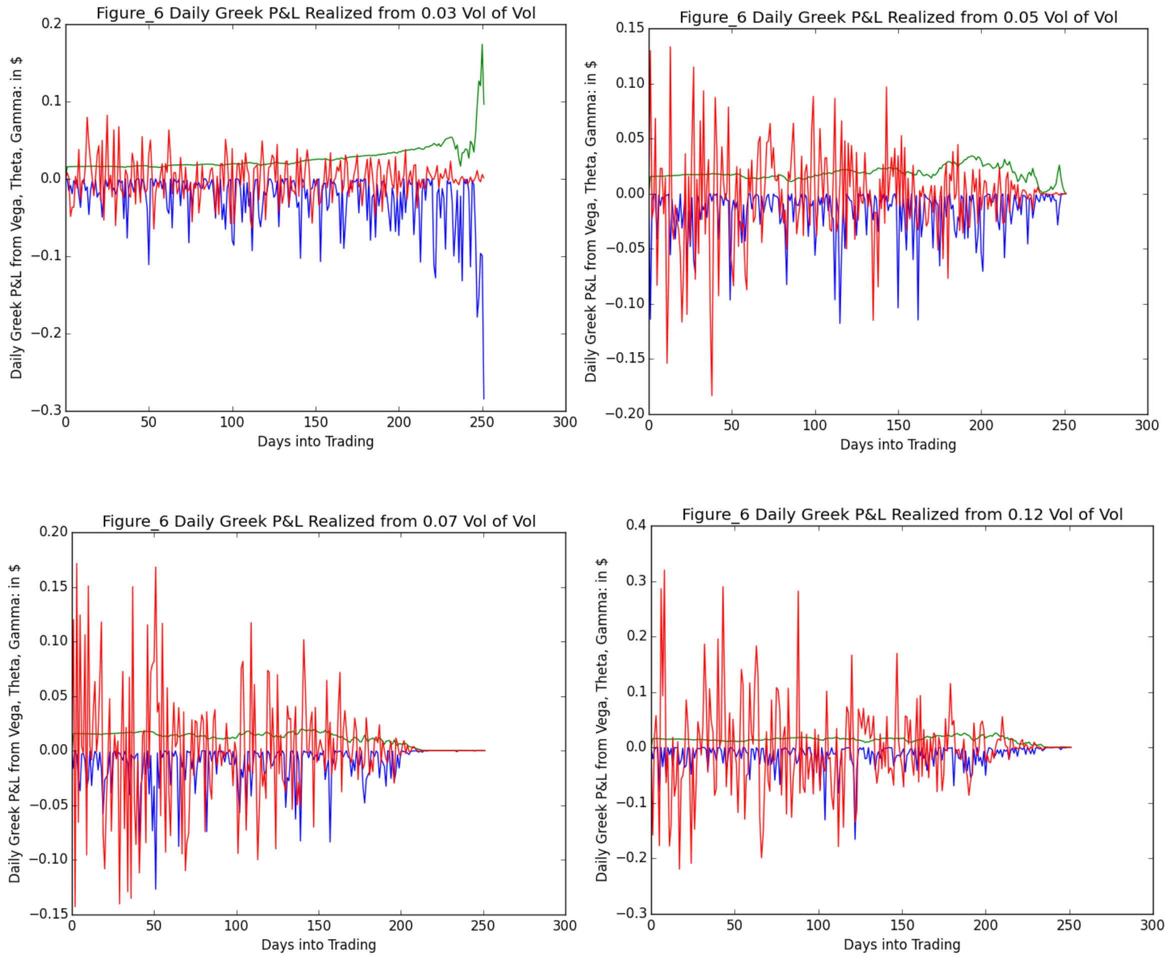

Figure 6. P&L Distribution of Vega, Theta, Gamma, with varying inherent volatility of volatility
Please note that their underlying price paths are randomly generated, so this is only one representation. They were created to illustrate the point of Vega risk, and the represented charts are reasonable realizations of random path.

It is obvious that once the volatility of volatility starts to go over 5% per annum, which is realistically speaking too low relative to data in reality, the Vega P&L starts to dominate the daily P&L in the replication portfolio. The following page shows a generalized representation of Replication Error in the Heston Model.

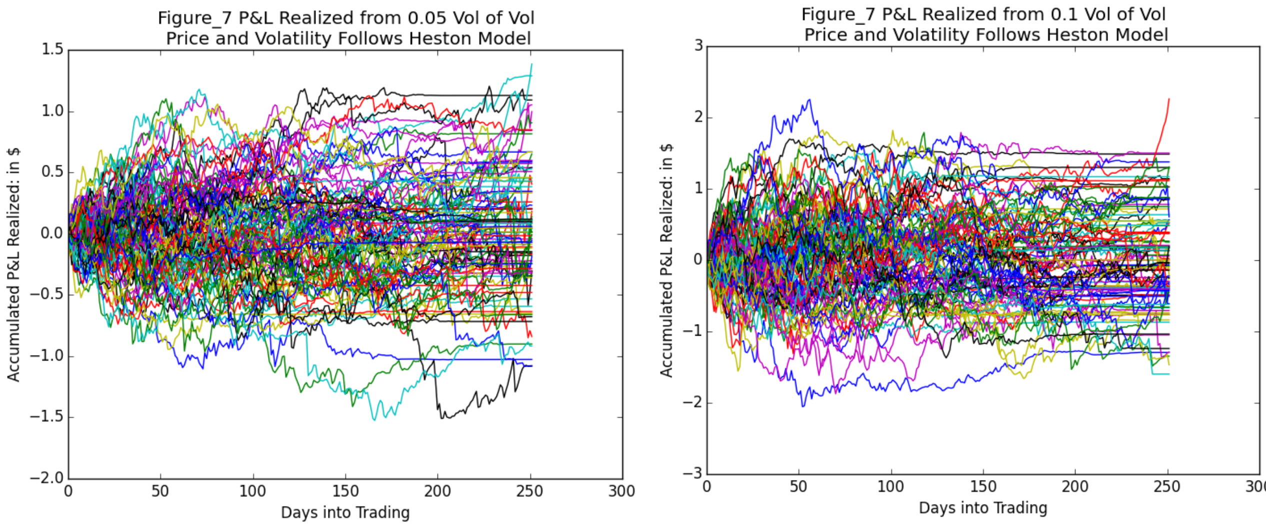

Figure 7, P&L Realized from Replicating Portfolio, total of 100 different paths per test,
Inputs given the same as that in producing Figure 5.
Left: Volatility of Volatility at 5% per year, Right: Volatility of Volatility at 10% per year

Compare Figure 7 to Figure 2 in page 2. It is obvious that the P&L realized by the Heston Model has much greater volatility, especially pronounced in the earlier period into trading by the effect of Vega. The replicating portfolio is no longer a genuine replication of a derivative product, a call option in this instance.

Is it to say that the Black Scholes Model is completely wrong given these results? No. Heston Model has a closed form solution, one that can be compared to the Black Scholes formula.[8] The solution of Heston Model is out of scope for this paper, but some points can be made. The difference in price computed with a calibrated Heston Model and the price in Black Scholes is minimal.[9] Even their deltas are considerably close, let alone that Vega risk cannot be wholly fixed by changing delta. The slight differences in delta arise from the fact that the correlation coefficient between the change in volatility and the change in spot price is negative in Heston Model, but surely this does not solve the whole problem.

[8] Nimalin Moodley, The Heston Model: A Practical Approach
[9] Yuan Yang, Valuing a European Option with Heston Model

Analysis of Distribution in Replication Error:

We have discussed two different models, namely Black Scholes Model and the Heston Model. In both cases, we have assumed that we would re-balance the portfolio once a day. What happens if we were to decrease the time step? How much of a reduction in variance of realized P&L will we observe? From here on, we focus only on Black Scholes Model.

Suppose we re-balance the portfolio 10 times a day, instead of once a day without taking into account the costs associated with the re-balance. The following is the result of such change on Black Scholes Model: Geometric Brownian motion $dS = \mu S dt + \sigma S dW$, spot = strike = \$100, $\sigma_{implied} = \sigma_{realized}$ = 20%, μ = dividend = 0%, time = 1 year

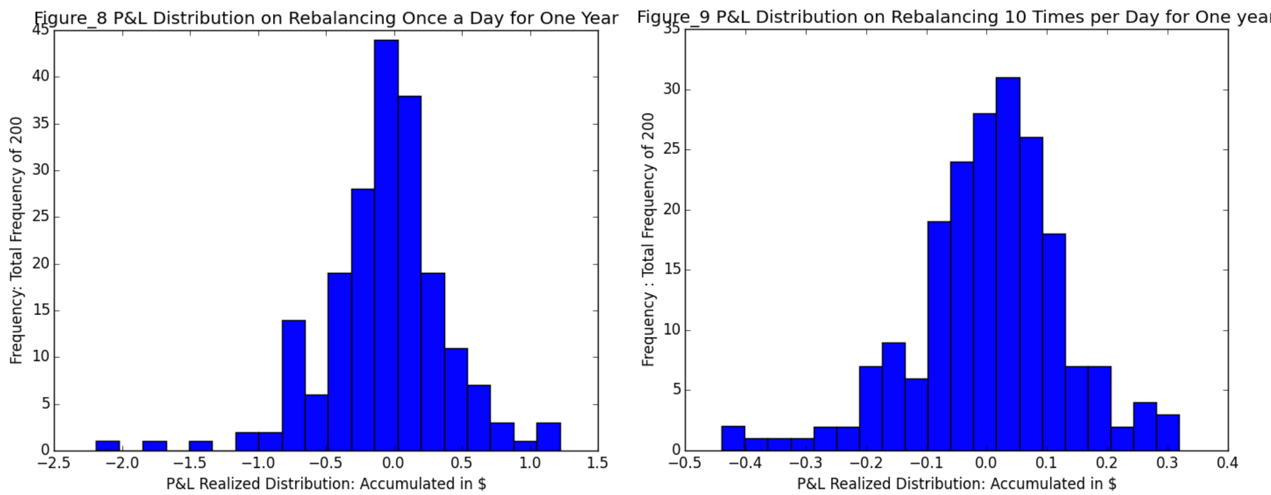

Figure 8: Left: Replication Error in once a day re-balancing, Right: 10 times a day re-balancing

Notice the significant difference in variances of distribution. While average of both charts are roughly about the same at \$0, variance of P&L differs heavily.

Once a day rebalancing resulted in:
```
Average of Distributed Final P&L is : -0.0629430976193
Varaince of Distributed Final P&L is : 0.205399853363
```

Ten times a day rebalancing resulted in:
```
Average of Distributed Final P&L is : 0.00286888631968
Varaince of Distributed Final P&L is : 0.015831059881
```
(4)

Paul Wilmott suggests a better way to approach this difference. For Equation (2), Wilmott has written it in a slightly different form[10]: $dP\&L = \frac{1}{2}\left(\sigma^2{}_{implied} - \widetilde{\sigma^2}\right)S^2\Gamma dt$

The only change is that he's replaced Theta with another sigma term such that Gammas of implied and realized are compared. Notice the one with tilde is realized when we are short the option. Therefore, both are really the same equation. Let's rewrite the equation to show that from each time step Δt in years:

$$P\&L\ of\ 1\ time\ step \cong \frac{1}{2}\Gamma(\sigma^2 S^2(1 - N(0,1)^2)\Delta t)$$

$$E[P\&L\ of\ 1\ time\ step] = E\left[\frac{1}{2}\Gamma(\sigma^2 S^2(1 - N(0,1)^2)\Delta t)\right] = \frac{1}{2}\Gamma\sigma^2 S^2 \Delta t E[(1 - N(0,1)^2)] = 0$$

$$Var[P\&L\ of\ 1\ time\ step] = \left(\frac{1}{2}\Gamma\sigma^2 S^2 \Delta t\right)^2 Var[(1 - N(0,1)^2)] = \frac{1}{2}(\Gamma\sigma^2 S^2 \Delta t)^2$$

$$where\ Var[(1 - N(0,1)^2)] = Var[N(0,1)^2] = 2$$

$$Var[P\&L\ of\ 1\ year] = \Delta t \frac{1}{2}(\Gamma\sigma^2 S^2 \Delta t)^2 = \frac{1}{2}\Gamma\sigma^2 S^2 \Delta t$$

We confirm our numerical results (4) above: variance of rebalancing 10 times per day is roughly a tenth of that of rebalancing once a day.

[10] Riaz Ahmad, Paul Wilmott: Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and optimal portfolios

Conclusion:

When the volatility is constant, replication error is easy to understand. We only worry about gamma risk. Even the distribution of replication error is easy to understand. It's simply a function of Gamma and volatility. We can change the variance by increasing the number of rebalancing per given time.

Now, our original question: What is the bid-ask spread necessary for a market maker to be making money 80% of the time if he rebalances his portfolio daily? Simple answer is, he can't get such wide bid-ask spread, unless he is in a highly illiquid market. We have seen in result (4) that the variance of replication error for once a day rebalance is roughly \$0.2, so standard deviation of distribution is roughly \$0.44. Given the conditions for computing result (4), option price is roughly \$8. If we add 1.5 standard deviation of \$0.66 to the price, the market maker is asking for a price of \$8.66. This is even higher than the price for an option with strike price at \$99 instead of \$100; the Black Scholes price is \$8.44 in this case. Unless one is in a very illiquid market, getting an order done with such a high premium will not be an easy task.

As soon as we let the volatility be time varying, which is true in reality, a number of things break down. Vega starts to generate replication errors in addition to gamma, and the mathematics behind the distribution of replication error is no longer so simple. To this regard, Wilmott has answered the question in his book - Paul Wilmott on Quantitative Finance[11].

Further study can be done respect to combining different options. Appendix A shows an example of two different options with different strike prices, but it assumes constant volatility. A naïve implementation of time varying volatility can also be done under the current data structure, but implementing volatility surface is certainly a better approach to generate more realistic backtesting.

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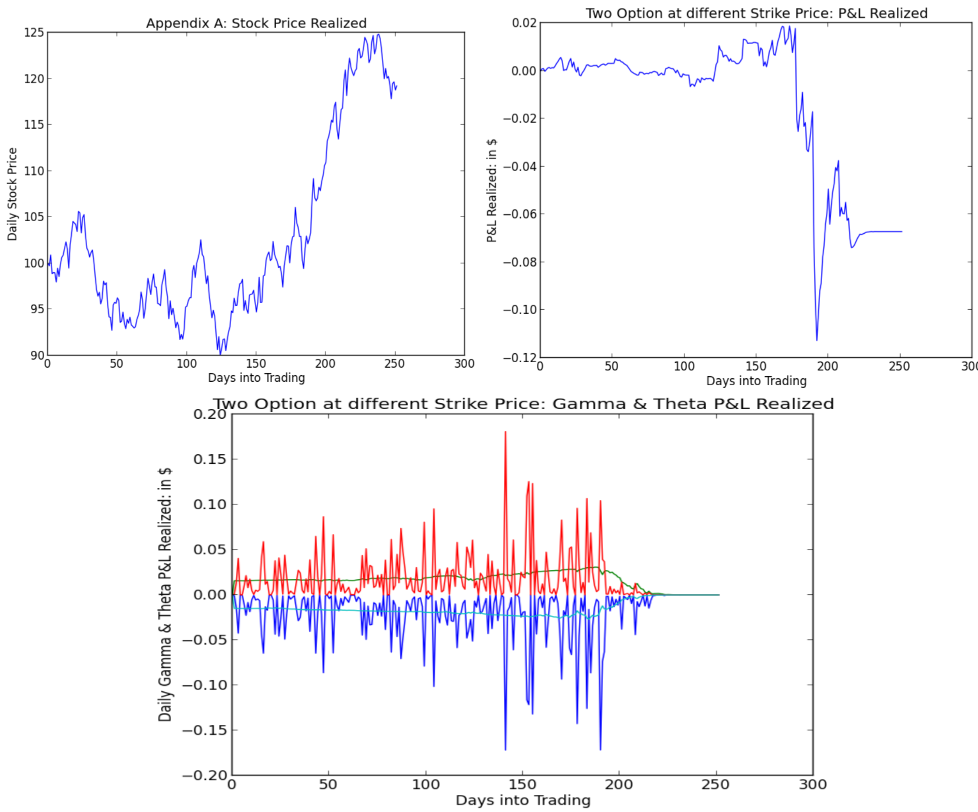
[11] Paul Wilmott, Paul Wilmott on Quantitative Finance

Appendix A - Combining Options at Different Strikes & Thoughts on Combination

Suppose we combine call options at different strike prices with same maturity. More specifically, suppose we sell a call option with strike \$100, and buy a call with strike \$95, and maintain a portfolio as we did previously by computing delta and rebalance it daily. Assume options on both strike prices have implied volatility of 20% per annum. Let the stock price follow GBM, Spot = *\$100,* $\sigma_{realized}$ = *20% per year,* μ = *dividend = 0%, time = 1 year = 252 days.* The following illustrates a result of this portfolio.

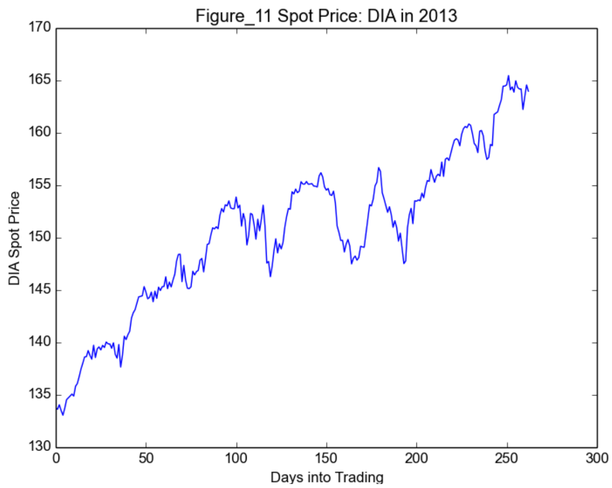


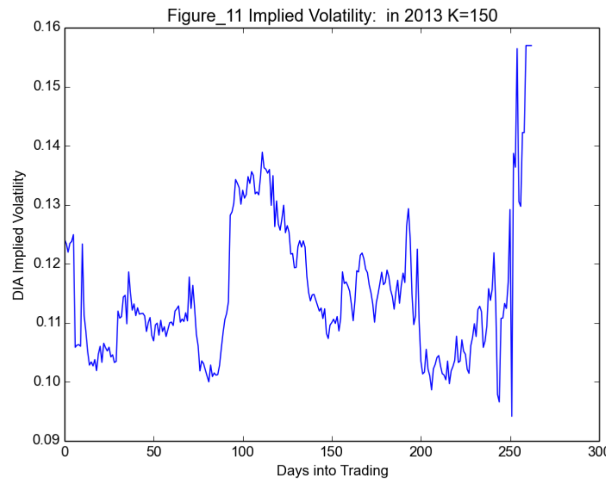
Top Row: Left: Realized Stock Price from GBM, right: Realized Replication Error
Bottom Row: Dissection of Daily Gamma and Theta P&L from two different options

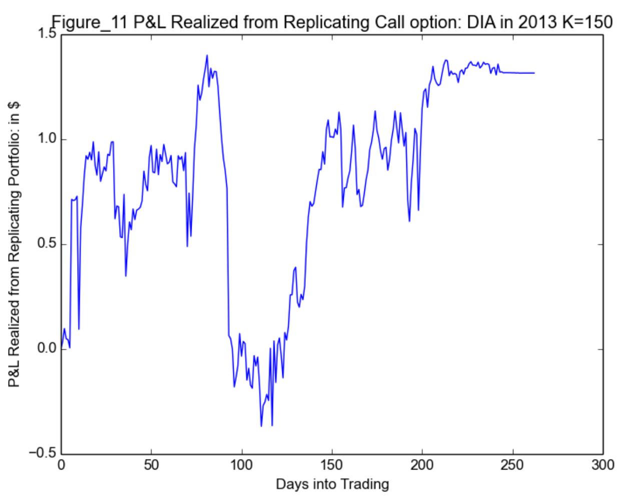
Intuitively, the result makes sense. Each Greek P&L tends to offset each of the option, and net delta is near zero until we get fairly close to the time to expiry, at which point the underlying price matters much because of Gamma. If volatility were to change over time, similar results of offsetting Vega P&L from each option is expected, but the change of volatility surface would be important in determining the extent of offsets, along with the realized underlying spot price.

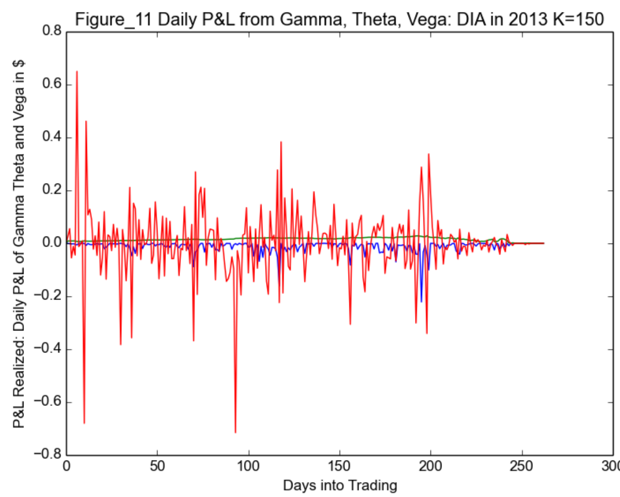
Appendix B – Backtest of Replication with data from 2013

Dow Jones Industrial Average: 01/02/2013 to 01/17/2014, Strike Price at 150



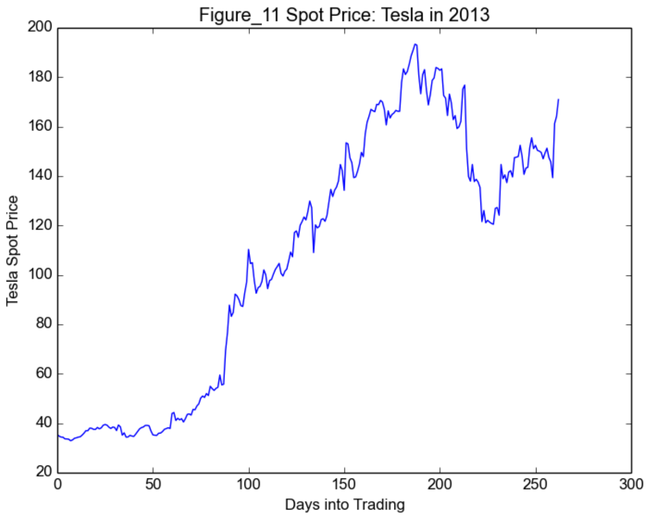


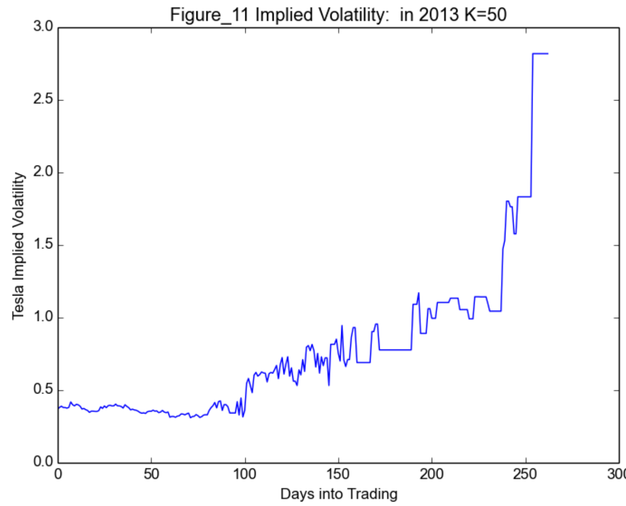


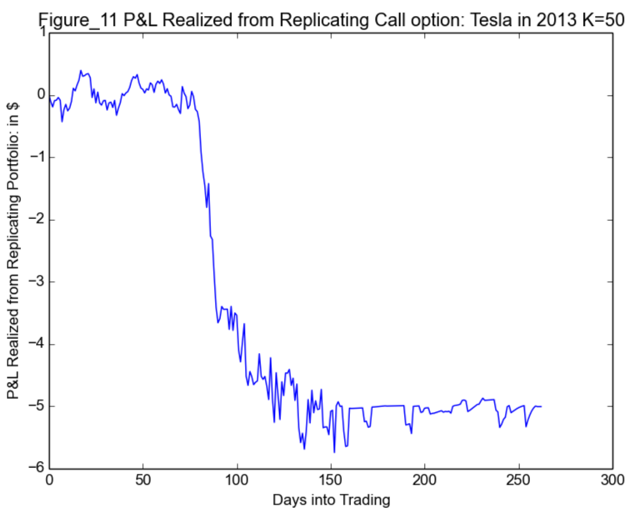


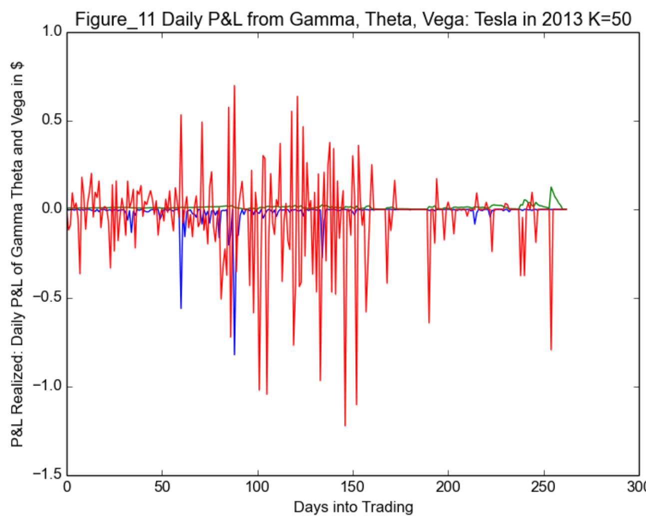
Appendix C – Backtest of Replication with data from 2013

Tesla: 01/02/2013 to 01/17/2014, Strike Price at $50



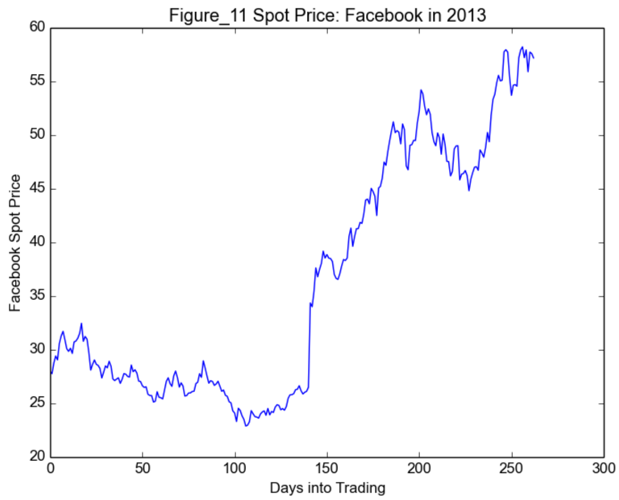


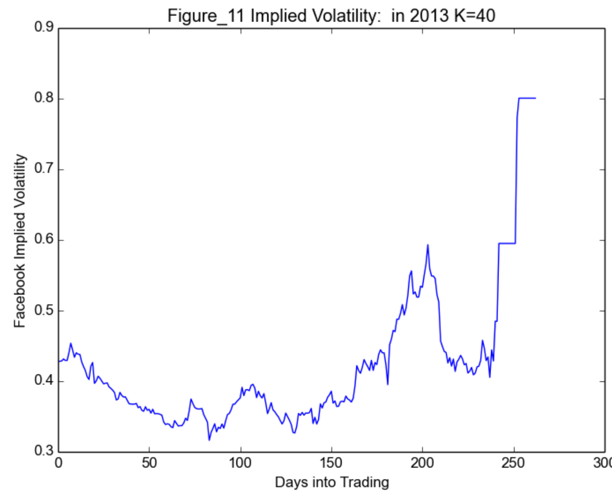


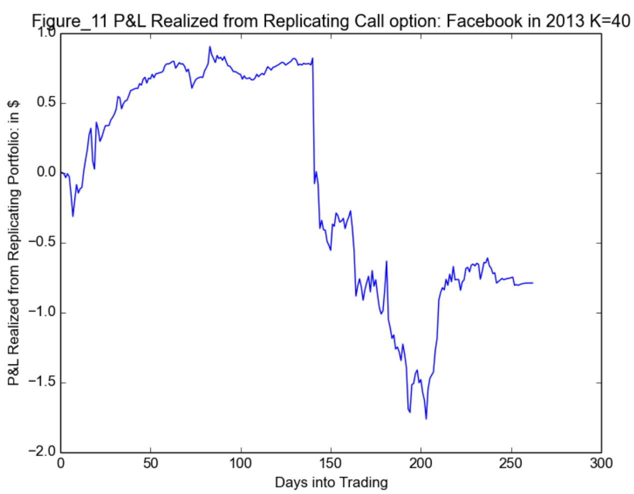


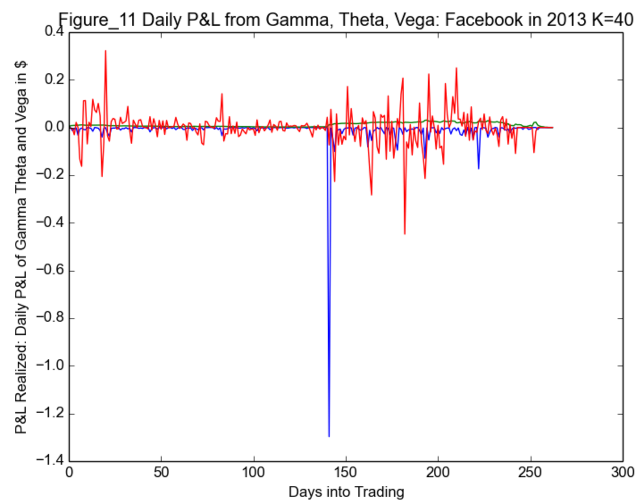
Appendix D – Backtest of Replication with data from 2013

Facebook: 01/02/2013 to 01/17/2014, Strike Price at $40









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