Homework 2 - Quantum Mechanics I

NAME: ____ **SCORE:** ________

Deadline: Wednesday 1 October 2025 by 13h00
Credits: 20 points → 20 credits **Number of problems:** 4
Type of evaluation: Formative Evaluation

This assignment is individual and consists of 4 problems related to unit 2 of quantum mechanics. Please justify all calculations and highlight the answers.

1. **(5 points) Probability densities**
   Consider the log-normal distribution:

   $\rho(x) = \frac{A}{x}\, e^{-\frac{(\ln(x)-\mu)^2}{2\,\sigma^2}}$,

   where A, μ, and σ are positive real constants.

   (a) Determine A.

   (b) Find $\langle x \rangle$.

   (c) Find $\langle x^2 \rangle$, and σ_x.

   (d) What do A, μ, and σ represent?

   (e) Plug some fiducial numbers for these constants, and sketch the graph of $\rho(x)$ using your favourite programming language.

2. **(5 points) Schrödinger equation and normalisation**
   Let a wave function of a particle be given by:

   $$\Psi(x,t) = \begin{cases} C\,\frac{1}{x}\,e^{-i\omega t}, & 1 \le x \le 2 \\ 0, & \text{otherwise,} \end{cases}$$

   where C and ω are constants and $i \equiv \sqrt{-1}$.

   (a) Find C so that $\Psi(x,t)$ is normalised.

   (b) Plug some fiducial energy and sketch the initial wave function, $\Psi(x,0)$, versus x.

   (c) Calculate the probability of finding the particle between $\frac{3}{2} \le x \le 2$.

   (d) Calculate the expectation values of position $\langle x \rangle$ and momentum $\langle p \rangle$.

   (e) Choose two times and sketch the normalised wave function $\Psi(x,t)$ at those times. How does the wave function evolve?

3. **(5 points) Wave packet and probability density**
A free particle of mass m has the following wave function at time $t = 0$:

$$\Psi(x,0) = \frac{\sqrt{\alpha}}{(2\pi)^{3/4}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{4}(k-k_0)^2} e^{ikx} dk$$

(a) Find an expression for $\phi(k)$.

(b) Calculate the time-dependent wave packet $\Psi(x,t)$.

(c) Calculate the probability density $|\Psi(x,t)|^2$.

(d) Use your favourite programming tool to plot the probability density for $t = 0$ and $t > 0$, and briefly explain your findings.

(e) Compute the uncertainties σ_x and σ_p. Verify hat $\sigma_x\,\sigma_p \geq \frac{\hbar}{2}$ and discuss when the equality holds.

4. **(5 points) Infinite square well potential and expectation values**
In class we solved the Schrödinger equation for an infinite square well potential of width L. Such potential allows for bound solutions only as the particle cannot escape from the well. The solutions we found had the following functional form:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\,\pi}{L}\,x\right) \tag{1}$$

For the n-th state given by the function above, calculate:

(a) The expectation values associated with the position x: $\langle x\rangle$, $\langle x^2\rangle$.

(b) The expectation values associated with the momentum p: $\langle p\rangle$, $\langle p^2\rangle$.

(c) The dispersions σ_x and σ_p, and their product $\sigma_x\,\sigma_p$.

(d) Use programming tools to make a plot of $(\sigma_x\,\sigma_p)$ vs. n.

(e) Is the uncertainty principle satisfied? Which of the $\psi_n(x)$ states comes closest to the uncertainty limit?