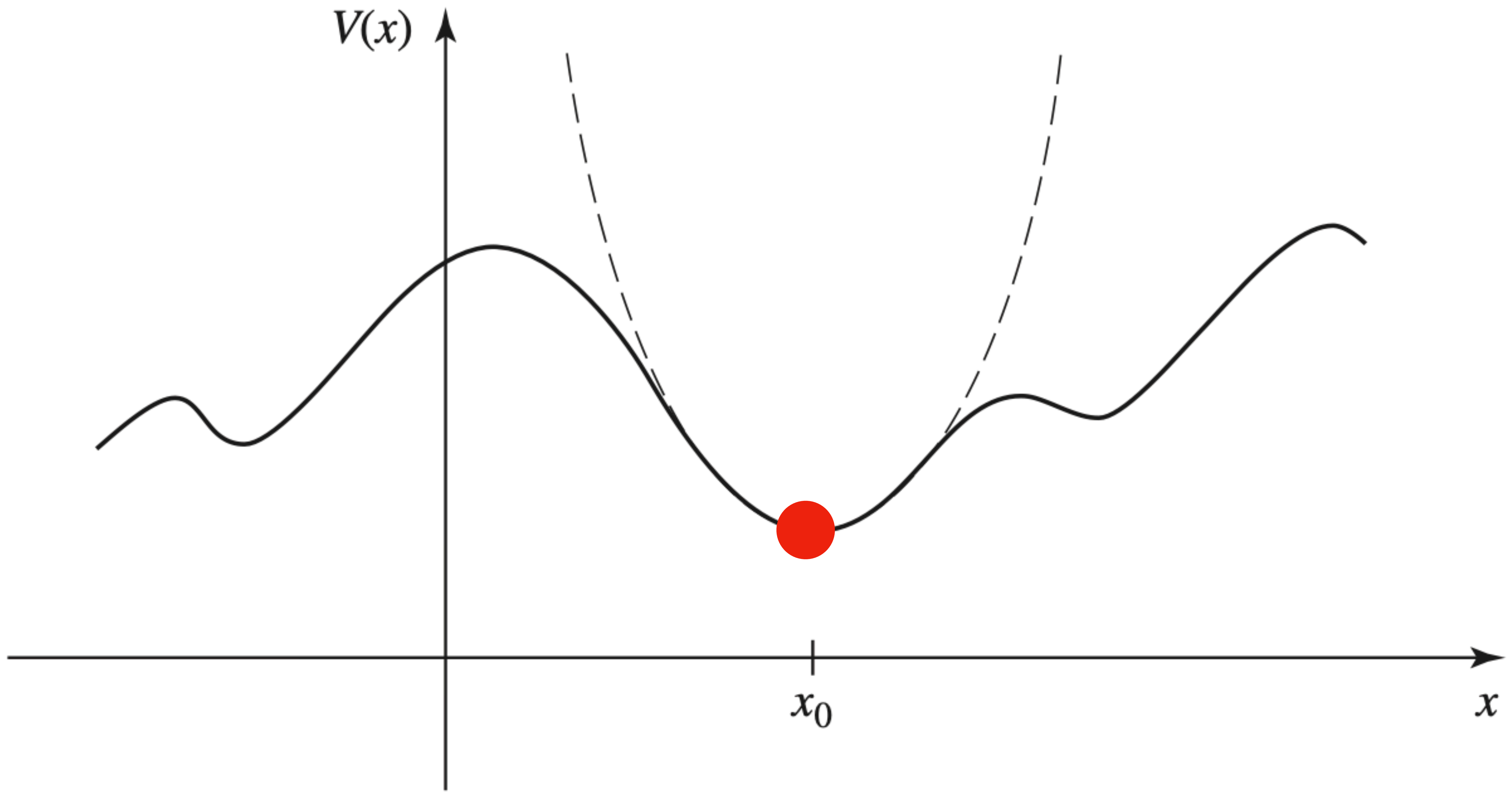
QM Harmonic Oscillator

General potential



Time-independent Sch. Eq:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi$$

1. Algebraic method (ladder ops)
2. Power series method

1. Algebraic method:

$$\frac{1}{2m}\left[\hat{p}^2+(m\omega x)^2\right]\psi=E\psi,$$

Ladder operators for E:

$$\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp i\,\hat{p}+m\omega x\right)$$

$$\hbar\omega\left(\hat{a}_\pm\hat{a}_\mp \pm \frac{1}{2}\right)\psi=E\psi$$

$$\hat{H}\left(\hat{a}_+\psi\right)=(E+\hbar\omega)\left(\hat{a}_+\psi\right)$$

$$\hat{H}\left(\hat{a}_-\psi\right)=(E-\hbar\omega)\left(\hat{a}_-\psi\right)$$

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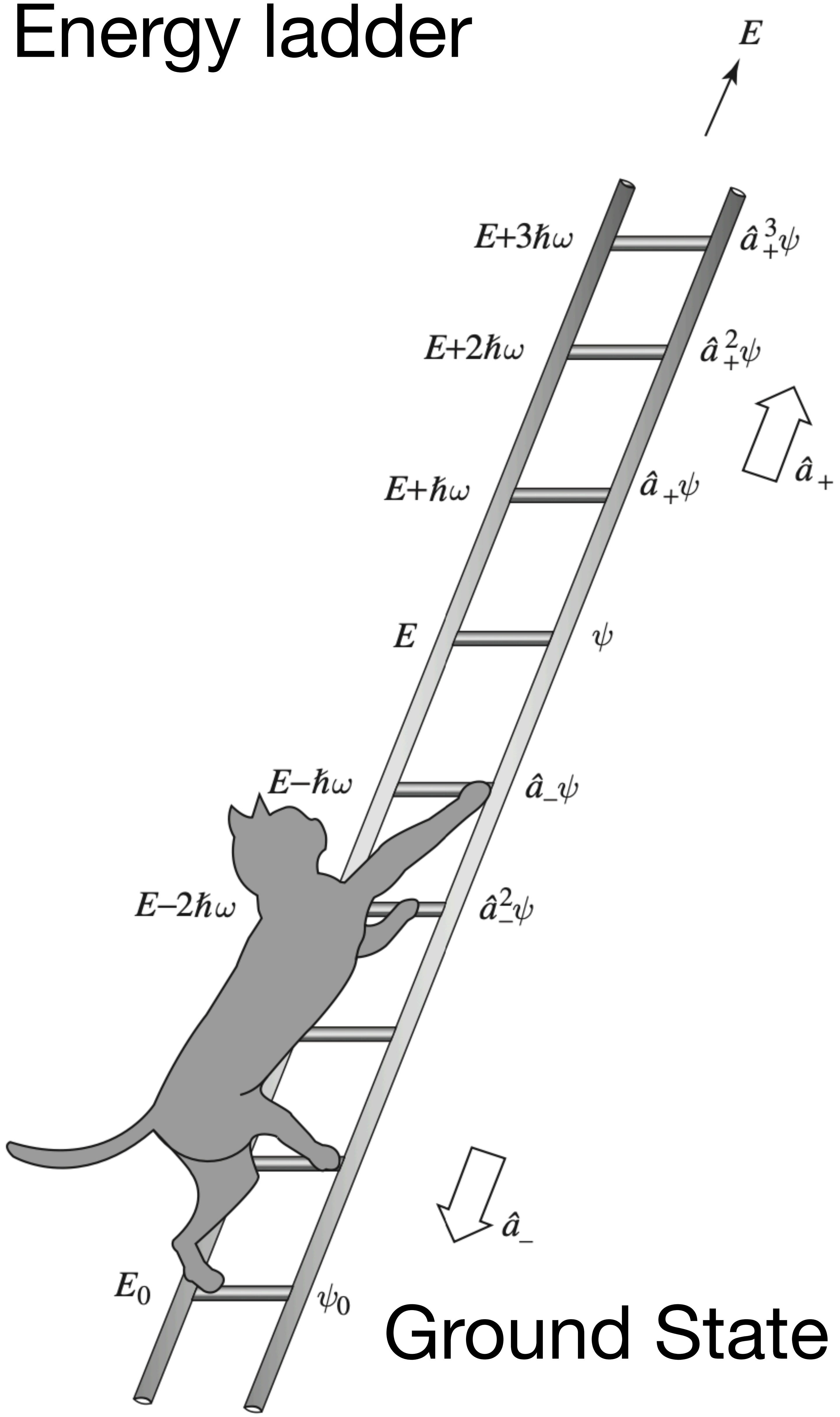
Energy ladder



Ladder operators for E:

$$\boxed{\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}, \quad \hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}.}$$

$$\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega x)$$

$$\boxed{\psi_n = \frac{1}{\sqrt{n!}}\left(\hat{a}_+\right)^n \psi_0,}$$

$$\hat{H}\left(\hat{a}_+\psi\right) = (E+\hbar\omega)\left(\hat{a}_+\psi\right)$$

$$\hat{H}\left(\hat{a}_-\psi\right) = (E-\hbar\omega)\left(\hat{a}_-\psi\right)$$

Excited States

$$\boxed{\psi_n(x) = A_n\left(\hat{a}_+\right)^n \psi_0(x)\,, \quad \text{with} \quad E_n = \left(n+\frac{1}{2}\right)\hbar\omega,}$$

Ground State

$$\hat{a}_-\psi_0 = 0.$$

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}.}$$

$$E_0 = \frac{1}{2}\hbar\omega.$$

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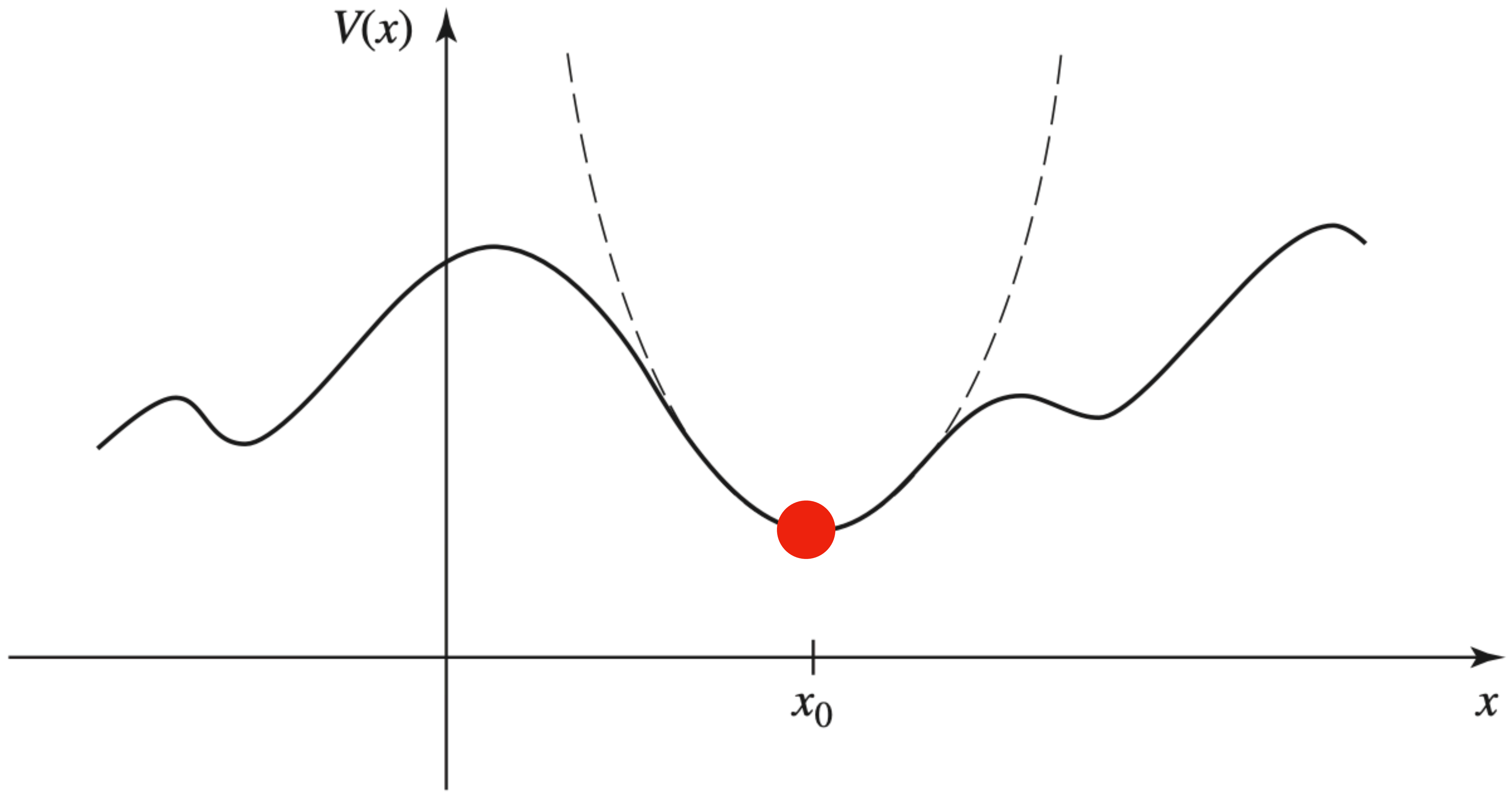

Time-independent Sch. Eq:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi$$

1. Algebraic method (ladder ops)
2. Power series method

2. Power series method

Variable changes -> ODE where you can guess power series solutions.

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x; \qquad K \equiv \frac{2E}{\hbar\omega}.$$

$$\frac{d^2\psi}{d\xi^2}=\left(\xi^2-K\right)\psi,$$

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Asymptotic behaviour: $x \to \infty$

$$\frac{d^2\psi}{d\xi^2}\approx \xi^2\psi,$$

$$\psi(\xi)\approx Ae^{-\xi^2/2}+\cancel{Be^{+\xi^2/2}}$$

$$\psi(\xi)=h(\xi)\,e^{-\xi^2/2},$$

Back to the Sch. Eq:

$$\frac{d^2h}{d\xi^2}-2\xi\frac{dh}{d\xi}+(K-1)\,h=0.$$

We assume power series solutions:

$$h(\xi)=a_0+a_1\xi+a_2\xi^2+\cdots=\sum_{j=0}^{\infty}a_j\xi^j.$$

$$\sum_{j=0}^{\infty}\left[(j+1)(j+2)\,a_{j+2}-2ja_j+(K-1)\,a_j\right]\xi^j=0.$$

Recursion formula:

$$a_{j+2}=\frac{(2j+1-K)}{(j+1)(j+2)}a_j.$$

$$h(\xi)=h_{\text{even}}(\xi)+h_{\text{odd}}(\xi),$$

$\xi \to \infty$

$$h(\xi)\approx C\sum\frac{1}{(j/2)!}\xi^j\approx C\sum\frac{1}{j!}\xi^{2j}\approx Ce^{\xi^2}.$$

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$$\frac{d^2\psi}{d\xi^2}=\left(\xi^2-K\right)\psi,$$

$$\xi\equiv\sqrt{\frac{m\omega}{\hbar}}x;$$

$$\sum_{j=0}^{\infty}\left[(j+1)(j+2)a_{j+2}-2ja_j+(K-1)a_j\right]\xi^j=0.$$

Recursion formula:

$$K=2n+1,$$

$$a_{j+2}=\frac{(2j+1-K)}{(j+1)(j+2)}a_j. \quad \Rightarrow \quad a_{j+2}=\frac{-2(n-j)}{(j+1)(j+2)}a_j.$$

$$h(\xi)=h_{\text{even}}(\xi)+h_{\text{odd}}(\xi),$$

$$\xi\to\infty$$

$$h(\xi)\approx C\sum\frac{1}{(j/2)!}\xi^j\approx C\sum\frac{1}{j!}\xi^{2j}\approx Ce^{\xi^2}.$$

$$\Rightarrow \quad \psi_n(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)\,e^{-\xi^2/2}.$$

Hermite polynomials

$$H_0=1,$$
$$H_1=2\xi,$$
$$H_2=4\xi^2-2,$$
$$H_3=8\xi^3-12\xi,$$
$$H_4=16\xi^4-48\xi^2+12,$$
$$H_5=32\xi^5-160\xi^3+120\xi.$$

We need to truncate the series at a value “n”:

$$K=2n+1,\qquad K\equiv\frac{2E}{\hbar\omega}.$$

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega,\qquad \text{for } n=0,1,2,\ldots.$$

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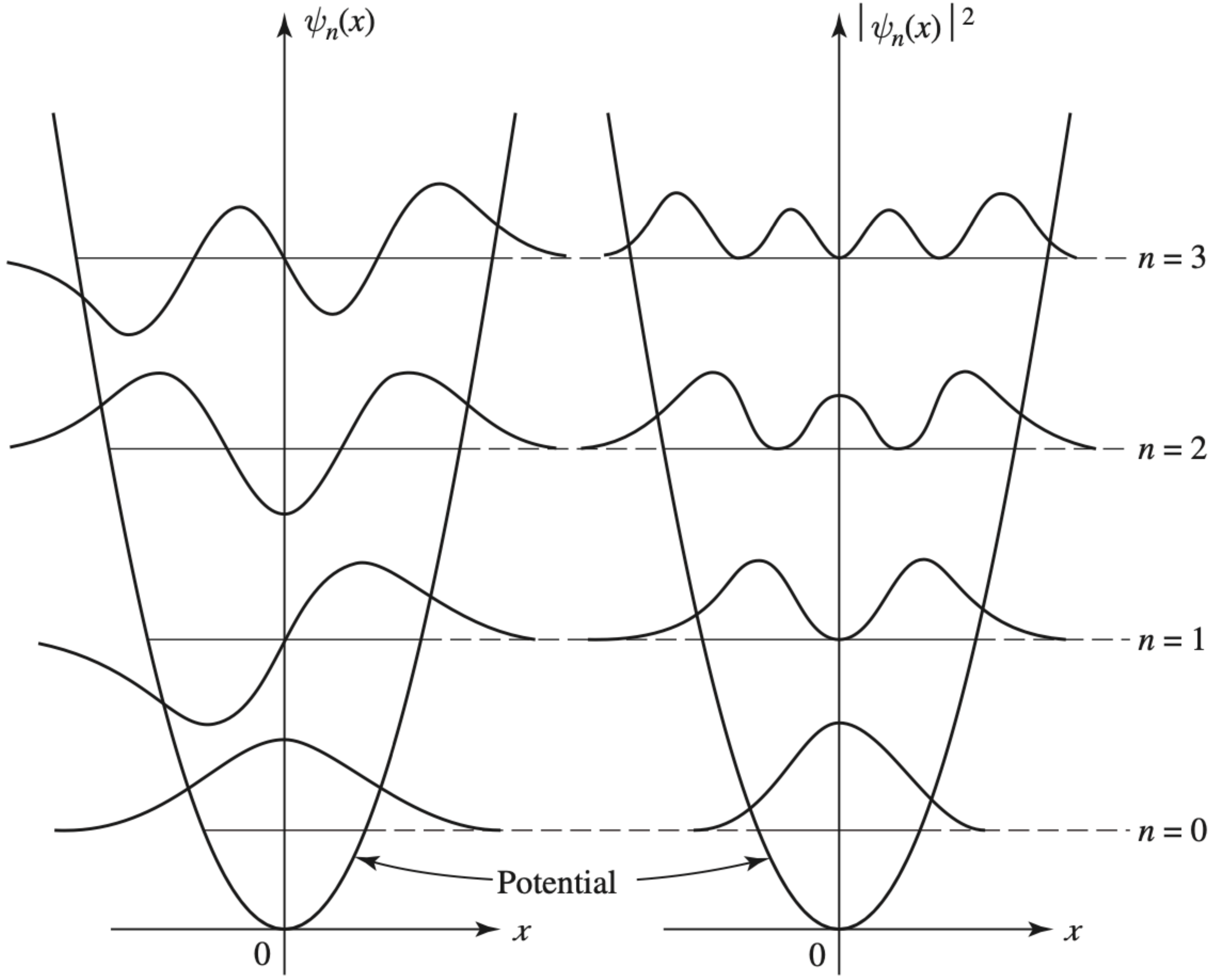
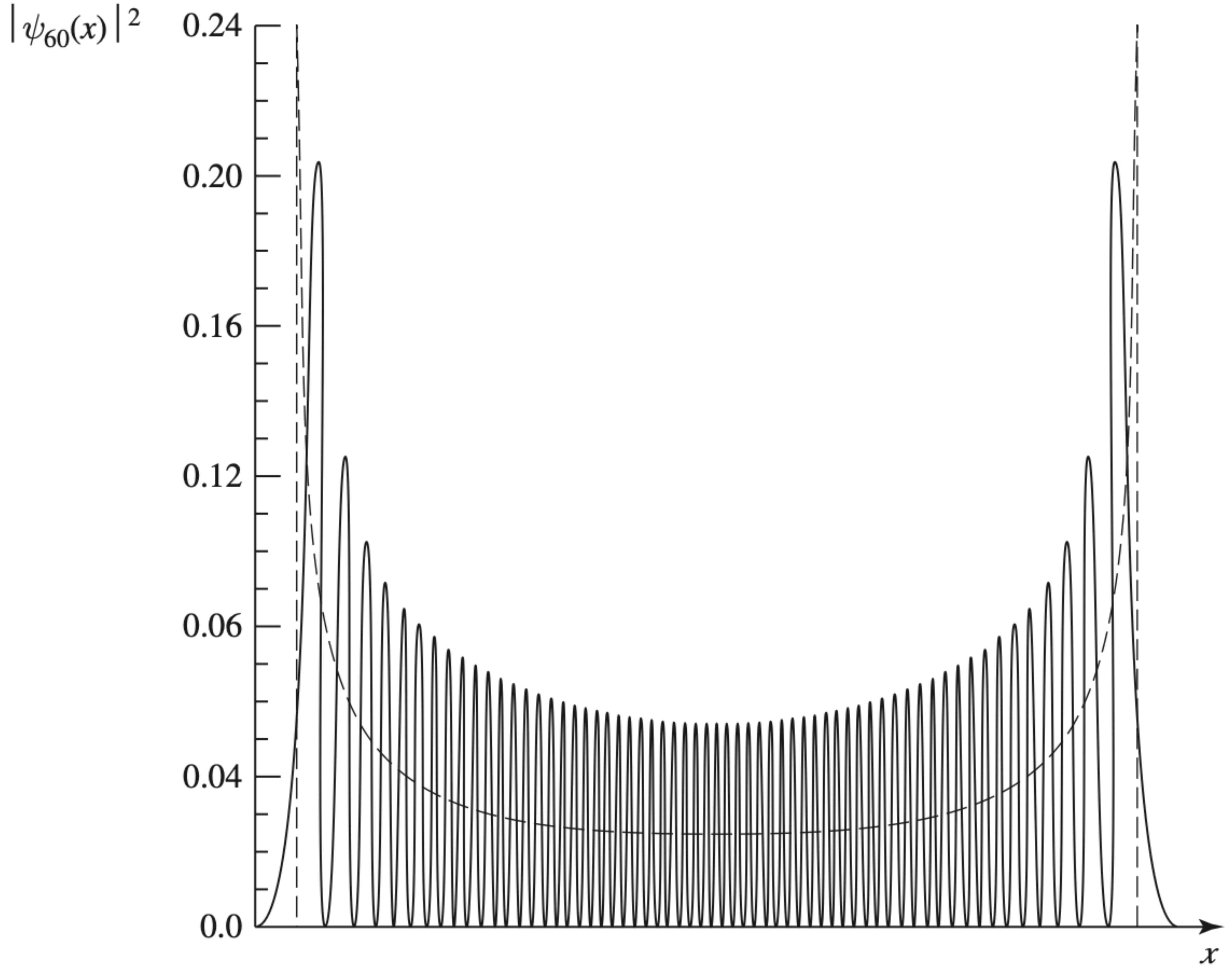


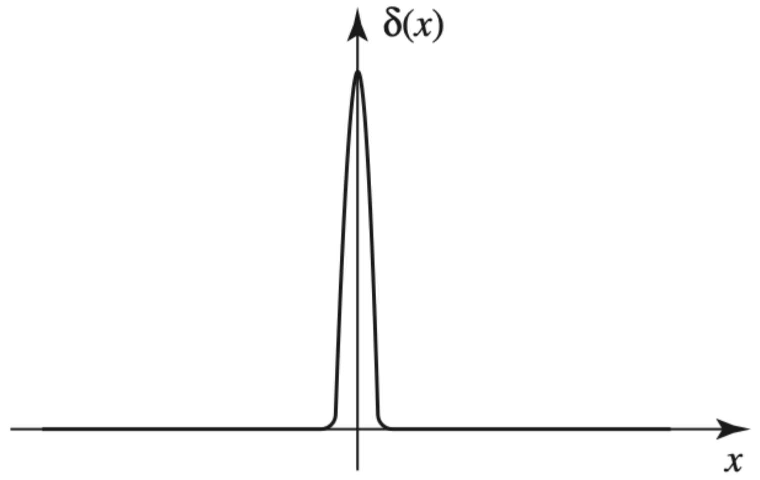


Delta function well

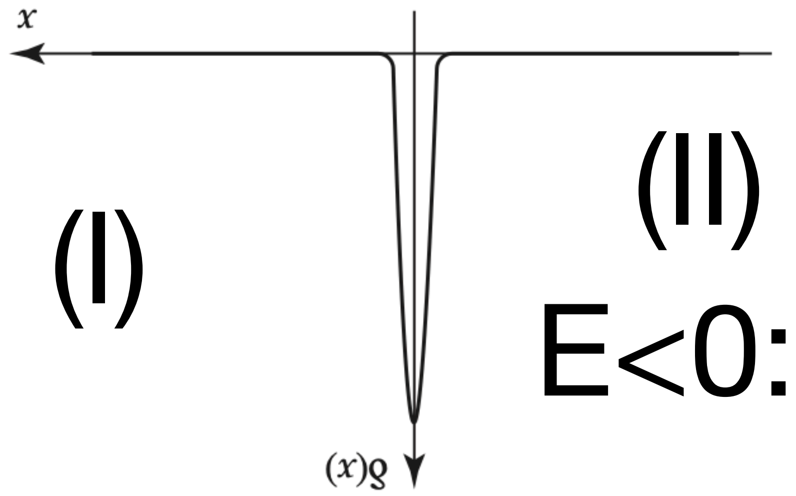
Delta functions:

$$\delta(x) \equiv \left\{ \begin{array}{ll} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{array} \right\}, \text{ with } \int_{-\infty}^{+\infty} \delta(x)\, dx = 1.$$

Barrier or well:



E>0: Scattering States



(I)

(II)

$$V(x) = -\alpha\delta(x)$$

E<0: Bound States

Time-ind Sch. Eq:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\delta(x)\,\psi = E\psi$$

A) Bound states

$$\kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

(I)

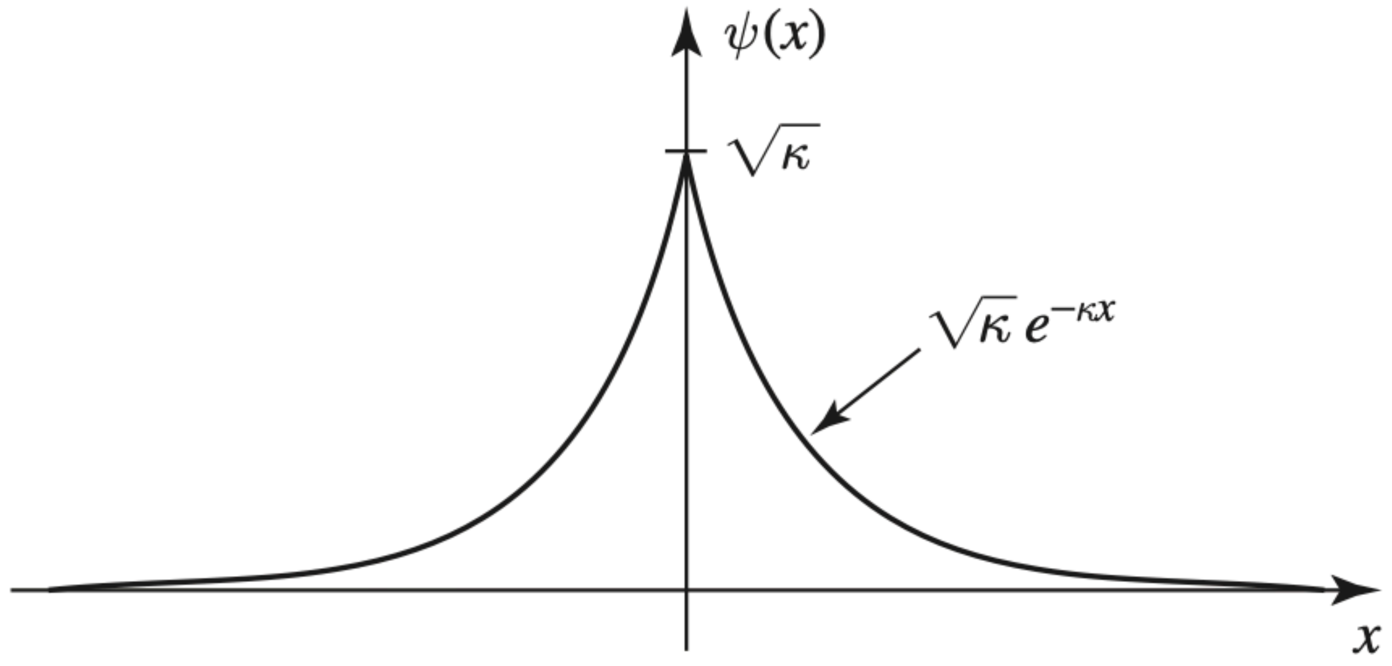
$$\psi(x) = \cancel{Ae^{-\kappa x}} + Be^{\kappa x}$$

$$\psi_I(x) = Be^{\kappa x}, \quad (x < 0).$$

(II) $$\psi_{II}(x) = Fe^{-\kappa x}, \quad (x > 0).$$

BCs:

$$\psi(x) = \begin{cases} Be^{\kappa x}, & (x \le 0), \\ Be^{-\kappa x}, & (x \ge 0); \end{cases}$$



Delta function well

$$\psi(x) = \begin{cases} Be^{\kappa x}, & (x \leq 0)\,, \\ Be^{-\kappa x}, & (x \geq 0)\,; \end{cases}$$

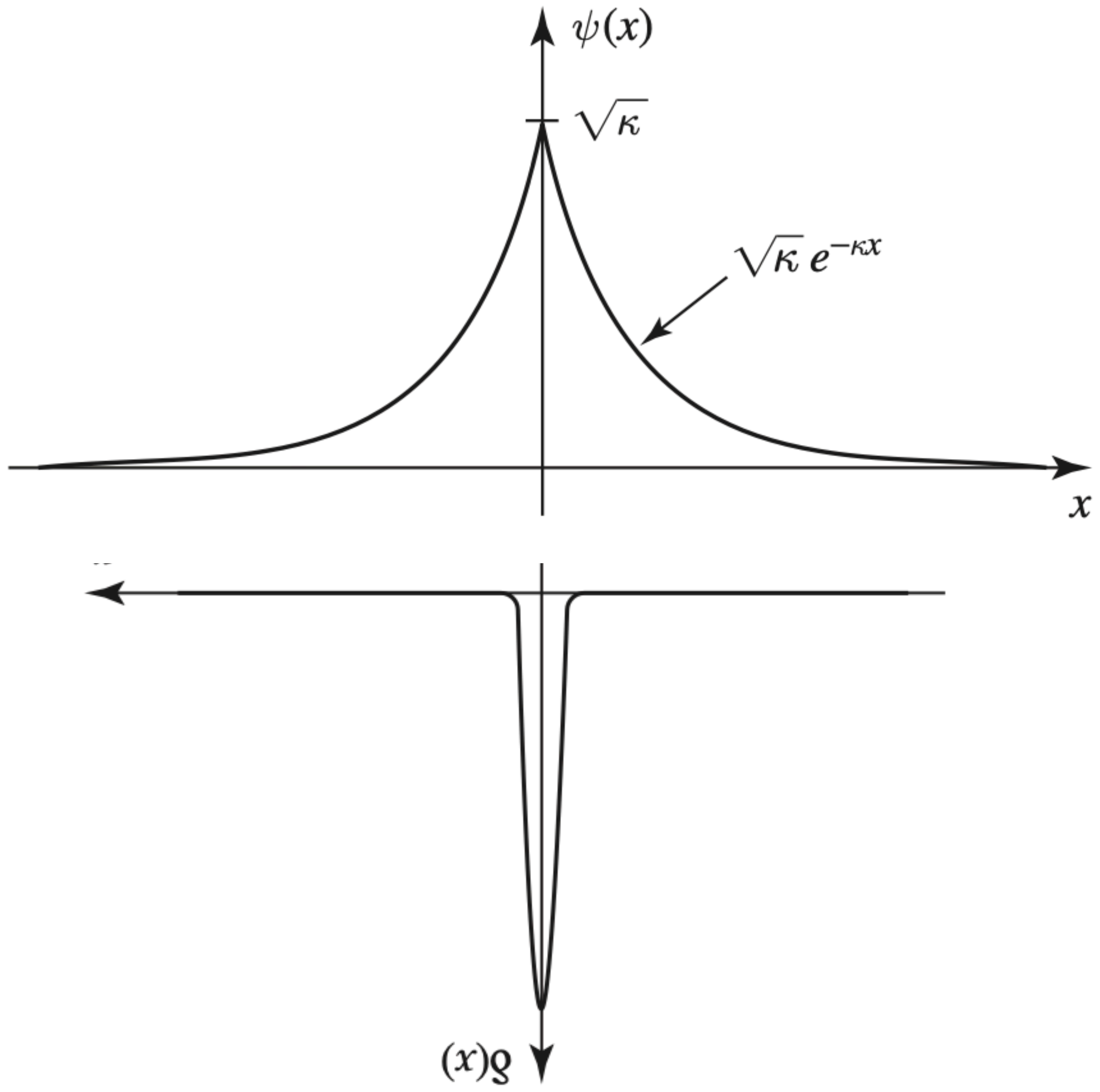
BCs on the derivative:

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{+\epsilon}\frac{d^2\psi}{dx^2}\,dx + \int_{-\epsilon}^{+\epsilon}V(x)\,\psi(x)\,dx = E\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx$$

$$\Delta\left(\frac{d\psi}{dx}\right) \equiv \lim_{\epsilon\to 0}\left(\left.\frac{\partial\psi}{\partial x}\right|_{+\epsilon} - \left.\frac{\partial\psi}{\partial x}\right|_{-\epsilon}\right) = \frac{2m}{\hbar^2}\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon}V(x)\,\psi(x)\,dx.$$

$$\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\psi(0)$$

$$\begin{cases} d\psi/dx = -B\kappa e^{-\kappa x}, \text{ for } (x > 0)\,, & \text{so } d\psi/dx\big|_{+} = -B\kappa, \\ d\psi/dx = +B\kappa e^{+\kappa x}, \text{ for } (x < 0)\,, & \text{so } d\psi/dx\big|_{-} = +B\kappa, \end{cases}$$



$$\boxed{\kappa = \frac{m\alpha}{\hbar^2},}$$

Quantisation!

$$\kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

$$\boxed{E = -\frac{\hbar^2\kappa^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}.}$$

Delta function well

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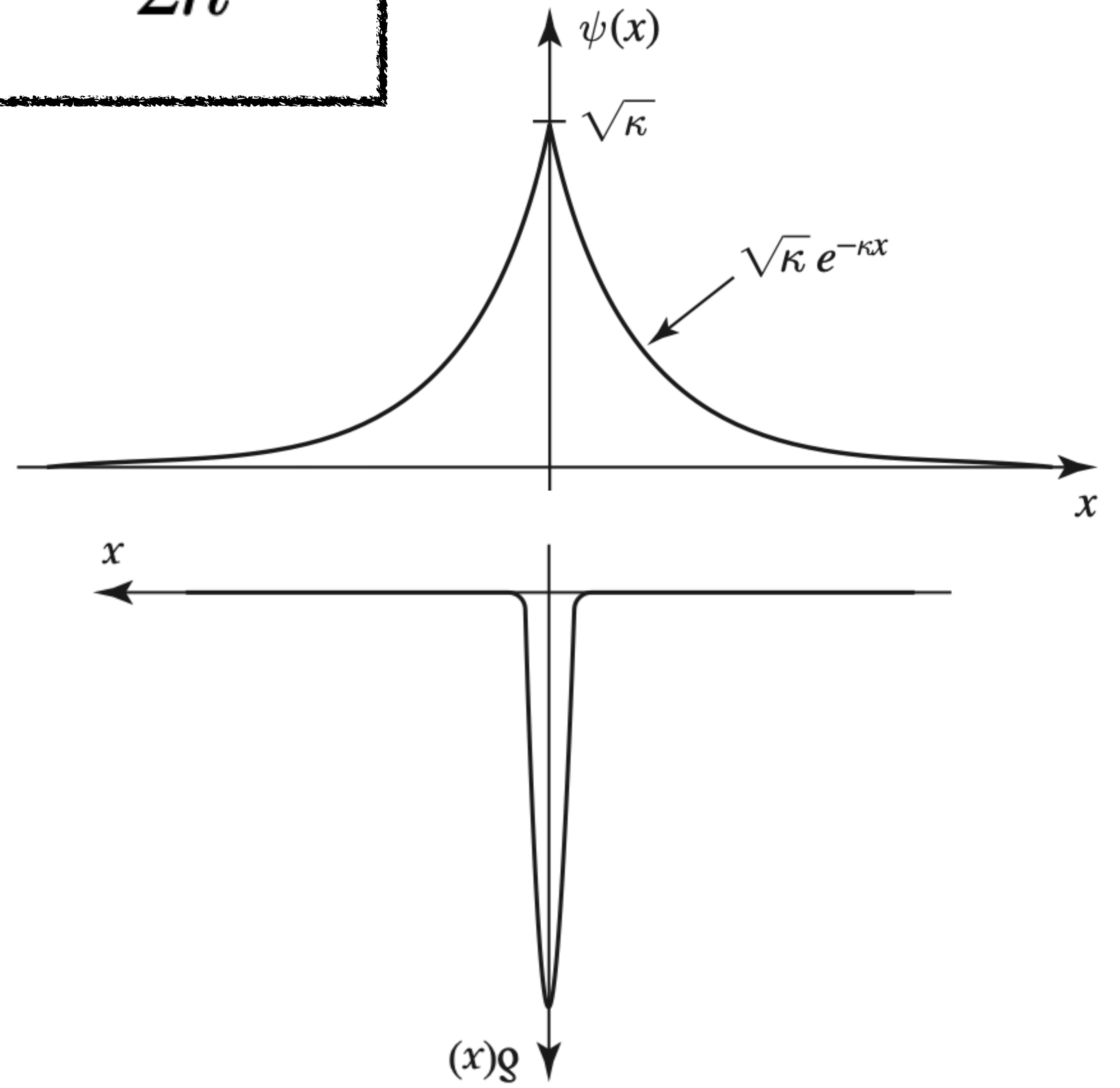


$$\psi(x) = \begin{cases} Be^{\kappa x}, & (x \le 0)\,, \\ Be^{-\kappa x}, & (x \ge 0)\,; \end{cases}$$

Normalisation:

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 \, dx = 2\,|B|^2 \int_0^{\infty} e^{-2\kappa x} \, dx = \frac{|B|^2}{\kappa} = 1,$$

$$\boxed{B = \sqrt{\kappa} = \frac{\sqrt{m\alpha}}{\hbar}.}$$

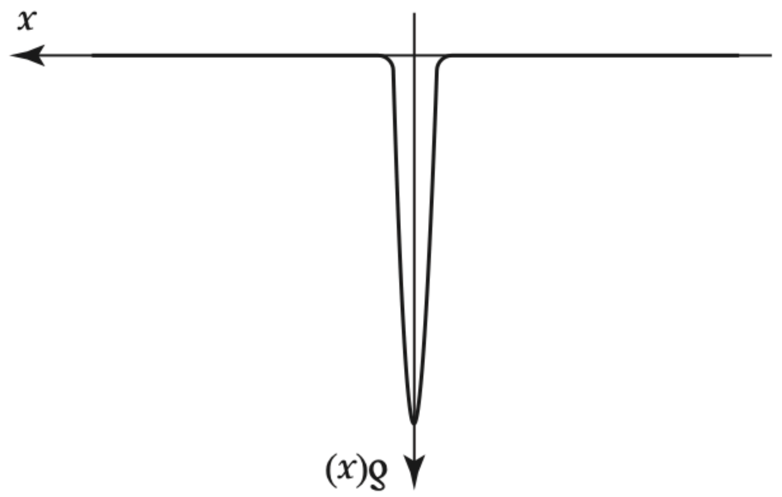
We have a single bound state:

$$\boxed{\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}; \quad E = -\frac{m\alpha^2}{2\hbar^2}.}$$

Delta function well

E>0: Scattering States

(I) (II)



(II) $\psi(x) = Fe^{ikx} + \cancel{Ge^{-ikx}}.$

Transmitted wave

Another particle

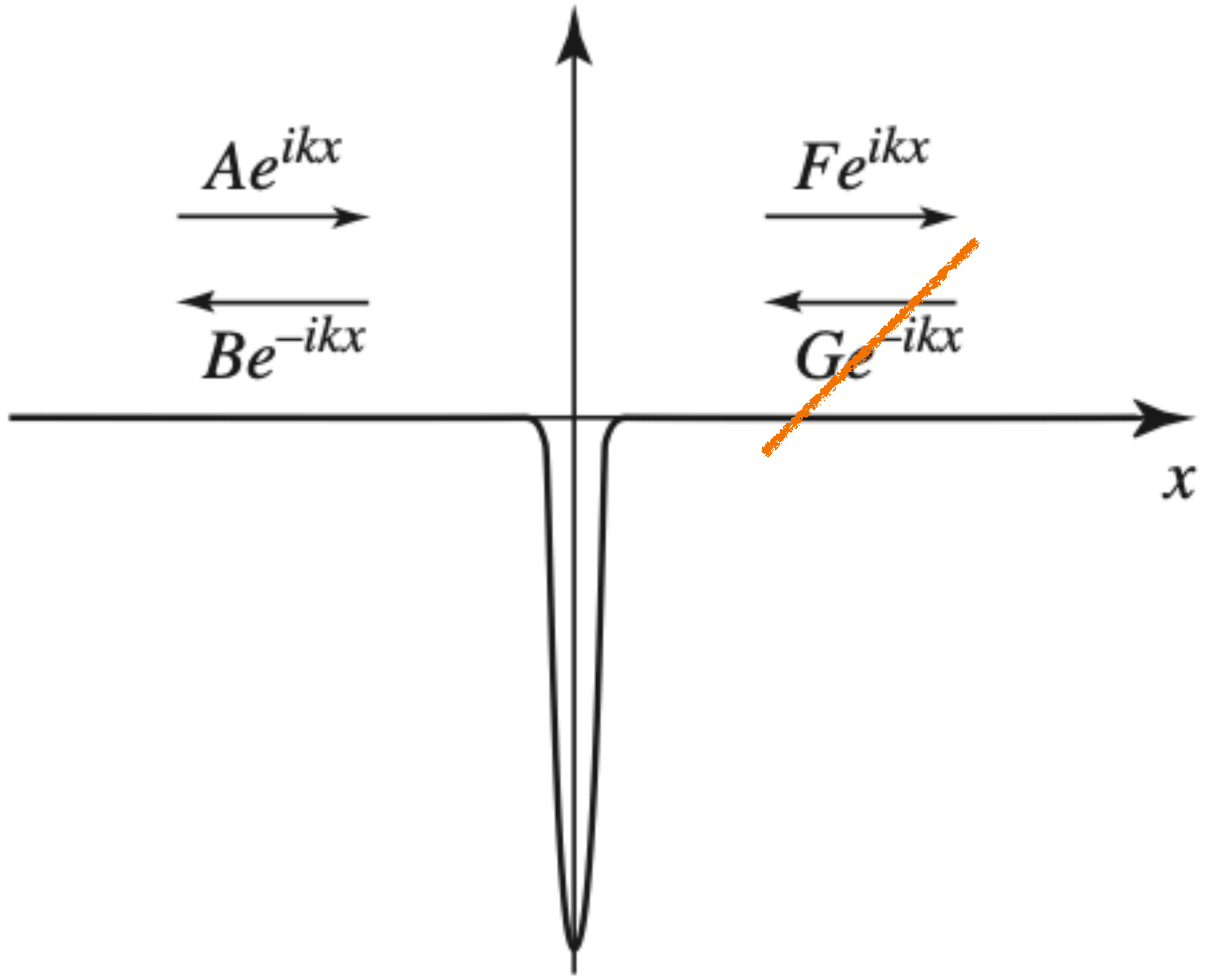
Stationary states (plane waves -> non-physical)

Physical states: Wave Packets!

Time-ind Sch. Eq:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\delta(x)\,\psi = E\psi$$

$$k \equiv \frac{\sqrt{2mE}}{\hbar}$$



(I) Stationary states (plane waves -> non-physical)

$$\psi(x) = Ae^{ikx} + Be^{-ikx},$$

Incident wave

Reflected wave

Delta function well

BCs:

Reflection coefficient
$$R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}.$$

Transmission coefficient
$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}.$$

$$R + T = 1.$$

$$\beta \equiv \frac{m\alpha}{\hbar^2 k}. \qquad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$\boxed{R = \frac{1}{1 + \left(2\hbar^2 E/m\alpha^2\right)}, \quad T = \frac{1}{1 + (m\alpha^2/2\hbar^2 E)}.}$$