

Homework 4 - Quantum Mechanics I

NAME: _____ SCORE: _____

Deadline: Tuesday 2 December 2025 by 21:00

Credits: 20 points **Number of problems:** 4

Type of evaluation: Formative Evaluation

This assignment is individual and consists of 4 problems related to units 3 and 4 of quantum mechanics. Please justify all calculations and highlight the answers.

1. (5 points) Spherical harmonics

- (a) A quantum system is known to be in the (unnormalised) state described by the wave function $\psi(\theta, \phi) = 4Y_4^{-4} - 2i Y_5^{-4} + 7Y_5^0$, where the $Y_\ell^m(\theta, \phi)$ are the spherical harmonics. What is the probability of finding the system in a state with quantum number $m = -4$?
- (b) What is the orbital angular momentum eigenfunction Y_ℓ^m in a state for which the operators L^2 and L_z have eigenvalues $6\hbar^2$ and $-\hbar$, respectively?
- (c) Construct all the possible spherical harmonics, $Y_\ell^m(\theta, \phi)$, for $\ell = 3$.
- (d) Using your favourite programming language, make 3D plots of all of them.
- (e) Choose two of the spherical harmonics constructed in part (b), and prove that they are normalised and orthogonal.

2. (5 points) Hydrogen atom

- (a) Construct all the possible spatial wave functions, $\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_\ell^m(\theta, \phi)$, of the hydrogen atom for $(n, \ell, m) = (2, 1, m)$.
- (b) Using your favourite programming language, make density plots of all of these states.
- (c) Calculate the energy level of these states in units of eV. Is there any degeneracy?
- (d) In terms of the Bohr radius, find $\langle r \rangle$, $\langle x \rangle$, $\langle r^2 \rangle$ and $\langle x^2 \rangle$ for an electron in the ground state of hydrogen.
- (e) Find $\langle x^2 \rangle$ in the state $(n, \ell, m) = (2, 1, m)$ with the highest possible value of m that is allowed. How different is the result with respect to that calculated in part (d) for the ground state of hydrogen?

3. (5 points) Spin

- (a) Suppose we have a spin- $\frac{1}{2}$ particle, whose quantum state can be represented by the eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of the S_z operator. Using the Pauli matrix σ_y , calculate a normalised eigenstate of the S_y operator with an eigenvalue $-\frac{\hbar}{2}$.
- (b) Consider the Pauli spin matrices σ_x , σ_y , and σ_z and the identity matrix, I . Calculate the commutators: $[\sigma_y, \sigma_x]$ and $[\sigma_y, \sigma_z]$.
- (c) Calculate the normalised spin eigenfunctions of a system with two particles, one has spin 1 and the other one has spin $\frac{3}{2}$. Pick one of the states and report the Clebsch-Gordan superposition.
- (d) Consider an electron in the spin state: $\chi = A \begin{bmatrix} 1+i \\ -1+2i \end{bmatrix}$. Normalise χ , and find the expectation values of S_x , S_y , and S_z .
- (e) Find the uncertainties σ_{S_x} , σ_{S_y} , and σ_{S_z} for χ in (d). Are the results consistent with all three uncertainty principles?

4. (5 points) Electron in a Magnetic Field

Consider an electron (at rest) embedded in an oscillating magnetic field:

$$\vec{B} = B_0 \cos(\omega t) \vec{j},$$

where B_0 and ω are constants.

- (a) Construct the Hamiltonian matrix for this system. Is this Hamiltonian time-dependent or time-independent?
- (b) The electron starts out (at $t = 0$) in the spin-down state with respect to the z axis (i.e., $\chi(0) = \chi_-^{(z)}$). Determine $\chi(t)$ at any subsequent time by solving the Schrödinger equation.
- (c) Find the probability of getting $+\frac{\hbar}{2}$, if you measure S_z .
- (d) What is the minimum field strength (B_0) required to force a complete flip in S_z ?
- (e) Does the oscillating field actually induce transitions between the eigenstates of S_y ? What modification to the applied field would be required to produce resonant spin flips between S_y eigenstates?