UNIT 3: Formalism of QM

- Linear algebra, Hermitian operators, and Hilbert space.
- Eigenfunctions, eigenvectors, and eigenvalues for discrete and continuous spectra.
- Dirac notation and the generalised statistical interpretation.
- Operators of position and momentum and the uncertainty principle.

Vectors and Operators in QM

QM theory is based on linear algebra:

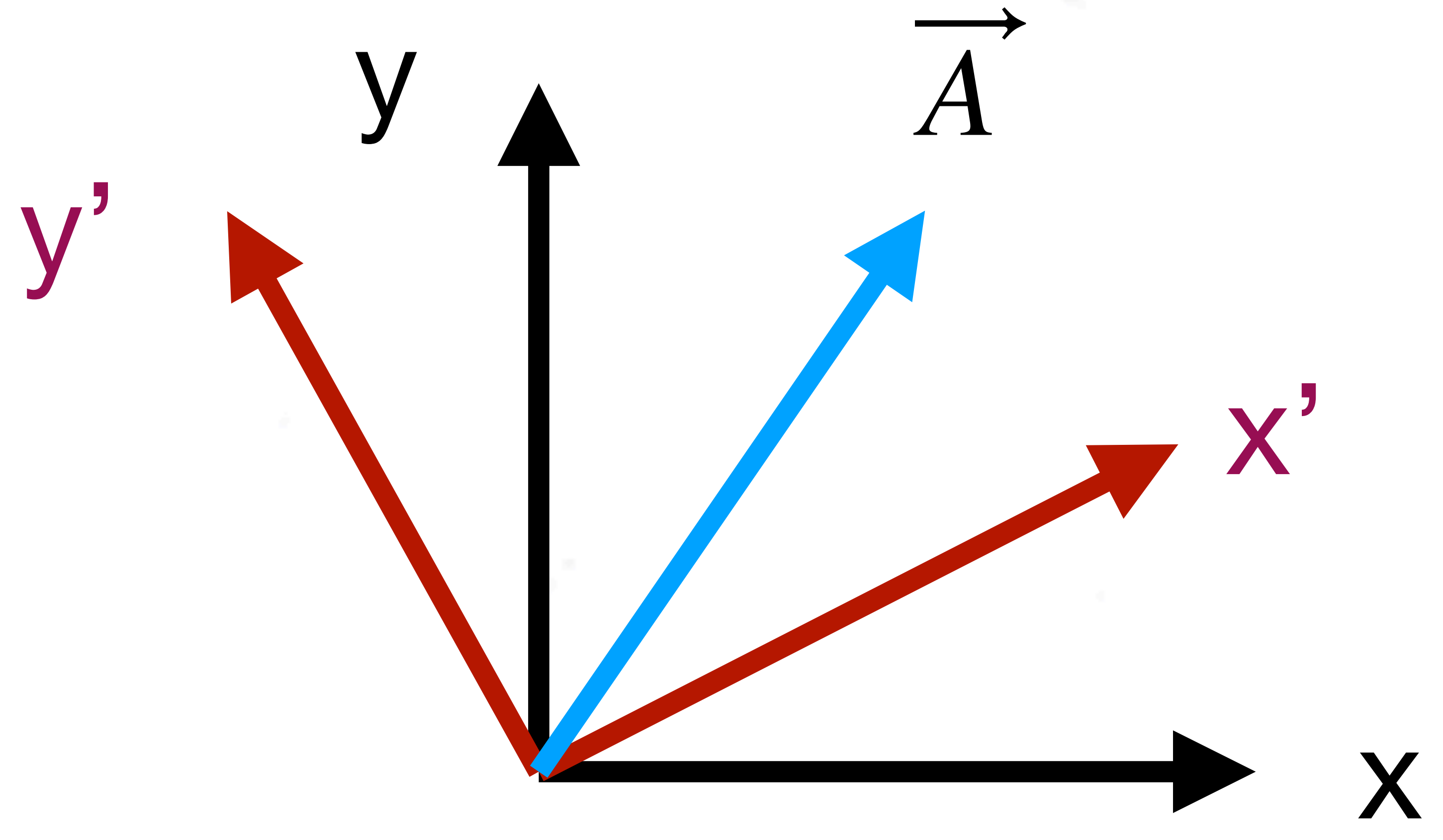
i) Wavefunctions : states → abstract vectors (functions in ∞-dim. spaces)

ii) Operators : observables → linear transformations

Vectors in QM: $|\alpha\rangle$

They are represented by the N-tuple of its components $\{a_n\}$ with respect to a specified orthonormal basis:

$$|\alpha\rangle = a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$



$$\vec{A} = A_x\vec{i} + A_y\vec{j}$$

$$\vec{A} = A'_x\vec{i}' + A'_y\vec{j}'$$

Vectors and Operators in QM

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Inner product: $\langle \alpha | \beta \rangle$

It is a complex number:

$$\langle \alpha | \beta \rangle = a_1^* b_1 + a_2^* b_2 + \cdots + a_N^* b_N$$

Linear transformations: T

They are represented by matrices (wrt the specified basis)

$$|\beta\rangle = \hat{T}|\alpha\rangle \rightarrow b = Ta = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1N} \\ t_{21} & t_{22} & & \\ \vdots & \vdots & \ddots & \\ t_{N1} & & & t_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

Vectors and Operators in QM

What is new/different in QM?

- Vectors live in ∞-dim. spaces
- manipulations that work in N-dim, may not work in ∞-dim.

Vector space:

It is the collection of all functions of x.
Wave functions must be normalized to represent possible physical states.

$$\int |\psi|^2 dx = 1$$

$\Rightarrow$ they must be square-integrable functions, on an interval.

Vectors and Operators in QM

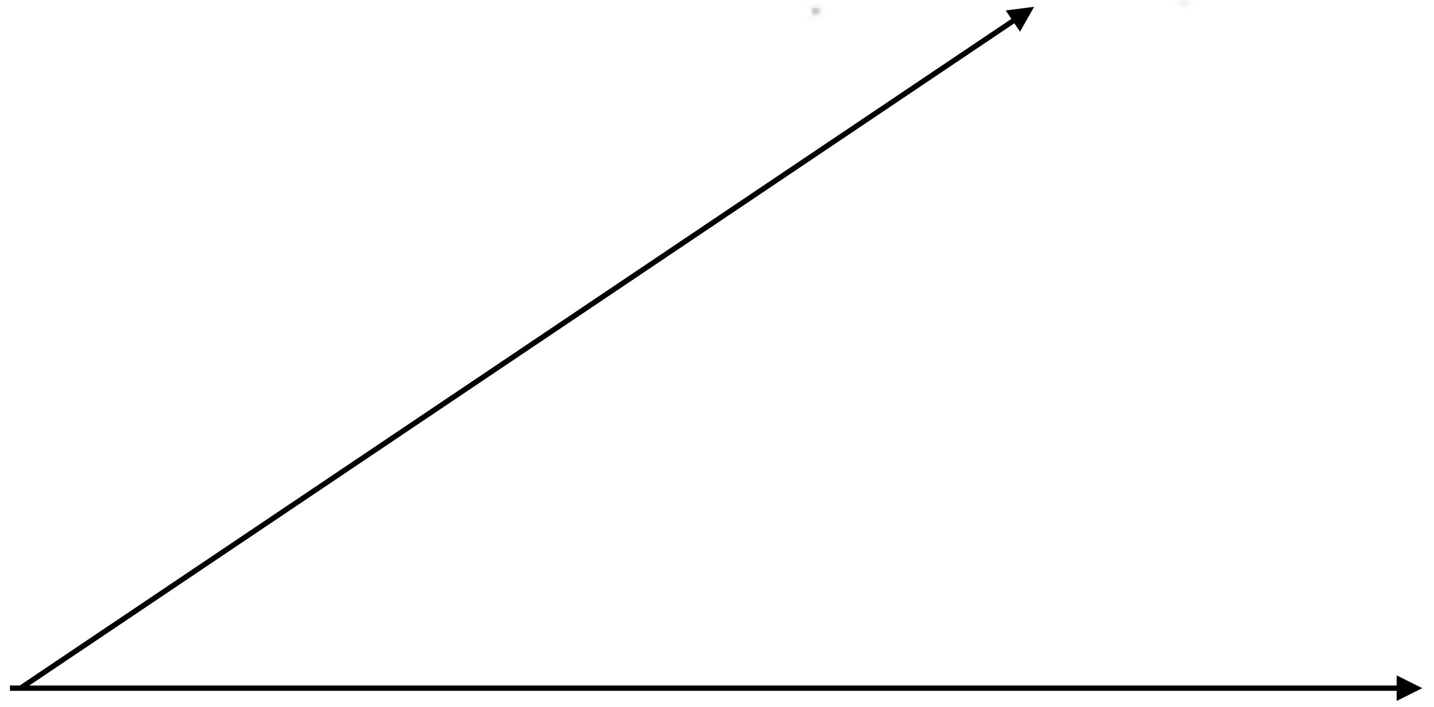
Hilbert space: $L^2(a,b)$

It is a vector space that contains the set of all square-integrable functions

Wave functions in QM $\in$ Hilbert space.

$$f(x) \rightarrow \int_a^b |f(x)|^2 dx < \infty$$

Inner products



Inner product of 2 functions: $f(x)$ & $g(x)$

$$\langle f|g\rangle \equiv \int_a^b f(x)^* g(x)\, dx$$

if $f, g \in H \Rightarrow \langle f|g\rangle$ is guaranteed to exist.
converges to a finite number.

Vectors and Operators in QM

if $f, g \in H \Rightarrow \langle f|g \rangle$ is guaranteed to exist.

converges to a finite number.

Schwarz inequality:

$$\left| \int_a^b f(x)^* g(x)\, dx \right| \leq \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx}$$

$$|\langle f|g \rangle|^2 \leq \langle f|f \rangle \langle g|g \rangle$$

Properties of inner products

Properties:

1) $\langle g|f\rangle = \langle f|g\rangle^*$

2) $\langle f|f\rangle = \int_a^b |f(x)|^2 dx. \Rightarrow \langle f|f\rangle \in \mathbb{R}$; >0 (unless $f(x)=0$)

3) A function is normalised if $\langle f|f\rangle = 1$

$$\langle \psi | \psi \rangle = \int \psi^* \psi dx = \int |\psi|^2 dx = 1$$

4) Two functions are orthogonal if $\langle f|g\rangle = 0$

Linear independence.

Properties of inner products

5) $\{f_n\}$ (a set of functions) is orthonormal if normalised and mutually orthogonal.

$$\langle f_m | f_n \rangle = \delta_{mn}$$ Kronecker delta.

$$\langle \psi_m | \psi_n \rangle = \int \psi_m{}^* \psi_n dx = \delta_{mn}$$ Infinite sq. potential

6) A set of functions is complete if any other function, $\in H$, can be expressed as a linear combination:

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

If $\{f_n\}$ are orthonormal, $c_n = \langle f_n | f \rangle$ (Fourier's trick)

Recap: Formalism of QM

Physical States: Vectors -> $\Psi(x, t)$

Observables: Operators / Matrices -> $\hat{H}$

Sch. Eq:

$$\hat{H}\psi = E\psi$$

Physical States: Vectors -> $\Psi(x, t)$

Hilbert space: $\Psi(x, t) \in H$

$$\int |\Psi(x, t)|^2 dx$$

Localisation, normalisation.

Inner product exists.

Sch. Eq:

$$\hat{H}\psi = E\psi$$

Recap: Formalism of QM

Observables: Operators / Matrices -> $\hat{H} = \hat{T} + \hat{V}$

Generally: $$\hat{Q}(\hat{x}, \hat{p})$$

They are measurable: x, p, T, H, L, etc.

Hermitian operators:

Expectation value of Q(x,p) is an inner product:

$$\langle Q \rangle = \int \psi^* \hat{Q} \psi \, dx = \langle \psi | \hat{Q} \psi \rangle$$

The possible outcome of a measurement in real: $\langle Q \rangle = \langle Q \rangle^*$

Hermitian operators:

Sch. Eq:

$$\hat{H}\psi = E\psi$$

Expectation value of $Q(x,p)$ is an inner product:

$$\langle Q \rangle = \int \Psi^* \hat{Q} \Psi \, dx = \langle \Psi | \hat{Q} \Psi \rangle$$

The possible outcome of a measurement in real: $\langle Q \rangle = \langle Q \rangle^*$

$$\Rightarrow \langle \Psi | \hat{Q} \Psi \rangle = \langle \hat{Q} \Psi | \Psi \rangle \quad \text{for any } \Psi.$$

Mathematical definition:

Operators representing observables:

$$\langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle \quad \text{for all } f(x).$$

We call $\hat{Q}$ Hermitian operators.

$$\langle f | \hat{Q} g \rangle = \langle \hat{Q} g | f \rangle \quad \text{for all } f(x) \wedge g(x).$$

Hermitian conjugate/adjoint:

$\langle f | \hat{Q} g \rangle = \langle \hat{Q}^\dagger f | g \rangle$ for all $f \wedge g$.

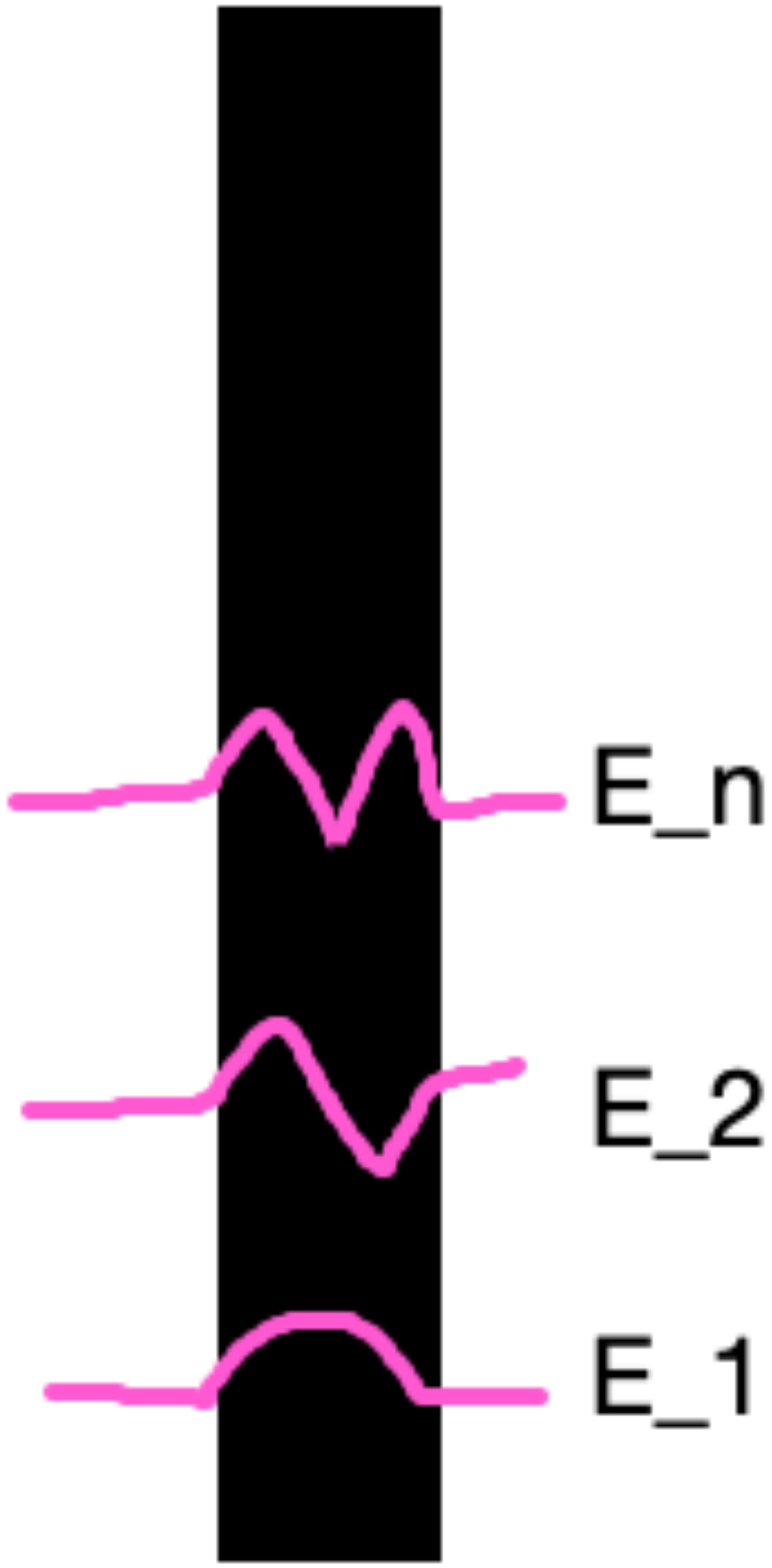
$\Rightarrow \hat{Q} = \hat{Q}^\dagger$

Determinate States:

- Ensemble of identically prepared systems, all in Ψ.
- We don't get the same answer due to Q.M. indeterminacy.

Can we prepare determinate states for Q?

- Stationary states are determinate states of H.
- A measurement of E for Ψ_n returns E_n.



Properties of determinate states:

1) σ of Q = 0 : (every measurement gives q) $\Rightarrow \langle Q \rangle = q$

$$\sigma^2 = \langle (Q - \langle Q \rangle)^2 \rangle = \langle \Psi | (\hat{Q} - q)^2 \Psi \rangle$$

$$= \langle (\hat{Q} - q)\Psi | (\hat{Q} - q)\Psi \rangle = 0$$

2) $\hat{Q}$ and $\hat{Q} - q$ are Hermitian operators.

3) The only vector whose inner product with itself vanishes is 0

$\Rightarrow \hat{Q}\psi = q\psi$ (eigenvalue eq.)

eigenvalue eigenfunction of $\hat{Q}$

Properties of determinate states:

4) The determinate states of Q are eigenfunctions of $\hat{Q}$

5) A measurement of Q on such states yields q.

6) $\Psi = 0$ excluded as eigenfunction. $\hat{Q}\,0 = q\,0 = 0$

7) q could be zero

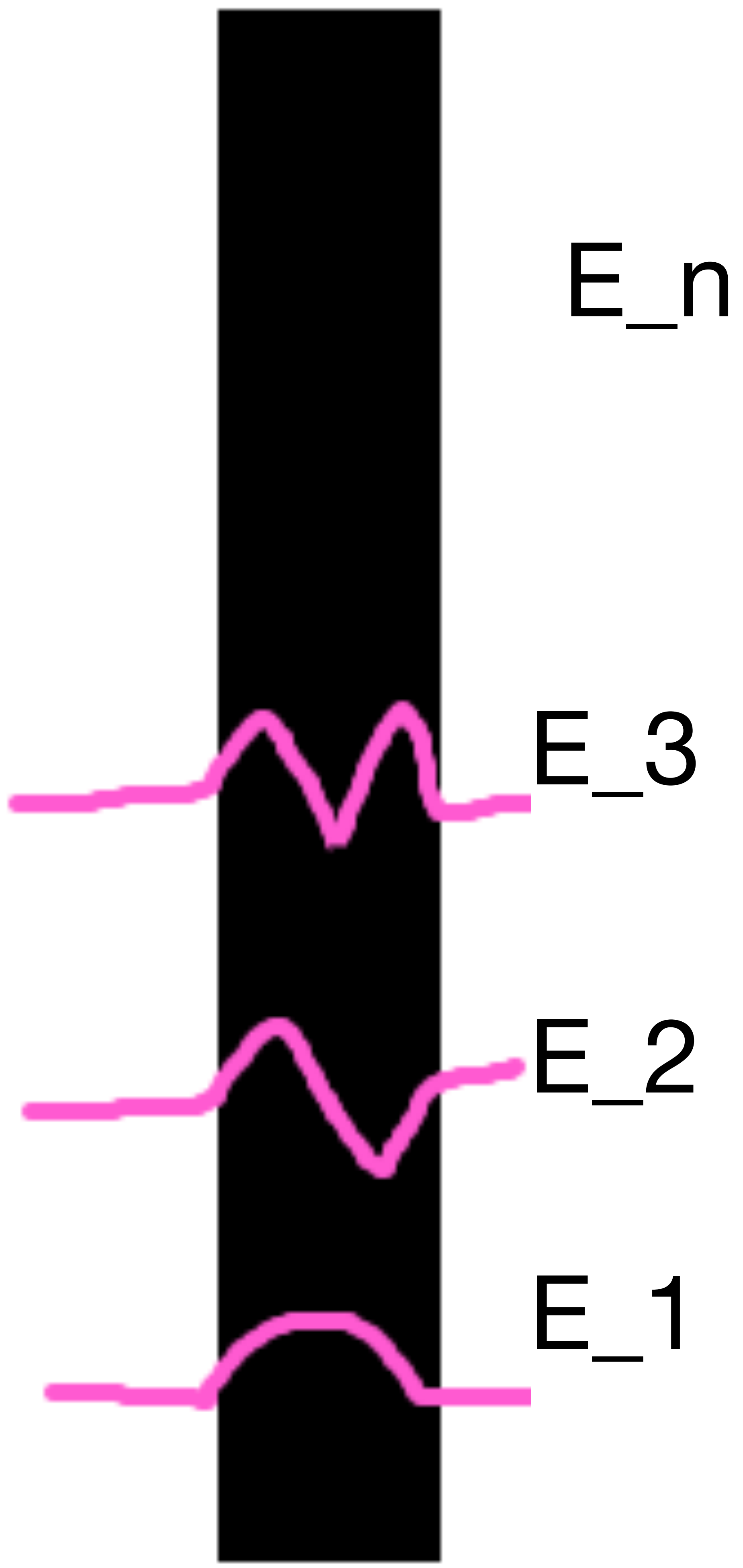
Spectrum:

It is the collection of all eigenvalues; $\{E_n\}$

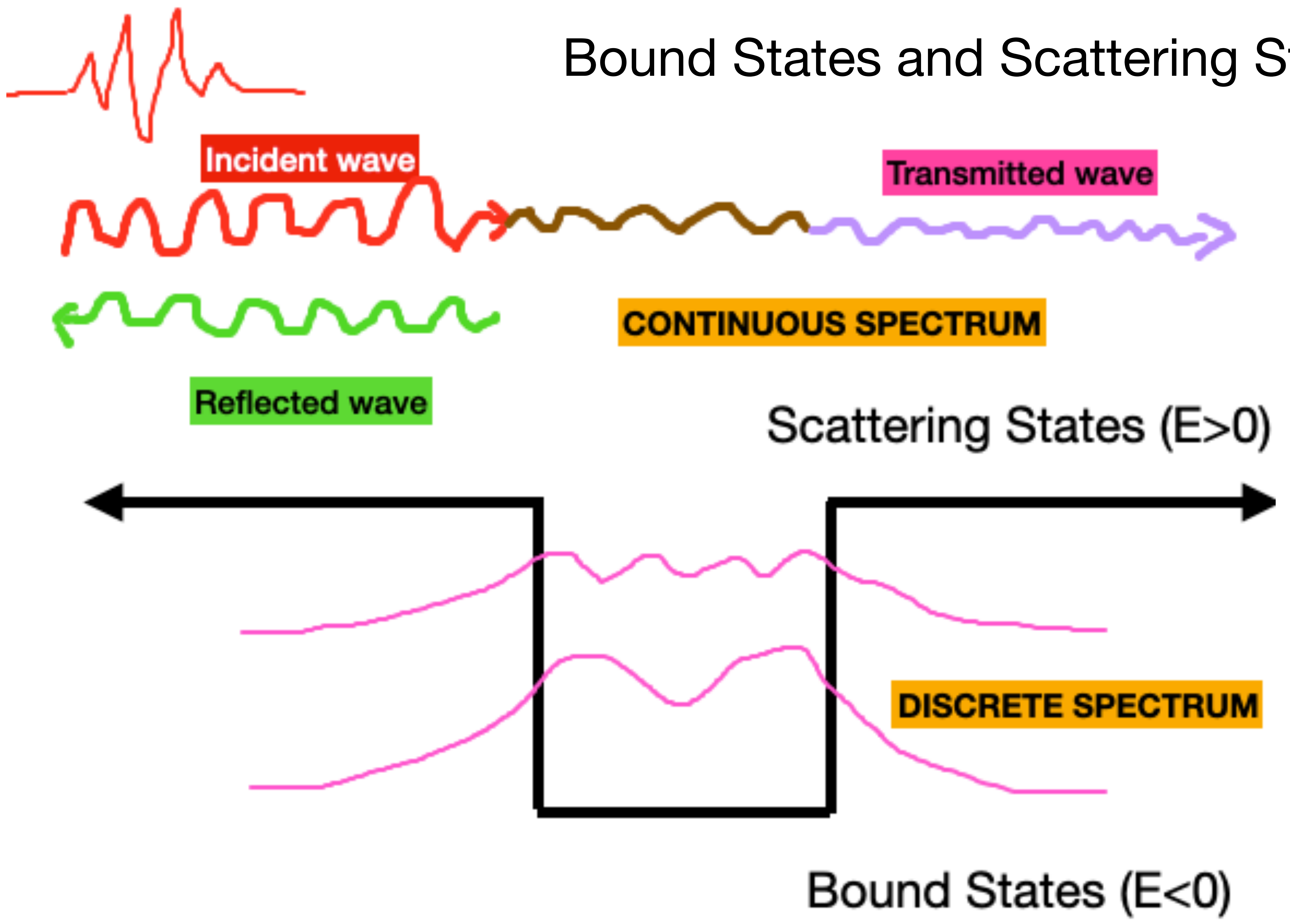
When 2 or more linearly independent share the same eigenvalue, spectrum is degenerate.

Determinate states of the total E are eigenfunctions of H

$$\hat{H}\psi = E\psi$$



Bound States and Scattering States



The stationary states are non-physical (plane waves)

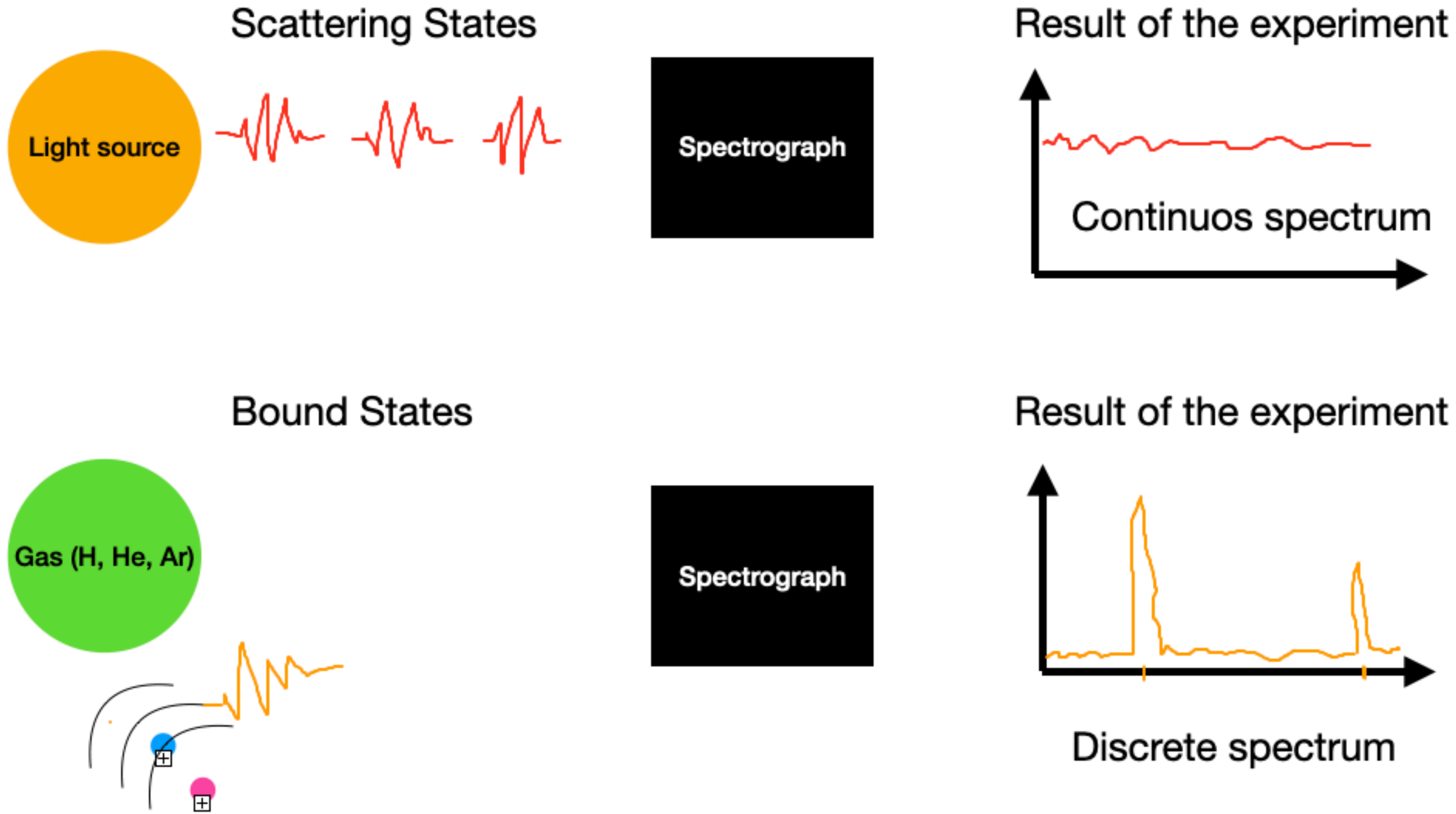
We need to construct wave packets -> continuous k(E).

No quantisation.

The stationary states are physical (localised!)

These are states on their own -> discrete n(E).

Quantisation.


Scattering States
Light source
Spectrograph
Result of the experiment
Continuos spectrum
Bound States
Gas (H, He, Ar)
Spectrograph
Result of the experiment
Discrete spectrum

Types of Spectra

① Discrete spectra ⇒ eigenfunctions $\in H$
they are physically realisable states.

② Continuous spectra ⇒ eigenfunctions are not normalisable ?
do not represent wave functions.
linear combinations may be normalisable

Example of 1: Hamiltonian for harmonic oscillator.

Example of 2 Hamiltonian for free particles.

Both 1 and 2: Hamiltonian for finite square well.

I) Discrete Spectra:

Inner products are guaranteed to exist.
We have eigenvectors of a Hermitian matrix.

2 properties:

1) Their eigenvalues are $\mathbb{R}$.

Suppose: $\hat{Q}f = qf$, $f(x)$ is an eigenfunction of $\hat{Q}$ with eigenvalue q.

$\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle$, $\hat{Q} \equiv$ Hermitian, $f(x) \in \mathbb{H}$

$$\langle f|\hat{Q}f\rangle = \int f(x)^* \hat{Q} f(x)\, dx = \int f^* q f\, dx =$$

$$= q\int f^* f\, dx = \boxed{q\langle f|f\rangle = \int [\hat{Q}f]^* f\, dx = \int q^* f^* f\, dx}$$

$$= q^* \int f^* f\, dx = \boxed{q^* \langle f|f\rangle}$$

$$\Rightarrow q\langle f|f\rangle = q^* \langle f|f\rangle$$

$$\Rightarrow (q - q^*)\langle f|f\rangle = 0 \Rightarrow \boxed{q = q^*} \Rightarrow q \in \mathbb{R}$$

Measurable!

2 properties:

2) Eigenfunctions belonging to distinct eigenvalues are orthogonal.

Suppose: $\hat{Q}f = qf$; $\hat{Q} \equiv$ Hermitian ; $f, g \in \mathbb{H}$

$\hat{Q}g = q'g$

$$\Rightarrow \langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$$

$$\langle f|\hat{Q}g\rangle = \int f^* \hat{Q}g\,dx = \int f^* q' g\,dx = q'\int f^* g\,dx = q'\langle f|g\rangle$$

$$\langle \hat{Q}f|g\rangle = \int [\hat{Q}f]^* g\,dx = \int q^* f^* g\,dx = q^* \int f^* g\,dx = q^* \langle f|g\rangle$$

$$\Rightarrow q'\langle f|g\rangle = q^*\langle f|g\rangle$$

$$\Rightarrow (q' - q^*)\langle f|g\rangle = 0 \Rightarrow \boxed{\langle f|g\rangle = 0} \Rightarrow f \wedge g \text{ are orthogonal.}$$

REMARKS:

① Stationary states of the ∞ □ well or harmonic oscillator are orthogonal because they are eigenfunctions of $\hat{H}$ with distinct eigenvalues.

② This holds for any determinate state of any observable.

③ If 2 or more eigenfunctions share the same eigenvalue ($q = q'$), any linear combination of them is itself an eigenfunction with the same eigenvalue (degenerate states)
We can use the Gram-Schmidt orthogonalisation procedure to construct orthogonal eigenfunctions within each degenerate subspace.

④ Eigenfunctions can always be chosen to be orthonormal, so we can use the Fourier's trick.

2) Continuous spectra => eigenfunctions are not normalisable
do not represent wave functions.
linear combinations may be normalisable

Reality, Orthogonality, Completeness?

We cannot prove theorems in the same way as for discrete spectra.

We need to analyse: **eigenvectors of x, eigenvectors of p.**

Problem:

Knowing that: $\int_{-\infty}^{+\infty} f(x) D_1(x)\,dx = \int_{-\infty}^{+\infty} f(x) D_2(x)\,dx$

where $D_1(x) \wedge D_2(x)$ are expressions involving delta functions.

show that: $\delta(cx) = \frac{1}{|c|}\delta(x)$; $c \in \mathbb{R}$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(cx)\,dx = \int_{-\infty}^{+\infty} \frac{1}{|c|} f\left(\frac{y}{c}\right)\delta(y)\,dy = \frac{1}{|c|} f(0) = \int_{-\infty}^{+\infty} f(x)\frac{1}{|c|}\delta(x)\,dx.$$

$y = cx$

$\Rightarrow dy = c\,dx$

$$\Rightarrow \boxed{\delta(cx) = \frac{1}{|c|}\delta(x)}$$

Continuous Spectra:

Exercise:

Find the eigenfunctions and eigenvalues of the momentum operator on the interval $-\infty < x < \infty$

Let $f_p(x)$ be the eigenfunction and p the eigenvalue:

$$-i\hbar \frac{d}{dx} f_p(x) = p f_p(x)$$

$$\Rightarrow \frac{df_p}{dx} = \frac{ip}{\hbar} f_p \Rightarrow \frac{df_p}{f_p} = \frac{ip}{\hbar} dx \Rightarrow \boxed{f_p = A e^{\frac{ip}{\hbar}x}}$$

This is not square-integrable for any $\mathbb{C}$ value of p.

The momentum operator has no eigenfunctions in $\mathbb{H}$ space.

Continuous Spectra:

$$\Rightarrow \boxed{f_p = A e^{\frac{ip}{\hbar}x}}$$

But, if we restrict ourselves to $\mathbb{R}$ eigenvalues:

$$\int_{-\infty}^{+\infty} f_{p'}^{*}(x) f_p(x)\,dx = |A|^2 \int_{-\infty}^{+\infty} e^{\frac{i(p-p')x}{\hbar}} dx = |A|^2 (2\pi)\hbar\, \delta(p-p')$$

Remember that:

$\hookrightarrow$ orthonormality

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} dk$$

$$\Rightarrow \boxed{A = \frac{1}{\sqrt{2\pi\hbar}}}$$

$$\delta(cx) = \frac{1}{|c|}\delta(x)$$

Therefore:

$$f_p = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ip}{\hbar}x}$$

$$\Rightarrow \boxed{\langle f_{p'} | f_p \rangle = \delta(p-p')}$$

Kronecker delta $\rightarrow$ Dirac delta

Dirac orthonormality

indices are continuous variables

Continuous Spectra:

$$f_p = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$$

Reality: real eigenvalues

$$\boxed{\langle f_{p'} | f_p \rangle = \delta(p-p')}$$

Kronecker delta → Dirac delta
Dirac orthonormality
indices are continuous variables

Dirac Orthonormality

What about completeness?

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x) \quad \Rightarrow \quad f(x) = \int_{-p}^{p} c(p) f_p(x) dp$$

$$f(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} c(p) e^{\frac{ipx}{\hbar}} dp$$

Fourier's trick:

$$\langle f_{p'} | f \rangle = \int_{-\infty}^{+\infty} c(p) \langle f_{p'} | f_p \rangle dp = \int_{-\infty}^{+\infty} c(p)\, \delta(p-p')\, dp = c(p')$$

Completeness!

Notes:

$$\langle f_{p'}|f\rangle = \int_{-\infty}^{+\infty} c(p)\langle f_{p'}|f_p\rangle\, dp = \int_{-\infty}^{+\infty} c(p)\,\delta(p-p')\, dp = c(p')$$

I) $$\int \psi_m(x)^* f(x)\, dx = \sum_{n=1}^{\infty} c_n \int \psi_m(x)^* \psi_n(x)\, dx = \sum_{n=1}^{\infty} c_n \delta_{nm} = c_m$$

with: $$f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

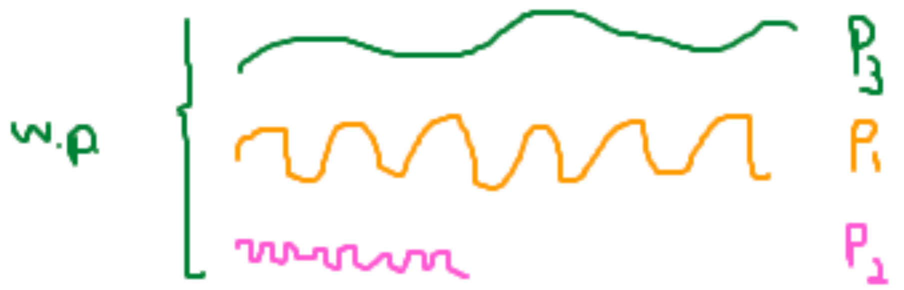
II) Plancherel's theorem:

F.T.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k)\, e^{ikx}\, dk \Leftrightarrow F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx$$

i.F.T.

III) The momentum eigenfunctions are sinusoidal:



$$f_p = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{ip}{\hbar}x}$$ With wavelength: $$\lambda = \frac{2\pi\hbar}{p}$$ de Broglie formula.

Exercise:

Find the eigenfunctions and eigenvalues of $\hat{x}$:

Let $g_y(x) \to$ eigenfunction

$y \to$ eigenvalue (fixed number)

$$\Rightarrow \hat{x} g_y(x) = \underset{\substack{\uparrow \\ \text{continuous}}}{x}\, g_y(x) = \underset{\substack{\uparrow \\ \text{fixed}}}{y}\, g_y(x)$$

$\hookrightarrow g_y(x) = 0$, except at $x = y$

$\Rightarrow g_y(x) = A\,\delta(x-y) \to$ Dirac delta

Notes:

① $y \in \mathbb{R}$ Reality!

② $g_y(x)$ are not square integrable.

③ they admit Dirac orthonormality, e.g. $\left\{ \begin{matrix} y \\ y' \end{matrix} \right\}$ are eigenvalues:

$$\Rightarrow \int_{-\infty}^{+\infty} g_{y'}^*(x)\, g_y(x)\, dx = |A|^2 \int_{-\infty}^{+\infty} \delta(x-y')\,\delta(x-y)\, dx$$

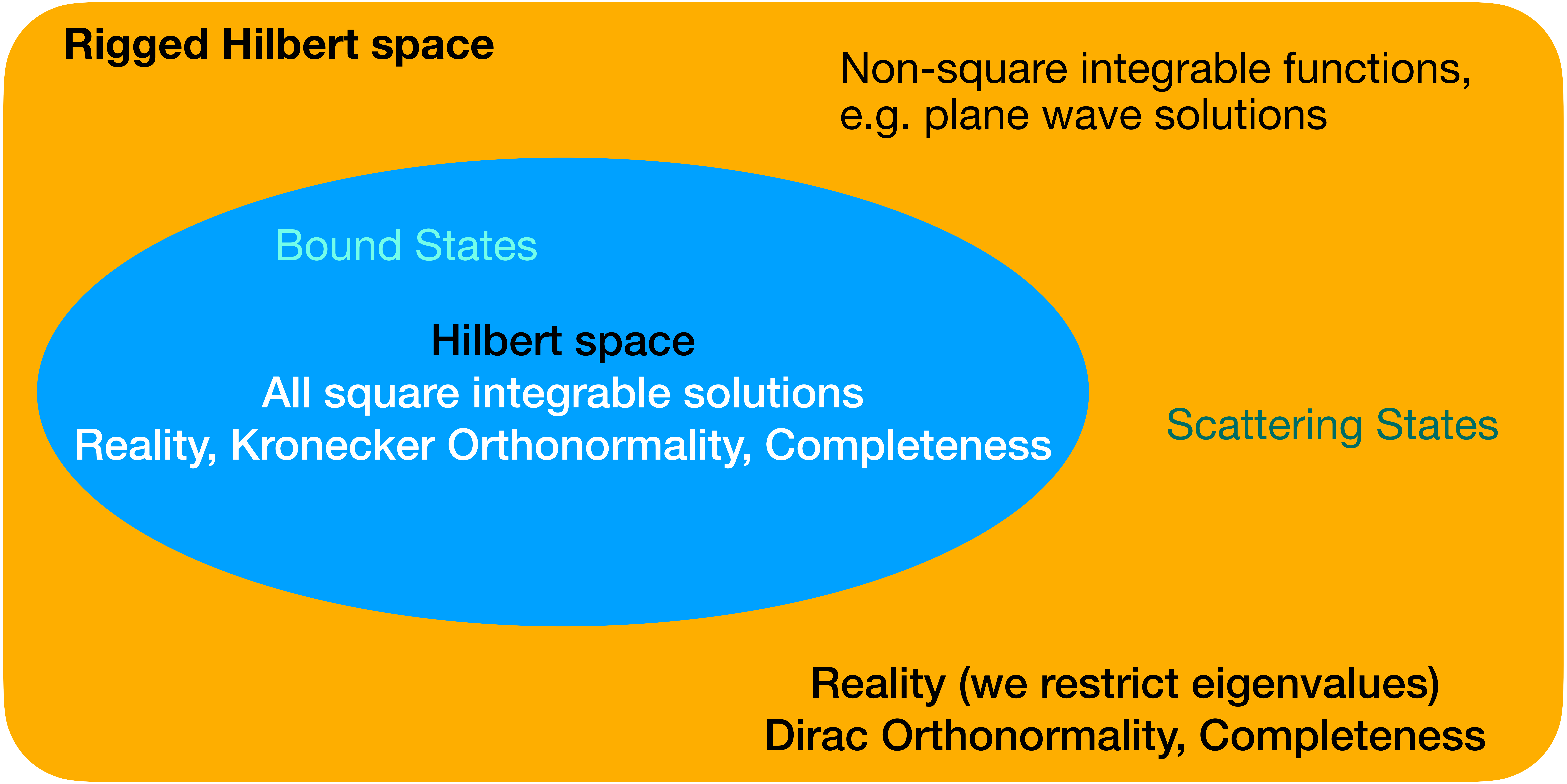
$$= |A|^2\, \delta(y-y')$$

Dirac orthonormality!

Let's pick $A = 1$: $g_y(x) = \delta(x-y)$

$$\Rightarrow \langle g_{y'} | g_y \rangle = \delta(y-y')$$

QM Vector Spaces



Generalised statistical interpretation

"If you measure an observable $Q(x,p)$ on a particle in state $\Psi(x,t)$, you are certain to get one of the eigenvalues of the Hermitian operator $\hat{Q}\left(x, -i\hbar\frac{d}{dx}\right)$. If the spectrum of $\hat{Q}$ is discrete, the probability of getting the particular q_n associated with the orthonormalised eigenfunction $f_n(x)$ is:

$$|c_n|^2, \quad \text{where} \quad c_n = \langle f_n | \Psi \rangle$$

If the spectrum is continuous, with $\mathbb{R}$ eigenvalues $q(z)$ and associated (Dirac-orthonormalised) eigenfunctions $f_z(x)$, the probability of getting a result in the range dz is:

$$|c(z)|^2 dz, \quad \text{where} \quad c(z) = \langle f_z | \Psi \rangle$$

Upon measurement, the Ψ collapses to the corresponding eigenstate."

Generalised statistical interpretation

The eigenfunctions of an observable operator are complete.

$$\Psi(x,t) = \sum_n c_n(t) f_n(x) \qquad (\text{when } \hat{Q} = \hat{H},\ c_n \equiv \text{constants})$$

The coefficients are given by Fourier's trick:

$$c_n(t) = \langle f_n | \Psi \rangle = \int f_n(x)^* \Psi(x,t)\, dx$$

c_n tells us how much f_n is contained in Ψ.

$$\boxed{\sum_n |c_n|^2 = 1}\ , \qquad |c_n|^2 \equiv \text{prob. that measuring } \hat{Q} \text{ gives } q_n.$$

particle in Ψ will be in f_n.

<u>Proof</u>:

$$\langle \Psi | \Psi \rangle = 1 \Rightarrow \left\langle \left(\sum_{n'} c_{n'} f_{n'} \right) \middle| \left(\sum_n c_n f_n \right) \right\rangle = \sum_{n'} \sum_n c_{n'}^* c_n \langle f_{n'} | f_n \rangle =$$

$$= \sum_{n'} \sum_n c_{n'}^* c_n \delta_{n'n} = \sum_n c_n^* c_n = \sum_n |c_n|^2 = 1 \quad //$$

Expectation value of Q:

$$\boxed{\langle Q \rangle = \sum_n q_n \underbrace{|c_n|^2}_{\text{prob. of getting } q_n}}$$

all possible outcomes of q_n

Generalised statistical interpretation -> Born's version

$|c(z)|^2 dz$, where $c(z) = \langle f_z | \Psi \rangle$? $|\Psi(x,t)|^2 dx$

Let's see the eigenvectors/eigenvalues of $\hat{x}$:

every $\mathbb{R}$ "y" is an eigenvalue of $\hat{x}$.

The Dirac orthonormalised eigenfunction is $g_y(x) = \delta(x-y)$

$$\Rightarrow c(y) = \langle g_y | \Psi \rangle = \int_{-\infty}^{+\infty} \delta(x-y)\, \Psi(x,t)\, dx = \Psi(y,t)$$

Generalised statistical interpretation

$|c(y)|^2 dy$ Prob.

$\longrightarrow$

Born's statistical interpretation

$|\Psi(y,t)|^2 dy$ Prob.

$\therefore$ the prob. of getting a result in the range dy is $|\Psi|^2 dy$ which is the original statistical interpretation.

Generalised statistical interpretation -> Born's version

$|c(z)|^2 dz$, where $c(z) = \langle f_z | \Psi \rangle$? $|\Psi(x,t)|^2 dx$

Let's see the eigenvectors/eigenvalues of $\hat{p}$:

$$\boxed{f_p = A e^{\frac{ip}{\hbar}x}}$$

For momentum?

Dirac orthonormalised eigenfunctions of p: $f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{px}{\hbar}}$

$$\Rightarrow c_p = \langle f_p | \Psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-\frac{ipx}{\hbar}} \Psi(x,t)\, dx$$

which is called the "momentum space wave function", $\Phi(p,t)$

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-\frac{ipx}{\hbar}} \Psi(x,t)\, dx$$

Momentum space F.T. Real space

Generalised statistical interpretation

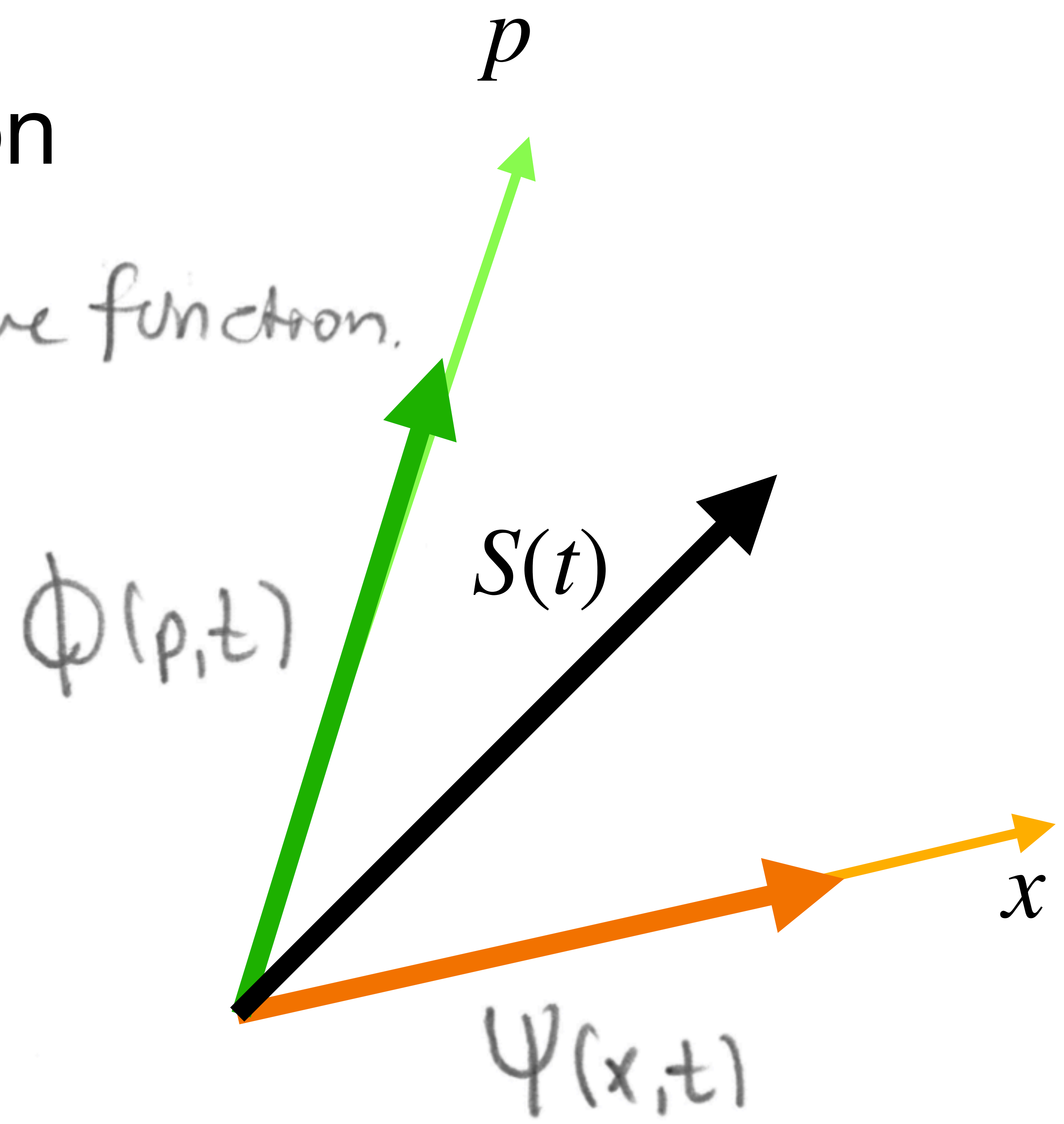
This is the Fourier transform of the (position space) wave function $\Psi(x,t)$, which is its inverse Fourier transform.

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-\frac{ipx}{\hbar}} \Psi(x,t)\, dx$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{+\frac{ipx}{\hbar}} \Phi(p,t)\, dp$$

The prob. that a measurement of p yields a result with dp,

$$|\Phi(p,t)|^2 dp$$



Fourier space (p-space, k-space)

$\Phi(p,t)$

$S(t)$

Real space (x-space)

$\Psi(x,t)$

Generalised uncertainty principle

Heisenberg's uncertainty principle: $\sigma_x \sigma_p \geq \frac{\hbar}{2}$

Is there generalised uncertainty principle?

Does the uncertainty principle apply to all QM operators?

Let's pick any two observables: $\hat{A}$ and $\hat{B}$ and let's compute their variances:

$\sigma_A^2 = \langle (\hat{A} - \langle A \rangle)\Psi | (\hat{A} - \langle A \rangle \Psi \rangle = \langle f | f \rangle$

where $f = (\hat{A} - \langle A \rangle)\Psi$

$\sigma_B^2 = \langle g | g \rangle$

$g = (\hat{B} - \langle B \rangle)\Psi$

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle$$

Schwarz inequality:

z

$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$

Schwarz inequality: $|z|^2$

$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle \langle g|g\rangle \geq |\langle f|g\rangle|^2$$

Recall: $|z|^2 = [Re(z)]^2 + [Im(z)]^2 \geq [Im(z)]^2 = \left[\frac{1}{2i}(z - z^*)\right]^2$

$$z = \langle f|g\rangle,$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\left[\langle f|g\rangle - \langle g|f\rangle\right]\right)^2$$

We need to write the inner products as functions of A and B.

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \left[\langle f|g\rangle - \langle g|f\rangle \right] \right)^2$$

We need to write the inner products as functions of A and B.

Hermiticity of $\hat{A} - \langle A\rangle$

$$\langle f|g\rangle = \langle (\hat{A} - \langle A\rangle)\psi | (\hat{B} - \langle B\rangle)\psi\rangle = \langle \psi | (\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)\psi\rangle$$

$$= \langle \psi | (\hat{A}\hat{B} - \hat{A}\langle B\rangle - \hat{B}\langle A\rangle + \langle A\rangle\langle B\rangle)\psi\rangle$$

$$= \langle \psi | \hat{A}\hat{B}\psi\rangle - \langle B\rangle\langle \psi | \hat{A}\psi\rangle - \langle A\rangle \langle \psi | \hat{B}\psi\rangle + \langle A\rangle\langle B\rangle \cancelto{1}{\langle \psi|\psi\rangle}$$

$$= \langle \hat{A}\hat{B}\rangle - \langle \hat{B}\rangle\langle A\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle$$

$$= \langle \hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle$$

$$\langle g|f\rangle = \langle \hat{B}\hat{A}\rangle - \langle A\rangle\langle B\rangle$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \left[\langle f|g \rangle - \langle g|f \rangle \right] \right)^2$$

We need to write the inner products as functions of A and B.

$$\langle f|g \rangle = \langle \hat{A}\hat{B} \rangle - \langle A \rangle \langle B \rangle$$

$$\langle g|f \rangle = \langle \hat{B}\hat{A} \rangle - \langle A \rangle \langle B \rangle$$

Substituting:

$$\langle f|g \rangle - \langle g|f \rangle = \langle \hat{A}\hat{B} \rangle - \langle \hat{B}\hat{A} \rangle = \langle [\hat{A},\hat{B}] \rangle,$$

$$\text{where } [\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Commutator of 2 operators:

$$\Rightarrow \quad \sigma_A^2 \sigma_B^2 \geq \underbrace{\left(\frac{1}{2i} \langle [\hat{A},\hat{B}] \rangle \right)^2}_{\in \mathbb{R} \quad > 0}$$

Uncertainty principle for every pair of observables whose operators do not commute **(incompatible observables).**

Commutator of 2 operators:

$$\Rightarrow \quad \sigma_A^2 \sigma_B^2 \geq \underbrace{\left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2}$$

Uncertainty principle for every pair of observables whose operators do not commute **(incompatible observables).**

Can we recover the original uncertainty principle?

$$\hat{A} = \hat{x} \quad , \quad \hat{B} = \hat{p} = -i\hbar \frac{d}{dx}$$

$$[\hat{x}, \hat{p}] = i\hbar$$

Canonical commutation relation!

$$\Rightarrow \quad \sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} i\hbar \right)^2 = \left(\frac{\hbar}{2} \right)^2$$

$$\Rightarrow \quad \boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$

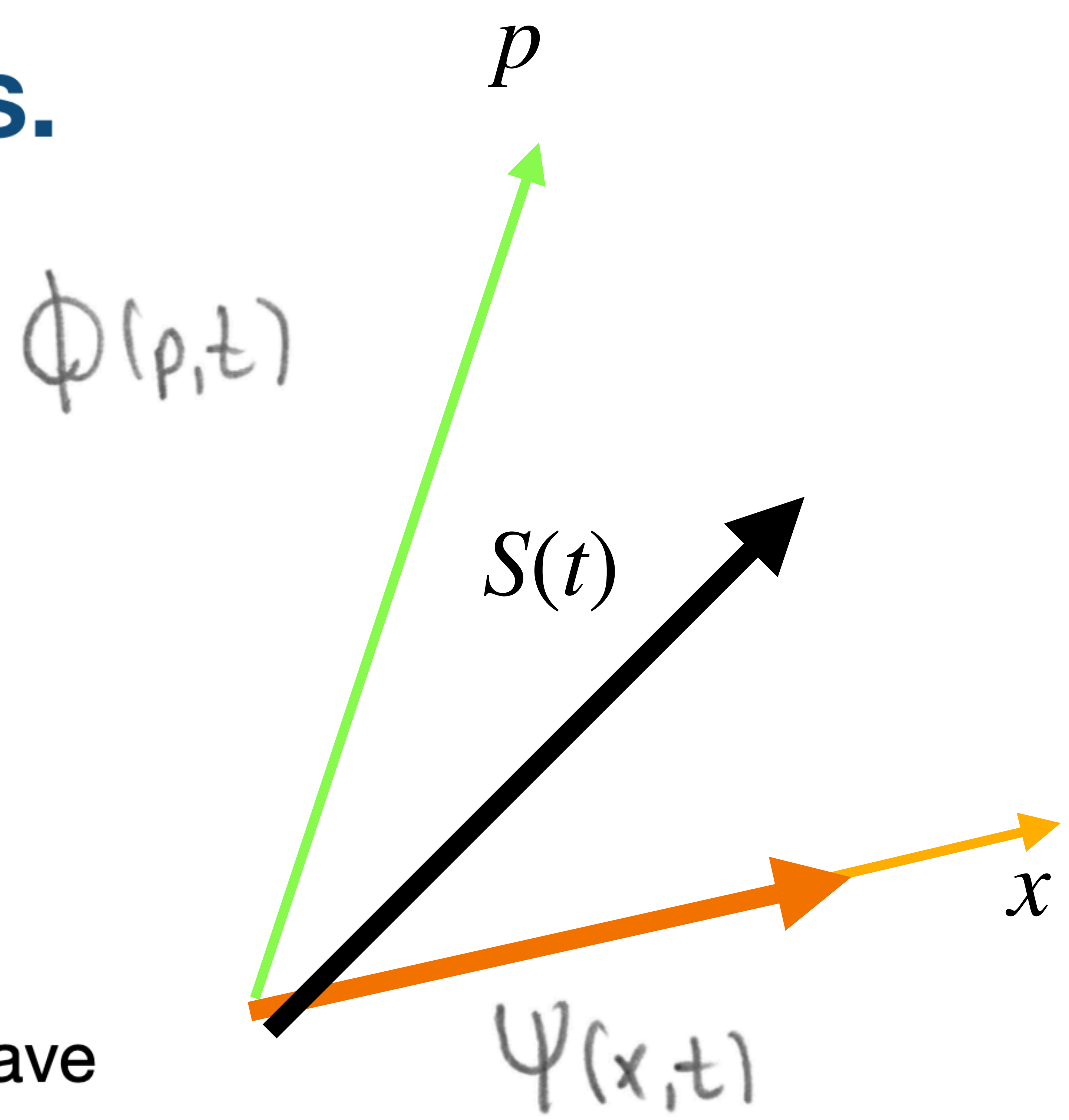
$\Rightarrow$ Heisenberg's uncertainty principle

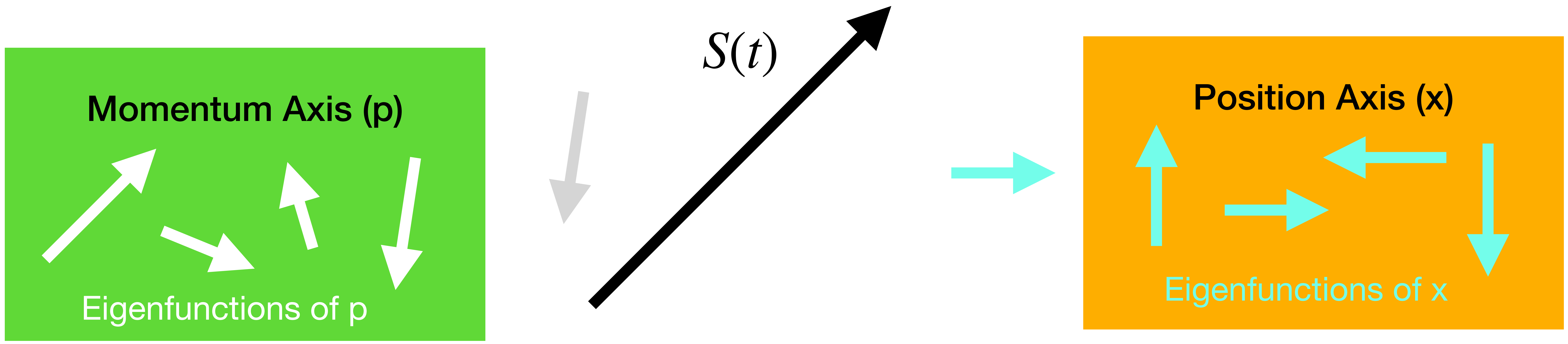
Compatible versus incompatible observables.

Incompatible observables:
They do not have a complete set of common eigenfunctions.

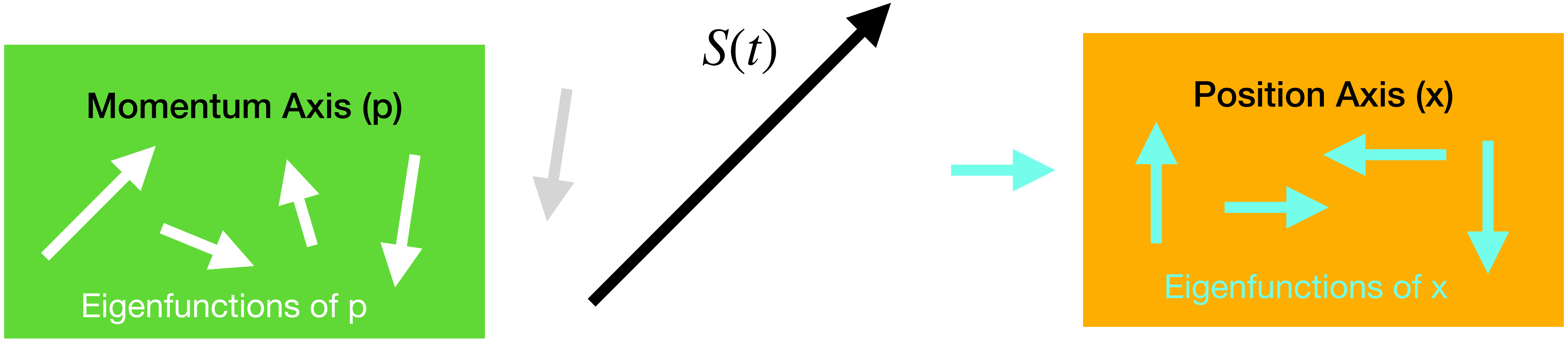
Compatible observables:
They do admit a complete set of simultaneous eigenfunctions (states that are determinate for both observables).

Notes:
- The uncertainty principle is a consequence of the statistical interpretation.
- A second measurement makes the first one obsolete because the wave function is not an eigenstate of the p and x operators.

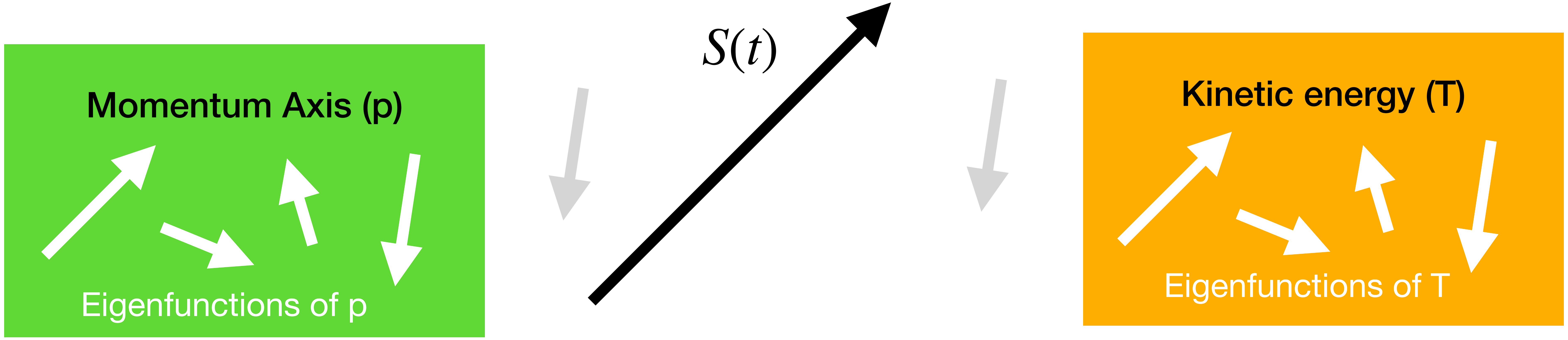




Incompatible observables:
They do not have a complete set of common eigenfunctions.



Compatible observables:
They do admit a complete set of simultaneous eigenfunctions (states that are determinate for both observables).



Minimum Uncertainty Wave Packet

The minimum-uncertainty wave packet

We have seen examples where $\sigma_x \sigma_p \geq \frac{\hbar}{2}$ $\left\{ \begin{array}{l} \text{ground state of H. osc.} \\ \text{Gaussian w.p. for free particle} \end{array} \right.$

What is the minimum uncertainty wp?

$$\left.\begin{array}{l} ① \ \sigma_A^2 \sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq \langle f|g\rangle^2 \\ ② \ |z|^2 = [Re(z)]^2 + [Im(z)]^2 \geq [Im(z)]^2 \end{array}\right\} \text{take eq., what happens to } \psi?$$

① $\sigma_A^2 \sigma_B^2 = \langle f|g\rangle^2$ if $g(x) = c f(x)$, $c \in \mathbb{C}$

② $|z|^2 = [Im(z)]^2$ if $Re(z) = 0 \Rightarrow Re\langle f|g\rangle = Re(\underset{\mathbb{C}}{c}\,\overset{\mathbb{R}}{\langle f|f\rangle}) = 0$

$\Rightarrow c = ia$

The necessary and sufficient condition for minimum uncertainty is

$g(x) = ia f(x)$, $a \in \mathbb{R}$

Minimum Uncertainty Wave Packet

$$g(x) = ia\, f(x) \quad , \quad a \in \mathbb{R}$$

$$f = (\hat{A} - \langle A \rangle)\psi \qquad g = (\hat{B} - \langle B \rangle)\psi$$

Position op (x) Momentum op (p)

$$\left(-i\hbar \frac{d}{dx} - \langle p \rangle\right)\psi = ia\left(x - \langle x \rangle\right)\psi$$

$$\Rightarrow \frac{d\psi}{dx} = \frac{i}{\hbar}\left(iax - ia\langle x \rangle + \langle p \rangle\right)\psi = \frac{a}{\hbar}\left(-x + \langle x \rangle + \frac{i}{a}\langle p \rangle\right)\psi$$

$$\Rightarrow \frac{d\psi}{\psi} = \frac{a}{\hbar}\left(-x + \langle x \rangle + \frac{i\langle p \rangle}{a}\right)dx$$

$$\Rightarrow \ln(\psi) = -\frac{a}{2\hbar}\left(x - \langle x \rangle\right)^2 + \frac{i\langle p \rangle x}{a} x + C_1$$

$$\Rightarrow \boxed{\psi = A\, e^{-a\frac{(x-\langle x \rangle)^2}{2\hbar}}\, e^{\frac{i\langle p \rangle x}{\hbar}}}$$

The minimum uncertainty w.p. is a Gaussian!

Energy-time uncertainty principle

Heisenberg's uncertainty principle: $\sigma_x \sigma_p \geq \frac{\hbar}{2}$

Energy-time uncertainty principle: $\boxed{\Delta E \, \Delta t \geq \frac{\hbar}{2}}$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

Let us pick $A = H$, $B = Q$, Q does not explicitly depend on t

$$\sigma_H^2 \sigma_Q^2 \geq \left(\frac{1}{2i} \langle [\hat{H}, \hat{Q}] \rangle \right)^2 = \left(\frac{1}{2i} \frac{\hbar}{i} \frac{d\langle Q \rangle}{dt} \right)^2$$

$$= \left(\frac{\hbar^2}{2} \right) \left(\frac{d\langle Q \rangle}{dt} \right)^2$$

$$\Rightarrow \sigma_H \sigma_Q \geq \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right|$$

Energy-time uncertainty principle

Heisenberg's uncertainty principle: $\sigma_x \sigma_p \geq \frac{\hbar}{2}$

Energy-time uncertainty principle: $\boxed{\Delta E \, \Delta t \geq \frac{\hbar}{2}}$

$$\Rightarrow \sigma_H \sigma_Q \geq \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right|$$

If $\Delta E \equiv \sigma_H$, $\quad \Delta t \equiv \dfrac{\sigma_Q}{\left| \frac{d\langle Q \rangle}{dt} \right|}$

$$\boxed{\Rightarrow \Delta E \, \Delta t \geq \frac{\hbar}{2}}$$

Energy-time uncertainty principle

Δt represents the amount of time it takes for the expectation value of Q to change by 1 standard deviation. (Mandelstam-Tamm formulation)

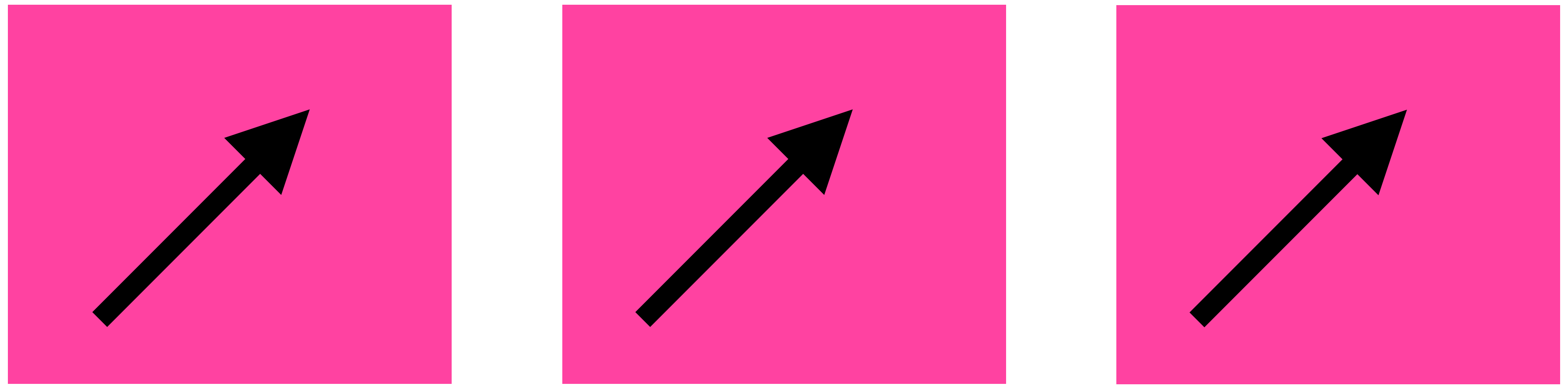
Energy-time uncertainty principle

$$\Rightarrow \Delta E\, \Delta t \geq \frac{\hbar}{2}$$

Δt represents the amount of time it takes for the expectation value of Q to change by 1 standard deviation. (Mandelstam – Tamm formulation)

System is changing slowly.

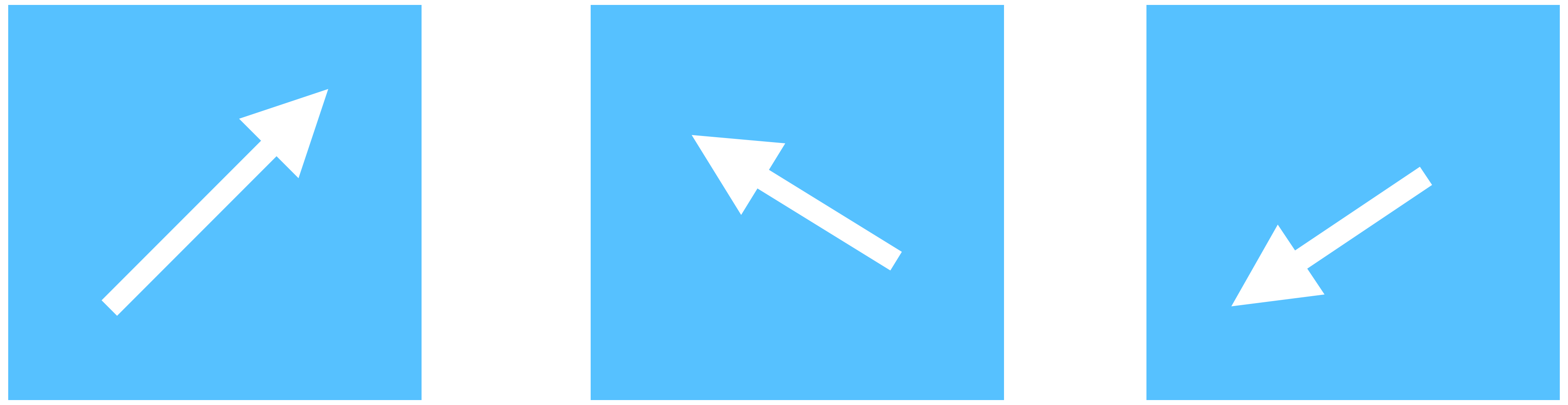
Long Δt



Uncertainty in E, small ΔE

System is changing fast

Short Δt



Uncertainty in E, large ΔE

Vectors and Operators

QM States

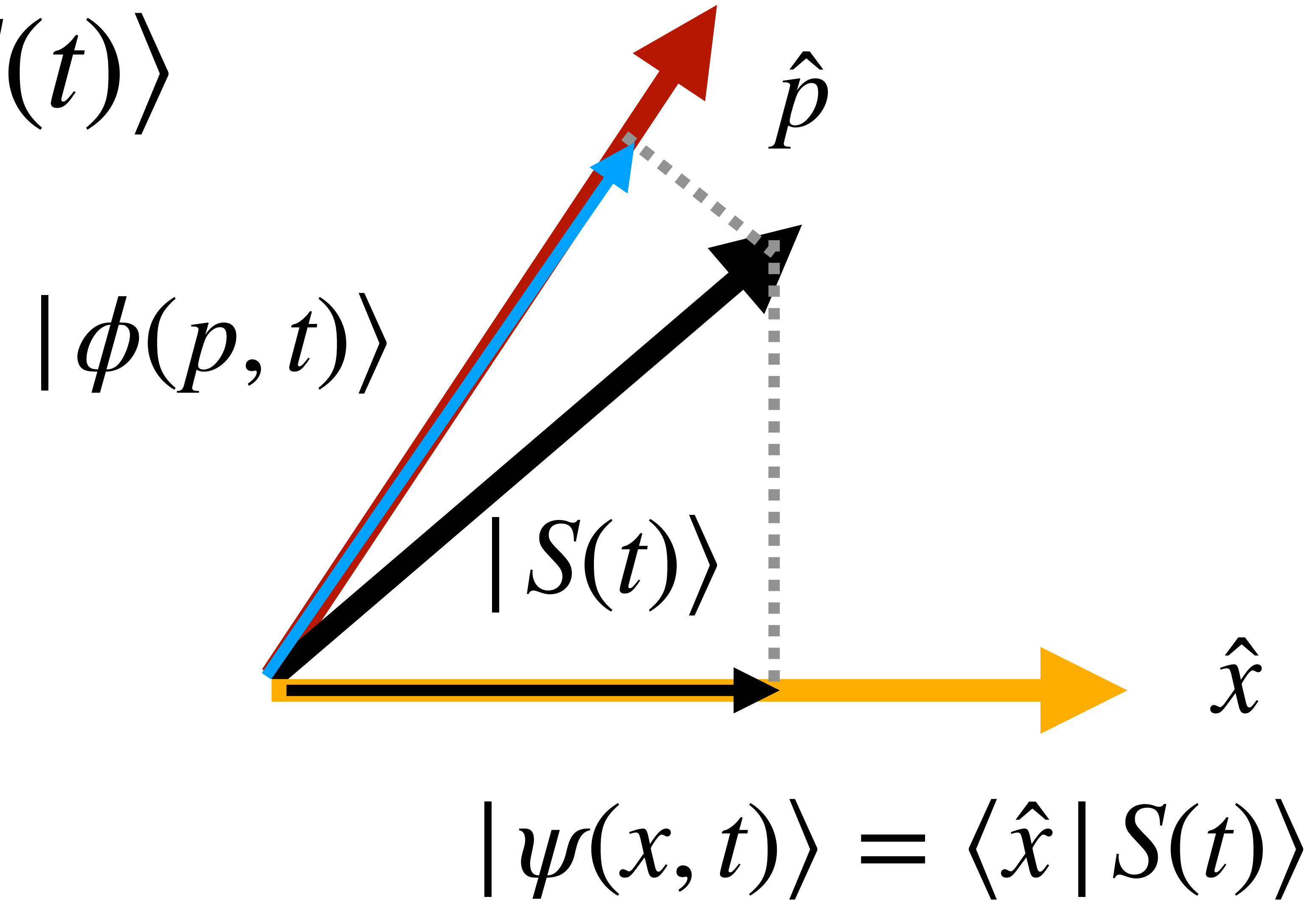
Belong Hilbert / rigged Hilbert spaces

Bound states (discrete spectrum)

Scattering states (continuous spectrum)

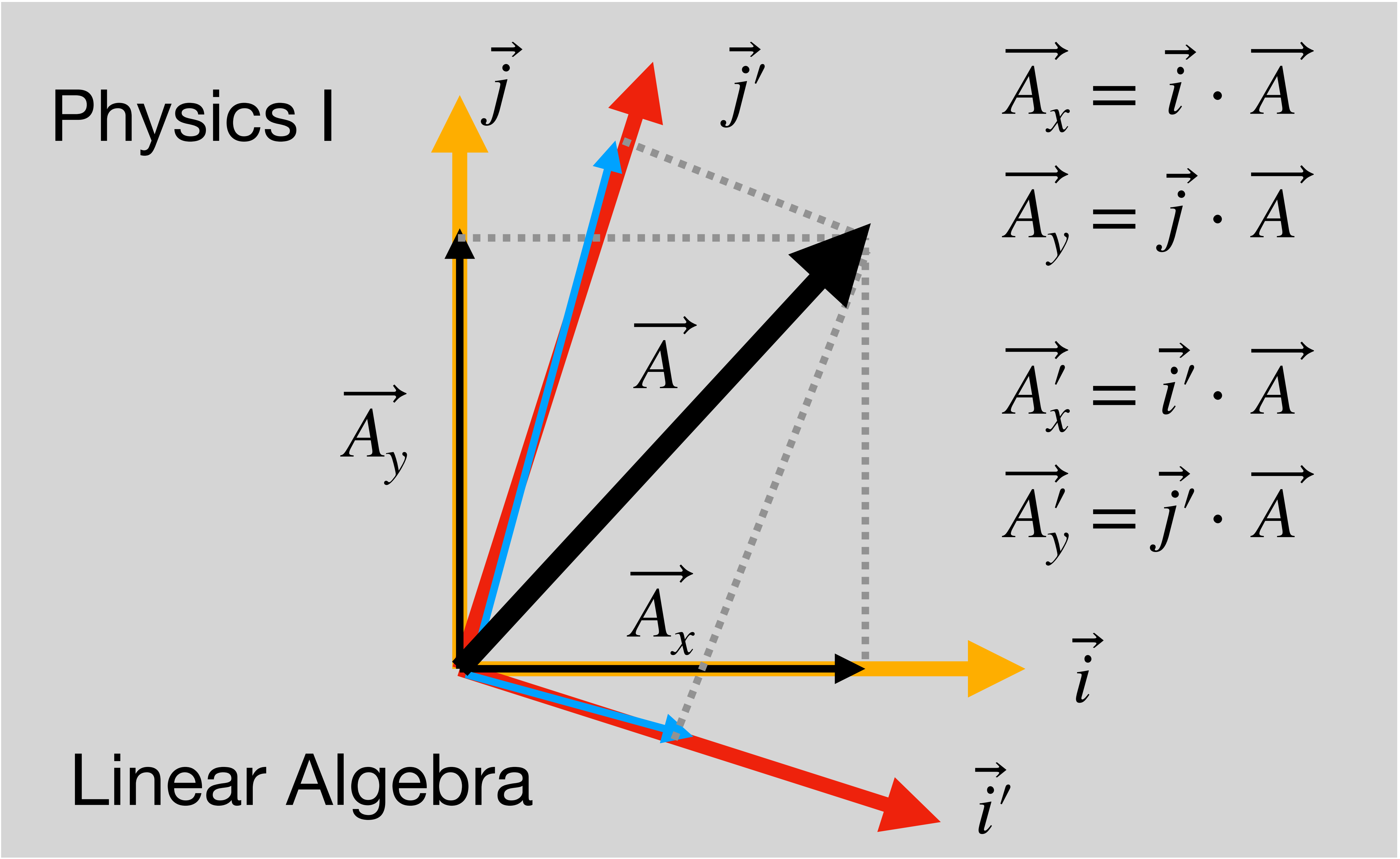
State of QM system (wave/particle):

$|\, S(t)\rangle$



Observables

Hermitian: reality, orthogonality, completeness



state of a system $\mapsto$ vector $|S(t)\rangle \in \mathbb{H}$

$$\boxed{\Psi(x,t) = \langle x | S(t) \rangle}$$ is the "x" component in the expansion of $S(t)$ in the basis of $\hat{x}$ eigenfunctions, $|x\rangle$.

↳ $\hat{i} \cdot \vec{A}$

$\Phi(p,t) = \langle p | S(t) \rangle$ → momentum eigenfunctions $|p\rangle$ with eigenvalue p.

$C_n(t) = \langle n | S(t) \rangle$ → energy eigenfunctions $|n\rangle$ of $\hat{H}$ for a discrete spectrum

↳ $C_n(t) = \langle f_n | \Psi \rangle = \int f_n^* \Psi \, dx$

Position space | Momentum space | Energy space

$$\int \Psi(y,t)\,\delta(x-y)\,dy = \int \Phi(p,t)\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{px}{\hbar}}\,dp = \sum c_n e^{-i\frac{E_n t}{\hbar}}\psi_n(x)$$

Delta function (eigenfunction of x) | Plane wave (eigenfunction of p) | Wiggle factor (E)

II) Operators: (represent observables)

$$|\beta\rangle = \hat{Q}|\alpha\rangle$$

linear transformations on $\mathbb{H}$ space

Mathematical representation of vectors and operators

Vectors are represented wrt an orthonormal basis $\{|e_n\rangle\}$ by comp.

discrete basis

$$|\alpha\rangle = \sum_n a_n |e_n\rangle , \qquad |\beta\rangle = \sum_n b_n |e_n\rangle$$

$$a_n = \langle e_n|\alpha\rangle \qquad b_n = \langle e_n|\beta\rangle$$

Operators are represented wrt a basis by matrix elements,

$$\langle e_m|\hat{Q}|e_n\rangle \equiv Q_{mn} ,$$ so $|\beta\rangle = \hat{Q}|\alpha\rangle$ becomes:

$$\Rightarrow \sum_n b_n |e_n\rangle = \sum_n a_n \hat{Q} |e_n\rangle$$

$$\Rightarrow \sum_n b_n \langle e_m|e_n\rangle = \sum_n a_n \langle e_m|\hat{Q}|e_n\rangle \qquad \langle e_m|e_n\rangle = \delta_{mn}$$

Mathematical representation of vectors and operators

$$\Rightarrow \sum_n b_n \langle e_m | e_n \rangle = \sum_n a_n \langle e_m | \hat{Q} | e_n \rangle$$

$$\langle e_m | e_n \rangle = \delta_{mn}$$

$$\Rightarrow \boxed{b_m = \sum_n Q_{mn} a_n} \qquad \vec{b} = Q\,\vec{a} \rightarrow \vec{b}_N = Q_{N\times N} \cdot \vec{a}_N$$

Conclusion:

Operators look different in different basis (matrices look different)

$\hat{x} \rightarrow \begin{cases} x & \text{Multiplication op.} \\ i\hbar \frac{\partial}{\partial p} & \text{Derivative op.} \end{cases}$ $\qquad \hat{p} \rightarrow \begin{cases} -i\hbar \frac{\partial}{\partial x} & \text{Derivative op.} \rightarrow \text{in } x \text{ space} \\ p & \text{Multiplication op.} \rightarrow \text{in } p \text{ space} \end{cases}$

Dirac Notation

Inner product: $\langle \alpha | \beta \rangle$

Bra-ket notation.

ket: $|\beta\rangle \to$ vector

bra: $\langle \alpha | \to$ linear function of vectors

$\langle \alpha | \beta \rangle = \mathbb{C}\,\#$

instruction to integrate $\int$.

$$\langle f | = \int f^* [\underset{\uparrow}{\cdots\cdots}] \, dx$$

any function

bra ket

$\langle \beta | \alpha \rangle$

ket expressed as column: $|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$\langle \beta | = (b_1^* \; b_2^* \; \cdots \; b_n^*)$, bra is a row.

$$\Rightarrow \langle \beta | \alpha \rangle = b_1^* a_1 + b_2^* a_2 + \cdots + b_n^* a_n$$

Dirac Notation

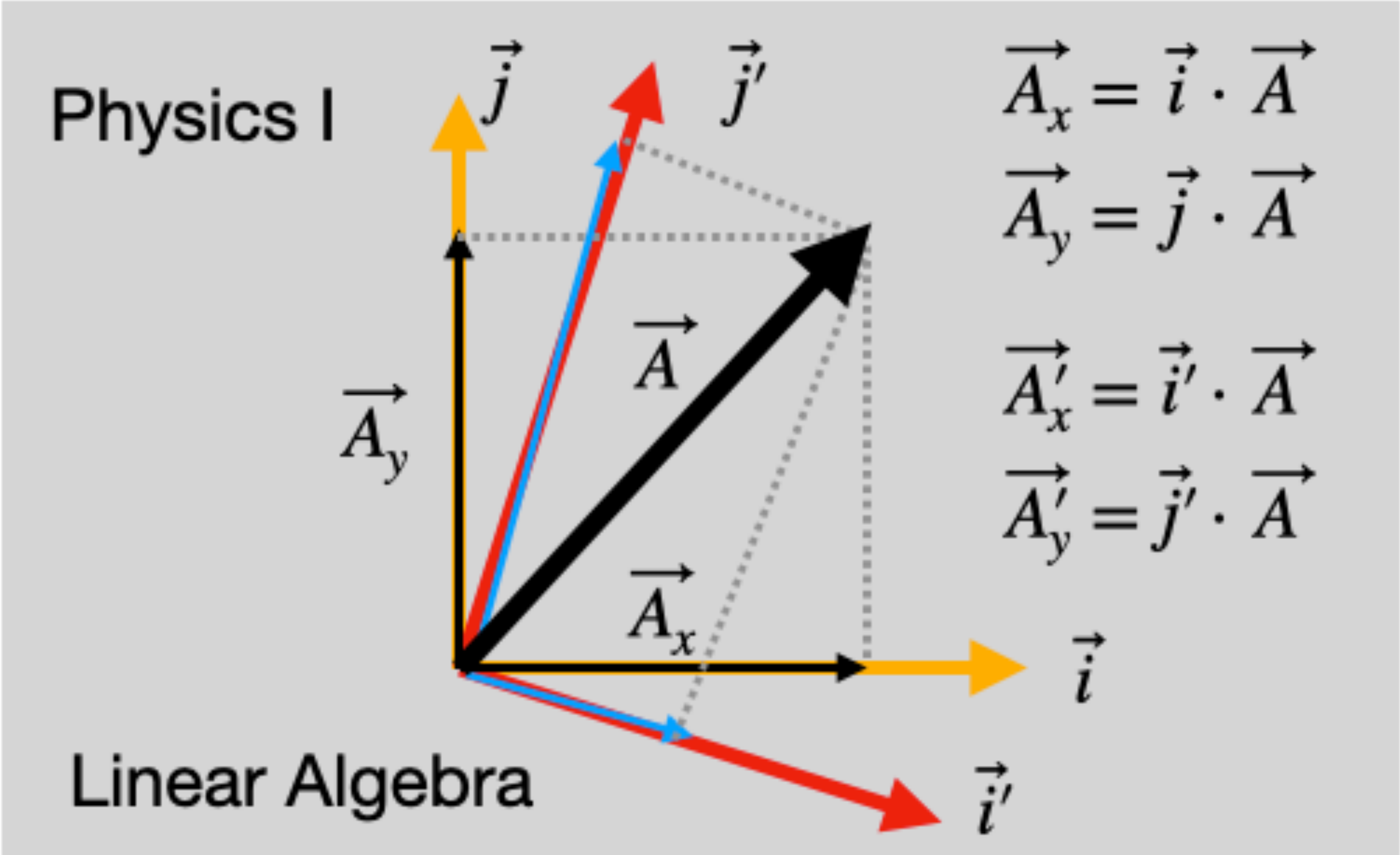
The collection of all bras constitutes the so-called dual space. Therefore, bras can be treated as separate entities, if $|\alpha\rangle$ is a normalised vector; the operator:

$$\boxed{\hat{P} \equiv |\alpha\rangle\langle\alpha|} \equiv \text{projection operator}$$

picks out the portion of any other vector that lies along $|\alpha\rangle$:

$$\hat{P}\,|\beta\rangle = \left(\langle\alpha|\beta\rangle\right)|\alpha\rangle$$

$$\vec{A}_x = (\vec{i} \cdot \vec{A})\;\vec{i}$$



$\hat{P}$ is the projection operator onto a 1D subspace spanned by $|\alpha\rangle$.

Set of basis vectors:

$$\{\vec{i}, \vec{i}, \vec{k}\}$$

$$\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$$

$$\langle \vec{i} \,|\, \vec{i} \rangle = 1$$

$$\langle \vec{i} \,|\, \vec{j} \rangle = 0$$

Set of basis vectors in QM:

$$\{\, |e_x\rangle, |e_y\rangle, |e_z\rangle \}$$

$$\langle e_x | e_x \rangle = 1$$

$$\langle e_x | e_y \rangle = 0$$

$$\hat{P} \equiv |\alpha\rangle\langle\alpha|$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\langle e_m | e_n \rangle = \delta_{mn} \Rightarrow \boxed{\sum_n |e_n\rangle\langle e_n| = 1} \equiv \text{identity operator.}$$

Discrete Case: $\{\,|e_n\rangle\}$

$$\langle e_m | e_n \rangle = \delta_{mn}$$

$\Rightarrow$ $\boxed{\sum_n |e_n\rangle\langle e_n| = 1}$ $\equiv$ identity operator.

Example: $\sum_n \left(\langle e_n|\alpha\rangle\right)|e_n\rangle = |\alpha\rangle$

Continuos Case: $\{\,|e_z\rangle\}$

$$\langle e_z | e_z' \rangle = \delta(z - z')$$

$\boxed{\int |e_z\rangle\langle e_z|\, dz = 1}$ $\equiv$ identity op.

Completeness!

Naming conventions:

Example: (Hermitian operator)

$$\langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle$$

In Dirac notation:

LHS: $\langle f | \hat{Q} f \rangle = \langle f | \hat{Q} | f \rangle$

RHS: $\langle \hat{Q} f |$ is the bra dual of $\hat{Q} | f \rangle$

$\langle \hat{Q} f | = \langle f | \hat{Q}^{+}$, remember $\langle f | \hat{Q} g \rangle = \langle \hat{Q}^{+} f | g \rangle$

Hermitian conjugate / adjoint

Operators acting on vectors that belong to H, then the new vector is also in H

$$\hat{Q} | \alpha \rangle = | \beta \rangle$$

Operations:

Sum of 2 operators: $(\hat{Q}+\hat{R})|\alpha\rangle = \hat{Q}|\alpha\rangle + \hat{R}|\alpha\rangle$

$$\hat{H} \equiv \hat{T} + \hat{V} \qquad \hat{H}|\psi\rangle = \hat{T}|\psi\rangle + \hat{V}|\psi\rangle$$

Product of 2 operators: $\hat{Q}\hat{R}|\alpha\rangle = \hat{Q}(\hat{R}|\alpha\rangle)$

$$|\beta\rangle$$

$$\hat{x}\hat{p} \neq \hat{p}\hat{x}$$

Functions of operators: typically defined by the power series expansion:

$$e^{\hat{Q}} \equiv 1 + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 + \cdots\cdots$$

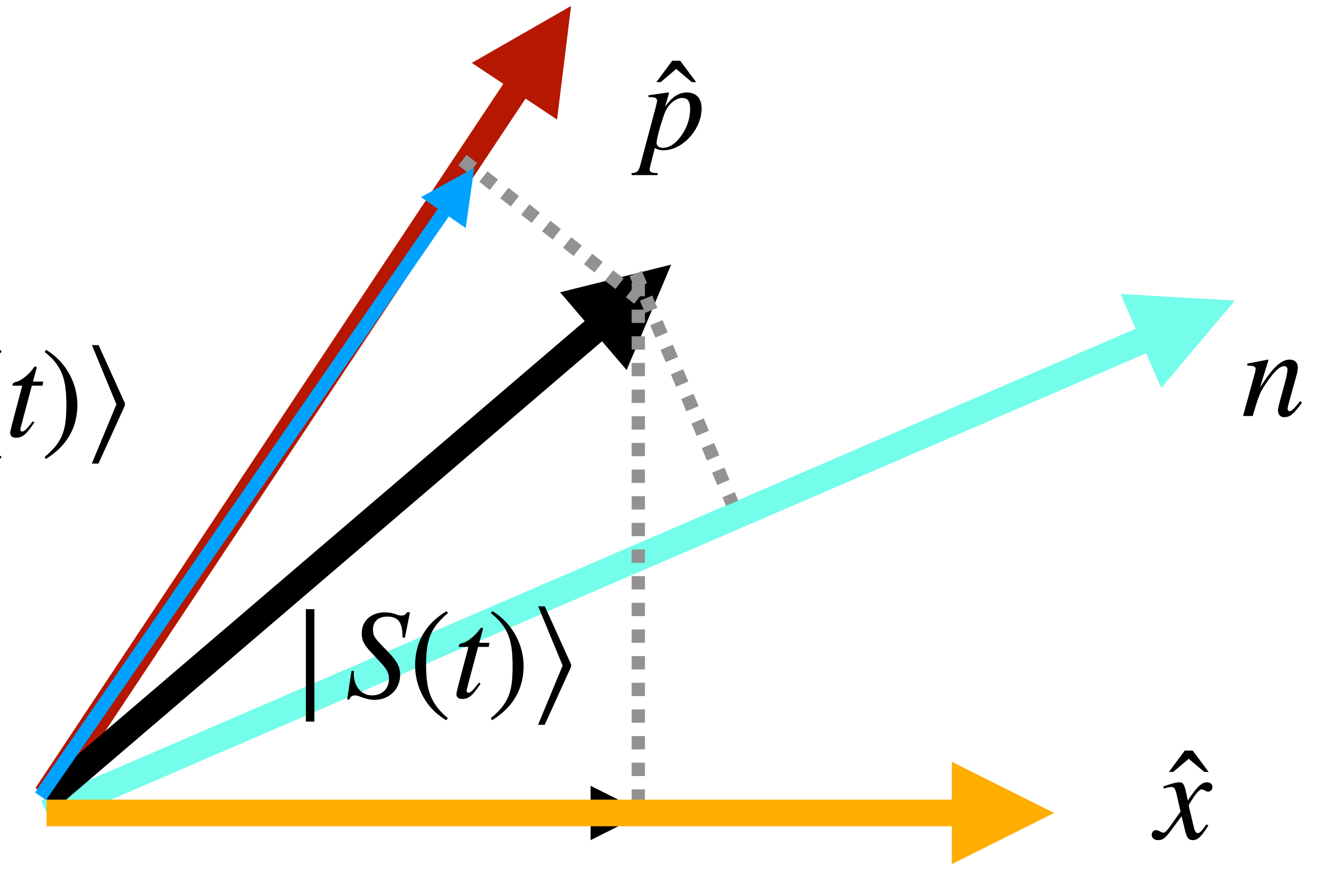
$$\frac{1}{1-\hat{Q}} \equiv 1 + \hat{Q} + \hat{Q}^2 + \hat{Q}^3 + \hat{Q}^4 + \cdots\cdots$$

$$\ln(1+\hat{Q}) \equiv \hat{Q} - \frac{1}{2}\hat{Q}^2 + \frac{1}{3}\hat{Q}^3 - \frac{1}{4}\hat{Q}^4 + \cdots\cdots$$

Change basis in Dirac notation, find S(t) from the projections.

State of QM system (wave/particle):

$|S(t)\rangle$



$|\phi(p,t)\rangle = \langle \hat{p} | S(t)\rangle$

$|c_n(t)\rangle = \langle n | S(t)\rangle$

$\Rightarrow \boxed{\sum_n |e_n\rangle\langle e_n| = 1}$ ≡ identity operator

$\boxed{\int |e_z\rangle\langle e_z| \, dz = 1}$ ≡ identity op.

$|\psi(x,t)\rangle = \langle \hat{x} | S(t)\rangle$

Position identity operator: $|x\rangle$ position eigenstates $\Rightarrow \int |x\rangle\langle x| \, dx = I$

Momentum identity operator: $|p\rangle$ Momentum eigenstates $\Rightarrow \int |p\rangle\langle p| \, dp = I$

Energy identity operator: $|n\rangle$ Energy eigenstates $\Rightarrow \sum |n\rangle\langle n| = I$

Position identity operator: $|x\rangle$ position eigenstates $\Rightarrow \int |x\rangle\langle x|\,dx = I$

Momentum identity operator: $|p\rangle$ Momentum eigenstates $\Rightarrow \int |p\rangle\langle p|\,dp = I$

Energy identity operator: $|n\rangle$ Energy eigenstates $\Rightarrow \sum |n\rangle\langle n| = I$

$$|S(t)\rangle = ?$$

We apply I on the State Vector:

$$|S(t)\rangle = I \cdot |S(t)\rangle = \int |x\rangle\langle x|S(t)\rangle\,dx = \int \Psi(x,t)\,|x\rangle\,dx$$

$$|S(t)\rangle = I \cdot |S(t)\rangle = \int |p\rangle\langle p|S(t)\rangle\,dp = \int \Phi(p,t)\,|p\rangle\,dp$$

$$|S(t)\rangle = I \cdot |S(t)\rangle = \sum |n\rangle\langle n|S(t)\rangle = \sum c_n(t)\,|n\rangle$$

Operators look different in different basis (matrices look different)

$\hat{x} \rightarrow \begin{cases} x & \text{Multiplication op.} \\ i\hbar \frac{\partial}{\partial p} & \text{Derivative op.} \end{cases}$ $\quad \hat{p} \rightarrow \begin{cases} -i\hbar \frac{\partial}{\partial x} & \text{Derivative op.} \rightarrow \text{in } x \text{ space} \\ p & \text{Multiplication op.} \rightarrow \text{in } p \text{ space} \end{cases}$

Operators act on kets ; eg $\hat{x}|S(t)\rangle$:

$\langle x|\hat{x}|S(t)\rangle$ = action of $\hat{x}$ operator in x basis $= x\Psi(x,t)$

$\langle p|\hat{x}|S(t)\rangle$ = action of $\hat{x}$ operator in p basis $= i\hbar \frac{\partial}{\partial p}\phi$

This notation allows us to transform operators between bases.

Problem 3.14

(a) Prove the following commutator identities:

$$\left[\hat{A}+\hat{B}, \hat{C}\right] = \left[\hat{A}, \hat{C}\right] + \left[\hat{B}, \hat{C}\right], \tag{3.64}$$

$$\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}. \tag{3.65}$$

(b) Show that

$$\left[x^n, \hat{p}\right] = i\hbar n x^{n-1}.$$

(c) Show more generally that

$$\left[f(x), \hat{p}\right] = i\hbar \frac{df}{dx}, \tag{3.66}$$

for any function $f(x)$ that admits a Taylor series expansion.

(d) Show that for the simple harmonic oscillator

$$\left[\hat{H}, \hat{a}_{\pm}\right] = \pm\hbar\omega\hat{a}_{\pm}. \tag{3.67}$$

Hint: Use Equation 2.54.

Problem 3.14

(a) Prove the following commutator identities:

$$\left[\hat{A}+\hat{B},\hat{C}\right]=\left[\hat{A},\hat{C}\right]+\left[\hat{B},\hat{C}\right], \tag{3.64}$$

$$\left[\hat{A}\hat{B},\hat{C}\right]=\hat{A}\left[\hat{B},\hat{C}\right]+\left[\hat{A},\hat{C}\right]\hat{B}. \tag{3.65}$$

3.64

$$[\hat{A}+\hat{B},\hat{C}]=[\hat{A},\hat{C}]+[\hat{B},\hat{C}]$$

$$LHS=[\hat{A}+\hat{B},\hat{C}]\,f=[(\hat{A}+\hat{B})\,\hat{C}-\hat{C}\,(\hat{A}+\hat{B})]\,f$$

$$=[\hat{A}\,\hat{C}+\hat{B}\,\hat{C}-\hat{C}\,\hat{A}-\hat{C}\,\hat{B}]\,f$$

$$=[[\hat{A}\,,\hat{C}]+[\hat{B}\,,\hat{C}]]\,f$$

$$LHS=[\hat{A}+\hat{B},\hat{C}]=[\hat{A}\,,\hat{C}]+[\hat{B}\,,\hat{C}]=RHS$$

Angular momentum

Problem 3.14

(a) Prove the following commutator identities:

$$\left[\hat{A}+\hat{B}, \hat{C}\right]=\left[\hat{A}, \hat{C}\right]+\left[\hat{B}, \hat{C}\right], \tag{3.64}$$

$$\left[\hat{A}\hat{B}, \hat{C}\right]=\hat{A}\left[\hat{B}, \hat{C}\right]+\left[\hat{A}, \hat{C}\right]\hat{B}. \tag{3.65}$$

3.65

$$[\hat{A}\,\hat{B}, \hat{C}] = \hat{A}\,[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\,\hat{B}$$

$$RHS = [\hat{A}\,[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\,\hat{B}]g = [\hat{A}[\hat{B}\hat{C} - \hat{C}\hat{B}] + [\hat{A}\hat{C} - \hat{C}\hat{A}]\hat{B}]g$$

$$= [\hat{A}\hat{B}\hat{C} - \cancel{\hat{A}\hat{C}\hat{B}}]g + [\cancel{\hat{A}\hat{C}\hat{B}} - \hat{C}\hat{A}\hat{B}]g$$

$$= [(\hat{A}\hat{B})\hat{C} - \hat{C}(\hat{A}\hat{B})]g$$

$$= (\hat{A}\hat{B})\hat{C} - \hat{C}(\hat{A}\hat{B}) = [\hat{A}\hat{B}, \hat{C}] = LHS$$

Angular momentum

(b) Show that

$$[x^n, \hat{p}] = i\hbar n x^{n-1}. \quad \Rightarrow \quad [\hat{x}, \hat{p}] = i\hbar$$

$$[\hat{p}, \hat{x}] = -i\hbar$$

$$LHS = [\hat{x}^n, \hat{p}]\, g = [\hat{x}^n, -i\hbar\frac{d}{dx}]\, g = [\hat{x}^n\,(-i\hbar\frac{d}{dx}) - (-i\hbar\frac{d}{dx})x^n]g$$

$$= -i\hbar\hat{x}^n\frac{dg}{dx} + i\hbar\frac{d(x^n g)}{dx}$$

$$= \cancel{-i\hbar\hat{x}^n\frac{dg}{dx}} + i\hbar(n)x^{n-1}g + \cancel{i\hbar x^n\frac{dg}{dx}}$$

$$= +i\hbar(n)x^{n-1}g$$

$$LHS = [\hat{x}^n, \hat{p}] = +i\hbar(n)x^{n-1} = RHS$$

Canonical Commutation relations