# Homework 4 - Quantum Mechanics I

**NAME:** ______________________________________________ **SCORE:** _____________

**Deadline:** Tuesday 2 December 2025 by 21:00
**Credits:** 20 points **Number of problems:** 4
**Type of evaluation:** Formative Evaluation

**This assignment is individual and consists of 4 problems related to units 3 and 4 of quantum mechanics. Please justify all calculations and highlight the answers.**

1. **(5 points) Spherical harmonics**

   (a) A quantum system is known to be in the (unnormalised) state described by the wave function $\psi(\theta, \phi) = 4Y_4^{-4} - 2i\, Y_5^{-4} + 7Y_5^0$, where the $Y_\ell^m(\theta, \phi)$ are the spherical harmonics. What is the probability of finding the system in a state with quantum number $m = -4$?

   (b) What is the orbital angular momentum eigenfunction $Y_\ell^m$ in a state for which the operators $L^2$ and $L_z$ have eigenvalues $6\hbar^2$ and $-\hbar$, respectively?

   (c) Construct all the possible spherical harmonics, $Y_\ell^m(\theta, \phi)$, for $\ell = 3$.

   (d) Using your favourite programming language, make 3D plots of all of them.

   (e) Choose two of the spherical harmonics constructed in part (b), and prove that they are normalised and orthogonal.

2. **(5 points) Hydrogen atom**

   (a) Construct all the possible spatial wave functions, $\psi_{nlm}(r, \theta, \phi) = R_{n\ell}(r)Y_\ell^m(\theta, \phi)$, of the hydrogen atom for $(n, \ell, m) = (2, 1, m)$.

   (b) Using your favourite programming language, make density plots of all of these states.

   (c) Calculate the energy level of these states in units of eV. Is there any degeneracy?

   (d) In terms of the Bohr radius, find $\langle r \rangle$, $\langle x \rangle$, $\langle r^2 \rangle$ and $\langle x^2 \rangle$ for an electron in the ground state of hydrogen.

   (e) Find $\langle x^2 \rangle$ in the state $(n, \ell, m) = (2, 1, m)$ with the highest possible value of $m$ that is allowed. How different is the result with respect to that calculated in part (d) for the ground state of hydrogen?

3. **(5 points) Spin**

(a) Suppose we have a spin-$\frac{1}{2}$ particle, whose quantum state can be represented by the eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of the $S_z$ operator. Using the Pauli matrix $\sigma_y$, calculate a normalised eigenstate of the $S_y$ operator with an eigenvalue $-\frac{\hbar}{2}$.

(b) Consider the Pauli spin matrices $\sigma_x$, $\sigma_y$, and $\sigma_z$ and the identity matrix, $I$. Calculate the commutators: $[\sigma_y, \sigma_x]$ and $[\sigma_y, \sigma_z]$.

(c) Calculate the normalised spin eigenfunctions of a system with two particles, one has spin 1 and the other one has spin $\frac{3}{2}$. Pick one of the states and report the Clebsch-Gordan superposition.

(d) Consider an electron in the spin state: $\chi = A \begin{bmatrix} 1+i \\ -1+2i \end{bmatrix}$. Normalise $\chi$, and find the expectation values of $S_x$, $S_y$, and $S_z$.

(e) Find the uncertainties $\sigma_{S_x}$, $\sigma_{S_y}$, and $\sigma_{S_z}$ for $\chi$ in (d). Are the results consistent with all three uncertainty principles?

4. **(5 points) Electron in a Magnetic Field**

Consider an electron (at rest) embedded in an oscillating magnetic field:

$$\vec{B} = B_0 \cos(\omega t)\, \vec{k},$$

where $B_0$ and $\omega$ are constants.

(a) Construct the Hamiltonian matrix for this system. Is this Hamiltonian time-dependent or time-independent?

(b) The electron starts out (at $t = 0$) in the spin-down state with respect to the $x$ axis (i.e., $\chi(0) = \chi_-^{(x)}$). Determine $\chi(t)$ at any subsequent time by solving the Schrödinger equation.

(c) Find the probability of getting $+\frac{\hbar}{2}$, if you measure $S_y$.

(d) What is the minimum field strength ($B_0$) required to force a complete flip in $S_y$?

(e) Does the oscillating field actually induce transitions between the eigenstates of $S_z$? What modification to the applied field would be required to produce resonant spin flips between $S_z$ eigenstates?