Mathematical tools for QM

- **Is QM a linear theory?**
- Why do we need complex numbers?

linear theory:

$$\left.\begin{array}{l}\text{Solution 1}\\ \text{Solution 2}\end{array}\right\} \Rightarrow \begin{array}{c}\text{New}\\ \text{Solution 3}\end{array}$$

We can create linear combinations of known solutions to get new solutions.

Linear Operators

- L.u =0
- L = linear operator, u = unknown
- Several operators applied to the same unknown: L1.u=0, L2.u=0
- Same operator applied to different unknowns: L(u1,u2,u3) =0

Properties of linear operators:

- Scale a solution: L(au) = a Lu
- Combine solutions: L(u1+u2) = L(u1) + L(u2)

EM theory is linear

Example: EM

$\frac{q}{V}$ $\quad$ $\frac{q}{A \cdot t}$

$(\vec{E}, \vec{B}, \rho, \vec{J})$ is a solution

Ⓛ

$\Rightarrow (\alpha \vec{E}, \alpha \vec{B}, \alpha \rho, \alpha \vec{J})$ is also a solution, $\alpha \in \mathbb{R}$

$$\left.\begin{array}{l} (E_1, B_1, \rho_1, J_1) \\ (E_2, B_2, \rho_2, J_2) \end{array}\right\} \text{are solutions:}$$

Ⓛ

$\Rightarrow (E_1 + E_2, B_1 + B_2, \rho_1 + \rho_2, J_1 + J_2)$ is a sln.

EM is linear

- Linear theory: Maxwell’s theory of EM
  - 2 1D plane waves propagating (do not touch each other, without affecting each other).
  - 3rd solution, 2 plane waves propagate simultaneously.
  - EM waves are all around - superposition, do not interfere with each other.

- E, B, charge density, current J (charge per unit area per unit time).
- Scale by #: Linearity x alpha (sln) is also a solution, alpha belong to R
- Sln1 + sln2 = sln

Is QM a linear theory?

Linear equation:

$$L \; u = 0$$

Linear operator (eq) $\swarrow$ ... $\searrow$ unknown (variable)

Properties: $\begin{cases} L(\alpha u) = \alpha L u \\ L(u_1 + u_2) = L u_1 + L u_2 \end{cases}$

Linear combinations:

$$L(\alpha u_1 + \beta u_2) = L(\alpha u_1) + L(\beta u_2) = \alpha L u_1 + \beta L u_2$$

If $u_1, u_2 \equiv$ sln $\Rightarrow$ $\alpha u_1 + \beta u_2 \Rightarrow$ sln

Linear vs. Non-linear Theories

linear & non-linear theories.

L } EM
QM

much simpler

NL } G.R.
C.M e.g. 3-body problem

very non-linear

QM is linear!

Linear vs. Non-linear Theories

linear & non-linear theories.

L $\begin{cases} \text{EM} \\ \text{QM} \end{cases}$

much simpler

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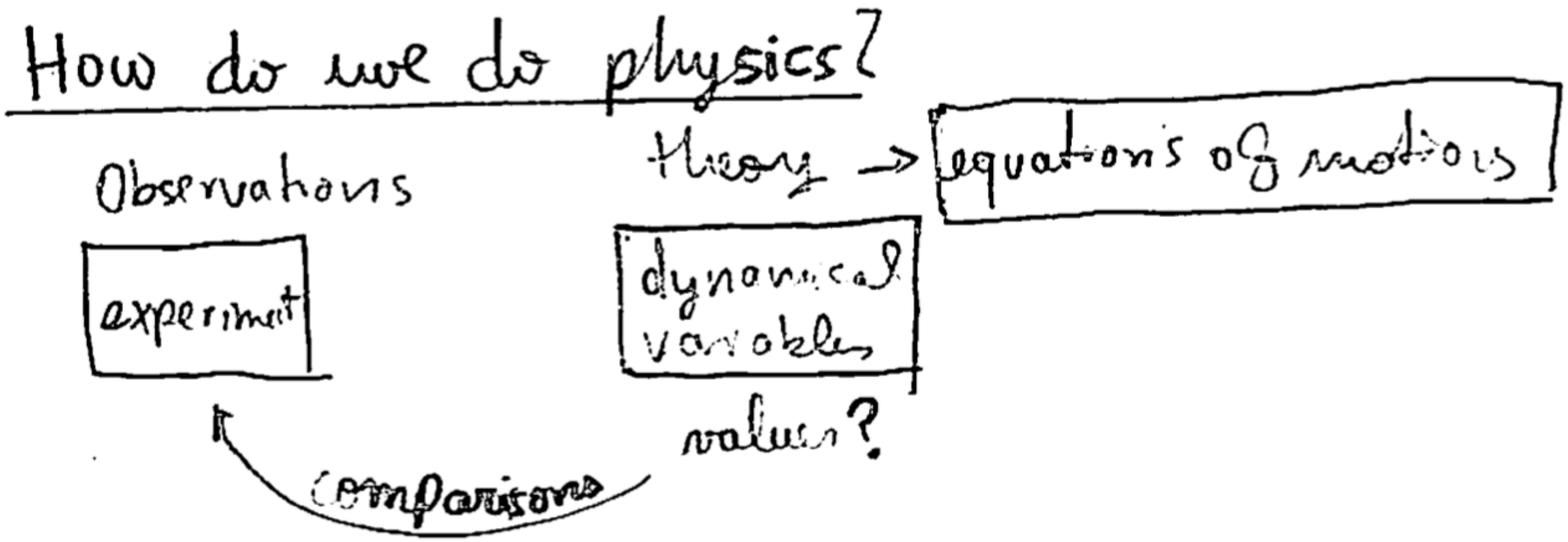
QM is linear!

How do we do physics? Scientific method

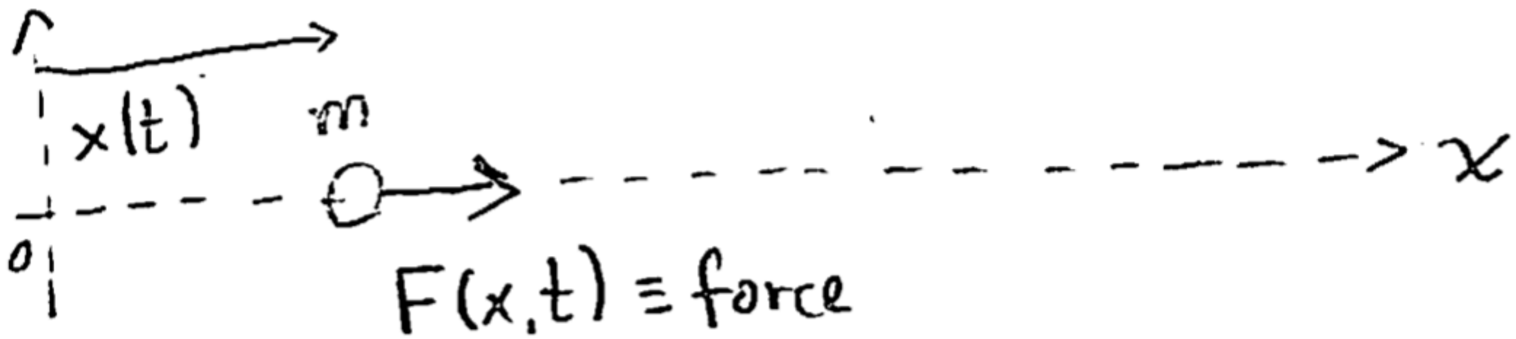
- Observations/experiments <-> dynamical variables (theory)
- Equations of motion solve for dynamical variables.



How do we do physics? Scientific method

- Motion of classical particles

In classical mechanics: 1D motion (non relativistic $v \ll c$)



$F(x,t) \equiv$ force

$$x(t) = ? \rightarrow v_x = \frac{dx}{dt} \rightarrow a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

$$\vec{p}_x = m\vec{v}_x$$

$$T = \frac{1}{2} m v_x^2$$

$\Bigg\}$ dynamical variables

How do we determine $x(t)$?

How do we do physics? Scientific method

- Motion of classical particles

Theory: Newton's 2nd law $F = ma = -\frac{\partial V}{\partial x}$ (conservative system)

$V \equiv$ potential energy function

$$\Rightarrow \boxed{m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x}}$$

If we know the initial conditions, eg $x(t=0)$, $v(t=0)$

$\Rightarrow x(t)$ ✓

e.g. elastic potential energy:

$$V = \frac{1}{2}kx^2 \Rightarrow \frac{\partial V}{\partial x} = kx$$

How do we do physics? Scientific method

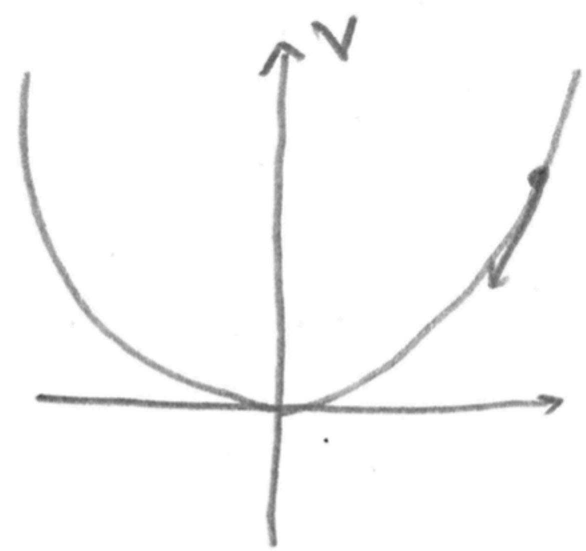
- Motion of classical particles

Motion in 1D : dynamical variable $x(t)$)

$$m\ddot{x} = -\frac{\partial V}{\partial x}$$

↳ derivatives are linear

non-linear



e.g. elastic potential energy:

$$V = \frac{1}{2} k x^2 \Rightarrow \frac{\partial V}{\partial x} = kx$$

$$\Rightarrow m\ddot{x} = -kx$$

$$\Rightarrow \boxed{m\ddot{x} + kx = 0} \quad \text{ODE}$$

Solution is:

$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$

$$\omega = \sqrt{\frac{k}{m}}$$

(Angular frequency)

How do we do quantum mechanics?

- Motion in quantum mechanics:

sai

QM is linear: dynamical variable Ψ (wavefunction)

↳ describes dynamics of the Q system

- The equation of motion is Schrödinger's equation:

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi}$$

Schrödinger's equation.

↳ Hamiltonian, linear operator.

$$\hat{H} = \hat{T} + \hat{V}$$

$$L\Psi = 0$$

$L \equiv i\hbar \frac{\partial}{\partial t} - \hat{H}$ is linear.

How do we do quantum mechanics?

- Schrödinger's equation: $$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$i = \sqrt{-1}$$

$$\hbar \equiv \text{Planck's constant} = \frac{h}{2\pi} = 1.054573 \times 10^{-34}\ J \cdot s$$

- QM is linear, so in some sense it is simpler than CM.
- We can scale solutions, and add/combine solutions to create superpositions, which become new solutions.

The necessity of complex numbers

- Why do we need complex numbers?

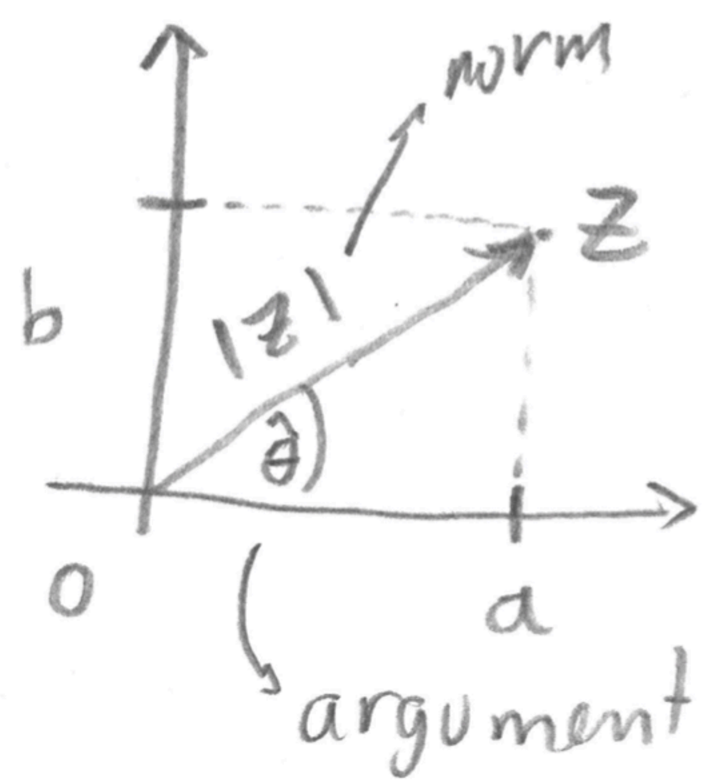
Complex numbers: $x^2 = 1 \Rightarrow x = \sqrt{-1}$

$i = \sqrt{-1}$

$\boxed{z = a + ib}$; $a, b \in \mathbb{R}$

$z \in \mathbb{C}$

$$\begin{cases} \mathrm{Re}(z) = a \\ \mathrm{Im}(z) = b \end{cases}$$



The necessity of complex numbers

- Why do we need complex numbers?

Complex conjugate of z:

$$\boxed{z^* = a - ib} \qquad (\bar{z}, \hat{z})$$

Norm of a complex #: $\in \mathbb{R}$

$$|z| = \sqrt{a^2+b^2}, \quad |z|^2 = a^2+b^2 = z z^*$$

Summation:

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

Multiplication:

$$(a+bi)(c+di) = (ac-bd) + (bc+ad)i$$

The necessity of complex numbers

- Why do we need complex numbers?

In polar coordinates:

$$a = |z| \cos\hat{\theta} \qquad \Rightarrow \qquad z = |z|(\cos\hat{\theta} + i \sin\hat{\theta}) = |z| e^{i\theta}$$

$$b = |z| \sin\hat{\theta}$$

Identity: $e^{i\hat{\theta}} = \cos\hat{\theta} + i \sin\hat{\theta}$ (Euler formula)

$$e^{-i\theta} = \cos\hat{\theta} - i \operatorname{sen}\hat{\theta}$$

In QM: $\Psi \in \mathbb{C}$

We need $\mathbb{C}$ numbers.

- The wave function, Ψ, has to be a complex number to satisfy Schrödinger equation.

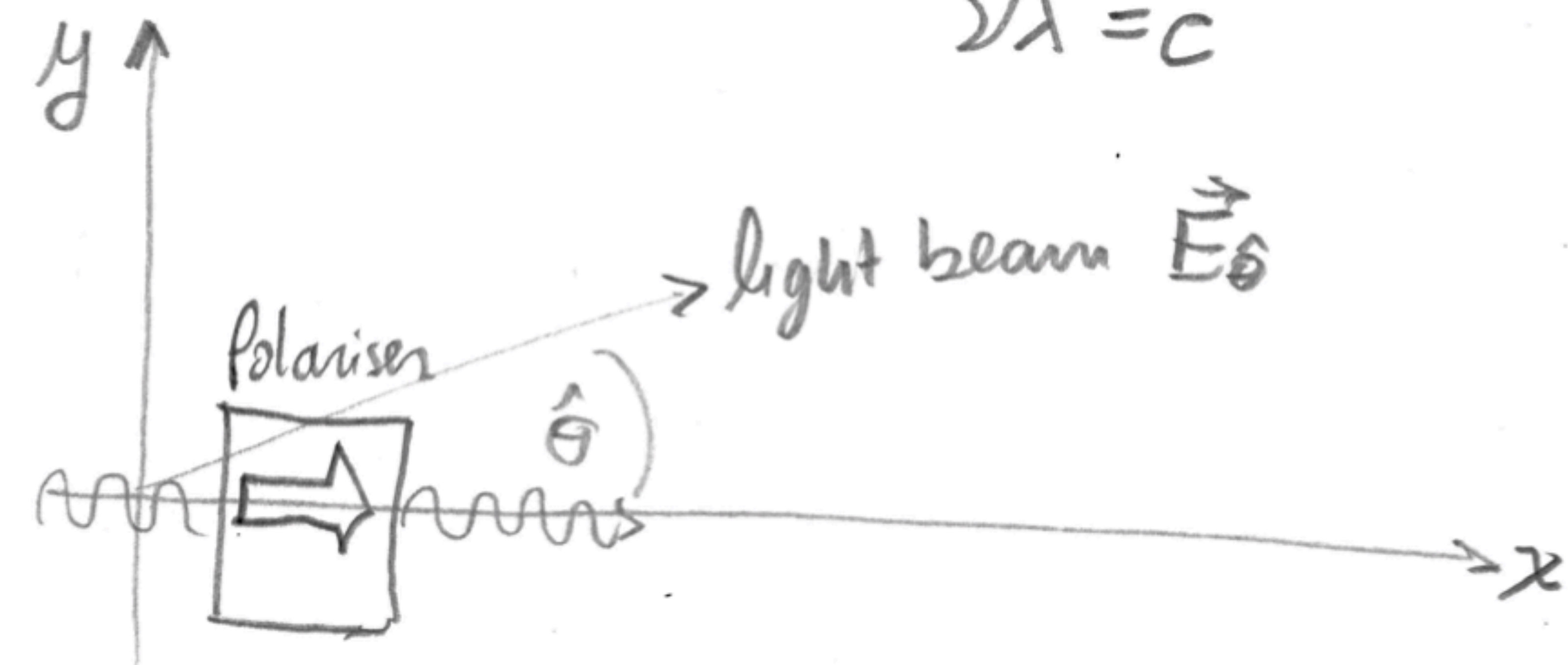
The loss of determinism

- **Classical Particles:** object with zero size with certain velocity at certain position.

- **Quantum Particles:** indivisible amount of energy that propagates.

γ: waves/particles ; $E = h\nu$

$\nu\lambda = c$



Same light, same energy. The colour does not change.

$$\vec{E}_\theta = E_0 \cos\hat{\theta}\,\hat{e}_x + E_0 \sin\hat{\theta}\,\hat{e}_y$$

Electric field before the polariser.

The loss of determinism

After the polariser:

$$\vec{E} = E_0 \cos\hat{\theta}\, \hat{e_x} \Rightarrow \left(\frac{E}{E_0}\right)^2 = \left(\cos^2\hat{\theta}\right)$$

magnitude of E field

fraction of $\vec{E}$ that goes thru.

γ

Now we send γ one by one.

In CM: identical γ should either get absorbed or go thru.

In QM: identical γ sometimes go thru / sometimes they don't

$\Rightarrow$ We lose predictability / we lose determinism