PLANE APPLICATION OF ORBIFOLD CLASSIFICATION

A thesis submitted in partial fulfillment of the requirements for the degree

MASTER OF SCIENCE

in

MATHEMATICS

by

WILLIAM THORN BAYNARD III

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at

THE GRADUATE SCHOOL OF THE COLLEGE OF CHARLESTON

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Abstract

The classification of objects and tilings of the plane through symmetry is an algebraic application used in chemistry, physics, and mathematics. In 1992, John H. Conway and William P. Thurston formulated a classification system based on the geometric notion of an "orbifold". The sophisticated orbifold approach to analyzing symmetry resulted in an elegant, more simplified classification scheme. The following is an exposition of the orbifold classification system with application to the well known 17 plane crystallographic groups.

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Introduction

Any solid object can be viewed microscopically having a chemical structure composed of atoms. The solid's underlying atomical structure may be characterized by a periodic repetition of a motif. Chemists refer to solids as crystals [**8**]. The classification of crystals by their structure began in a branch of chemistry called crystallography. Crytallographers observe periodic structure of a motif represented in a lattice with a unit cell. This is done in two dimensions by applying a coordinate system, or lattice base to the motif, taking parallelograms to be unit cells, the motif that is periodically repeated in the lattice. The lattice describes the symmetry properties of the chemical structure [**8**].

Any periodic repetition of a motif implies the existence of some form of symmetry observed in the overall structure. The classification of crystal structures by symmetry properties enables the deduction of physical properties such as hardness, rate of growth, electrical and thermal conductivity, index of refraction, elasticity, etc; each of which are orientation dependent [**8**]. Furthermore, observation and classification of objects' symmetry is an application for many other scientific realms including molecular biology, materials science, mineral-pertrography, physics, and mathematics.

Crystallographers have a labeling, or notation they employ in their classifcation system of crystals. Though, the classifcation is built on observing mathematically algebraic and topological properties of symmetry groups acting on surfaces. In 1992, John H. Conway and William P. Thurston formalized mathematically the chemistry based classification of objects through symmetry, developing the orbifold notation. Utilizing the topological concept of an orbit manifold, Conway and Thurston formulated a mathematical classification system. The system is called the orbifold (orbit manifold) classifcation system.

The orbifold classification system joins algebraic concepts from group theory classifying symmetries observed on topological surfaces: an equivalence relation arises as we relate the repeated motif, and therefore symmetry on a surface giving way to a modular space formed from the group of symmetries acting on a surface. Tentatively we say: if G is a group and X is a surface topological space, then a group action of G on X defines an equivalence relation with equivalence classes called orbits. The orbifold O is a quotient topological space, surface obtained by mapping all the elements in each orbit of X to a single point [**3**] [**4**] [**6**] [**9**]. The sophisticated orbifold approach to analyzing and classifying symmetries on surfaces, motivated by chemical crystral structure, resulted in a more elegant, simplified classification scheme.

The following is an exposition of Conway's orbifold classification system. We begin developing the foundational algebraic and topological theory that authorizes the system mathematically, considering definitions and concepts suitable for study at an advanced undergraduate or graduate level.

First, we develop the topological structure on a set called a space, beginning with the notion of distance, or metric in a space. We focus on the case of a topological space called a metric space and address functions on metric spaces. Our application is on the Euclidean plane metric space. Then we propose the algebraic group theory and group action on a space, examining rigid motions and symmetries on the plane. The orbifold classification system applies to three metric spaces called surfaces: the sphere, the hyperbolic plane, and the Euclidean plane; and their corresponding surface groups. We concentrate on discrete groups of motions on the plane pertinent to our application, the 17 plane crystallographic surface groups. The preliminaries conclude observing the orbifold quotient space.

Part 2 is our application. We establish the orbifold classification theory, announcing the orbifold properties observed in symmetries on the plane, Conway's orbifold notation, and the Euler characteristic of an orbifold. The orbifold Euler characteristic gives a way to enumerate all possible symmetry groups by calculating all possible characteristics [**3**]. Continuing, we lay out an 8 step procedure for application of the orbifold classification scheme on the Euclidean plane. We utilize our procedure to classify an example of each of the 17 plane crystallographic groups and calculate the orbifold Euler characteristic for each.

A crystallography assertion that all two dimensional crystallographic patterns belong to one of the 17 plane crystallographic groups [**10**], is a result from algebra, based on the crystallographic restriction and completeness of four rigid motions in the plane. Thus, we have an exhaustive list, illustrating an example of each, for our classification in two dimensions on the 17 plane groups; also referred to as the wallpaper groups for appearing in patterns on wall, textile, and floor decorations, etc.

Part 1

Preliminaries

CHAPTER 1

The topology

We introduce several definitions and concepts on topology from [**6**]. Since we are classifying symmetries of objects, we develop mathematical structure for spaces where these objects occur, mathematically speaking. So, we begin introducing the structure of topological spaces, particularly metric spaces.

DEFINITION 1.1 (Topology). A *Topology* on a set X is a collection $\mathfrak{T}$ of subsets of X having the following properties [**6**]:

- $\emptyset$ and X are in $\mathfrak{T}$.
- The union of the elements of any subcollection of $\mathfrak{T}$ is in $\mathfrak{T}$.
- The intersection of the elements of any finite subcollection of $\mathfrak{T}$ is in $\mathfrak{T}$.

DEFINITION 1.2 (Basis for a topology). If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X called *basis elements* such that [**6**]

- For each $x \in X$, there is at least one basis element B containing x.
- If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the *topology* $\mathfrak{T}$ *generated by* $\mathcal{B}$ as follows:

A subset U of X is said to be an element of $\mathfrak{T}$ if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathfrak{T}$ [**6**].

EXAMPLE 1.1. If we let $\mathcal{B}$ be the collection of all interiors of circles in the Euclidean plane $\mathbb{R}^2$, then $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^2$ [**6**].

DEFINITION 1.3 (Topological Space). A *Topological Space* is a set X with a specified topology $\mathfrak{T}$. If U is a subset of X and U belongs to the collection $\mathfrak{T}$, then we call U an *open set* [**6**].

EXAMPLE 1.2. The Euclidean plane $\mathbb{R}^2$ is a topological space with the topology in Example 1.1 [**6**].

We are primarily concerned with the case of a metric space.

1.1. Metric and metric space

The spaces we consider herein all have a notion of distance. We call this distance measure a metric. The metric will define the topology.

1.1.1. Metric.

DEFINITION 1.4 (Metric). A *metric* on a set X is a function

$$d : X \times X \to \mathbb{R} \tag{1.1}$$

having the following properties [**6**]:

- $d(x, y) \geq 0$ for all x, $y \in X$; equality holds if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all x, $y \in X$.
- (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$, for all x, y, $z \in X$.

Familiar sets that have defined metrics are sets of points on the sphere, the hyperbolic plane, and the Euclidean plane. For our application on the Euclidean plane, we use the following metric.

EXAMPLE 1.3 (The Euclidean plane). Consider the Euclidean plane E^2, which is the familiar $\mathbb{R}^2$. The Euclidean plane has the following Euclidean metric [**6**]:

$$d(x, y) = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{\frac{1}{2}}. \tag{1.2}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

EXAMPLE 1.4 (The sphere). Consider the sphere $S^2 = \{\ x \in \mathbb{R}^3 \mid d(x, 0) = 1\}$. The sphere has the following Euclidean metric [**6**]:

$$d(x, y) = ((x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2)^{\frac{1}{2}}. \tag{1.3}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$.

EXAMPLE 1.5 (The hyperbolic plane). For $x = (x_1, x_2) \in \mathbb{R}^2$, we denote $|x| = |(x_1, x_2)| = d(x_1, x_2)$. Consider the hyperbolic plane $H^2 = \{\ x \in \mathbb{R}^2 \mid |x| < 1\}$. The hyperbolic plane has the following hyperbolic metric:

$$d(x,y) = \frac{((x_1-y_1)^2+(x_2-y_2)^2)^{\frac{1}{2}}}{1-(x_1^2+x_2^2)-(y_1^2+y_2^2)} \tag{1.4}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

The following definition is a description of what we call an open ball, defined in terms of a metric.

Definition 1.5 (ϵ-balls). Given a metric d on X, the number $d(x,y)$ is often called the *distance* between x and y in the metric d. Given $\epsilon > 0$, consider the set

$$B_d(x,\epsilon) = \{y \mid d(x,y) < \epsilon\} \tag{1.5}$$

of all points y whose distance from x is less than ϵ. It is called the *ϵ-ball centered at x* [**6**]. If the metric d is understood, then we write $B_\epsilon(x)$.

Example 1.6. Consider the real line $\mathbb{R}$ with metric $d(x,y) = |x-y|$ for $x, y \in \mathbb{R}$. For $\epsilon > 0$, the ball of radius ϵ is the open interval $|x| < \epsilon$; i.e. $(x-\epsilon, x+\epsilon)$.

Example 1.7. Consider the Euclidean plane $\mathbb{R}^2$ with the Euclidean metric. For $\epsilon > 0$, the ball of radius ϵ is the interior of the circle of radius ϵ centered at $x \in \mathbb{R}^2$.

Now, we have our definition of an open set.

Definition 1.6 (Open set). Let X be a metric space and let U be a subset of X. U is an *open set* if for each $x \in U$ there exists an $r > 0$

such that [**7**]

$$d(x, y) < r \quad \Rightarrow \quad y \in U. \tag{1.6}$$

DEFINITION 1.7 (Metric topology). Let d be a metric on the set X. The collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology X, called the *metric topology* induced by d [**6**].

NOTE. Example 1.1 illustrates a metric topology.

If the collection U of ϵ-balls in $\mathbb{R}^2$ satisfies Definition 1.2, then the metric topology on $\mathbb{R}^2$ is by definition generated by U. We prove this assertion.

PROOF. Let U be the set of of ϵ-balls $B_d(x, \epsilon)$ in $\mathbb{R}^2$. Given any $x \in \mathbb{R}^2$ there exists $B_d(x, 1) \subset U \subset \mathbb{R}^2$. Let $x \in \mathbb{R}^2$. Assume for $u, v \in \mathbb{R}^2$ and $\epsilon_1, \epsilon_2 > 0$ we have

$$x \in B_d(u, \epsilon_1) \cap B_d(v, \epsilon_2). \tag{1.7}$$

Then

$$\epsilon_1 - d(u, x) = r_1 > 0 \quad \text{and} \tag{1.8}$$

$$\epsilon_2 - d(v, x) = r_2 > 0. \tag{1.9}$$

Choose $\epsilon > 0$ such that $\epsilon < \min\{r_1, r_2\}$. We claim $B_d(x, \epsilon) \subset B_d(u, \epsilon_1)$. Let $y \in B_d(x, \epsilon)$ and observe

$$\begin{aligned} d(u, y) &\leq d(u, x) + d(x, y) \\ &< d(u, x) + \epsilon \qquad (1.10) \\ &< d(u, x) + r_1 = \epsilon_1. \end{aligned}$$

Hence, $y \in B_d(u, \epsilon_1)$. Similarly we have $y \in B_d(v, \epsilon_2)$ and thus conclude

$$B_d(x, \epsilon) \subset B_d(u, \epsilon_1) \cap B_d(v, \epsilon_2). \tag{1.11}$$

$\square$

1.1.2. Metric space.

Having the concept of metric, we introduce the metric space.

DEFINITION 1.8 (Metric space). Let $\mathfrak{T}$ be the topology induced by a metric d on the set X. A *metric space* is the space $(X, \mathfrak{T}_d)$ defined by the topology $\mathfrak{T}$ having metric d on the set X [**6**].

Recall Examples 1.3, 1.4, and 1.5 of metrics on the sphere, hyperbolic plane, and Euclidean plane. Familiar examples of metric spaces are the Euclidean plane $\mathbb{R}^2$, the sphere S^2, and the hyperbolic plane H^2. These metric spaces are called surfaces which we will define later.

EXAMPLE 1.8 (Euclidean plane). Our application is on the metric space the Euclidean plane $\mathbb{R}^2$ together with the Euclidean metric. That is, for d_E the Euclidean metric with topology $\mathfrak{T}_{d_E}$ induced by d_E, we have $(\mathbb{R}^2, \mathfrak{T}_{d_E})$ is a metric space. The open sets $W \in \mathfrak{T}_{d_E}$ are arbitrary unions and finite intersections of ϵ-balls in the plane. These ϵ-balls are the basis, comprised of planar open circles centered at points $(x, y) \in \mathbb{R}^2$ having radius some positive number, $\epsilon > 0$. Consider the 2-dimensional unit ball in Figure 1.1, an open ball centered at the origin $(0, 0)$ containing all points x such that $|x| < 1$.

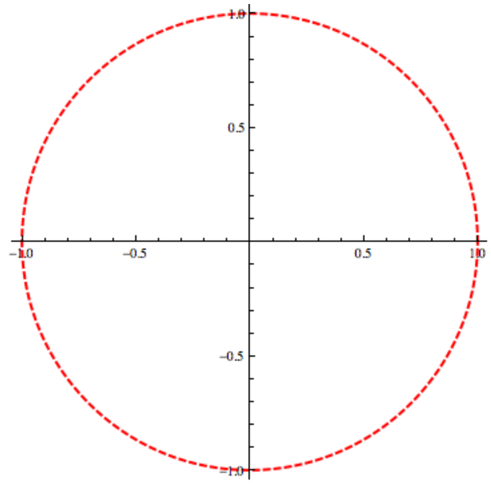

FIGURE 1.1. A 2-dimensional unit ball with dashed red line indicating non-inclusion of the boundary.

EXAMPLE 1.9 (Sphere). The sphere S^2 together with the Euclidean metric defined on it above is a metric space. That is, for d_E the Euclidean metric and topology $\mathfrak{T}_{d_E}$ induced by d_E, we have $(S^2, \mathfrak{T}_{d_E})$ is a metric space. The basis $U \in \mathfrak{T}_{d_E}$ may be taken as ϵ-balls centered at any point x, for all $\epsilon > 0$, on the sphere. Since $(S^2, \mathfrak{T}_{d_E})$ has the Euclidean metric d_E the ϵ-balls are just circles, not including their boundary, on the sphere. Consider Figures 1.2 and 1.3 of a sphere and a cut out of an open ball on a sphere.

EXAMPLE 1.10 (Hyperbolic plane). The hyperbolic plane H^2 together with the hyperbolic metric defined above is a metric space. That is, for d_H the hyperbolic metric with topology $\mathfrak{T}_{d_H}$ induced by d_H, we have $(H^2, \mathfrak{T}_{d_H})$ is a metric space. Like on the sphere, the open sets $V \in \mathfrak{T}_{d_H}$ are arbitrary unions and finite intersections of ϵ-balls centered at any point x, for all $\epsilon > 0$, on the hyperbolic plane. We may think of these balls as, what we will now call open circles, on the

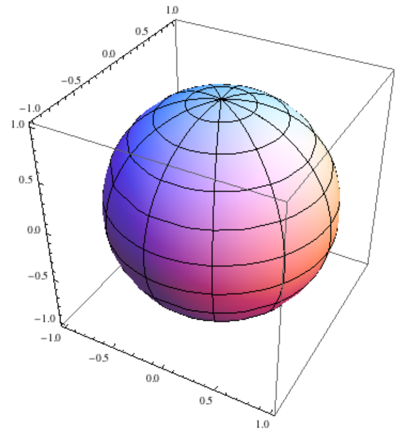

Figure 1.2. A sphere.

Figure 1.3. A cut out of an open ball on a sphere with dashed red line indicating non-inclusion of the boundary.

hyperbolic plane. However, the hyperbolic metric d_H warps the shape (see Figure 1.4) of these open circles.

Figure 1.4. The open set bounded by dashed red lines (indicated for non-inclusion of the boundary) is "warped inward" under the hyperbolic metric.

Definition 1.9 (Subspace of $\mathbb{R}^2$). Let A be a subset of the Euclidean plane. Then $A \subset \mathbb{R}^2$ becomes a metric space when we declare the distance between two points of A to be their Euclidean distance. That is, we declare $A \subset \mathbb{R}^2$ to have the Euclidean metric d_E. We say that A *inherits* its metric from $\mathbb{R}^2$ and is a *subspace* of $\mathbb{R}^2$ [**7**].

Remark. Definition 1.9 is specific to the Euclidean plane. We may extend the definition to include all spaces $\mathbb{R}^n$. However, we only provide the definition of the Euclidean plane subspace since the plane is our focus.

EXAMPLE 1.11. Let $A = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$ with the Euclidean metric d_E. Then A is a subspace of the Euclidean plane $\mathbb{R}^2$ inheriting the Euclidean metric.

EXAMPLE 1.12. Let $B = \{x \in \mathbb{R}^2 \mid d(x, 0) < 1\}$ with the Euclidean metric d_E. Then $B \subset \mathbb{R}^2$ is the open unit ball in the Euclidean plane. It is obvious that B is a subspace of the Euclidean plane.

REMARK. With respect to our ϵ-ball definition (Definition 1.5) of an open set, we may let each open set inherit the metric from its overall space, thinking of each open set as a subspace.

1.2. Functions and continuity

Now we examine functions and continuity, following the exposition of [**6**] and [**7**].

DEFINITION 1.10 (ϵ, δ condition continuity). Let X and Y be metric spaces. A function $f : X \to Y$ is *continuous* if it satisfies:

$$\begin{aligned} &\forall \epsilon > 0 \quad \text{and} \quad \forall x \in X \quad \exists \delta \quad \text{such that} \\ &y \in X \quad \text{and} \quad d(x, y) < \delta \quad \Rightarrow \quad d(f(x), f(y)) < \epsilon \end{aligned} \tag{1.12}$$

For functions of a real variable $f : (a, b) \to \mathbb{R}$ where (a, b) is an open interval on the real line $\mathbb{R}$ we have f is contiuous when:

$$\begin{aligned} &\forall \epsilon > 0 \quad \text{and} \quad \forall x \in (a, b) \quad \exists \delta > 0 \quad \text{such that} \\ &y \in (a, b) \quad \text{and} \quad |x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon \end{aligned} \tag{1.13}$$

Another (equivalent) definition of continuity is the following.

DEFINITION 1.11 (Open set continuity). Let X and Y be metric spaces having topologies $\mathfrak{T}_x$ and $\mathfrak{T}_y$, respectively, induced by some metric. A map $f : X \to Y$ is said to be *continuous* if for each open subset V of Y, the set $f^{-1}(V) = U$ is an open subset of X. That is, $f^{-1}(V) = U \in \mathfrak{T}_x$ for all $V \in \mathfrak{T}_y$. We say a function is continuous if the inverse image of an open set is an open set [**6**].

PROPOSITION 1.1. *If a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the open set condition on continuity, then f satisfies the ϵ, δ condition on continuity .*

PROOF. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy the open set condition on continuity. Let $x \in \mathbb{R}^2$. Then $f(x) \in \mathbb{R}^2$. Let $\epsilon > 0$ and consider $B_\epsilon(f(x))$, the open ball of radius ϵ centered at $f(x)$. Then we have that

$$f^{-1}(B_\epsilon(f(x))) = \{u \in \mathbb{R}^2 \mid f(u) \in B_\epsilon(f(x))\}. \tag{1.14}$$

is an open set. Now, $x \in \mathbb{R}^2$ satisfying $f(x) \in B_\epsilon(f(x))$ implies $x \in f^{-1}(B_\epsilon(f(x))$. Furthermore, by Definition 1.6 we know there exists $\delta > 0$ such that

$$B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)). \tag{1.15}$$

For any y where $d(x, y) < \delta$ we have $y \in B_\delta(x)$ implying

$$y \in B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)), \tag{1.16}$$

so that $y \in f^{-1}(B_\epsilon(f(x))$. Then $f(y) \in B_\epsilon(f(x))$. Thus, $d(f(x), f(y)) < \epsilon$. $\square$

DEFINITION 1.12 (Homeomorphism). Let $(X, \mathfrak{T}_{d_x})$ and $(Y, \mathfrak{T}_{d_y})$ be metric spaces. Let $f : (X, \mathfrak{T}_{d_x}) \to (Y, \mathfrak{T}_{d_y})$ be a bijective map. That is, let $f : (X, \mathfrak{T}_{d_x}) \to (Y, \mathfrak{T}_{d_y})$ be both injective (or one-to-one) and surjective (or onto). If the function f and the inverse function $f^{-1} : (Y, \mathfrak{T}_{d_y}) \to (X, \mathfrak{T}_{d_x})$ are continuous, then f is called a *homeomorphism* [**6**].

Consider the following example familiar from Calculus.

EXAMPLE 1.13. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 12x + 5$. Then define $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(y) = \frac{1}{12}(y - 5). \tag{1.17}$$

Notice that f and g are both continuous functions for all real numbers x and y. Observe,

$$f(g(y)) = f(\frac{1}{12}(y-5)) = 12(\frac{1}{12}(y-5)) + 5 = y, \tag{1.18}$$

and

$$g(f(x)) = g(12x + 5) = \frac{1}{12}(12x + 5 - 5) = x. \tag{1.19}$$

for all $x, y \in \mathbb{R}$. If we set $g = f^{-1}$, then f is a homeomorphism [**6**].

EXAMPLE 1.14. If we remove the point $(0, 0, 1) \in \mathbb{R}^3$ from the sphere S^2, then it is homeomorphic to $\mathbb{R}^2$ since we can project

$(S^2 - \{(0,0,1)\})$ to $\mathbb{R}^2$ by the continuous map

$$f(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right). \tag{1.20}$$

We call f the *stereographic projection* having inverse [**6**]:

$$f^{-1}(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 1 - \frac{2}{1+x^2+y^2}\right) \tag{1.21}$$

A homeomorphism that preserves distances is called an isometry. Since we are developing topology with respect to metrics, we define isometry here.

Definition 1.13 (Isometry). Let (X,d) be a metric space. If $f : X \to X$ satisfies the condition

$$d(f(x), f(y)) = d(x,y)$$

for all x, $y \in X$, then f is called an *isometry* of X [**6**].

Example 1.15. Consider the plane as the space $\mathbb{R}^2$ of column vectors with coordinate system (x,y) having the Euclidean metric. Let $t : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+a \\ y+a \end{pmatrix}. \tag{1.22}$$

for all $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^2$ and some constant $a \in \mathbb{R}$. We show that t is continuous and a homeomorphism preserving distances.

PROOF. Let $\binom{u}{v} \in \mathbb{R}^2$. For all $\epsilon > 0$ and $\forall \binom{x}{y} \in \mathbb{R}^2$ $\exists \delta$ such that

$$\left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} \right| = ((x-u)^2 + (y-v)^2)^{\frac{1}{2}} = d((x,y),(u,v)) < \delta. \tag{1.23}$$

Therefore, choose $\delta = \epsilon$ and we have

$$\begin{aligned} \left| t \begin{pmatrix} x \\ y \end{pmatrix} - t \begin{pmatrix} u \\ v \end{pmatrix} \right| &= \left| \begin{pmatrix} x+a \\ y+a \end{pmatrix} - \begin{pmatrix} u+a \\ v+a \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} x-u \\ y-v \end{pmatrix} \right| = d((x,y),(u,v)) < \epsilon. \end{aligned} \tag{1.24}$$

Thus, the function t is continuous. Now define $t^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$t^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-a \\ y-a \end{pmatrix}, \tag{1.25}$$

and observe

$$t^{-1}t \begin{pmatrix} x \\ y \end{pmatrix} = t^{-1} \begin{pmatrix} x+a \\ y+a \end{pmatrix} = \begin{pmatrix} (x+a)-a \\ (y+a)-a \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.26}$$

and

$$tt^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} x-a \\ y-a \end{pmatrix} = \begin{pmatrix} (x-a)+a \\ (y-a)+a \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1.27}$$

Therefore, the function t is a homeomorphism having inverse t^{-1}.

Now, we check that t preserves distances.

$$\begin{aligned} d(t(x,y),t(u,v)) &= \left| t\begin{pmatrix} x \\ y \end{pmatrix} - t\begin{pmatrix} u \\ v \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} x+a \\ y+a \end{pmatrix} - \begin{pmatrix} u+a \\ v+a \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} x-u \\ y-v \end{pmatrix} \right| = d((x,y),(u,v)). \end{aligned} \tag{1.28}$$

Thus, the function t is is an isometry. □

Remark. We call the previously mentioned function t a translation. Later, we explore this and other motions in the plane, showing that a rotation is an isometry.

CHAPTER 2

The group action

We will be acting on topological spaces with algebraic groups.

2.1. Groups and group action

We define some concepts from group theory.

2.1.1. Group theory.

DEFINITION 2.1 (Equivalence relation). An *equivalence relation* $\backsim$ on a set X is a subset A of $X \times X$ written as $a \backsim b$ if $(a, b) \in A$, satisfying the following properties [**4**]:

- *Reflexive* if $a \backsim a$, for all $a \in X$
- *Symmetric* if $a \backsim b$ implies $b \backsim a$ for all $a, b \in X$,
- *Transitive* if $a \backsim b$ and $b \backsim c$ implies $a \backsim c$ for all $a, b, c \in X$.

DEFINITION 2.2 (Equivalence class). Let $\backsim$ define an equivalence relation on a set X. The *equivalence class* of $a \in X$ is defined to be the set $\{x \in X \mid x \backsim a\}$. Elements of the equivalence class of a are said to be *equivalent* to a. If C is an equivalence class, then any element of C is called a *representative* of the class C [**4**].

REMARK. In our application, we observe 17 types of equivalence relations on the plane and identify representatives of each equivalence class. All elements of an equivalence class will be mapped to a point in the "quotient space".

DEFINITION 2.3 (Partition). A *partition* of the set X is any collection $\{A_i \; i \in I\}$ of nonempty subsets of X (where I denotes some indexing set) such that [**4**]

- $X = \cup_{i \in I} A_i$, and
- $A_i \cap A_j = \emptyset$, for all $i, j \in I$ where $i \neq j$.

That is, we say X is the disjoint union of the sets A_i in the partition.

REMARK. Every partition defines an equivalence relation and every equivalence relation defines a partition. In our application, we select a particular set A_k of $\mathbb{R}^2$ in the partition. Then we map all points to our chosen set A_k, that will be our orbifold.

We continue with more group theory.

DEFINITION 2.4 (Binary operation). A *binary operation* $\circ$ on a set G is a function $\circ : G \times G \to G$. For any $a, b \in G$ we write $a \circ b$ for $\circ(a, b)$. A binary operation $\circ$ on a set G is *associative* if for all $a, b, c \in G$ we have $a \circ (b \circ c) = (a \circ b) \circ c$.
If $\circ$ is a binary operation on a set G we say elements a and b of G *commute* if $a \circ b = b \circ a$. We say $\circ$ (or G) is *commutative* if for all $a, b \in G$, we have $a \circ b = b \circ a$ [**4**].

Consider a few examples of binary operations.

Example 2.1 (Addition). Our usual addition $+$ is a commutative binary operation on sets such as the integers, $\mathbb{Z}$, and the reals $\mathbb{R}$ [**4**]. For instance, $3+5=8=5+3$ and $1.67+3.14=4.81=3.14+1.67$.

Example 2.2 (Multiplication). Our usual multiplication $\times$ is a commutative binary operation on sets such as the integers $\mathbb{Z}$ and the reals $\mathbb{R}$. For instance, $3\times 5=15=5\times 3$ and $2.3\times .3=.69=.3\times 2.3$.

Example 2.3 (Function composition). The composition of functions is an associative binary operation. For functions f and g where $f : X \to Y$ and $g : Y \to Z$, the composition $g \circ f : X \to Z$ is defined by $(g\circ f)(x) = g(f(x))$. If we have another function $h : Z \to W$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Before giving the definition of a group, we need the concept of closure under a binary operation.

Definition 2.5 (Closure). Suppose that $\circ$ is a binary operation on a set G. Let H be a subset of G. If the restriction of $\circ$ of H is a binary operation on H, then H is said to be *closed* under the operation $\circ$. That is, for all $a, b \in H$ we have $a \circ b \in H$ implies H is closed under $\circ$. Observe that if $\circ$ is an associative (respectively, commutative) binary operation on G and $\circ$ restricted to some subset H of G is a binary operation on H, then $\circ$ is automatically associative (respectively, commutative) on H as well [**4**].

Now we furnish the definition of a group.

Definition 2.6 (Group). A *group* is an ordered pair $(G, \circ)$ where G is a set and $\circ$ is a binary operation on G satisfying the following axioms [**4**]:

- (Associativity) For all $a, b, c \in G$ we have $(a \circ b) \circ c = a \circ (b \circ c)$.
- (Identity) There exists an element e in G, called an *identity* of G, such that for all $a \in G$ we have $a \circ e = e \circ a = a$.
- (Inverses) For each $a \in G$ there is an element a^{-1} of G, called an *inverse* of a, such that $a \circ a^{-1} = a^{-1} \circ a = e$.

Note, we say G is a *finite group* if G is a finite set [**4**].

Remark. In our application, we will only consider finite groups. Notationally speaking, henceforth we denote a group G dropping the binary operation; and write ab for $a \circ b$, where $a, b \in G$. Also, we use 1 for the identity e.

Recall Definition 1.12 of homeomorphism and consider the following example of a group.

Example 2.4. Let X be a topological space. Let $\mathcal{H}$ be the set of all homeomorphisms. We claim that $\mathcal{H}$ forms a group. Let us prove this assertion.

Proof. Let X and $\mathcal{H}$ be as above in our hypothesis. The map $I : X \to X$ defined by $I(x) = x$ for all $x \in X$, where $I^{-1} = I$, is a homeomorphism satisfying $f \circ I = I \circ f = f$ for all $f \in \mathcal{H}$. So, $I \in \mathcal{H}$. We call I the identity map. Now, let $f \in \mathcal{H}$. Then f is continuous and there exists f^{-1}, also continuous, such that $ff^{-1} = I \in \mathcal{H}$

implying $f^{-1} \in \mathcal{H}$. Thus, any element of $\mathcal{H}$ has an inverse. Since function composition is associative we have that
$f \circ (g \circ h) = (f \circ g) \circ h$ for $g, h \in \mathcal{H}$. Also, given $f, g \in \mathcal{H}$ we know from Calculus that the composition $f \circ g$ is continuous. Further, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ is the composition of continuous functions and therefore continuous, so $f \circ g \in \mathcal{H}$. Hence, $\mathcal{H}$ is a group. □

The dihedral groups are class of groups whose elements are symmetries of geometric objects. For every $n \in \mathbb{N}$ and $n \geq 3$ we call D_n the set of symmetries of a regular n-gon. A symmetry is any rigid motion of the n-gon, that is moving a copy of the n-gon in any manner in 3-space so it exactly covers the original n-gon [**4**]. (We discuss rigid motions and symmetry later.) The size, or order, of the set of rigid motions of a dihedral group is

$$|D_n| = 2n. \tag{2.1}$$

Therefore, D_n is called the *dihedral group of order* $2n$ [**4**]. We will call D_2 the plane symmetry group, of a nonsquare rectangle and D_1 the plane symmetry group of the letter "V" [**5**].

Remark. For our application, we observe symmetries of a hexagon D_6, square D_4, triangle D_3, rectangle D_2 and D_1.

The following example of dihedral group D_4 consists of symmetries of a square.

EXAMPLE 2.5 (Dihedral group D_4). First, assume we are in the plane $\mathbb{R}^2$ with the usual (x, y) coordinate system. Consider the square

$$s = \boxed{\begin{matrix} b & a \\ c & d \end{matrix}} \tag{2.2}$$

centered at the origin $(0, 0)$.

The group D_4 has rotations $\{R_0, R_{90}, R_{180}, R_{270}\}$ that are counterclockwise rotations of angles $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, respectively, about the origin. The element R_0 is the identity element of D_4. Additionally, D_4 has x-axis horizontal, y-axis vertical, $y = x$ diagonal, and $y = -x$ diagonal reflections $\{H, V, D_x, D_{-x}\}$. That is, $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D_x, D_{-x}\}$. View the group actions on the square:

$$R_0 : s \longrightarrow \boxed{\begin{matrix} b & a \\ c & d \end{matrix}} \tag{2.3}$$

$$R_{90} : s \longrightarrow \boxed{\begin{matrix} a & d \\ b & c \end{matrix}} \tag{2.4}$$

$$R_{180} : s \longrightarrow \boxed{\begin{matrix} d & c \\ a & b \end{matrix}} \tag{2.5}$$

$$R_{270} : s \longrightarrow \boxed{\begin{matrix} c & b \\ d & a \end{matrix}} \tag{2.6}$$

$$H : s \longrightarrow \boxed{\begin{matrix} c & d \\ b & a \end{matrix}} \tag{2.7}$$

$$V : s \longrightarrow \boxed{\begin{matrix} a & b \\ d & c \end{matrix}} \tag{2.8}$$

$$D_x : s \longrightarrow \boxed{\begin{matrix} d & a \\ c & b \end{matrix}} \tag{2.9}$$

$$D_{-x} : s \longrightarrow \boxed{\begin{matrix} b & c \\ a & d \end{matrix}} \tag{2.10}$$

DEFINITION 2.7 (Subgroup). Let G be a group. The subset H of G is a *subgroup* of G if H is nonempty and H is closed under products and inverses. That is, $x, y \in H$ implies $x^{-1} \in H$ and $xy \in H$. If H is a subgroup of G, we write $H \leq G$.

Examples of subgroups of the dihedral groups are called the cyclic rotation groups.

DEFINITION 2.8 (Cyclic rotation group). A symmetry group consisting of the rotational symmetries of angles $\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{(n-1)2\pi}{n}$, and no other symmetries is called a *cyclic rotation group of order* n and is denoted by $C_n = \{R_0, R_{\frac{2\pi}{n}}, \ldots, R_{\frac{(n-1)2\pi}{n}}\}$ [**5**].

REMARK. We will refer to the cyclic rotation groups of order n as the cyclic groups denoted by C_n. The rotations of the dihedral groups comprise the cyclic groups.

EXAMPLE 2.6 (Cyclic group C_4). Recall our example of D_4 (Example 2.5) and reconsider the square in $\mathbb{R}^2$

$$s = \boxed{\begin{array}{cc} b & a \\ c & d \end{array}}$$

centered at the origin $(0,0)$. The rotations of this square $\{R_0, R_{90}, R_{180}, R_{270}\}$ forms the cyclic group C_4.

EXAMPLE 2.7. The set $\mathcal{I}$ of isometries from a topological space X to X is a subgroup of the group $\mathcal{H}$ of homeomorphisms.

We recall from abstract algebra the Subgroup Criterion.

PROPOSITION 2.1 (The Subgroup Criterion). *A subset H of a group G is a subgroup if and only if* [**4**]

- *$H \neq \emptyset$, and*
- *for all $x, y \in H$, $xy^{-1} \in H$.*

Furthermore, if H is finite, then it suffices to check that H is nonempty and closed under multiplication [**4**].

PROOF. Let H be a subgroup of G. By Definition 2.7, H contains the inverse of each of its elements, is closed under products, and is nonempty. Therefore, $xy^{-1} \in H$ for all $x, y \in H$. Now, assume $H \neq \emptyset$ and $xy^{-1} \in H$ for all $x, y \in H$. Since H is nonempty, let $x \in H$. If $y = x$, then $xy^{-1} = xx^{-1} = 1 \in H$. Since $1, x \in H$, we have $1x^{-1} = x^{-1} \in H$ implying H is closed under inverses. Further, for any $x, y \in H$ we have $x, y^{-1} \in H$. Thus, $x(y^{-1})^{-1} = xy \in H$ implying H is closed under products. So, $H \leq G$. This completes the proof due to [**4**]. $\square$

Lemma 2.1. *The cyclic rotation group C_4 is a subgroup of the dihedral group D_4.*

Proof. We prove Lemma 2.1 by checking the Subgroup Criterion (Proposition 2.1). Notice that $C_4 \subset D_4$ and $C_4 \neq \emptyset$ since we have defined that $C_4 = \{R_0, R_{90}, R_{180}, R_{270}\}$. Further notice we have inverses $R_0^{-1} = R_0$, $R_{90}^{-1} = R_{270}$, $R_{180}^{-1} = R_{180}$, and $R_{270}^{-1} = R_{90}$. So, C_4 is closed under inverses and for any $R_i, R_j \in C_4$ we have $R_i R_j^{-1} \in C_4$ since any rotation gives another rotation. Thus, $C_4 \leq D_4$. □

2.1.2. Group action.

Orbifold classification involves group action on metric spaces. We now define what it means for a group to act on a set.

Definition 2.9 (Group action). Suppose G is a group and A is a set. A *group action* of G on A is a map from $G \times A$ to A, written as $g \cdot a$ for all $g \in G$ and $a \in A$, satisfying the following properties [**4**]:

- $g_1 \cdot (g_2 \cdot a) = (g_1 \cdot g_2) \cdot a$, for all $g_1, g_2 \in G$ and $a \in A$.
- $1 \cdot a = a$, for all $a \in A$.

Remark. If the set A is a topological space we require the group action $g \cdot a$ to be continuous.

Let us formalize group action on a topological space.

Definition 2.10 (Group action on a topological space). Suppose G is a group and X is a topological space. Let $g \in G$ and $x \in X$. A *group action on a topological space* of G on X is a continuous map $\alpha : G \times X \to X$, where we denote $\alpha(g \times x)$ by $g \cdot x$ satisfying [**6**]:

- $e \cdot x = x$ for all $x \in X$.
- $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $x \in X$ and $g_1, g_2 \in G$.

Remark. The map α defines an equivalence relation on X, whereby $x, y \in X$ are equivalent if and only if $y = g \cdot x$ for some $g \in G$. The equivalence classes of this equivalence relation are called the "orbits" of G [**9**].

2.2. Rigid motion and symmetry

Our concentration is group symmetry described by rigid motions of objects in the plane. The concepts and definitions presented in this section are from [**1**].

2.2.1. Rigid motion.

Definition 2.11 (Rigid motion). A map $m : \mathbb{R}^2 \to \mathbb{R}^2$ from the Euclidean plane to itself is a *rigid motion* if it distance-preserving. In other words, a rigid motion is an isometry. Recall, that is, for any two points $x, y \in \mathbb{R}^2$ the distance from x to y is equal to the distance from $m(x)$ to $m(y)$ [**1**].

Example 2.8. We will show that the rigid motions are translations, rotations, reflections, and glide reflections. Recall, the set $\mathcal{I}$ of isometries forms a subgroup of the group $\mathcal{H}$ of homeomorphisms. So also, these rigid motions form a subgroup M of $\mathcal{H}$, having the law of composition of functions [**1**].

Remark. In $\mathbb{R}^2$, we say motions are *orientation-preserving* if the motion does not flip the plane over, and *orientation-reversing* if the motion flips the plane over [**1**].

Definition 2.12 (The 4 rigid motions). Specifically, we classify motions as follows [**1**]:

- Orientation-preserving motions:
  - *Translation*: parallel motion of the plane by a vector $a : p \rightsquigarrow p + a$. (Note: the notation $\rightsquigarrow$ indicates a translation).
  - *Rotation*: rotates the plane by an angle $\theta \neq 0$ about some point.
- Orientation-reversing motions:
  - *Reflection* about a line ℓ.
  - *Glide reflection*: obtained by reflecting about a line ℓ, and then translating by a nonzero vector a parallel to ℓ.

Each of the four rigid motions is a homeomorphism that preserves distances. That is, each is an isometry. We prove that any rotation in the plane is an isometry.

Proof. Consider the plane as the space $\mathbb{R}^2$ of column vectors with coordinate system (x, y) having the Euclidean metric. Let ρ_θ be a rotation by an angle θ about the origin $(0, 0)$ defined by

$$\rho_\theta(x) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.11}$$

First, we show that ρ_θ is continuous. For all $\epsilon > 0$ and $\forall x = (x_1, x_2) \in \mathbb{R}^2$ there exists $\delta > 0$ such that for $y = (y_1, y_2) \in \mathbb{R}^2$ we have

$$d(x, y) = |x - y| = \big((x_1 - y_1)^2 + (x_2 - y_2)^2\big)^{\frac{1}{2}} < \delta \tag{2.12}$$

Choose $\delta = \epsilon$ to obtain

$$\begin{aligned}
d(\rho_\theta(x), \rho_\theta(y)) &= |\rho_\theta(x) - \rho_\theta(y)| \\
&= \left| \begin{pmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_1 \sin\theta + x_2 \cos\theta \end{pmatrix} - \begin{pmatrix} y_1 \cos\theta - y_2 \sin\theta \\ y_1 \sin\theta + y_2 \cos\theta \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} (x_1 - y_1)\cos\theta - (x_2 - y_2)\sin\theta \\ (x_1 - y_1)\sin\theta + (x_2 - y_2)\cos\theta \end{pmatrix} \right| \\
&= \big(((x_1 - y_1)\cos\theta - (x_2 - y_2)\sin\theta)^2 + ((x_1 - y_1)\sin\theta + (x_2 - y_2)\cos\theta)^2\big)^{\frac{1}{2}} \\
&= \big((x_1 - y_1)^2(\sin^2\theta + \cos^2\theta) + (x_2 - y_2)^2(\sin^2\theta + \cos^2\theta)\big)^{\frac{1}{2}} \\
&= \big((x_1 - y_1)^2 + (x_2 - y_2)^2\big)^{\frac{1}{2}} \\
&= \left| \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \right| \\
&= |x - y| = d(x, y) < \epsilon
\end{aligned} \tag{2.13}$$

observing that

$$\begin{aligned}
|x - y|^2 = (x_1 - y_1)^2 \sin^2\theta - 2(x_1 - y_1)(x_2 - y_2)\sin\theta\cos\theta + (x_2 - y_2)^2\cos^2\theta) \\
+ (x_1 - y_1)^2\cos^2\theta + 2(x_1 - y_1)(x_2 - y_2)\sin\theta\cos\theta + (x_2 - y_2)^2\cos^2\theta.
\end{aligned}$$

Thus, a rotation is continuous. Now we show that ρ_θ is a homeomorphism by defining the inverse of ρ_θ as

$$\rho_\theta^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2.14}$$

Notice that

$$\begin{aligned} \rho_\theta^{-1}\rho_\theta &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \quad \text{the identity matrix,} \end{aligned} \tag{2.15}$$

and similarly $\rho_\theta\rho_\theta^{-1} = I$. Therefore, ρ_θ is a homeomorphism. Furthermore, since $d(\rho_\theta(x), \rho_\theta(y)) = d(x, y)$ we have that a rotation is an isometry. □

We have the following examples of rigid motions.

Example 2.9 (Translation). A translation is a function that carries all points the same distance in the same direction [**5**]. Let $(1, 1)$ and $(0, 0)$ be points in the plane. Let t_a be a translation defined by the vector

$$a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.16}$$

translating our points $(1, 1)$ to $(2, 1)$ and $(0, 0)$ to $(1, 0)$. That is, $t_a(1, 1) = (2, 1)$ and $t_a(0, 0) = (1, 0)$. In general, a translation t_a by the vector

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{2.17}$$

is defined by

$$t_a(x) = x + a = \begin{pmatrix} x_1 + a_1 \\ x_2 + a_2 \end{pmatrix} \tag{2.18}$$

for all $x \in \mathbb{R}^2$.

EXAMPLE 2.10 (Reflection). A reflection across a line ℓ is a transformation that leaves every point of ℓ fixed and takes every point p, not lying on ℓ, to the point $p\prime$ so that ℓ is the perpendicular bisector of the line segment from p to $p\prime$. The line ℓ is called the axis of reflection. In the plane, we have the reflection $(x, y) \to (x, -y)$ is a reflection across the x-axis. The reflection $(x, y) \to (y, x)$ is a reflection across the line $y = x$ [**5**].

EXAMPLE 2.11 (Rotation). A rotation in the plane fixes a point [**5**]. Let s be the square in the plane centered at the origin $(0, 0)$ having sides of length 2. Rotate the square by the angle $\theta = \frac{\pi}{2}$ about the origin. This rotation fixes the origin, rotating all other points 180 degrees counterclockwise. For example, this rotation takes the point $(1, 1)$, the upper right hand vertex of the square, to the point $(-1, -1)$, the lower left hand vertex of the square.

EXAMPLE 2.12 (Glide reflection). A glide reflection is the product of a translation and a reflection across the line containing the translation vector. We can consider successive footprints in wet sand are related by a glide reflection [**5**]. For example, let s_1 and s_2 be the squares in the plane having sides of length 1 and vertices:

$$\begin{aligned} &s_1 \text{ vertices } \{(0,1),(-1,1),(-1,0),(0,0)\}. \\ &s_2 \text{ vertices } \{(1,4),(0,4),(0,3),(1,3)\}. \end{aligned} \tag{2.19}$$

The squares are related by the glide reflection defined by the translation t_a given by the vector

$$a = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{2.20}$$

followed by the reflection r across the y-axis such that, for the point $(-1,0)$, we have

$$rt_a(-1,0) = (0,3) \tag{2.21}$$

Notice that the vector a is parallel to the y-axis of reflection r.

The proof of the following theorem ensures there are only four rigid motions.

THEOREM 2.1. *The list of 4 rigid motions is complete. That is, every rigid motion is a translation, a rotation, a reflection, a glide reflection, or the identity* [**1**].

PROOF. We identify the plane as the space $\mathbb{R}^2$ of column vectors, by choosing a coordinate system, say (x, y). Recall, this space has the Euclidean metric. Choose generators of the translations, the rotations about the origin (0, 0), and the reflection about the x-axis as follows:

- *Translation* t_a by a vector a defined by

$$t_a(x) = x + a = \begin{pmatrix} x_1 + a_1 \\ x_2 + a_2 \end{pmatrix}. \tag{2.22}$$

- *Rotation* ρ_θ by an angle θ about the origin defined by

$$\rho_\theta(x) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.23}$$

- *Reflection* r about the x_1-axis defined by

$$r(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} \tag{2.24}$$

  (We refer to r as the *standard reflection.*)

Geometrically, we can see that every rigid motion m can be obtained by composing the functions of translation, rotation, and reflection defined above. For instance, we can view any rigid motion in the plane by applying the functions of rotation and/or reflection (above) and then translating by a vector a. That is, we have either

$$m = t_a\rho_\theta \quad \text{or else} \quad m = t_a\rho_\theta r, \tag{2.25}$$

for some vector a and angle θ, possibly zero. Now, let m be an orientation-preserving rigid motion that is not a translation. We will show that m is a rotation about some point. Notice that an orientation-preserving rigid motion that fixes a point p in the plane $\mathbb{R}^2$ must be a rotation about p. Therefore, we need show that every orientation-preserving rigid motion m that is not a translation fixes some point. We find the fixed point by solving the equation $x = t_a\rho_\theta(x)$ algebraically for x. By definition of a translation (2.22), we have that $t_a(\rho_\theta(x)) = \rho_\theta(x) + a$. Thus, we need to solve the following equation.

$$x - \rho_\theta(x) = a, \tag{2.26}$$

or

$$\begin{pmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{2.27}$$

Observe that $\det(1-\rho_\theta) = 2 - 2\cos\theta$. If $\theta \neq 0$, then the determinant is nonzero. Hence, there is a unique solution for x. Now, to proceed we need the following corollary.

Corollary 2.1. *The motion $m = t_a\rho_\theta$ is the rotation through the angle θ about its fixed point.*

Proof. We just observed that the fixed point of motion m satisfies $p = \rho_\theta(p) + a$. So, for any x we have

$$\begin{aligned} m(p+x) &= t_a\rho_\theta(p+x) \\ &= \rho_\theta(p+x) + a \\ &= \rho_\theta(p) + \rho_\theta(x) + a \\ &= p + \rho_\theta(x). \end{aligned} \tag{2.28}$$

Therefore, $m : p + x \to p + \rho_\theta(x)$. Whence, m is the rotation about p through the angle θ. $\square$

Continuing, we need to show that any orientation-reversing motion $m = t_a\rho_\theta r$ is a reflection or a glide reflection. Here, we will consider a reflection to be a glide reflection having glide vector zero. We will show this by finding a line ℓ that is sent to itself by the motion m, and so that the motion of m on line ℓ is a translation. Note that an orientation-reversing motion acting in this manner on a line is a glide reflection. We have the following two steps. First, we notice that the motion $\rho_\theta r = r\prime$ is a reflection about a line that intersects the x-axis at an angle of $\frac{1}{2}\theta$ at the origin. Thus, $m = t_a r\prime$. If we rotate the coordinates so that the x-axis becomes the line of reflection of $r\prime$, then $r\prime = r$, the standard reflection (2.24). The translation t_a remains a translation, but the coordinates of the vector a change. So, we have a new coordinate system, where the motion is $m = t_a r$ acting as

$$m \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + a_1 \\ -x_2 + a_2 \end{pmatrix}. \tag{2.29}$$

This motion maps the line $x_2 = \frac{1}{2}a_2$ to itself by the translation

$$(x_1, \frac{1}{2}a_2)\prime \rightsquigarrow (x_1 + a_1, \frac{1}{2}a_2)\prime. \tag{2.30}$$

Hence, m is a glide reflection along the line $x_2 = \frac{1}{2}a_2$. This completes the proof due to [**1**] that the list of 4 rigid motions is complete. $\square$

2.2.2. Symmetry.

In our application we inspect repeated patterns on the plane, tilings. Each tiling has a figure repeated making a tiling motif. We classify tilings by observing the four rigid motions determining symmetries and symmetry groups that we define now.

Definition 2.13 (Symmetry). If a rigid motion m carries a subset A of the Euclidean plane $\mathbb{R}^2$ to itself, then we call it a *symmetry*. That is, for $A \subset \mathbb{R}^2$, if $m : A \to A$, then m is a symmetry [**1**].

Example 2.13. Let s be our square in the plane $\mathbb{R}^2$. The rigid motion $m_1 = R_{90}$, that is a rotation we described above as an element of D_4, is a symmetry. The rigid motion $m_2 = H$, that is the horizontal reflection in D_4 is a symmetry.

Definition 2.14 (Group of symmetries). Let A be a figure in the Euclidean plane $\mathbb{R}^2$. The set of all symmetries of A is called the *group of symmetries* of a figure A [**1**].

Lemma 2.2. *The set G of symmetries of a figure A in $\mathbb{R}^2$ forms a subgroup of the group M of rigid motions.*

Proof. Let A be a figure in $\mathbb{R}^2$. Let G be the set of symmetries of A. By Definition 2.13, $G \subset M$. The motion 1 that leaves A fixed is in G, so $G \neq \emptyset$. Let $m_1, m_2 \in G$. Since $m_2 : A \to A$ by Definition 2.13, we have that $m_1 m_2(A) = m_1(A)$. Also, $m_1(A) = A$, so $m_1 m_2(A) = A$. Hence, $G \leq M$. □

Example 2.14. The dihedral group D_4 and the cyclic group C_4 (Examples 2.5, 2.6) of symmetries of the figure $A = s$, our square in $\mathbb{R}^2$, are groups of symmetries.

Later we look at examples of groups of symmetries on the Euclidean plane $\mathbb{R}^2$.

2.3. Discrete groups of motions and surface groups

The groups of symmetries on the plane that we examine in our application are examples of discrete groups of motions.

2.3.1. Discrete groups of motions.

The concepts and definitions presented in this section are from [**1**].

Definition 2.15 (Discrete group of motions). Let M be the group of motions. We call $G \leq M$ a *discrete group of motions* if there exists $\epsilon > 0$ such that [**1**]

- a translation $t_a \in G$ by a nonzero vector a has length $|a| \geq \epsilon$;
- a rotation $\rho \in G$ about a point by a nonzero angle θ has size $|\theta| \geq \epsilon$.

In the introduction we talked of lattices in terms of Crystallography. Now, we define lattice mathematically.

Definition 2.16 (Plane lattice). Let L be a subgroup of $\mathbb{R}^2$. If L is generated by two linearly independent vectors a, as

$$L = \{ma + nb \mid m, n \in \mathbb{Z}\}, \tag{2.31}$$

then L is called a *plane lattice*. The generating set (a, b) is called a *lattice basis* [**1**].

Any tiling motif is obtained by translating a figure to be repeated by two linearly independent translations and is therefore a lattice. Our application encompasses the following type of lattice. Let

$G \leq M$ be a discrete group of motions. Let L_G be the "translation group" of G, defined as the set of vectors $a \in \mathbb{R}^2$ such that $t_a \in G$ [**1**].

Definition 2.17 (Two-dimensional crystallographic group). If L_G is a lattice, then we call G a *two dimensional crystallographic group*, or *lattice group* [**1**].

Two dimensional crystallographic groups are groups of symmetries of tiling motifs (wallpaper patterns) and of two dimensional crystals. There are 17 crystallographic groups. The following Proposition 2.2, proof omitted, limits the possible rigid motions in a plane lattice so one can classify lattice groups into only 17 types (for proof see [**1**]).

Proposition 2.2 (Crystallographic Restriction). *Let H be a finite subgroup of the group G of symmetries of a lattice L. Then*

- *Every rotation in H has order $1, 2, 3, 4,$ or 6.*
- *H is one of the groups C_n, D_n, where $n = 1, 2, 3, 4,$ or 6.*

2.3.2. Surface groups.

We now describe a surface and surface groups. Surfaces are topological spaces with special properties. Orbifold classification is applied to three surfaces, each having specific surface groups.

Definition 2.18 (Surface). A *surface* is a topological space X that satisfies the following properties:

- For each pair x_1, x_2 of distinct points of X, there exist open sets U_1, and U_2 containing x_1 and x_2, respectively, that are disjoint (i.e. Hausdorff space).

- Each point x of X has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^2$ (i.e. 2-manifold).
- Every open covering, the union of a collection $\mathcal{A}$ of open subsets of X that is equal to X, contains a finite subcollection that also covers X (i.e. compact).

That is, a *surface* is a compact 2-manifold.

EXAMPLE 2.15. The metric spaces the sphere, the hyperbolic plane, and the Euclidean plane are surfaces. We will apply orbifold classification on the Euclidean plane surface.

Our group action on surfaces consists of surface groups.

DEFINITION 2.19 (Surface group). A *surface group* is a discrete group of isometries of one of the three surfaces: the sphere, the hyperbolic plane, and the Euclidean plane.

EXAMPLE 2.16. The surface groups of the sphere S^2 consist of the *orthogonal groups*, where the symmetry group G, of a finite physical object, preserves some sphere centered at the object's center of gravity [**3**].

EXAMPLE 2.17. The surface groups of the Euclidean plane $\mathbb{R}^2$ consist of the *17 plane crystallographic groups*. These groups are our application focus.

REMARK. The surface groups of the hyperbolic plane H^2 consist of the *non-Euclidean crystallographic groups* [**3**].

2.4. Quotient space and orbifold quotient space

Acting on a topological space X by a group G yields a quotient topological space that is an orbifold.

2.4.1. Quotient space.

DEFINITION 2.20 (Orbit). Let G be a finite group acting on a topological space X. An *orbit* O_p of a point p under a group G is the set of all images of p under the elements $g \in G$ [**3**]. That is,

$$O_p = \{ \ g \cdot p \mid g \in G\}. \tag{2.32}$$

EXAMPLE 2.18. Recall Example 2.5 of D_4 acting on the square s. Let us put a restriction on the size of s, declaring s has sides of length 2. The point $(1, 1) \in s$. The orbit $O_{(1,1)}$ is the set $\{(-1, 1), (-1, -1), (1, -1), (1, 1)\}$ of all images of $(1, 1)$ under the elements of D_4 acting on s. Notice that the vertices of s are in the same orbit having $(1, 1)$ as a representative.

An orbit is an equivalence class and a point in the orbit is a representative. The quotient map sends all elements to their respective orbit.

DEFINITION 2.21 (Quotient map). Suppose $P : X \to Y$ is a surjective map between topological spaces. We say P is a *quotient map* if a subset U of Y is open in Y if and only if $P^{-1}(U)$ is open in X [**6**].

EXAMPLE 2.19. Let the square $s \in \mathbb{R}^2$, having sides of length 2, be centered at the the point $(4,4) \in \mathbb{R}^2$. Translate s by the vectors

$$a_1 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{2.33}$$

so that a_1 and a_2 generate a plane lattice
$L_s = \{ma_1 + na_2 \mid m, n \in \mathbb{Z}\}$ such as

$$\begin{array}{ccc} \boxed{\begin{array}{cc} f & e \\ g & h \end{array}} & & \boxed{\begin{array}{cc} b & a \\ c & d \end{array}} \\ & \bullet & \\ \boxed{\begin{array}{cc} q & p \\ r & s \end{array}} & & \boxed{\begin{array}{cc} x & w \\ y & z \end{array}} \end{array} \tag{2.34}$$

where the origin in $(0,0) \in \mathbb{R}^2$ is labeled by $\bullet$.

Let the dihedral group D_4 act on our lattice L_s. The square s, having vertices $\{a, b, c, d\}$, maps to the square s''', having vertices $\{w, x, y, z\}$, by the rotation $R_{\frac{3\pi}{2}}$ of angle $\frac{3\pi}{2}$ about the origin. Also, s maps to s''' by the standard reflection (2.24) through the origin. Similarly, s maps to the squares s' and s'' having vertices $\{e, f, g, h\}$ and $\{p, q, r, s\}$, respectively. So, s is in same the equivalence class as s', s'', and s''', under elements of D_4. Also, if we narrow our focus of L_s to only the picture (2.34), then $O_s = \{s, s', s'', s'''\}$ is the orbit of our squares. Further, if we choose s to be our representative of O_s then this is the quotient map $\{s, s', s'', s'''\} \to s$, obtaining the orbifold s.

Definition 2.22 (Quotient topology). Let X be a topological space and A a set. If $P : X \to A$ is a surjective map, then there exists exactly one topology on A relative to which P is a quotient map. It is called the *quotient topology* induced by P [**6**].

Remark. The quotient topology on the set O_{p_i} of orbits of a topological space X is realized by proclaiming that any set $S \subset O_{p_i}$ is open if and only if the union of its inverse image in X is open [**9**].

Example 2.20. Recall Example 2.19 of a quotient map. The set A corresponds to our square s (i.e. the orbifold s). So, s inherits the Euclidean metric topology of the Euclidean plane.

The space of orbits with the quotient topology is the "quotient space" of a topological space by the group action [**9**].

Definition 2.23 (Quotient space). Let X be the topological space above, and let A be a partition of X into disjoint subsets whose union is X. Let $P : X \to A$ be the surjective map that carries each point of X to the element of A containing it. In the quotient topology induced by P, the space A is called a *quotient space* of X. A is obtained by mapping (identifying) all the elements in each partition (conjugacy) class to a single point [**6**].

Example 2.21. An example of a quotient space is an orbifold determined by a symmetry group action on the plane $\mathbb{R}^2$.

2.4.2. The orbifold quotient space.

Our application is the the orbifold quotient space, formally defined as follows.

Definition 2.24. Consider a topological space X, a set A that is a partition of X into disjoint subsets having union X, and orbits O_p as in the quotient space. The *orbifold* is defined as

$$P : X \to A \quad \text{such that} \quad P(p) = O_p \quad \text{where} \quad A = \{\, O_p \mid p \in X\}. \tag{2.35}$$

Define $x \backsim g \cdot x$ for all x and g. The resulting quotient space is denoted X/G and called the *orbit space* [**6**], or *orbifold quotient space* [**3**].

A subspace metric space inherits the metric from the overall space. An orbifold is a type of subspace determined by a quotient map with group elements that are isometries. Therefore, an orbifold inherits a metric from the overall surface, implying angles are defined on the orbifold [**3**]. Next, we view examples of orbifolds in $\mathbb{R}^2$.

Before proceeding with examples, we consider the geometric techniques called cutting and pasting. Cut and paste techniques show how to obtain a topological space X by pasting together edges of one or more polygonal regions [**6**].

Definition 2.25 (Polygonal region). Let c be a point in $\mathbb{R}^2$. Given $a > 0$, consider the circle of radius a in $\mathbb{R}^2$ centered at c. Let $\theta_0 < \theta_1 < \cdots < \theta_n$ be a sequence of real numbers, where $n \geq 3$ and $\theta_n = \theta_0 + 2\pi$. Consider the points $p_i = c + a(\cos\theta_i, \sin\theta_i)$, that lie on the circle. These points are numbered in counterclockwise order around the circle, and $p_n = p_0$. The line through p_{i-1} and p_i splits the plane into two closed half-planes. Let H_i be the half-plane that contains the points p_k. Then the space

$$P = H_1 \cap \cdots \cap H_n \tag{2.36}$$

is called the *polygonal region* determined by the points p_i. The points p_i are called the *vertices* of P. The line segment joining p_{i-1} and p_i is called an *edge* of P [**6**].

Definition 2.26 (Labelling). Let P be a polygonal region in the plane $\mathbb{R}^2$. A *labelling* of the edges of P is a map from the set of edges of P to a set S called the set of *labels.* Given an orientation of each edge of P, and given a labelling of the edges of P, we define an equivalence relation on the points of P as follows [**6**]:

Each point in the interior of the polygonal region P is equivalent only to itself. Given any two edges of P that have the same label, let h be the positive linear map of one onto the other. Define each point x of the first edge to be equivalent to the point $h(x)$ of the second edge. This relation generates an equivalence relation on P. The quotient space X obtained from this equivalence relation is said to have been obtained by *pasting the edges of P together* according to the given orientations and labelling [**6**].

We have examples of cut and pasting and labelling of polygonal regions that describe orbifolds observed in orbifold classification on the plane. In our application we see the following examples, together with a triangle and a square/rectangle, describe orbifolds on the plane acted on by the 17 plane groups.

Example 2.22 (Annulus). We obtain an annulus by gluing the ends of a rectangular region together, keeping, say, the upper boundary as the inner circle and the lower boundary as the outer circle bounding the inside of the annulus. The polygonal region (2.37) is homeomorphic to an annulus.

$$\text{Annulus} \approx \boxed{\begin{array}{ccc} \downarrow & \qquad\qquad & \downarrow \\ a & & a \\ \downarrow & & \downarrow \end{array}} \tag{2.37}$$

Example 2.23 (Cone). The polygonal region (2.38) becomes a cone when we apply the following procedure. We cut along the dotted line and wrap the polygonal region around the center point, or "cone point" A identifying the corner points together as $a = a' = a'' = a'''$.

$$\text{Cone} \approx \boxed{\begin{array}{ccc} a''' & \qquad & a \\ & & \\ & \mathrm{A} & \\ & | & \\ a'' & | & a' \end{array}} \tag{2.38}$$

This can be done similarly beginning with a triangular region rather than a square.

Example 2.24 (Sphere). The fundamental polygonal region (2.39) is homeomorphic to the sphere [**6**].

$$S^2 \approx \boxed{\begin{array}{ccc} \leftarrow & b & \leftarrow \\ \uparrow & & \downarrow \\ b & & a \\ \uparrow & & \downarrow \\ \rightarrow & a & \rightarrow \end{array}} \tag{2.39}$$

Remark. We will see in our examples of the 17 plane groups that by taking a polygonal region, a square or triangle, and, for expediency, gluing each of the corners together we obtain something that is homeomorphic to a sphere. In practice, it looks like a pillow with 4 or 3 corners respectively. Furthermore, cutting off the top of a sphere gives an open sphere. We will see that one of the 17 plane groups has an obifold homeomorphic to an open sphere (or open pillow). The open sphere can be viewed by identifying the cut off boundary of the sphere with a "mirror line" on a boundary of the polygonal region.

Example 2.25 (Projective plane). The *projective plane* P^2 is the quotient space obtained from the sphere S^2 by identifying each point x of S^2 with its antipodal point $-x$ [**6**]. It can be specified by the scheme *abab* [**6**]. Observe the polygonal region (2.40) that is homeomorphic to the projective plane [**6**].

$$P^2 \approx \boxed{\begin{array}{ccc} \leftarrow & a & \leftarrow \\ \downarrow & & \uparrow \\ b & & b \\ \downarrow & & \uparrow \\ \rightarrow & a & \rightarrow \end{array}} \tag{2.40}$$

Example 2.26 (Torus). We obtain a torus by gluing parallel ends together. First, we glue the top and bottom together, say, obtaining a cylinder. Then we glue the ends of the cylinder together to make a torus. Observe the polygonal region (2.41) that is homeomorphic to a torus [**6**].

$$T^1 \approx \begin{array}{|ccc|}\hline \rightarrow & a & \rightarrow \\ \uparrow & & \uparrow \\ b & & b \\ \uparrow & & \uparrow \\ \rightarrow & a & \rightarrow \\ \hline \end{array} \tag{2.41}$$

Example 2.27 (Möbius strip). The Möbius strip is similar to the annulus, however we orient one side in the opposite direction of the other. We twist the rectangular region and glue the ends together so that the arrows are glued facing the same direction. Observe the polygonal region (2.42) that is homeomorphic to a Möbius strip.

$$\text{Möbius strip} \approx \begin{array}{|ccc|}\hline \downarrow & & \uparrow \\ a & & a \\ \downarrow & & \uparrow \\ \hline \end{array} \tag{2.42}$$

Example 2.28 (Klein bottle). Observe the polygonal region (2.43) that is homeomorphic to the Klein bottle [**6**].

$$K \approx \begin{array}{|ccc|}\hline \rightarrow & a & \rightarrow \\ \downarrow & & \uparrow \\ b & & b \\ \downarrow & & \uparrow \\ \rightarrow & a & \rightarrow \\ \hline \end{array} \tag{2.43}$$

Part 2

Application

CHAPTER 3

Orbifold classification on $\mathbb{R}^2$

We now introduce the orbifold classification concept and provide a procedure for applying the classification system on the Euclidean plane. Since our application is on the plane, we will be examining plane lattices and classifying lattice groups. First, let us consider the properties of an orbifold.

3.1. Properties of an orbifold on $\mathbb{R}^2$

Recall the four rigid motions in the plane: reflections, rotations, glide reflections, and translations. The properties of an orbifold on $\mathbb{R}^2$ correspond to the four rigid motions that preserve symmetry in a lattice group. In Conway's language, the four rigid motions correspond to mirrors, gyrations, miracle crosses, and wonderful wanderings, respectively. We introduce this terminology here.

3.1.1. Mirrors.

We think of a mirror as a reflection. Consider the following on mirrors and observe Figures 3.1 and 3.2.

Definition 3.1 (Mirror). A *mirror*, or *mirror line*, of a lattice group is the line fixed by some reflection in the group. Mirror Lines correspond to *boundary curves* on the orbifold [**3**].

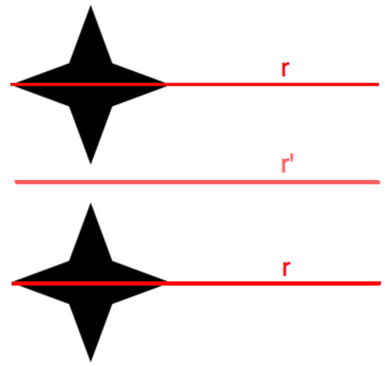

FIGURE 3.1. Two distinct mirrors r and r'.

DEFINITION 3.2 (Mirror point). A *mirror point* is a point that lies on a mirror of a lattice group. Mirror Points correspond to *boundary points* on the orbifold [**3**].

DEFINITION 3.3 (Ordinary mirror point). An *ordinary mirror point* is a mirror point that lies on just one mirror. An *ordinary boundary point* is the orbifold image of ordinary mirror points [**3**].

DEFINITION 3.4 (m-fold mirror point). An *m-fold mirror point* is a mirror point that lies on exactly m mirrors. A *type-m corner point* is the orbifold image of m-fold mirror points. At a type-m corner point, the boundary has angle $\frac{\pi}{m}$ [**3**].

FIGURE 3.2. The point c is a 2-fold mirror point.

3.1.2. Gyrations.

Gyrations are a type of rotation. Consider the following on gyrations and observe Figure 3.3.

DEFINITION 3.5 (Gyration). A *gyration* is a rotation in the lattice group whose center does not lie on any mirror [**3**].

DEFINITION 3.6 (m-fold gyration point). An *m-fold gyration point* is a point in the lattice that is the center of some gyration of order m, but not of any gyration of higher order. A *cone point of order m* is the orbifold image of an m-fold gyration point [**3**].

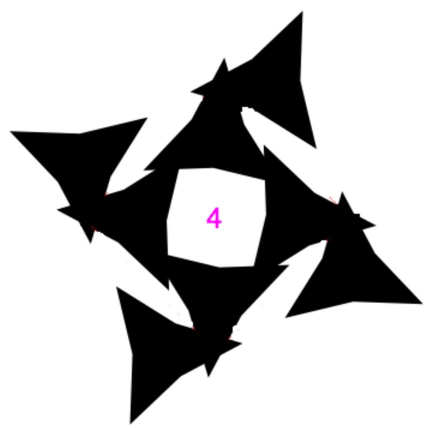

FIGURE 3.3. A 4-fold gyration point labeled by the number 4.

3.1.3. The miracle.

We think of a glide reflection as a miracle cross, or simply a miracle (see Figure 3.4). Observation of miracles are necessary in our classification procedure when some lattice symmetry is unexplained by the prior two properties, mirrors and gyrations.

DEFINITION 3.7 (Miracle). A *miracle cross*, or *miracle*, occurs when we find a path from one part of the lattice to a mirror image of itself that does not cross a mirror line. We may observe this property as glide reflection or mirrorless reflection [**3**].

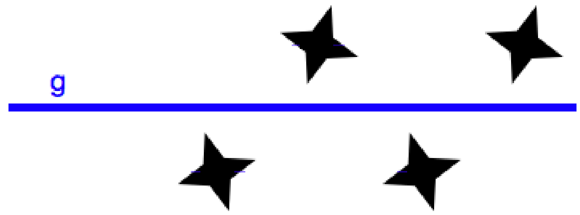

FIGURE 3.4. The glide reflection g indicates the existence of a miracle.

3.1.4. The wonder.

A wonder corresponds to translations (see Figure 3.5). We narrow the orbifold properties down to a wonder when all previous properties are not present in explaining symmetry in a lattice.

DEFINITION 3.8 (Wonder). A *wonderful wandering*, or *wonder*, occurs when we find two linearly independent nontrivial translations from one part of the lattice to two nonreflected images of itself. We may observe this property as a two distinct translations that cannot be explained by mirrors, gyrations, or miracles.

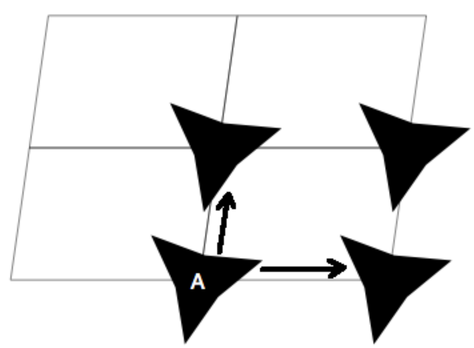

FIGURE 3.5. A wonder represented by two arrows that are two distinct translations of A to two nonreflected images of A.

3.2. The orbifold notation

Conway introduces a formal alphabet to create notation of the orbifold classification system. We present Conway's orbifold notation, specific to classification on $\mathbb{R}^2$, here.

We will use the following *digits* [**3**]:

$$A, B, \ldots, C \qquad \text{and} \qquad a, b, \ldots, c \tag{3.1}$$

together with the following *symbols* [**3**]:

$$*, \times, \circ. \tag{3.2}$$

The digits and symbols go together in the following way [**3**]:

$$AB \ldots C$$

$$*ab \ldots c$$

$$AB \ldots C * ab \ldots c$$

$$*\times$$

$$AB \ldots C\times$$

$$\times\times$$

$$\circ$$

We have the following explanation of the orbifold notation (specific to $\mathbb{R}^2$) [**3**]:

- A wonder is represented by a circle, $\circ$.
- A miracle is represented by each final cross, $\times$.

- The digits $A, B, \ldots, C$ not preceded by stars are the orders of distinct cone points.
- A star $*$ with all digits $a, b, \ldots, c$ that follow indicates the "type" of boundary curve; whereby the digits $a, b, \ldots, c$ written as $a_{m_1}, b_{m_2}, \ldots, c_{m_k}$ are the values m_i of distinct type-m corner points having boundary angles $\frac{\pi}{m_i}$.

The *classification theorem for 2-manifolds* asserts that a quotient space obtained from a polygonal region in the plane by pasting its edges together in pairs is homeomorphic either to the sphere, to the n-fold torus, or to the m-fold projective plane [**6**]. This assures that any connected surface can be obtained by adding handles and crosscaps to a sphere and then punching a hole in it for each boundary curve. Accounting for possibilities of local collapse, we can see that the orbifold notation covers all possibilities for the orbifold of a surface group. In fact, it trivializes their enumeration [**3**].

3.3. Classification procedure on $\mathbb{R}^2$

How do we find the orbifold notation and orbifold for a tiling (lattice) of the Euclidean plane? We have a procedure that identifies which of the 17 plane crystallographic groups (listed in Table 4.1) is the lattice group of the plane tiling. Given a tiling of the plane, our procedure is the following:

Step 1: Draw lines on all mirror lines (reflections) and circle their intersections (if any).

Step 2: Draw numbers on all gyration points (centers of rotation) corresponding to the point's highest order of symmetry preserving rotation (if any). (Erase or disregard numbers on mirrors.)

Step 3: Draw dotted lines on all miracles (glide reflections, if any).

Step 4: For each circle (of mirror line intersection), determine the number of intersecting lines m at that circled point a and write that number after the symbol $*$. Repeat for all distinct circled points, writing the numbers as $*a_{m_1}b_{m_2}\ldots c_{m_k}$, or simply $*ab\ldots c$ (if any). If no circles are drawn, then write $*$ for each distinct mirror line.

Step 5: For each number drawn (on gyration points), determine the numbers $AB\ldots C$ that are distinct and write those numbers before $*$ as $AB\ldots C*$. Put this with the symbol from Step 4 to have $AB\ldots C * ab\cdots c$.

Step 6: For dotted lines (miracles), write $\times$ following the other symbols or numbers. If no other symbols or numbers appear, write $\times\times$.

Step 7: If none of the above is determined, write $\circ$.

Step 8: Take the orbifold to be any fundamental polygonal region of the tiling that contains or has boundary containing each distinct orbifold property, corresponding to its orbifold symbol found by Steps 1-7, needed to re-tile the original tiling motif when applying its orbifold properties.

Remark. When choosing the orbifold image we must remember to choose a fundamental polygonal region of the tiling; that is, a region that contains all distinct orbifold properties corresponding to its orbifold symbol, necessary to recreate the original tiling. So we choose an image that can re-tile the plane motif by only applying its orbifold properties. For instance, any combination of mirrors, rotations, miracles, and wonder must achieve the same re-tiling, beginning from the orbifold image. Otherwise, it will not be a “fundamental” region.

3.4. The orbifold Euler characteristic

We recall from topology that the *Euler characteristic* of a surface is: $V - E + F$ (i.e. *vertices* - *edges* + *faces*). For example, the torus has $V = 1$, $E = 2$, and $F = 1$ implying its Euler characteristic is $1 - 2 + 1 = 0$. Given an orbifold O we can compute the Euler characteristic of an orbifold, call it $ch(O)$, by the following *defect formula* [**2**]:

$$ch(O) = 2 - \sum \text{defect(s)} \tag{3.3}$$

where the *defect(s)* of the orbifold are computed from the orbifold notation as follows [**2**] [**3**]:

- A digit A before the symbol $*$ (corresponds to cone points) counts as $\frac{(A-1)}{A}$.
- A digit a following the symbol $*$ (corresponds to corner points) counts as $\frac{(a-1)}{2a}$.
- The symbols $*$ and $\times$ both count as 1.
- The symbol $\circ$ counts as 2.

According to Conway, the defect formula corresponds with the Euler characteristic in the following way [**2**]:

$$ch(O) = 2 - \sum \text{defect(s)} = \frac{V - E + F}{|G|} \tag{3.4}$$

for a finite group G determining O by the group action under the quotient map. We assume (3.4) based on Conway's proof in [**2**]. So, now we define the Euler characteristic of an orbifold.

Definition 3.9 (Orbifold Euler characteristic). Let X be a surface and G a finite group. Suppose O is an orbifold of X, determined by the group action of G, under the quotient map. The *orbifold Euler characteristic* of O is obtained by dividing the Euler characteristic of X by the order of G [**3**].

Conway utilizes the orbifold Euler characteristic to enumerate the surface groups. That is, Conway calculates all possible orbifold Euler characteristics on the three surfaces of Example 2.15. The characteristics that are 0 arise from the Euclidean lattice groups. The characteristics that are positive arise from the spherical groups and the characteristics that are negative arise from the hyperbolic groups [**2**] [**3**]. As for the Euclidean groups, Conway's orbifold Euler characteristic technique for enumerating the groups assures there are 17 types [**2**] [**3**]; because there are 17 ways of obtaining $ch(0) = 0$. Of course, this agrees with the Crystallographic Restriction (Proposition 2.2).

The lattice groups on the Euclidean plane all have orbifold Euler characteristic 0 because the orbifolds of the lattice groups are all quotient spaces of a torus [**2**]. That is, we observe lattice groups in lattices that are generated by two linearly independent translations. So, if we view our lattice as a grid of translations generated by two linearly independent translations, then we can see two distinct edges of our lattice corresponding to our two distinct translations (see Figure 3.5). These two distinct edges glued to their corresponding parallel edges becomes a torus (see Figure 2.26). Notice the parallel edges have the same orientation as the two distinct edges since they are generated by the same translation; hence they are non-distinct.

NOTE. When we compute the orbifold Euler characteristic on our lattice groups we get characteristic 0. However, some of the orbifolds of the lattice groups are homeomorphic to surfaces that do not have ordinary Euler characteristic 0. For instance, there are orbifolds of lattice groups that are homeomorphic to a sphere, which has ordinary Euler characteristic 2. So, the orbifold Euler characteristic and the Euler characteristic do not necessarily agree in the quotient space.

CHAPTER 4

17 examples

Recall that the Crystallographic Restriction (Proposition 2.2) assures the lattice groups in $\mathbb{R}^2$ are classified into only 17 types: the 17 plane crystallographic groups. These 17 groups are listed in Table 4.1 below with their corresponding orbifold notation. We apply orbifold classification on the 17 plane groups, presenting examples of plane tilings (lattices) with enlarged figures illustrating their orbifold properties. We utilize the orbifold notation (from Section 3.2) and our classification procedure (from Section 3.3) to classify 17 examples, one for each of the 17 plane groups. We also calculate for each example the orbifold Euler characteristic using the defect formula (3.3).

NOTE. In the figures pertaining to the examples below, we use red lines to indicate mirror lines, blue lines to indicate miracles, and purple numbers to indicate gyration points of a particular order. Distinct mirrors and distinct miracles are indicated by lighter color red and blue lines contrasted by darker color red and blue lines, respectively. Our classification procedure is written to enable its use without color, if necessary. We proceed with our examples in the order of our list of the 17 plane crystallographic groups (Table 4.1).

English name	orbifold notation
discope group	$*2222$
triscope group	$*333$
tetrascope group	$*442$
hexascope group	$*632$
ditrope group	2222
tritrope group	333
tetratrope group	442
hexatrope group	632
digyro group	$22*$
dirhomb group	$2*22$
trigyro group	$3*3$
tetragyro group	$4*2$
monoscope group	$**$
monorhomb group	$*\times$
diglide group	$22\times$
monoglide group	$\times\times$
monotrope group	$\circ$

TABLE 4.1. The 17 plane crystallographic groups

4.1. Discope group example

Consider the tiling in Figure 4.1 having orbifold symbol $*2222$. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

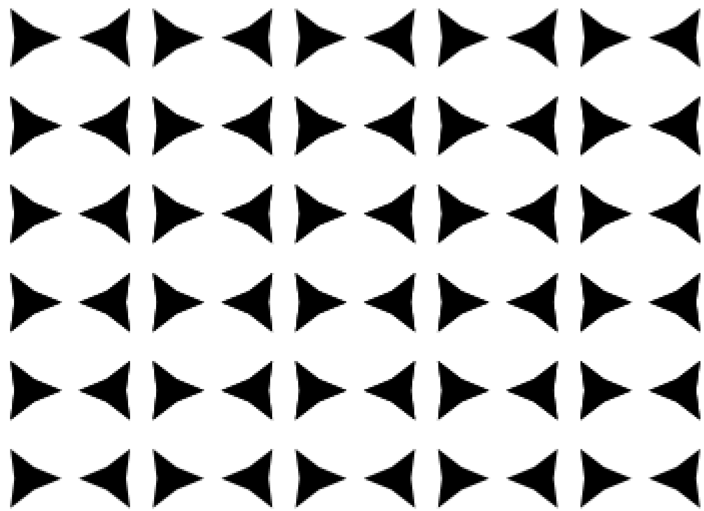

FIGURE 4.1. Plane tiling having orbifold symbol $*2222$.

Step 1: Draw lines on all mirror lines and consider their intersections. Consider Figure 4.2 of our tiling with mirror lines marked in red.

Step 2: Does not apply since no gyrations.

Step 3: Does not apply since no miracles.

Step 4: For each mirror line intersection, the number of intersecting lines is 2 for 4 distinct points so we write $*2222$.

Step 5: Does not apply since no gyrations.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 1 applies.

Step 8: Take the orbifold to be any rectangle bounded by distinct mirror lines. Observe Figures 4.3 and 4.4 of the orbifold.

We calculate the orbifold Euler Characteristic of the $*2222$ tiling using the defect formula, accounting for the following defects:

- The 2's following the $*$ each count as $\frac{2-1}{2(2)} = \frac{1}{4}$.
- The $*$ counts as 1.

Therefore, the $*2222$ orbifold Euler Characteristic is

$$2 - \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + 1\right) = 0 \tag{4.1}$$

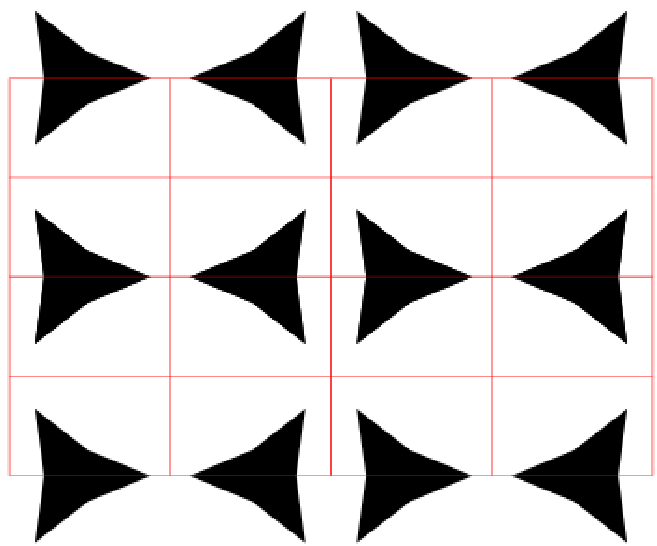

FIGURE 4.2. The $*2222$ plane tiling with mirror lines marked.

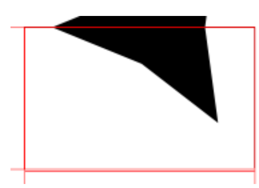

FIGURE 4.3. A $*2222$ rectangular orbifold region bounded by mirror lines.

Figure 4.4. The $*2222$ orbifold is topologically a rectangle.

4.2. Triscope group example

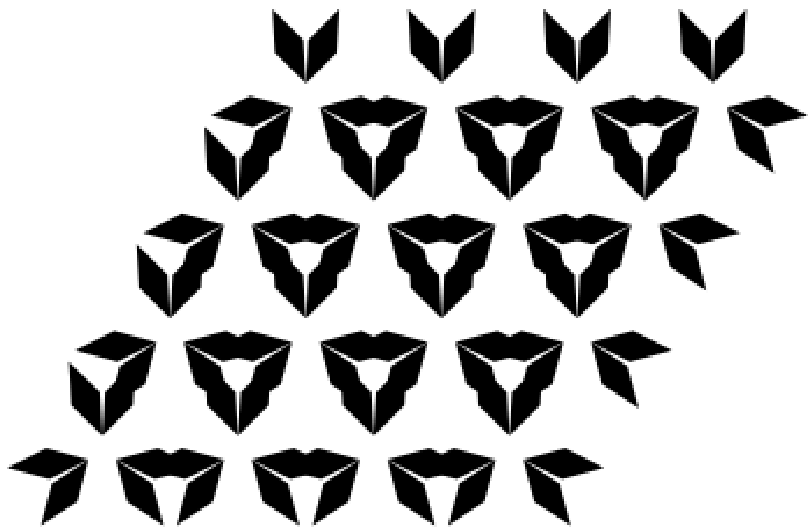

FIGURE 4.5. Plane tiling having orbifold symbol $*333$.

Consider the tiling in Figure 4.5 having orbifold symbol $*333$. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Draw lines on all mirror lines and consider their intersections. Consider Figure 4.6 of our tiling with mirror lines marked in red.

Step 2: Does not apply since no gyrations.

Step 3: Does not apply since no miracles.

Step 4: For each mirror line intersection, the number of intersecting lines is 3 for 3 distinct points so we write $*333$.

Step 5: Does not apply since no gyrations.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 1 applies.

Step 8: Take the orbifold to be any triangle bounded by distinct mirror lines. Observe Figures 4.7 and 4.8 of the orbifold.

We calculate the orbifold Euler Characteristic of the $*333$ tiling using the defect formula, accounting for the following defects:

- The 3's following the $*$ each count as $\frac{3-1}{2(3)} = \frac{1}{3}$.
- The $*$ counts as 1.

Therefore, the $*333$ orbifold Euler Characteristic is

$$2 - \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + 1\right) = 0 \tag{4.2}$$

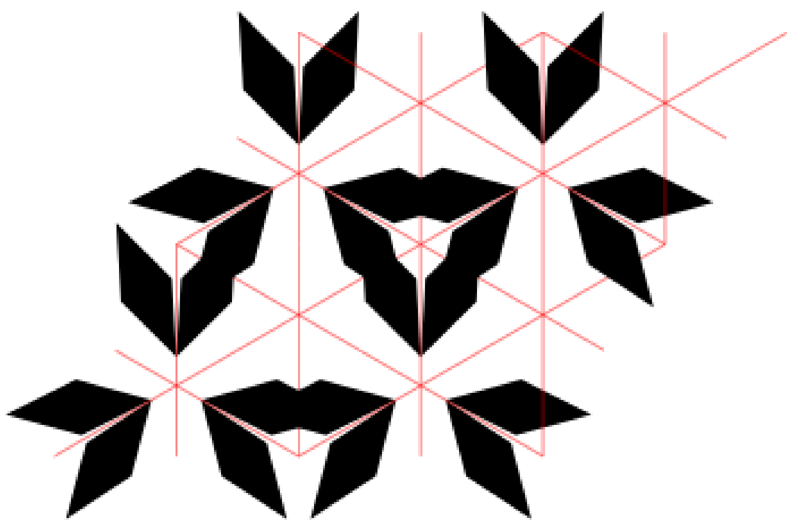

FIGURE 4.6. The $*333$ plane tiling with mirror lines marked.

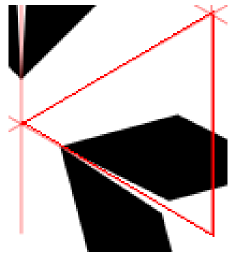

FIGURE 4.7. A $*333$ triangular orbifold region bounded by mirror lines.

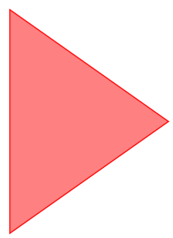

FIGURE 4.8. The $*333$ orbifold is topologically a triangle.

4.3. Tetrascope group example

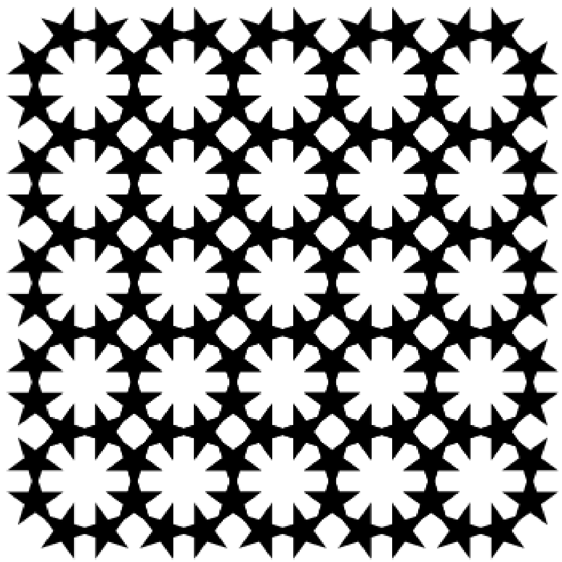

FIGURE 4.9. Plane tiling having orbifold symbol $*442$.

Consider the tiling in Figure 4.9 having orbifold symbol $*442$. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Draw lines on all mirror lines and consider their intersections. Consider Figure 4.10 of our tiling with mirror lines marked in red.

Step 2: Does not apply since no gyrations.

Step 3: Does not apply since no miracles.

Step 4: For each mirror line intersection, the number of intersecting lines is either: 4 for 2 distinct points; or 2 for 1 distinct point so we write $*442$.

Step 5: Does not apply since no gyrations.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 1 applies.

Step 8: Take the orbifold to be any triangle bounded by distinct mirror lines. Observe Figures 4.11 and 4.12 of the orbifold.

We calculate the orbifold Euler Characteristic of the $*442$ tiling using the defect formula, accounting for the following defects:

- The 4's following the $*$ each count as $\frac{4-1}{2(4)} = \frac{3}{8}$. The 2 following the $*$ counts as $\frac{2-1}{2(2)} = \frac{1}{4}$.
- The $*$ counts as 1.

Therefore, the $*442$ orbifold Euler Characteristic is

$$2 - \left(\frac{3}{8} + \frac{3}{8} + \frac{1}{4} + 1\right) = 0 \tag{4.3}$$

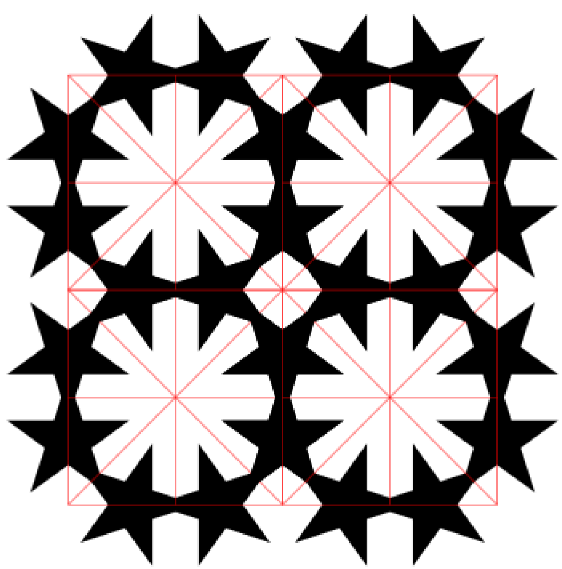

FIGURE 4.10. The $*442$ plane tiling with mirror lines marked.

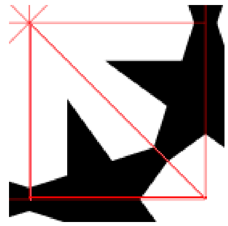

FIGURE 4.11. A $*442$ triangular orbifold region bounded by mirror lines.

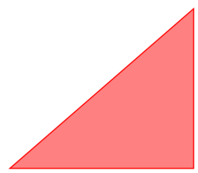

FIGURE 4.12. The $*442$ orbifold is topologically a triangle.

4.4. Hexascope group example

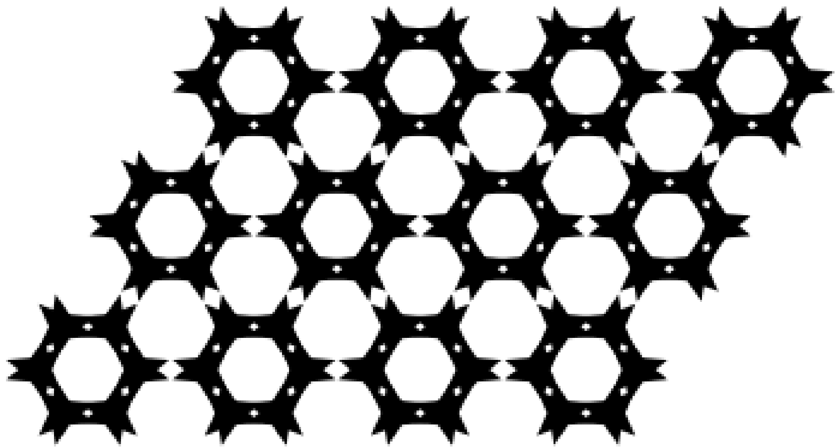

FIGURE 4.13. Plane tiling having orbifold symbol $*632$.

Consider the tiling in Figure 4.13 having orbifold symbol $*632$. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Draw lines on all mirror lines and consider their intersections. Consider Figure 4.14 of our tiling with mirror lines marked in red.

Step 2: Does not apply since no gyrations.

Step 3: Does not apply since no miracles.

Step 4: For each mirror line intersection, the number of intersecting lines is either: 6 for 1 distinct point; 3 for 1 distinct point; or 2 for 1 distinct point so we write $*632$.

Step 5: Does not apply since no gyrations.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 1 applies.

Step 8: Take the orbifold to be any triangle bounded by distinct mirror lines. Observe Figures 4.15 and 4.16 of the orbifold.

We calculate the orbifold Euler Characteristic of the $*632$ tiling using the defect formula, accounting for the following defects:

- The 6 following the $*$ counts as $\frac{6-1}{2(6)} = \frac{5}{12}$. The 3 following the $*$ counts as $\frac{3-1}{2(3)} = \frac{1}{3}$. The 2 following the $*$ counts as $\frac{2-1}{2(2)} = \frac{1}{4}$.
- The $*$ counts as 1.

Therefore, the $*632$ orbifold Euler Characteristic is

$$2 - \left(\frac{5}{12} + \frac{1}{3} + \frac{1}{4} + 1\right) = 0 \tag{4.4}$$

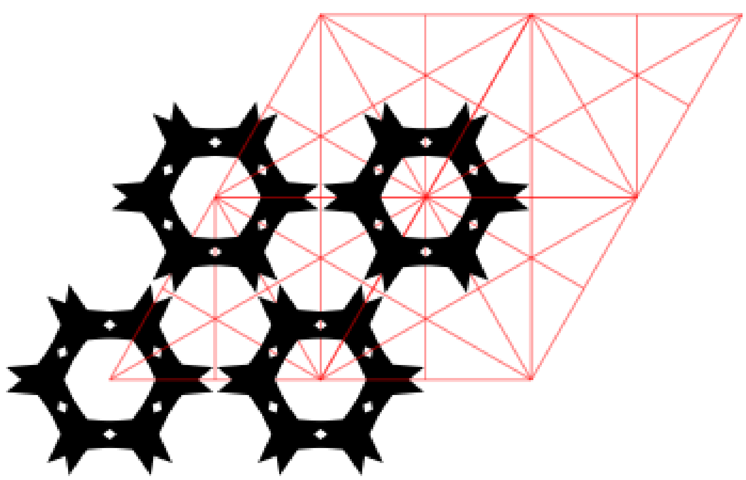

FIGURE 4.14. The $*632$ plane tiling with mirror lines marked.

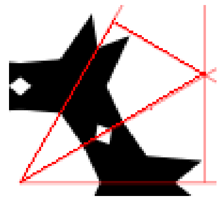

FIGURE 4.15. A ∗632 triangular orbifold region bounded by mirror lines.

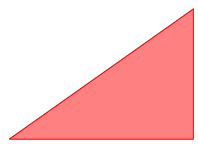

FIGURE 4.16. The ∗632 orbifold is topologically a triangle.

4.5. Ditrope group example

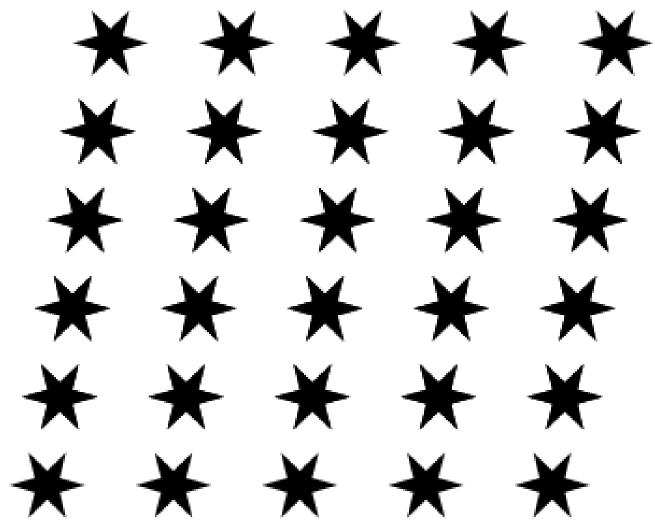

FIGURE 4.17. Plane tiling having orbifold symbol 2222.

Consider the tiling in Figure 4.17 having orbifold symbol 2222. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Does not apply since no mirrors.

Step 2: Draw the numbers 2 on all gyration points, corresponding to these point's highest order of symmetry preserving rotation, which is 2. Consider Figure 4.18 of our tiling with numbers on gyrations points marked in purple.

Step 3: Does not apply since no miracles.

Step 4: Does not apply since no mirrors.

Step 5: For each number 2 drawn on gyration points, there are 4 distinct numbers on gyration points so we write 2222.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 2 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains 4 distinct 2-fold gyration points, which is homeomorphic to a sphere. Recall Example 2.24 and observe Figures 4.19 and 4.20 of the orbifold. We choose a fundamental polygonal region that is a square and glue the gyration points together to obtain a sphere (or pillow with 4 corners).

We calculate the orbifold Euler Characteristic of the 2222 tiling using the defect formula, accounting for the following defects:

- The 2's each count as $\frac{2-1}{2} = \frac{1}{2}$.

Therefore, the 2222 orbifold Euler Characteristic is

$$2 - \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = 0 \tag{4.5}$$

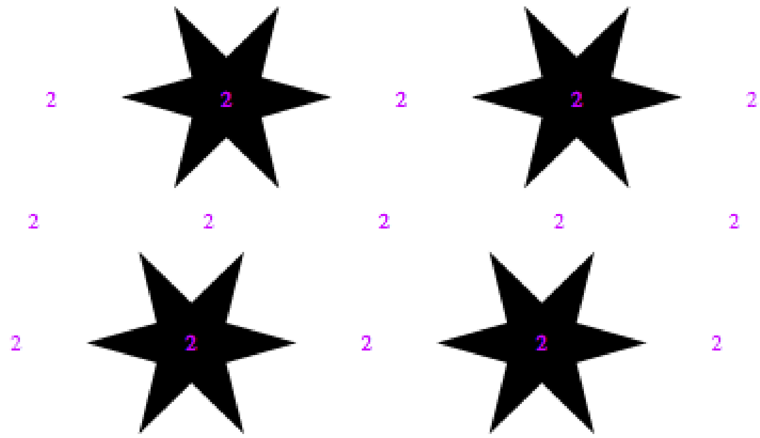

FIGURE 4.18. The 2222 plane tiling with 2-fold gyration points marked.

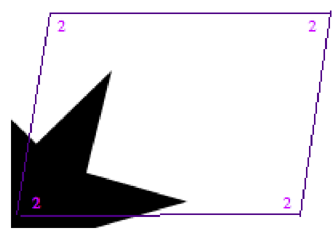

FIGURE 4.19. A 2222 rectangular (parallelogram) orbifold region with corners as 2-fold gyration points.

FIGURE 4.20. The 2222 orbifold is topologically a sphere (or 4-cornered pillow).

4.6. Tritrope group example

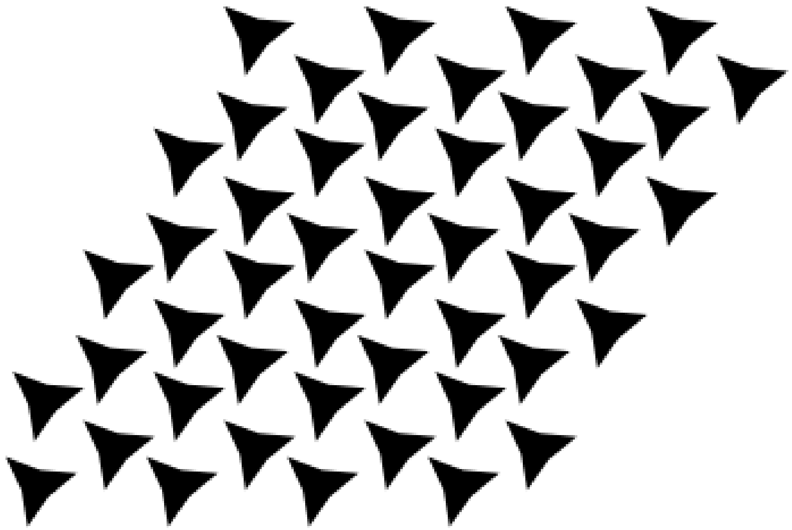

FIGURE 4.21. Plane tiling having orbifold symbol 333.

Consider the tiling in Figure 4.21 having orbifold symbol 333. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Does not apply since no mirrors.

Step 2: Draw the numbers 3 on all gyration points, corresponding to these point's highest order of symmetry preserving rotation, which is 3. Consider Figure 4.22 of our tiling with numbers on gyrations points marked in purple.

Step 3: Does not apply since no miracles.

Step 4: Does not apply since no mirrors.

Step 5: For each number 3 drawn on gyration points, there are 3 distinct numbers on gyration points so we write 333.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 2 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains 3 distinct 3-fold gyration points, which is homeomorphic to a sphere. Recall Example 2.24 and observe Figures 4.23 and 4.24 of the orbifold. We choose a fundamental polygonal region that is a triangle and glue the gyration points together to obtain a sphere (or pillow with 3 corners).

We calculate the orbifold Euler Characteristic of the 333 tiling using the defect formula, accounting for the following defects:

- The 3's each count as $\frac{3-1}{3} = \frac{2}{3}$.

Therefore, the 333 orbifold Euler Characteristic is

$$2 - \left(\frac{2}{3} + \frac{2}{3} + \frac{2}{3}\right) = 0 \tag{4.6}$$

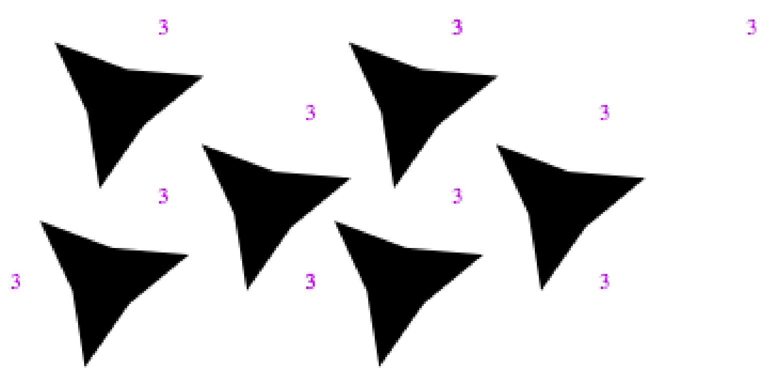

FIGURE 4.22. The 333 plane tiling with 3-fold gyration points marked.

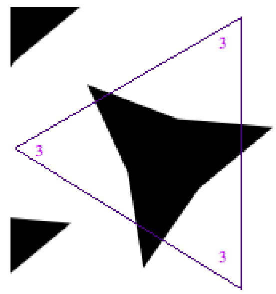

Figure 4.23. A 333 triangular orbifold region with corners as 3-fold gyration points.

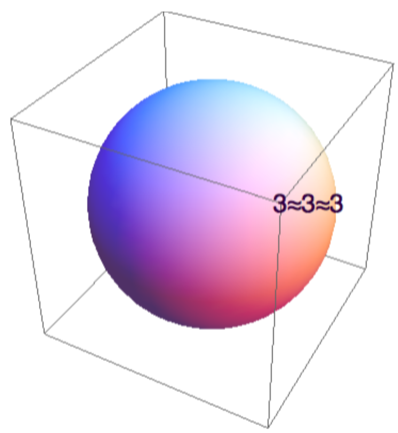

Figure 4.24. The 333 orbifold is topologically a sphere (or 3-cornered pillow).

4.7. Tetratrope group example

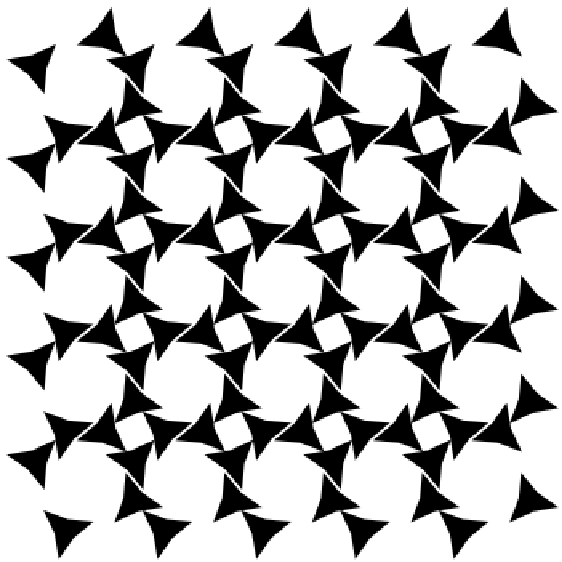

FIGURE 4.25. Plane tiling having orbifold symbol 442.

Consider the tiling in Figure 4.25 having orbifold symbol 442. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Does not apply since no mirrors.

Step 2: Draw the numbers 4 and 2 on gyration points, corresponding to these point's highest order of symmetry preserving rotation, which is 4 and 2 respectively. Consider Figure 4.26 of our tiling with numbers on gyrations points marked in purple.

Step 3: Does not apply since no miracles.

Step 4: Does not apply since no mirrors.

Step 5: For each number 4 and 2 drawn on gyration points, there are 2 and 1, respectively, distinct numbers on gyration points so we write 442.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 2 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains a 2-fold gyration point and 2 distinct 4-fold gyration points, which is homeomorphic to a sphere. Recall Example 2.24 and observe Figures 4.27 and 4.28 of the orbifold. We choose a fundamental polygonal region that is a triangle and glue the gyration points together to obtain a sphere (or pillow with 3 corners).

We calculate the orbifold Euler Characteristic of the 442 tiling using the defect formula, accounting for the following defects:

- The 4's each count as $\frac{4-1}{4} = \frac{3}{4}$. The 2 counts as $\frac{2-1}{2} = \frac{1}{2}$.

Therefore, the 442 orbifold Euler Characteristic is

$$2 - \left(\frac{3}{4} + \frac{3}{4} + \frac{1}{2}\right) = 0 \tag{4.7}$$

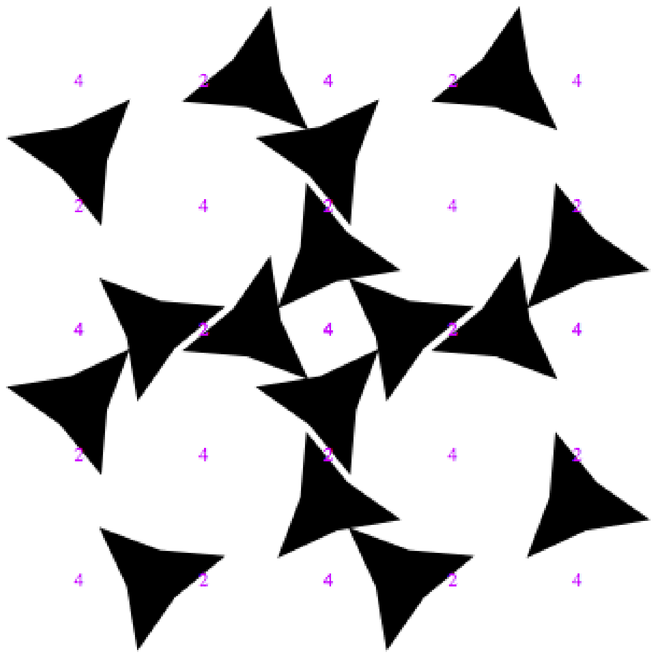

FIGURE 4.26. The 442 plane tiling with 4-fold and 2-fold gyration points marked.

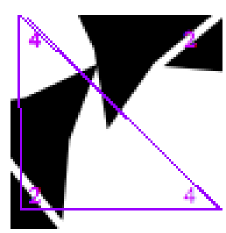

FIGURE 4.27. A 442 triangular orbifold region with corners as two 4-fold gyration points and one 2-fold gyration point.

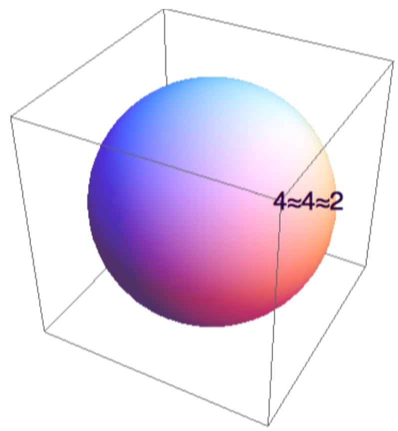

FIGURE 4.28. The 442 orbifold is topologically a sphere (or 3-cornered pillow).

4.8. Hexatrope group example

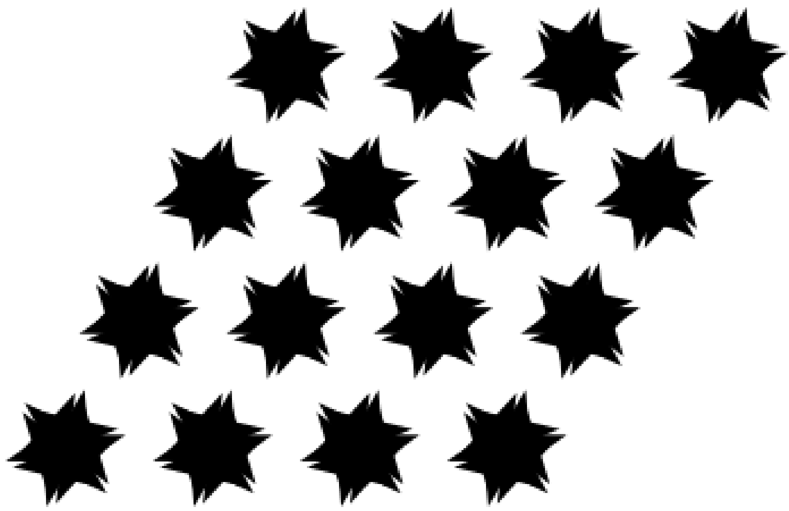

FIGURE 4.29. Plane tiling having orbifold symbol 632.

Consider the tiling in Figure 4.29 having orbifold symbol 632. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Does not apply since no mirrors.

Step 2: Draw the numbers 6, 3, and 2 on gyration points, corresponding to these point's highest order of symmetry preserving rotation, which is 6, 3 and 2 respectively. Consider Figure 4.30 of our tiling with numbers on gyrations points marked in purple.

Step 3: Does not apply since no miracles.

Step 4: Does not apply since no mirrors.

Step 5: For each number 6, 3, and 2 drawn on gyration points, there is 1, 1 and 1, respectively, distinct numbers on gyration points so we write 632.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 2 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains a 6, a 3, and a 2-fold gyration point, which is homeomorphic to a sphere. Recall Example 2.24 and observe Figures 4.31 and 4.32 of the orbifold. We choose a fundamental polygonal region that is a triangle and glue the gyration points together to obtain a sphere (or pillow with 3 points).

We calculate the orbifold Euler Characteristic of the 632 tiling using the defect formula, accounting for the following defects:

- The 6 counts as $\frac{6-1}{6} = \frac{5}{6}$. The 3 counts as $\frac{3-1}{3} = \frac{2}{3}$. The 2 counts as $\frac{2-1}{2} = \frac{1}{2}$.

Therefore, the 632 orbifold Euler Characteristic is

$$2 - \left(\frac{5}{6} + \frac{2}{3} + \frac{1}{2}\right) = 0 \tag{4.8}$$

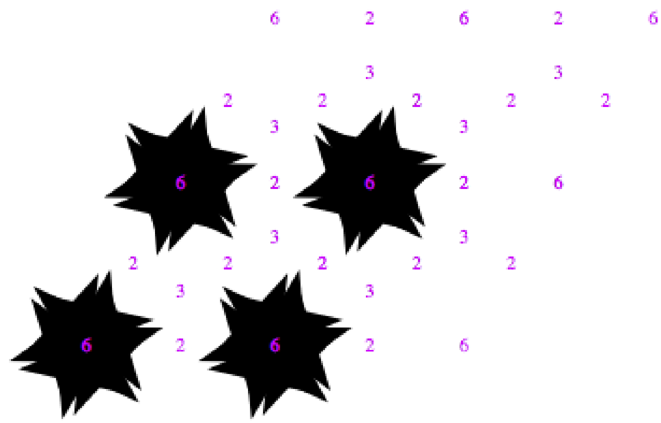

FIGURE 4.30. The 632 plane tiling with 6-fold, 3-fold, and 2-fold gyration points marked.

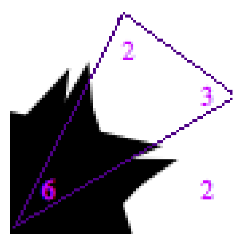

FIGURE 4.31. A 632 triangular orbifold region with corners as one 6-fold, one 3-fold, and one 2-fold gyration points.

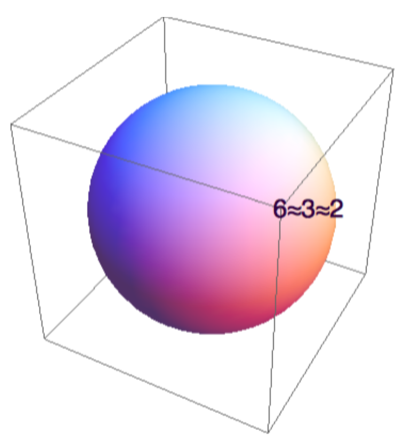

FIGURE 4.32. The 632 orbifold is topologically a sphere (or 3-cornered pillow).

4.9. Digryo group example

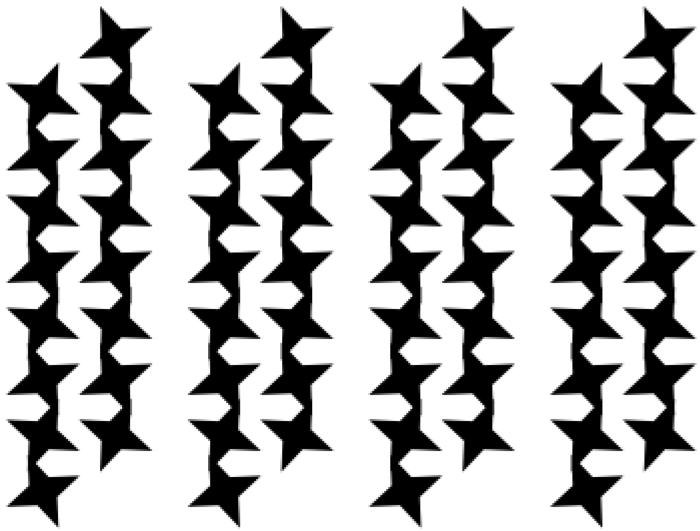

FIGURE 4.33. Plane tiling having orbifold symbol 22∗.

Consider the tiling in Figure 4.33 having orbifold symbol 22∗. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Draw lines on mirror lines. Consider Figure 4.34 of our tiling with mirror lines marked in red.

Step 2: Draw the numbers 2 on gyration points, corresponding to these point's highest order of symmetry preserving rotation, which is 2. Consider Figure 4.34 with numbers on gyrations points marked in purple.

Step 3: Does not apply since no miracles.

Step 4: There is 1 distinct mirror line so we write ∗.

Step 5: For each number 2 drawn on gyration points, there is 2 distinct numbers on gyration points so we write $22*$.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 1 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains the distinct mirror line and 2 distinct 2-fold gyration points, which is homeomorphic to an open sphere (or open pillow). Recall our discussion of an open sphere (see Example 2.24). Notice here, choosing a polygonal region to be our obifold we have a mirror that accounts for the cut off top edge of the sphere. Observe Figures 4.35 and 4.36 of the orbifold.

We calculate the orbifold Euler Characteristic of the $22*$ tiling using the defect formula, accounting for the following defects:

- The 2's before the $*$ each count as $\frac{2-1}{2} = \frac{1}{2}$.
- The $*$ counts as 1.

Therefore, the $22*$ orbifold Euler Characteristic is

$$2 - \left(\frac{1}{2} + \frac{1}{2} + 1\right) = 0 \tag{4.9}$$

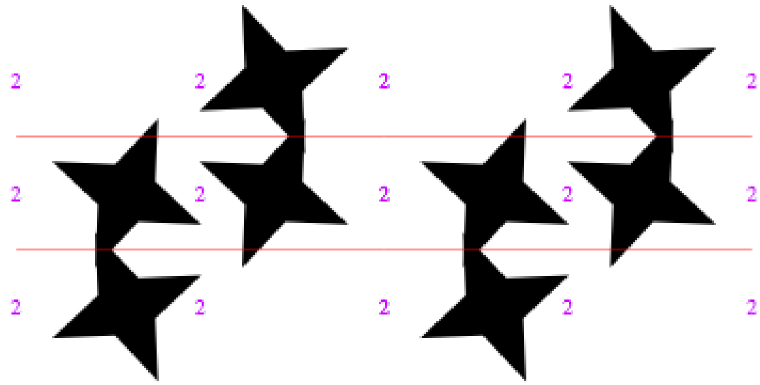

FIGURE 4.34. The 22∗ plane tiling with mirror lines and 2-fold gyration points marked.

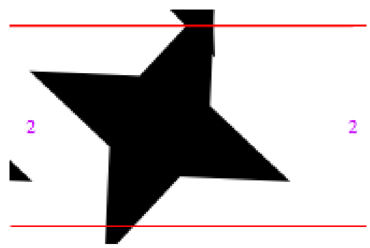

FIGURE 4.35. A $22*2$ rectangular orbifold region with two corners as 2-fold gyration points and non-distinct mirror line top and bottom boundary.

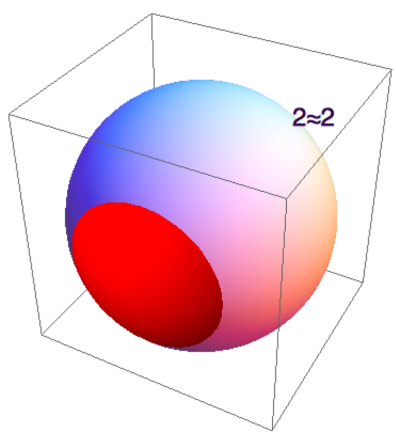

FIGURE 4.36. The 22∗ orbifold is topologically an open sphere (or open pillow).

4.10. Dirhomb group example

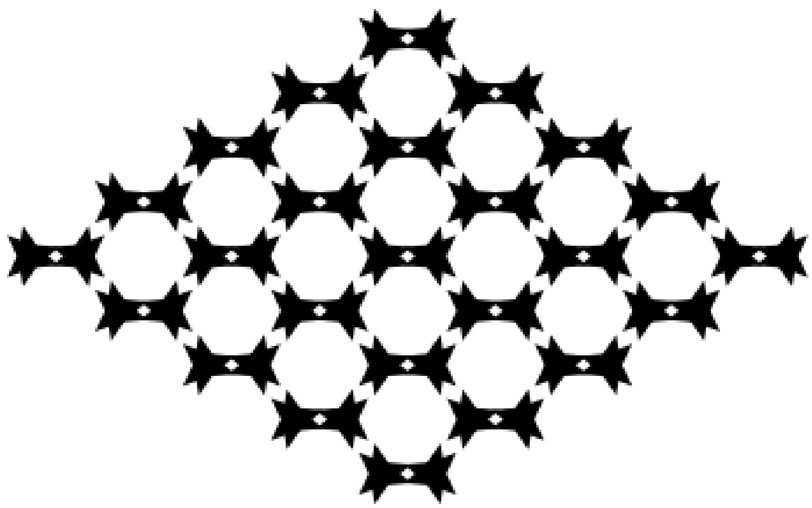

FIGURE 4.37. Plane tiling having orbifold symbol $2 * 22$.

Consider the tiling in Figure 4.37 having orbifold symbol $2 * 22$. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Draw lines on mirror lines. Consider Figure 4.38 of our tiling with mirror lines marked in red.

Step 2: Draw the numbers 2 on gyration points, corresponding to these point's highest order of symmetry preserving rotation, which is 2. Consider Figure 4.38 with numbers on gyrations points marked in purple.

Step 3: Does not apply since no miracles.

Step 4: For each mirror line intersection, the number of intersecting lines is 2 for 2 distinct points so we write $*22$.

Step 5: For each number 2 drawn on gyration points, there is 1 distinct number on gyration points so we write $2 * 22$.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 1 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains the distinct mirror lines and a 2-fold gyration point, which is topologically a cone. Recall Example 2.23 and observe Figures 4.39, 4.40 and 4.41 of the orbifold.

We calculate the orbifold Euler Characteristic of the $2 * 22$ tiling using the defect formula, accounting for the following defects:

- The 2 before the $*$ counts as $\frac{2-1}{2} = \frac{1}{2}$.
- The 2's following the $*$ each count as $\frac{2-1}{2(2)} = \frac{1}{4}$.
- The $*$ counts as 1.

Therefore, the $2 * 22$ orbifold Euler Characteristic is

$$2 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{4} + 1 \right) = 0 \tag{4.10}$$

FIGURE 4.38. The $2 * 22$ plane tiling with mirror lines and 2-fold gyration points marked.

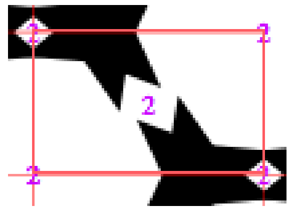

FIGURE 4.39. A $2 * 22$ rectangular orbifold with a cone point of order 2 in the center bounded by mirror lines, where the vertical mirror lines are distinct from the horizontal mirror lines.

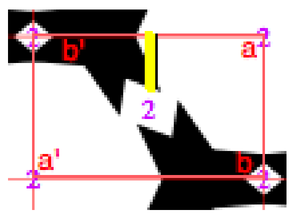

FIGURE 4.40. The $2 * 22$ orbifold is the cone obtained by cutting along the yellow line and then wrapping the flaps to identify $a \approx a'$ and $b \approx b'$.

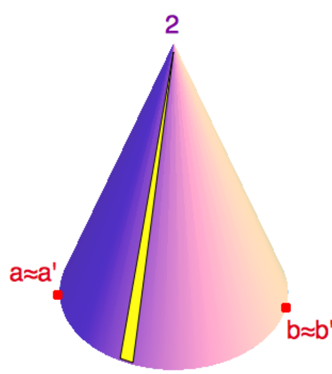

FIGURE 4.41. The $2 * 22$ orbifold is topologically a conical surface.

4.11. Trigyro group example

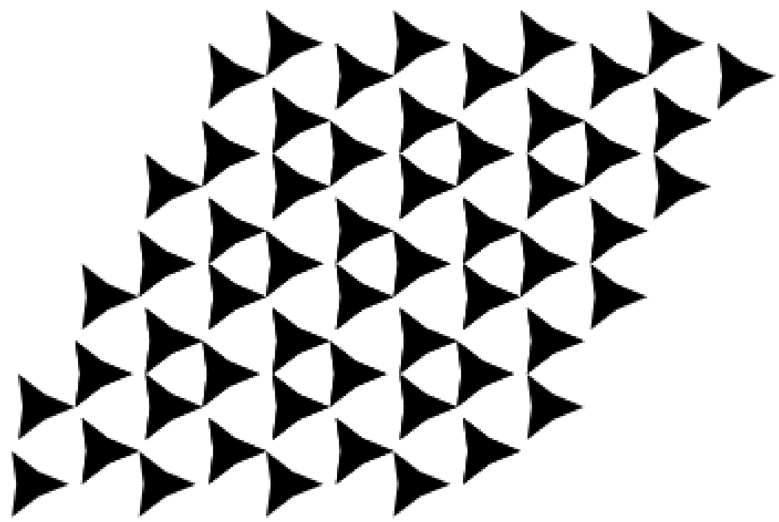

FIGURE 4.42. Plane tiling having orbifold symbol $3 * 3$.

Consider the tiling in Figure 4.42 having orbifold symbol $3 * 3$. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Draw lines on mirror lines. Consider Figure 4.43 of our tiling with mirror lines marked in red.

Step 2: Draw the numbers 3 on gyration points, corresponding to these point's highest order of symmetry preserving rotation, which is 3. Consider Figure 4.43 with numbers on gyrations points marked in purple.

Step 3: Does not apply since no miracles.

Step 4: For each mirror line intersection, the number of intersecting lines is 3 for 1 distinct point so we write $*3$.

Step 5: For each number 3 drawn on gyration points, there is 1 distinct number on gyration points so we write $3 * 3$.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 1 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains the distinct mirror line and a 3-fold gyration point, which is topologically a cone. Recall Example 2.23 and observe Figures 4.44, 4.45, and 4.46 of the orbifold.

We calculate the orbifold Euler Characteristic of the $3 * 3$ tiling using the defect formula, accounting for the following defects:

- The 3 before the $*$ counts as $\frac{3-1}{3} = \frac{2}{3}$.
- The 3 following the $*$ counts as $\frac{3-1}{2(3)} = \frac{1}{3}$.
- The $*$ counts as 1.

Therefore, the $3 * 3$ orbifold Euler Characteristic is

$$2 - \left(\frac{2}{3} + \frac{1}{3} + 1\right) = 0 \tag{4.11}$$

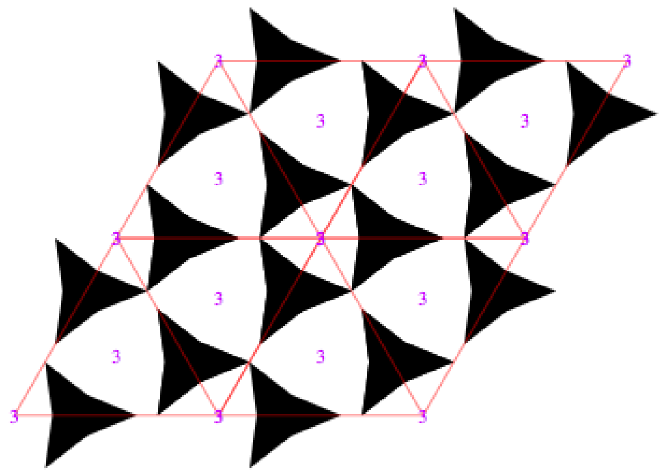

FIGURE 4.43. The $3*3$ plane tiling with mirror lines and 3-fold gyration points marked.

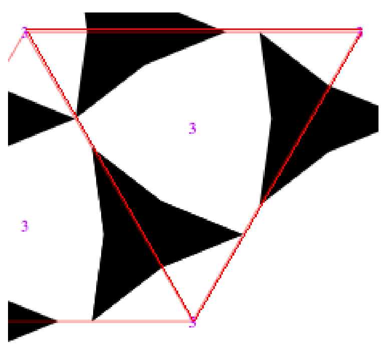

FIGURE 4.44. A $3*3$ triangular orbifold with a cone point of order 3 in the center bounded by non-distinct mirror lines.

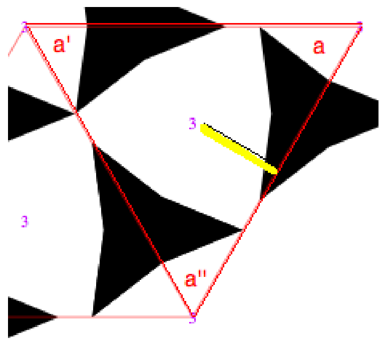

FIGURE 4.45. The $3*3$ orbifold is the cone obtained by cutting along the yellow line and then wrapping the flaps to identify $a \approx a' \approx a''$.

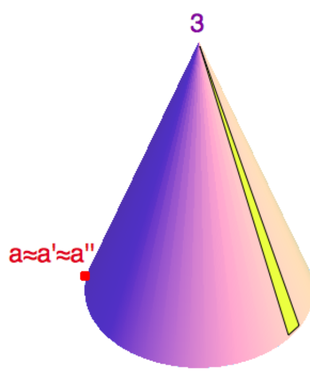

FIGURE 4.46. The $3*3$ orbifold is topologically a conical surface.

4.12. Tetragyro group example

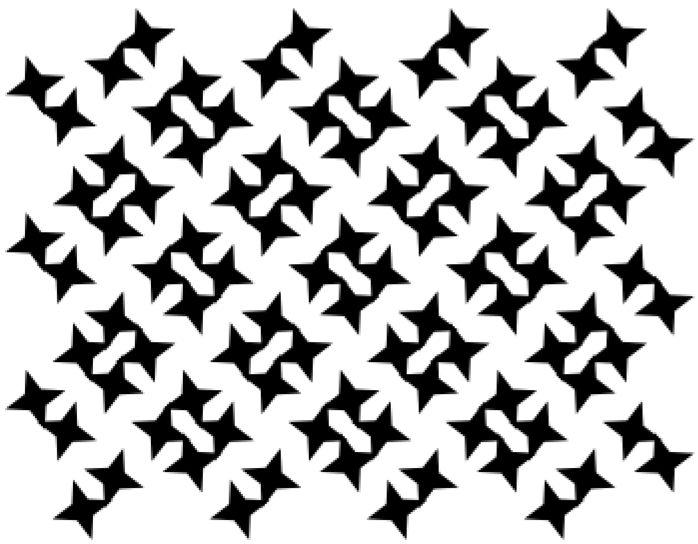

FIGURE 4.47. Plane tiling having orbifold symbol $4 * 2$.

Consider the tiling in Figure 4.47 having orbifold symbol $4 * 2$. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Draw lines on mirror lines. Consider Figure 4.48 of our tiling with mirror lines marked in red.

Step 2: Draw the numbers 4 on gyration points, corresponding to these point's highest order of symmetry preserving rotation, which is 4. Consider Figure 4.48 with numbers on gyrations points marked in purple.

Step 3: Does not apply since no miracles.

Step 4: For each mirror line intersection, the number of intersecting lines is 2 for 1 distinct point so we write $*2$.

Step 5: For each number 4 drawn on gyration points, there is 1 distinct number on gyration points so we write $4 * 2$.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 1 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains the distinct mirror line and a 4-fold gyration point, which is topologically a cone. Recall Example 2.23 and observe Figures 4.49, 4.50, and 4.51 of the orbifold.

We calculate the orbifold Euler Characteristic of the $4 * 2$ tiling using the defect formula, accounting for the following defects:

- The 4 before the $*$ counts as $\frac{4-1}{4} = \frac{3}{4}$.
- The 2 following the $*$ counts as $\frac{2-1}{2(2)} = \frac{1}{4}$.
- The $*$ counts as 1.

Therefore, the $4 * 2$ orbifold Euler Characteristic is

$$2 - \left(\frac{3}{4} + \frac{1}{4} + 1\right) = 0 \tag{4.12}$$

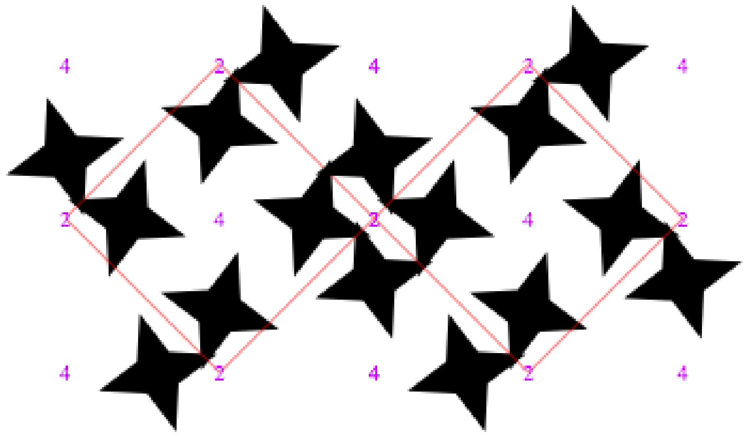

FIGURE 4.48. The $4 * 2$ plane tiling with mirror lines and 4-fold gyration points marked.

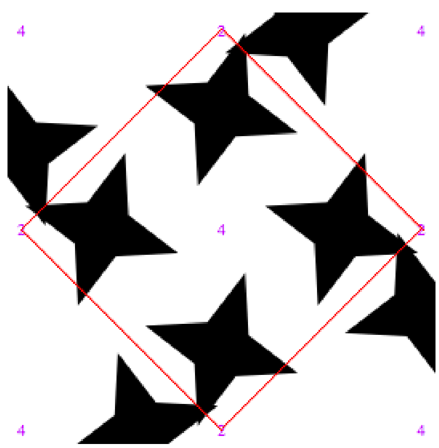

FIGURE 4.49. A $4 * 2$ square orbifold with a cone point of order 4 in the center bounded by non-distinct mirror lines.

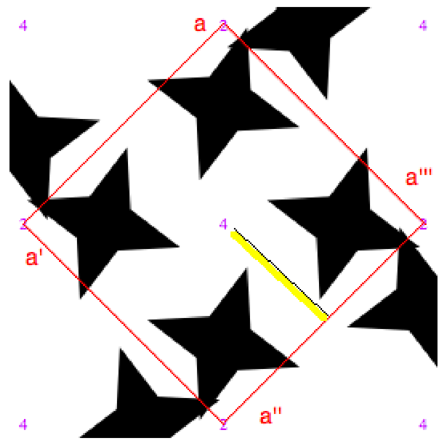

FIGURE 4.50. The $4*2$ orbifold is the cone obtained by cutting along the yellow line and then wrapping the flaps to identify $a \approx a' \approx a'' \approx a'''$.

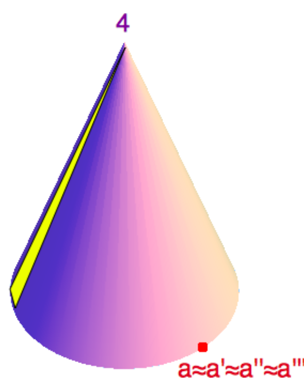

FIGURE 4.51. The $4*2$ orbifold is topologically a conical surface.

4.13. Monoscope group example

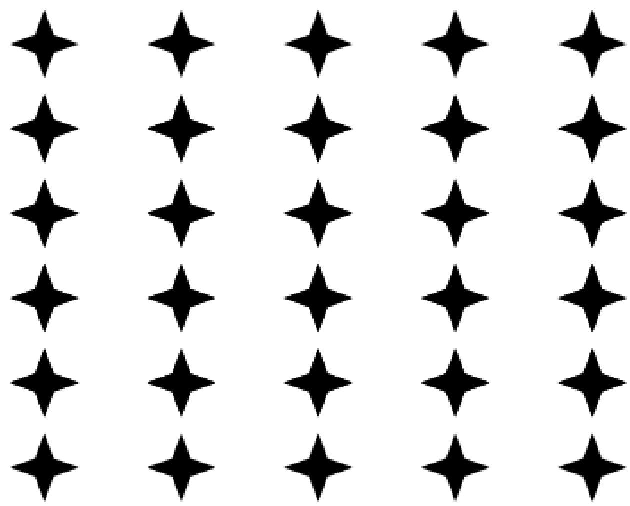

FIGURE 4.52. Plane tiling having orbifold symbol $**$.

Consider the tiling in Figure 4.52 having orbifold symbol $**$. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Draw lines on mirror lines. Consider Figure 4.53 of our tiling with mirror lines marked in red.

Step 2: Does not apply since no rotations.

Step 3: Does not apply since no miracles.

Step 4: For each of the 2 distinct mirror lines, there are no mirror line intersections so we write $**$.

Step 5: Does not apply since no rotations.

Step 6: Does not apply since no miracles.

Step 7: Does not apply since Step 1 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains the distinct mirror lines, which is homeomorphic to an annulus. Recall Example 2.22 and observe Figures 4.54, 4.55, and 4.56 of the orbifold.

We calculate the orbifold Euler Characteristic of the $**$ tiling using the defect formula, accounting for the following defects:

- Each $*$ counts as 1.

Therefore, the $**$ orbifold Euler Characteristic is

$$2 - (1 + 1) = 0 \tag{4.13}$$

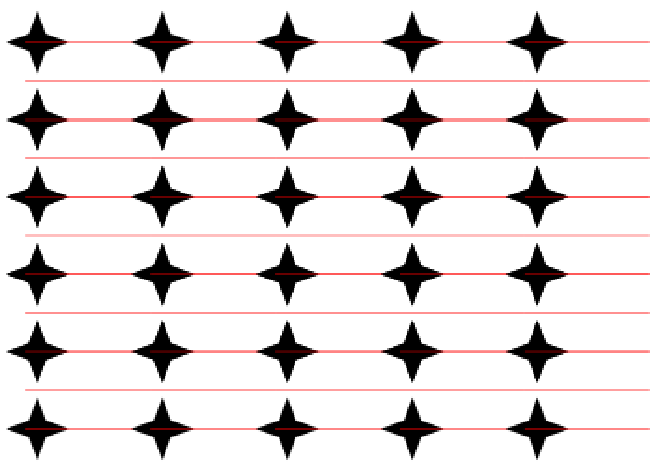

FIGURE 4.53. The $**$ plane tiling with mirror lines marked.

FIGURE 4.54. A $**$ rectangular orbifold bounded on top and bottom by two distinct mirror lines.

FIGURE 4.55. The $**$ orbifold is homeomorphic to an annulus obtained by wrapping the rectangular region upward, keeping the upper mirror line on the inner part of the loop and the bottom mirror line on the outer part of the loop, gluing the yellow colored edges together.

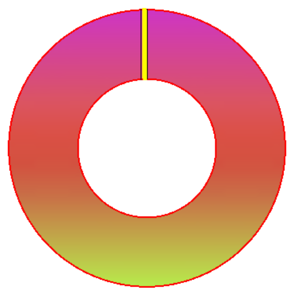

FIGURE 4.56. The $**$ orbifold is topologically an annulus.

4.14. Monorhomb group example

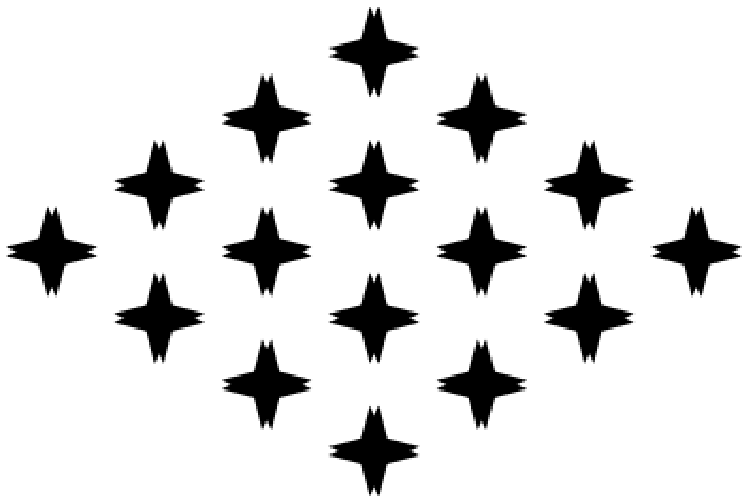

FIGURE 4.57. Plane tiling having orbifold symbol $*\times$.

Consider the tiling in Figure 4.57 having orbifold symbol $*\times$. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Draw lines on mirror lines. Consider Figure 4.58 of our tiling with mirror lines marked in red.

Step 2: Does not apply since no rotations.

Step 3: Draw (dotted) lines on lines of wonders. Consider Figure 4.58 with wonders (glide reflections) marked in blue.

Step 4: For each of the 1 distinct mirror lines, there are no mirror line intersections so we write $*$.

Step 5: Does not apply since no rotations.

Step 6: We have miracles and the symbol $*$, so we write $*\times$.

Step 7: Does not apply since Step 1 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains the distinct mirror line and miracle, which is homeomorphic to a Möbius strip. Recall Example 2.27 and observe Figures 4.59, 4.60, and 4.61 of the orbifold.

We calculate the orbifold Euler Characteristic of the $*\times$ tiling using the defect formula, accounting for the following defects:

- The $*$ counts as 1, and the $\times$ counts as 1.

Therefore, the $*\times$ orbifold Euler Characteristic is

$$2 - (1 + 1) = 0 \tag{4.14}$$

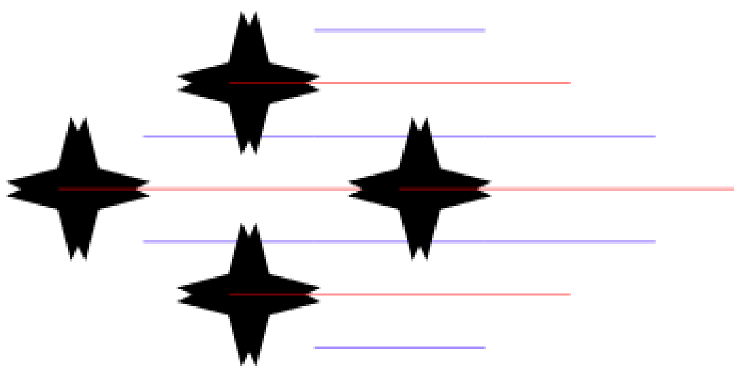

FIGURE 4.58. The $*\times$ plane tiling with mirror lines and glide reflection lines marked.

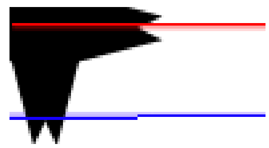

FIGURE 4.59. A $*\times$ rectangular orbifold bounded on top by a mirror line and bounded on bottom by a glide reflection line.

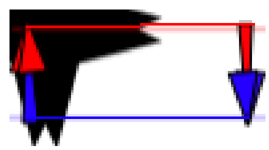

FIGURE 4.60. The $*\times$ orbifold is homeomorphic to a Móbius strip obtained by twisting the rectangular region and gluing the ends so that the arrows become aligned in the same direction.

FIGURE 4.61. The $*\times$ orbifold is topologically a Möbius strip.

4.15. Diglide group example

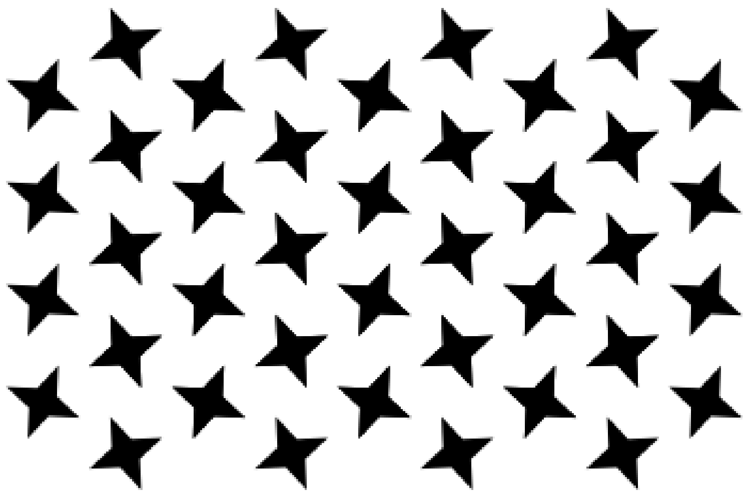

FIGURE 4.62. Plane tiling having orbifold symbol 22×.

Consider the tiling in Figure 4.62 having orbifold symbol 22×. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Does not apply since no mirror lines.

Step 2: Draw the numbers 2 on gyration points, corresponding to these point's highest order of symmetry preserving rotation, which is 2. Consider Figure 4.63 of our tiling with numbers on gyrations points marked in purple.

Step 3: Draw (dotted) lines on lines of wonders. Consider Figure 4.63 with wonders (glide reflections) marked in blue.

Step 4: Does not apply since no mirror lines.

Step 5: For each number 2 drawn on gyration points, there is 2 distinct numbers on gyration points so we write 22.

Step 6: We have miracles and the numbers 22, so we write 22×

Step 7: Does not apply since Step 2 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains the distinct gyrations and miracle, which is homeomorphic to the projective plane. Recall Example 2.25 and observe Figures 4.64 and 4.65 of the orbifold.

We calculate the orbifold Euler Characteristic of the 22× tiling using the defect formula, accounting for the following defects:

- The 2's before the $*$ each count as $\frac{2-1}{2} = \frac{1}{2}$.
- The $\times$ counts as 1.

Therefore, the 22× orbifold Euler Characteristic is

$$2 - \left(\frac{1}{2} + \frac{1}{2} + 1\right) = 0 \tag{4.15}$$

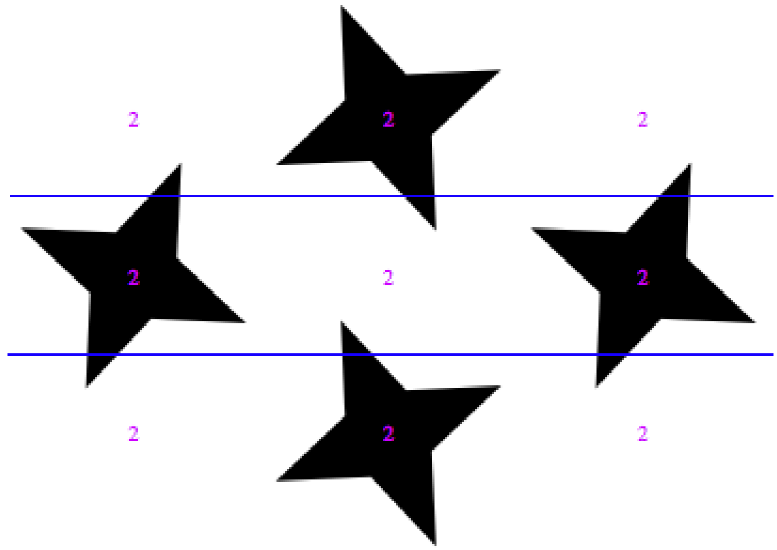

FIGURE 4.63. The 22× plane tiling with 2-fold gyration points and glide reflection lines marked.

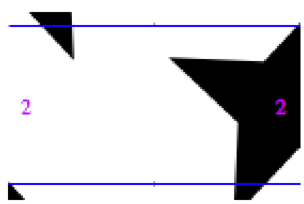

FIGURE 4.64. A 22× rectangular orbifold region bounded top and bottom by non-distinct glide reflection lines and having two distinct 2-fold gyration points.

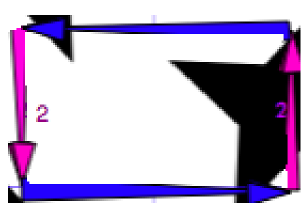

FIGURE 4.65. The 22× orbifold is homeomorphic to the real projective plane by orienting the edges in this way, where the blue arrows and purple arrows are like sides, respectively, in a labelling scheme.

4.16. Monoglide group example

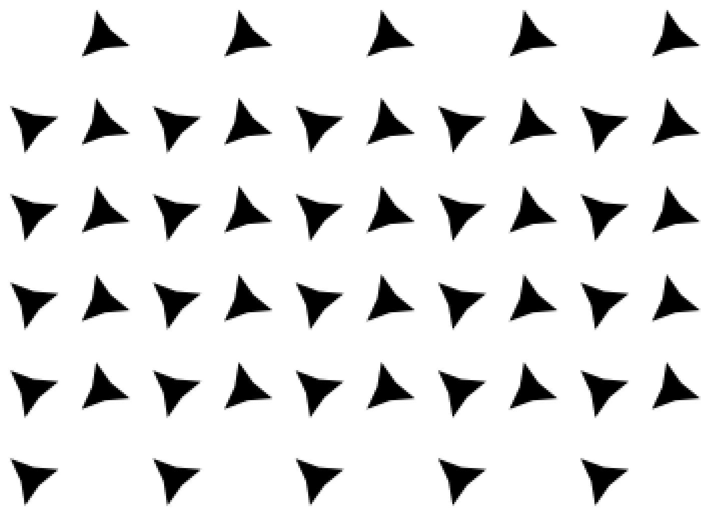

FIGURE 4.66. Plane tiling having orbifold symbol $\times\times$.

Consider the tiling in Figure 4.66 having orbifold symbol $\times\times$. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Step 1: Does not apply since no mirror lines.

Step 2: Does not apply since no gyrations.

Step 3: Draw (dotted) lines on all miracles. Consider Figure 4.67 of our tiling with miracles marked in blue.

Step 4: Does not apply since no mirror lines.

Step 5: Does not apply since no gyrations.

Step 6: We have miracles and no other symbols or numbers, so we write $\times\times$.

Step 7: Does not apply since Step 3 applies.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains the distinct miracles, which is homeomorphic to a Klein bottle. Recall Example 2.28 and observe Figures 4.68, 4.69, and 4.70 of the orbifold.

We calculate the orbifold Euler Characteristic of the $\times\times$ tiling using the defect formula, accounting for the following defects:

- Each $\times$ counts as 1.

Therefore, the $\times\times$ orbifold Euler Characteristic is

$$2 - (1 + 1) = 0 \tag{4.16}$$

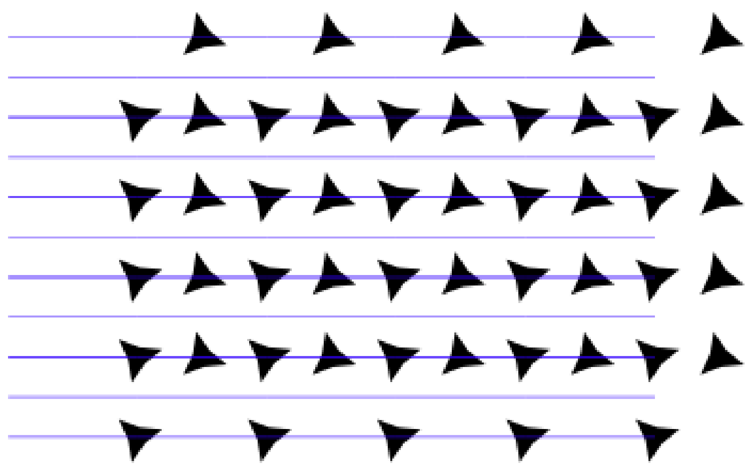

FIGURE 4.67. The $\times\times$ plane tiling with glide reflection lines marked.

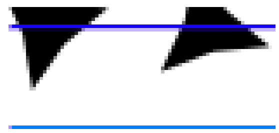

FIGURE 4.68. A ×× rectangular orbifold region bounded top and bottom by distinct glide reflection lines.

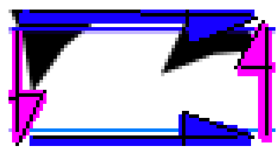

FIGURE 4.69. The ×× orbifold is homeomorphic to a Klein by orienting the edges in this way, where the blue arrows and purple arrows are like sides, respectively, in a labelling scheme.

FIGURE 4.70. The ×× orbifold is topologically a Klein bottle.

4.17. Monotrope group example

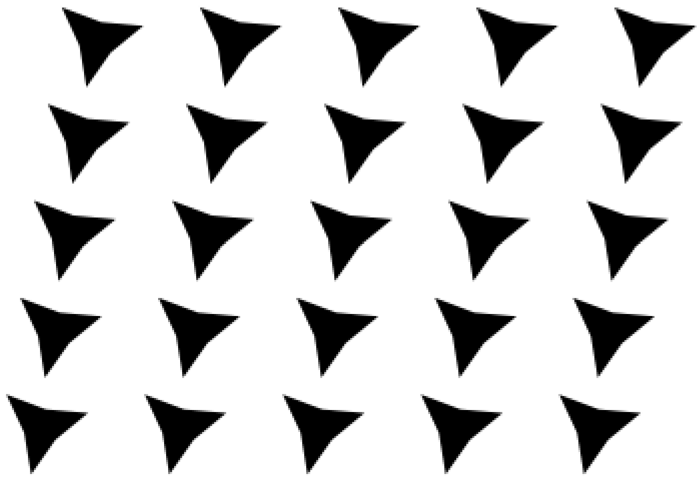

FIGURE 4.71. Plane tiling having orbifold symbol $\circ$.

Consider the tiling in Figure 4.71 having orbifold symbol $\circ$. To find this tiling's orbifold symbol we apply our procedure as follows, observing the tiling's orbifold properties and orbifold in the figures below:

Steps 1-6: Do not apply since no mirror lines, gyrations, or miracles.

Step 7: None of the above Steps apply, so we write $\circ$. Notice our tiling in Figure 4.72 with translation grid marked.

Step 8: Take the orbifold to be any fundamental region of the tiling that contains a translation grid, which is homeomorphic to a torus. Recall Example 2.26 and observe Figures 4.73, 4.74, and 4.75 of the orbifold.

We calculate the orbifold Euler Characteristic of the $\circ$ tiling using the defect formula, accounting for the following defects:

- The $\circ$ counts as 2.

Therefore, the $\circ$ orbifold Euler Characteristic is

$$2 - (2) = 0 \tag{4.17}$$

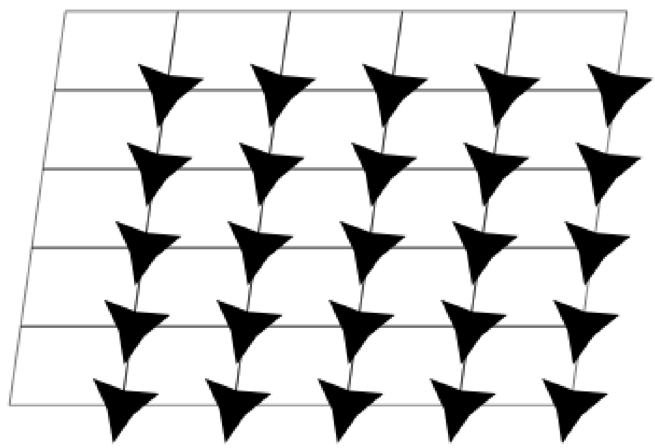

FIGURE 4.72. The $\circ$ plane tiling with translation grid marked.

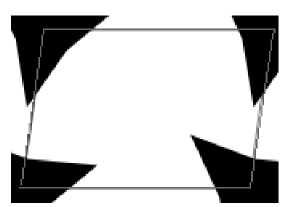

FIGURE 4.73. A $\circ$ rectangular (parallelogram) orbifold region bounded by translation lines.

FIGURE 4.74. The ∘ orbifold is homeomorphic to a torus by orienting the edges in this way, where the blue arrows and purple arrows are like sides, respectively, in a labelling scheme. We glue the top and bottom (blue arrows) together making a cylinder, then glue the ends of the cylinder (red arrows) obtaining a torus.

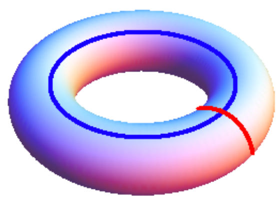

FIGURE 4.75. The ∘ orbifold is topologically a torus.

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