Exercise(s) and final project of TMR4515

ADVANCED MODEL-BASED DESIGN AND TESTING OF MARINE CONTROL SYSTEM

Application of hybrid multiple model control to an inverted pendulum

The home work should be delivered in IFAC style document; it will make 50% of your final grade; 6-10 pages; you could work in team of maximum 4 members (Deadline: Exam date)

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An inverted pendulum (also known as cart and pole) consists of a thin rod attached at its bottom to a moving cart. Whereas a normal pendulum is stable when hanging downwards, a vertical inverted pendulum is inherently unstable, and must be actively balanced in order to remain upright, typically by moving the cart horizontally as part of a feedback system.

The inverted pendulum is a classic problem in dynamics and control theory and widely used as benchmark for testing control algorithms. The inverted pendulum is related to rocket or missile guidance, where thrust is actuated at the bottom of a tall vehicle.

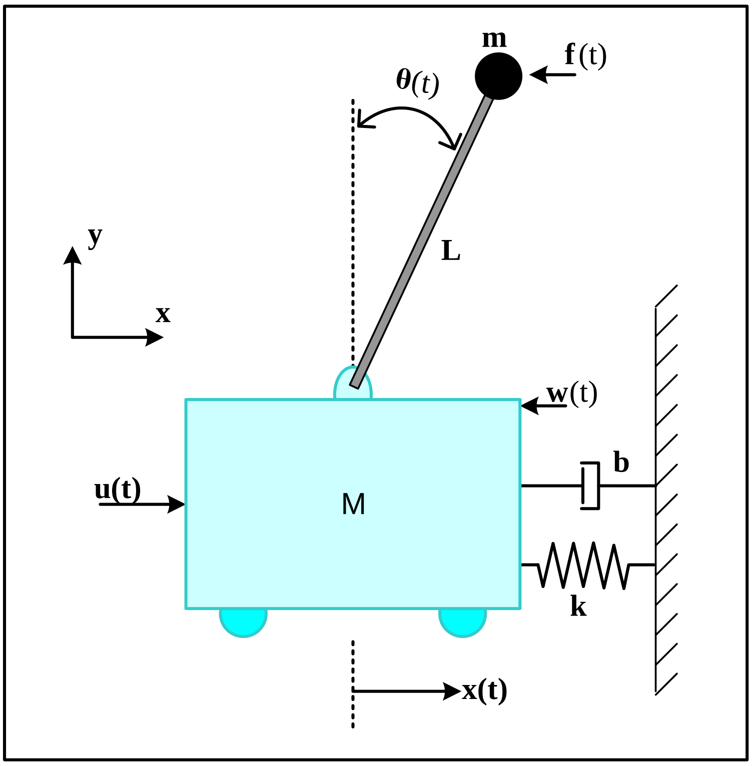

Figure 1: Inverted Pendulum

0.1 Equations of motion

The differential equations describing the motion of the system may be obtained by writing the sum of forces in the horizontal direction and the sum of the moments about the pivot point. The mass of the stick is small compared with mass m and will be neglected. The length of the stick is L. The effect of friction is also neglected.

Non-linear model: The sate vector is defined as

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{pmatrix} \tag{1}$$

We bring in free body diagram:

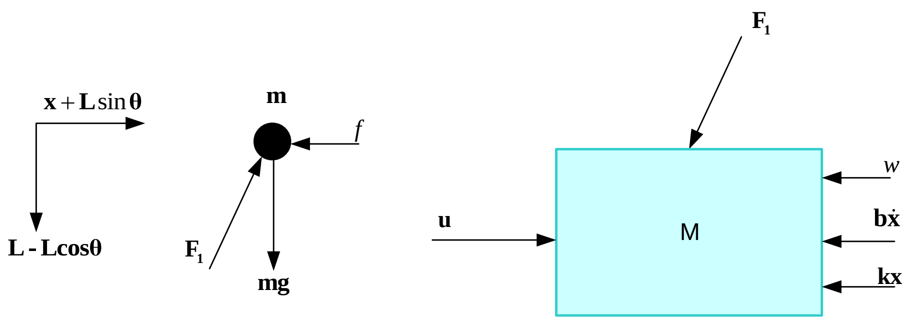

Figure 2: Free Body for Ball and Cart

The sum of the forces in the horizontal direction considering external disturbance effecting cart and pendulum,

$$\begin{cases} m\frac{d^2}{dt^2}(x + L\sin\theta) = F_1\sin\theta - f \\ m\frac{d^2}{dt^2}(L - L\cos\theta) = mg - F_1\cos\theta \\ M\ddot{x} = u - F_1\sin\theta - w - b\dot{x} - kx \end{cases} \tag{2}$$

These are 3 equation in 4 signals x, θ, u, F_1. By noting that ($\frac{d^2}{dt^2}\sin\theta = \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta$) and ($\frac{d^2}{dt^2}\cos\theta = -\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta$), we get

$$\begin{cases} m\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = F_1\sin\theta - f \\ mL\ddot{\theta}\sin\theta + mL\dot{\theta}^2\cos\theta = mg - F_1\cos\theta \\ M\ddot{x} = u - F_1\sin\theta - w - b\dot{x} - kx \end{cases} \tag{3}$$

We can eliminate F_1 by adding 1^{st} and 3^{rd} to get

$$(m + M)\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = u - f - w - b\dot{x} - kx \tag{4}$$

multiply 1^{st} by $\cos\theta$, 2^{nd} by $\sin\theta$, add, divide by m to get

$$\ddot{x}\cos\theta + L\ddot{\theta} = g\sin\theta - \frac{f\cos\theta}{m}. \tag{5}$$

Solve the latter two equation for $\ddot{x}$ and $\ddot{\theta}$:

$$\begin{bmatrix} m+M & mL\cos\theta \\ \cos\theta & L \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u - f - w + mL\dot{\theta}^2\sin\theta - b\dot{x} - kx \\ g\sin\theta - \frac{f\cos\theta}{m} \end{bmatrix} \tag{6}$$

$$\ddot{x} = \frac{u - f - w + f\cos^2\theta + mL\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta - b\dot{x} - kx}{M + m\sin^2\theta} \tag{7}$$

$$\ddot{\theta} = \frac{-u\cos\theta + w\cos\theta - mL\dot{\theta}^2\sin\theta\cos\theta + (m+M)g\sin\theta - \frac{M}{m}f\cos\theta + b\dot{x}\cos\theta + kx\cos\theta}{L(M + m\sin^2\theta)} \tag{8}$$

The system is highly nonlinear, but it can be approximated by a linear system for $|\theta|$ small enough, say $|\theta| < 5^{\circ}$

$$\ddot{x} = \frac{u - w - mg\theta - b\dot{x} - kx}{M} \tag{9}$$

$$\ddot{\theta} = \frac{-u + w + (m+M)g\theta - \frac{M}{m}f + b\dot{x} + kx}{LM} \tag{10}$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{M} & -\frac{mg}{M} & -\frac{b}{M} & 0 \\ \frac{k}{ML} & \frac{m+M}{ML}g & \frac{b}{ML} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{LM} \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{1}{M} & 0 \\ \frac{1}{LM} & -\frac{1}{mL} \end{bmatrix} \begin{bmatrix} w \\ f \end{bmatrix} \tag{11}$$

We assumed that $M = 1\,kg$ $m = 3/4\,kg$ $b = 0.1\,N.sec/m$, $k = 0.15\,N/m.$ and we Assume that L is uncertain and $L \in [0.9 \;\; 1.9]$

Homework I:

Build a simulation model of the nonlinear and linear system in Simulink (use a fixed sampling time of one millisecond).

Homework II:

- Design an LQR controller based on linearized model (Assume that $L = 1.4(m)$).
- Apply the designed controller to the both Linear and Nonliner simulink model (remember to select an appropriate initial values for you nonlinear system such that the pendulum is vertical upward in the start)
- By using cheap and expensive LQR control, investigate how the controller gain varies (also look at the control action).
- By changing the value of real L in the system, check the stability of the closed loop system (Linear system+LQR controller) and find out for which parametric uncertainty interval, the LQR controller (designed for $L = 1.4(m)$) can stabilize the system.
- Through a simulation, repeat the question above for the nonlinear model. Remember that this will depend on your choice of initial conditions and disturbances (pick all the initial values as zero and select Band-Limited White Noise for w and f with power noise 10 and 10^{-4}, respectively. Use one millisecond sampling time; filter w and f with a low pass filter $\frac{0.64}{s+0.8}$. For the measurement noise, use Band-Limited White Noise with power noise 10^{-7}.)

Homework III:

- Assume that you can only measure x and θ. Design a Kalman Filter and estimate the velocities. (Use one millisecond sampling time and select Band-Limited White Noise for w and f with power noise 10 and 10^{-4}, respectively. Filter w and f with a low pass filter $\frac{0.64}{s+0.8}$. Remember to include the dynamic of the filters in your KF. For the measurement noise, use Band-Limited White Noise with power noise 10^{-7}.)

- Apply the designed LQR gain to estimated state from KF and build an LQG controller; Apply the designed LQG controller to the both Linear and Nonliner simulink model (remember to select an appropriate initial values for you nonlinear system such that the pendulum is vertically upward in the start)

Homework IV:

Attached to this report, you receive matlab files for state space matrices of 4 continuous-time controllers (the controllers are two input one output dynamic systems; the order of the input to the controllers is $[\theta \;\; x]^T$). These controllers are robust and provide guaranteed stability and performance according to the table 1.

Table 1: Summary of Designing Controllers

Controller	Uncertainty Interval
Local Controller#1	[0.90 1.15]
Local Controller#1	[1.15 1.40]
Local Controller#3	[1.40 1.65]
Local Controller#3	[1.65 1.90]

- Discretize the given controllers with sampling time $Ts = 0.001$.
- Design 4 KF based on nominal L values $\{1.0250\ ,\ 1.2750\ ,\ 1.5250\ ,\ 1.7750\}$
- Design a hybrid controller using dynamic hypothesis testing (you can merge the control signals using the conditional probabilities or you could switch among them and use a dwell time of $\tau = 0.005$)