HW3 MATH 7570 Fall 2023

Wenjuan Bian

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1. In the textbook, Section 14.7, problem 20.
2. In the textbook, Section 14.7, problem 22.
3. In the textbook, Section 14.7, problem 24.
4. In the textbook, Section 14.7, problem 25.

1. T20. Suppose that $x \sim N_n(a, \sigma^2 I)$ where a is a nonzero n-vector of constants. Let b represent another nonzero n-vector. Define $P_a = \frac{aa^T}{a^T a}$ and $P_b = \frac{bb^T}{b^T b}$.

   (a) Determine the distribution of $\frac{x^T P_a x}{\sigma^2}$. (Simplify as much as possible here and in all parts of this exercise.)

   Solution:

   Let $A = \dfrac{P_a}{\sigma^2}$, then $x^T A x \backsim \chi^2_{(n,\lambda)}$,

   where,

$$\begin{aligned}\lambda &= \frac{a^T A a}{2} \\ &= \frac{a^T a a^T a}{2\sigma^2 a^T a} \\ &= \frac{a^T a}{2\sigma^2}\end{aligned}$$

   (b) Determine the distribution of $\frac{x^T P_b x}{\sigma^2}$.

   Solution:

   Let $B = \dfrac{P_b}{\sigma^2}$, then $x^T B x \backsim \chi^2_{(n,\lambda)}$,

   where,

$$\lambda = \frac{a^T B a}{2} = \frac{a^T b b^T a}{2\sigma^2 b^T b}$$

(c) Determine, in as simple a form as possible, a necessary and sufficient condition for the two quadratic forms in parts (a) and (b) to be independent.

Solution:

For the quadratic forms $x^T A x$ and $x^T B x$ to be independent, they should be uncorrelated. A necessary and sufficient condition for this is:

$$B(\sigma^2 I)A = 0$$

Plugging in the values for A and B:

$$\frac{bb^T}{\sigma^2 b^T b} \cdot \sigma^2 I \cdot \frac{aa^T}{\sigma^2 a^T a} = 0$$

This simplifies to:

$$\frac{bb^T aa^T}{\sigma^2 (b^T b)(a^T a)} = 0$$

For this to be true, the vectors a and b must be orthogonal. Thus, the necessary and sufficient condition is:

$$ba^T = 0$$

(d) Under the condition in part (c), can the expressions for the parameters in either or both of the distributions in parts (a) and (b) be simplified further? If so, how?

Solution:

Under the condition in part (c), $ba^T = 0$, this implies vectors a and b are orthogonal.

Given that a and b are orthogonal, $a^T b = 0$ and $b^T a = 0$. Therefore, the term $a^T b b^T a$ is zero.

Thus:

$$\lambda = 0$$

for the distribution in (b), and the distribution in part (b) simplifies to:

$$x^T Bx \sim \chi^2_{(n,0)}$$

The distribution in part (a) remains unchanged, as the orthogonality condition does not affect the term $a^T a$.

2. T22. Suppose that $x \sim N_n\left(\mu, \sigma^2\left[(1-\rho)I + \rho J\right]\right)$ where $n \geq 2$, $-\infty < \mu < \infty$, $\sigma^2 > 0$, and $0 < \rho < 1$. Define $A = \frac{1}{n}J$ and $B = \frac{1}{n-1}(I - \frac{1}{n}J)$.

   (a) Determine $\mathrm{var}(x^T Ax)$.

   Solution:

$$\begin{aligned}
\mathrm{var}(x^T Ax) &= 2tr(A\Sigma)^2 + 4\mu^T A\Sigma A\mu \\
&= 2tr\{\frac{1}{n}J\sigma^2\left[(1-\rho)I + \rho J\right]\}^2 + 4\mu^T\frac{1}{n}J\sigma^2\left[(1-\rho)I + \rho J\right]\frac{1}{n}J\mu \\
&= \frac{2\sigma^4}{n^2}tr\{[(1-\rho)J + n\rho J]\}^2 + \frac{4\sigma^2}{n^2}\mu^T[(1-\rho)J^2 + \rho J^3]\mu \\
&= \frac{2\sigma^4}{n^2}\{n^2[1+(n-1)\rho]^2\} + \frac{4\sigma^2}{n^2}\cdot n[1+(n-1)\rho]\mu^T J\mu \\
&= 2\sigma^4[1+(n-1)\rho]^2 + \frac{4\sigma^2}{n}[1+(n-1)\rho]\mu^T J\mu
\end{aligned}$$

   (b) Determine $\mathrm{var}(x^T Bx)$.

   Solution:

$$\mathrm{var}(x^T Bx) = 2tr(B\Sigma)^2 + 4\mu^T B\Sigma B\mu$$

$$
\begin{aligned}
B\Sigma &= \frac{1}{n-1}(I-\frac{1}{n}J)\sigma^2\left[(1-\rho)I+\rho J\right] \\
&= \frac{\sigma^2}{n-1}(I-\frac{1}{n}J)\left[(1-\rho)I+\rho J\right] \\
&= \frac{\sigma^2}{n-1}[(1-\rho)I+\rho J-\frac{1}{n}(1-\rho)J-\frac{\rho}{n}J^2] \\
&= \frac{\sigma^2}{n-1}[(1-\rho)I+\frac{1}{n}(1-\rho)J] \\
&= \frac{\sigma^2(1-\rho)}{n-1}[I-\frac{1}{n}J]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
tr(B\Sigma)^2 &= \frac{\sigma^4(1-\rho)^2}{(n-1)^2}tr(I-\frac{1}{n}J)^2 \\
&= \frac{\sigma^4(1-\rho)^2}{(n-1)^2}tr(I-\frac{2}{n}J+\frac{1}{n^2}J^2) \\
&= \frac{\sigma^4(1-\rho)^2}{(n-1)^2}tr(I-\frac{1}{n}J) \\
&= \frac{\sigma^4(1-\rho)^2}{(n-1)^2}n(1-\frac{1}{n}) \\
&= \frac{\sigma^4(1-\rho)^2}{n-1}
\end{aligned}
$$

$$
\begin{aligned}
B\Sigma B &= \frac{\sigma^2(1-\rho)}{n-1}[I-\frac{1}{n}J]\sigma^2\left[(1-\rho)I+\rho J\right] \\
&= \frac{\sigma^4(1-\rho)}{n-1}[(1-\rho)I+\rho J-\frac{1-\rho}{n}J-\frac{1}{n}\rho J^2] \\
&= \frac{\sigma^4(1-\rho)^2}{n-1}(I-\frac{1}{n}J)
\end{aligned}
$$

$$
4\mu^T B\Sigma B\mu = \frac{4\sigma^4(1-\rho)^2}{n-1}\mu^T(I-\frac{1}{n}J)\mu
$$

Thus,

$$\begin{aligned}\text{var}(x^T Bx) &= 2tr(B\Sigma)^2 + 4\mu^T B\Sigma B\mu \\ &= \frac{\sigma^4(1-\rho)^2}{n-1} + \frac{4\sigma^4(1-\rho)^2}{n-1}\mu^T(I - \frac{1}{n}J)\mu\end{aligned}$$

(c) Are $x^T Ax$ and $x^T Bx$ independent? Explain.

Solution:

$$\begin{aligned}B\Sigma A &= \frac{\sigma^2(1-\rho)}{n-1}[I - \frac{1}{n}J]\frac{1}{n}J \\ &= \frac{\sigma^2(1-\rho)}{n-1}(\frac{1}{n}J - \frac{1}{n^2}J^2) \\ &= \frac{\sigma^2(1-\rho)}{n-1}(\frac{1}{n}J - \frac{1}{n}J) \\ &= 0\end{aligned}$$

Therefore, $x^T Ax$ and $x^T Bx$ are independent.

(d) Suppose that n is even. Let x_1 represent the vector consisting of the first $\frac{n}{2}$ elements of x and let x_2 represent the vector consisting of the remaining $\frac{n}{2}$ elements of x. Determine the distribution of $x_1 - x_2$.

Solution:

Split μ into two equal length vectors μ_1, μ_2, then distribution for x_1 is $N_{\frac{n}{2}}(\mu_1, \sigma^2[(1-\rho)I + \rho J])$ and the distribution for x_2 is $N_{\frac{n}{2}}(\mu_2, \sigma^2[(1-\rho)I + \rho J])$

Then the distribution of $x_1 - x_2$ is $N_{\frac{n}{2}}(\mu_1 - \mu_2, 2\sigma^2[(1-\rho)I + \rho J])$, since:

$$\begin{aligned}M_{X_1 - X_2}(t) &= \exp(t^T\mu_1 + \frac{1}{2}t^T\Sigma t)\exp(-t^T\mu_2 + \frac{1}{2}t^T\Sigma t) \\ &= \exp(t^T\mu_1 + \frac{1}{2}t^T\Sigma t)\exp(-t^T\mu_2 + \frac{1}{2}t^T\Sigma t) \\ &= \exp(t^T(\mu_1 - \mu_2) + \frac{t^T 2\Sigma t}{2})\end{aligned}$$

(e) Suppose that $n = 3$, in which case $x = (x_1, x_2, x_3)^T$. Determine the conditional distribution of $(x_1, x_2)^T$ given that $x_3 = 1$.

Solution:

$(x_1, x_2)^T \sim N[\mu_{(1,2)|3}, \Sigma_{(1,2)|3}]$

Where

$$\begin{aligned}\Sigma_{(1,2)|1} &= \begin{pmatrix} \sigma^2[1+(n-1)\rho] & \rho \\ \rho & \sigma^2[1+(n-1)\rho] \end{pmatrix} - \frac{1}{\sigma^2[1+(n-1)\rho]} \begin{pmatrix} \rho \\ \rho \end{pmatrix} \begin{pmatrix} \rho & \rho \end{pmatrix} \\ &= \begin{pmatrix} \sigma^2[1+(n-1)\rho] & \rho \\ \rho & \sigma^2[1+(n-1)\rho] \end{pmatrix} - \frac{\rho^2}{\sigma^2[1+(n-1)\rho]} J\end{aligned}$$

and

$$\mu_{(1,2)|3} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \frac{1-\mu_3}{\sigma^2[1+(n-1)\rho]} \begin{pmatrix} \rho \\ \rho \end{pmatrix}$$

3. T24. Suppose that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right),$$

where x_1 and μ_1 are n_1-vectors, x_2 and μ_2 are n_2-vectors, Σ_{11} is $n_1 \times n_1$, and Σ is nonsingular. Let A represent an $n_1 \times n_1$ symmetric matrix of constants and let B be a matrix of constants having n_2 columns.

(a) Determine, in as simple a form as possible, a necessary and sufficient condition for $x_1^T A x_1$ and $B x_2$ to be uncorrelated.

Solutions:

For $x_1^T A x_1$ and Bx_2 to be uncorrelated, the necessary and sufficient condition is $cov(Bx_2, x_1^T AX_1)$.

If they are uncorrelated then:

$$\begin{aligned}cov(Bx_2, x_1^T AX_1) =& E(Bx_2 x_1^T AX_1) - E(Bx_2)E(x_1^T AX_1) \\ &= E(Bx_2)E(x_1^T AX_1) - E(Bx_2)E(x_1^T AX_1) \\ &= 0\end{aligned}$$

If the $cov(Bx_2, x_1^T AX_1) = 0$, we can get:

$$
\begin{aligned}
cov(Bx_2, x_1^T AX_1) &= E(Bx_2 x_1^T AX_1) - E(Bx_2)E(x_1^T AX_1) \\
&= 0
\end{aligned}
$$

Thus we have:

$$E(Bx_2 x_1^T AX_1) = E(Bx_2)E(x_1^T AX_1)$$

Therefore, they are uncorrelated.

For $cov(Bx_2, x_1^T AX_1)$, we have

$$
\begin{aligned}
cov(Bx_2, x_1^T AX_1) =& E\{(Bx_2 - B\mu_2)[x_1^T AX_1 - E(x_1^T AX_1)]\} \\
=& E\{(Bx_2 - B\mu_2)[x_1^T AX_1 - tr(A\Sigma_{11}) - \mu_1^T A\mu_1)]\} \\
=& E[(Bx_2 - B\mu_2)x_1^T AX_1] - E(Bx_2 - B\mu_2)[tr(A\Sigma_{11}) - \mu_1^T A\mu_1] \\
=& E[(Bx_2 - B\mu_2)x_1^T AX_1] \\
=& E[(Bx_2 - B\mu_2)(x_1 - \mu_1 + \mu_1)^T A(x_1 - \mu_1 + \mu_1)] \\
=& E[(Bx_2 - B\mu_2)(x_1 - \mu_1)^T A(x_1 - \mu_1) + (Bx_2 - B\mu_2)\mu_1^T A(x_1 - \mu_1) \\
& + (Bx_2 - B\mu_2)(x_1 - \mu_1)^T A\mu_1 + (Bx_2 - B\mu_2)\mu_1^T A\mu_1] \\
=& E[(Bx_2 - B\mu_2)\mu_1^T A(x_1 - \mu_1) + (Bx_2 - B\mu_2)(x_1 - \mu_1)^T A\mu_1] \\
=& E[(Bx_2 - B\mu_2)\mu_1^T A(x_1 - \mu_1)] + E[(Bx_2 - B\mu_2)(x_1 - \mu_1)^T A\mu_1] \\
=& 2B\Sigma_{21}A\mu_1
\end{aligned}
$$

So, for $x_1^T Ax_1$ and Bx_2 to be uncorrelated, we should have:

$$cov(Bx_2, x_1^T AX_1) = 2B\Sigma_{21}A\mu_1 = 0$$

So the condition is $B\Sigma_{21}A\mu_1 = 0$

(b) Determine, in as simple a form as possible, a necessary and sufficient condition for $x_1^T Ax_1$ and Bx_2 to be independent.

Solutions:

For $x_1^T Ax_1$ and Bx_2 to be independent, the covariance between them should be zero.

same as above. The necessary and sufficient condition is:

$$cov(Bx_2, x_1^T AX_1) = 2B\Sigma_{21}A\mu_1 = 0$$

(c) Using your answer to part (a), give a necessary and sufficient condition for $x_1^T x_1$ and x_2 to be uncorrelated.

Solution:

For the $cov(x_1^T x_1, x_2)$, this means $A = I_{n1}$ and $B = I_{n2}$, thus

$$2B\Sigma_{21}A\mu_1 = 2\Sigma_{21}\mu_1$$

Then we get $cov(x_1^T x_1, x_2) = 2\Sigma_{21}\mu_1$.

Therefore we get:

$$\Sigma_{21}\mu_1 = 0$$

(d) Using your answer to part (b), give a necessary and sufficient condition for $x_1^T x_1$ and x_2 to be independent.

Solution:

Similar to (c), $\Sigma_{21}\mu_1 = 0$ is necessary and sufficient condition for $x_1^T x_1$ and x_2 to be independent.

(e) Consider the special case in which $\Sigma = (1-\rho)I + \rho J$, where $0 < \rho < 1$. Can $x_1^T x_1$ and x_2 be uncorrelated? Can $x_1^T x_1$ and x_2 be independent? If the answer to either or both of these questions is yes, give a necessary and sufficient condition for the result to hold.

Solution:

Given that

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

and the special form of Σ is:

$$\Sigma = (1-\rho)\begin{pmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \end{pmatrix} + \rho \begin{pmatrix} J_{n_1} & J_{12} \\ J_{21} & J_{n_2} \end{pmatrix},$$

From the above, we deduce:

$$\Sigma_{12} = \rho J_{12},$$

making Σ_{12} a matrix where all elements have value ρ.

For $x_1^T x_1$ and x_2 to be uncorrelated, the condition derived from part (c) was $\Sigma_{21}\mu_1 = 0$. Given that all elements of Σ_{21} have value ρ, it's necessary for $\rho J_{12}\mu_1 = 0$ to meet this condition. Similarly, for $x_1^T x_1$ and x_2 to be independent, the condition was $\Sigma_{21}\mu_1 = \rho J_{12}\mu_1 = 0$.

4. T25. Suppose that $x \sim N_n(\mu, \Sigma)$, where $\text{rank}(\Sigma) = n$. Let A represent a nonnull symmetric idempotent matrix such that $A\Sigma = kA$ for some $k \neq 0$.

(a) $x^T Ax/k \sim \chi^2(\nu, \lambda)$ for some ν and λ, and determine ν and λ.

solution:

Given a normally distributed vector $x \sim N_n(\mu, \Sigma)$ and an idempotent matrix A with rank n, the quadratic form $Q = x^T Ax$ follows a non-central chi-squared distribution:

$$Q \sim \chi^2(n, \lambda')$$

Where

$$\lambda = \frac{\mu^T A\mu}{2}$$

If we scale x by a factor of $\frac{1}{\sqrt{k}}$ and then consider the quadratic form:

$$Q' = \left(\frac{x}{\sqrt{k}}\right)^T A \left(\frac{x}{\sqrt{k}}\right)$$

Then the non-centrality parameter is adjusted to:

$$\lambda = \frac{1}{2k}\mu^T A\mu$$

Thus, the distribution of Q' is:

$$Q' \sim \chi^2(n, \frac{1}{2k}\mu^T A\mu)$$

Where:

n is again the degrees of freedom, equal to the rank of A. $\frac{1}{k}\mu^T A\mu$ is the adjusted non-centrality parameter.

(b) $x^T Ax$ and $x^T(I - A)x$ are independent.

solution:

$$A\Sigma(I - A) = A\Sigma - A\Sigma A \qquad\qquad = kA - kAA = 0$$

Sincere $A\Sigma(I - A) = 0$, $x^T Ax$ and $x^T(I - A)x$ are independent

(c) Ax and $(I - A)x$ are independent.

solution:

$$\begin{aligned} cov(Ax, (I - A)x) &= Acov(x)(I - A) \\ &= A\Sigma(I - A) \\ &= 0 \end{aligned}$$

Thus, Ax and $(I - A)x$ are independent.

(d) $k = \frac{\text{tr}(A\Sigma A)}{\text{rank}(A)}$.

solution:

$$\begin{aligned} \text{tr}(A\Sigma A) &= \text{tr}(kAA) \\ &= \text{tr}(kA) \\ &= k \cdot \text{tr}(A) \end{aligned}$$

Since A is idempotent, $\text{tr}(A) = \text{rank}(A)$, thus,

$$\begin{aligned} \frac{\text{tr}(A\Sigma A)}{\text{rank}(A)} &= \frac{k \cdot \text{tr}(A)}{\text{rank}(A)} \\ &= k \end{aligned}$$