HW3 MATH 7570 Fall 2023

Wenjuan Bian

September 2023

1. In the textbook, Section 4.3, problem 11. No skewness and kurtosis are considered.
2. In the textbook, Section 4.3, problem 12.
3. In the textbook, Section 4.3, problem 13.
4. In the textbook, Section 4.3, problem 14.

1. (T11) Let x be a random n-vector with mean μ and positive definite variance– covariance matrix Σ. Define $\Sigma^{-\frac{1}{2}} = (\Sigma^{\frac{1}{2}})^{-1}$

   (a) Show that $\Sigma^{-\frac{1}{2}}\Big(x - \mu)$ has mean 0 and variance–covariance matrix I.

   Solution:

   Let

$$z = \Sigma^{-\frac{1}{2}}(x - \mu)$$

$$\begin{aligned} E(z) &= E\left(\Sigma^{-\frac{1}{2}}(x - \mu)\right) \\ &= \Sigma^{-\frac{1}{2}} E(x - \mu) \\ &= \Sigma^{-\frac{1}{2}}(E(x) - \mu) \\ &= \Sigma^{-\frac{1}{2}}(\mu - \mu) \\ &= \Sigma^{-\frac{1}{2}}(0) \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}(z) &= \mathrm{Var}\left(\Sigma^{-\frac{1}{2}}(x-\mu)\right) \\
&= E[\left(\Sigma^{-\frac{1}{2}}(x-\mu)\right)\left((x-\mu)^T\Sigma^{-\frac{1}{2}}\right)] \\
&= \Sigma^{-\frac{1}{2}}E[(x-\mu)(x-\mu)^T]\Sigma^{-\frac{1}{2}} \\
&= \Sigma^{-\frac{1}{2}}[\mathrm{Var}(x-\mu)]\Sigma^{-\frac{1}{2}} \\
&= \Sigma^{-\frac{1}{2}}\Sigma\Sigma^{-\frac{1}{2}} \\
&= I
\end{aligned}
$$

So, z has a variance-covariance matrix I.

Thus, $z = \Sigma^{-\frac{1}{2}}(x-\mu)$ has mean 0 and variance-covariance matrix I.

(b) Determine $E[(x-\mu)^T\Sigma^{-1}(x-\mu)]$ and $var[(x-\mu)^T\Sigma^{-1}(x-\mu)]$, assuming for the latter that the skewness matrix Λ and excess kurtosis matrix Ω of x exist.

Solution:

Let

$$
y = x - \mu
$$

Then t has mean 0 and variance-covariance matrix Σ

By Theorem 4.2.4, $E(x^TAx) = \mu^TA\mu + tr(A\Sigma)$, we have:

$$
\begin{aligned}
E[(x-\mu)^T\Sigma^{-1}(x-\mu)] &= E[y^T\Sigma^{-1}y] \\
&= 0^T\Sigma^{-1}0 + tr(\Sigma^{-1}\Sigma) \\
&= n
\end{aligned}
$$

var$[(x-\mu)^T\Sigma^{-1}(x-\mu)]$**:**

Let

$$
y = \Sigma^{-\frac{1}{2}}(x-\mu)
$$

Where y follows a multivariate normal distribution $y \sim N(0, I)$ by (a).

The quadratic form of y:

$$y^T y = (x - \mu)^T \Sigma^{-1} (x - \mu)$$

If $y \sim N(0, I)$, then the quadratic form $y^T y$ follows a chi-squared distribution with n degrees of freedom, i.e., $y^T y \sim \chi_n^2$.

For a chi-squared distribution with n degrees of freedom, the variance is $2n$. Thus,

$$\text{var}(y^T y) = \text{var}[(x - \mu)^T \Sigma^{-1} (x - \mu)] = 2n$$

Hence, the variance of the quadratic form is $2n$.

According to $var(X^T AX) = 2tr(A\sigma)^2 + 4\mu^T AA\mu$,

$$\begin{aligned} var[(x - \mu)^T \Sigma^{-1} (x - \mu)] &= 2tr(\Sigma\Sigma^{-1})^2 + 0 \\ &= 2n + 0 \\ &= 2n \end{aligned}$$

This is consistent with the result above.

2. (T12) Let x be a random n-vector with variance–covariance matrix $\sigma^2 I$, where $\sigma^2 > 0$, and let Q represent an n $\times$ n orthogonal matrix. Determine the variance– covariance matrix of Qx.

Solution:

By the property of orthogonal matrices, $Q^T Q = I$.

$$\begin{aligned} \text{Cov}(Qx) &= E[(Qx - E(Qx))(Qx - E(Qx))^T] \\ &= E[(Qx)(Qx)^T] - E[Qx]E[(Qx)^T] \end{aligned}$$

Given that x has mean 0 (not specified), we have:

$$\begin{aligned} &E[x] = 0 \\ &E[Qx] = QE[x] = 0 \\ &E[(Qx)^T] = 0 \end{aligned}$$

and $E[xx^T] = \sigma^2 I$, thus

$$\begin{aligned}
\text{Cov}(Qx) &= E[(Qx)(Qx)^T] \\
&= E[Qxx^TQ^T] \\
&= Q(\sigma^2 I)Q^T \\
&= \sigma^2 I
\end{aligned}$$

Thus, the variance-covariance matrix of Qx is $\sigma^2 I$.

3. (13) Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ have mean vector $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and variance–covariance matrix $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$, where x_1 and μ_1 are m-vectors and Σ_{11} is $m \times m$. Suppose that Σ is positive definite, in which case both Σ_{22} and $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ are positive definite by Theorem 2.15.7a.

(a) Determine the mean vector and variance–covariance matrix of $x_{1.2} = x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$.

Solution:

$$\begin{aligned}
E[x_{1.2}] &= E\left[x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)\right] \\
&= E[x_1] - E[\mu_1] - \Sigma_{12}\Sigma_{22}^{-1}E[x_2 - \mu_2] \\
&= \mu_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(\mu_2 - \mu_2) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
var(x_{1.2}) &= var(x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)) \\
&= var(var(x_1) + var(\Sigma_{12}\Sigma_{22}^{-1}x_2) - 2cov((x_1)(\Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2))
\end{aligned}$$

$$\begin{aligned}
\text{var}(\Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)) &= \text{var}(\Sigma_{12}\Sigma_{22}^{-1}(x_2)) \\
&= \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}\Sigma_{22}^{-1T}\Sigma_{12}^T \\
&= \Sigma_{12}\Sigma_{22}^{-1T}\Sigma_{12}^T \\
&= \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}
\end{aligned}$$

$$
\begin{aligned}
cov((x_1)(\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)) &= cov((x_1)(x_2))(\Sigma_{12}\Sigma_{22}^{-1})^T \\
&= \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\text{var}(x_{1.2}) &= \Sigma_{11} + \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - 2(\Sigma_{12}(\Sigma_{22}^{-1})\Sigma_{21}) \\
&= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}
\end{aligned}
$$

(b) Show that $\text{cov}(x_1 - \Sigma_{12}\Sigma_{22}^{-1}x_2, x_2) = 0$.

Solution:

$$
\begin{aligned}
cov(x_1 - \Sigma_{12}\Sigma_{22}^{-1}x_2, x_2) &= E[(x_1 - \Sigma_{12}\Sigma_{22}^{-1}x_2)x_2] - E(x_1 - \Sigma_{12}\Sigma_{22}^{-1}x_2)E(x_2) \\
&= E[(x_1x_2 - \Sigma_{12}\Sigma_{22}^{-1}x_2x_2] - E(x_1 - \Sigma_{12}\Sigma_{22}^{-1}x_2)E(x_2) \\
&= E(x_1x_2) - E(x_1)E(x_2) - [E(\Sigma_{12}\Sigma_{22}^{-1}x_2x_2) - E(\Sigma_{12}\Sigma_{22}^{-1}x_2)E(x_2)] \\
&= \Sigma_{12} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22} \\
&= \Sigma_{12} - \Sigma_{12} \\
&= 0
\end{aligned}
$$

(c) Determine $E\left\{x_{1.2}^T[\text{var}(x_{1.2})]^{-1}x_{1.2}\right\}$.

Solution:

By Theorem 4.2.4, we have $E(x^TAx) = \mu^TA\mu + tr(A\Sigma)$, and by (a) we have $E(x_{1.2}^T) = 0$ and $\text{var}(x_{1.2}) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Thus,

$$
\begin{aligned}
E\left\{x_{1.2}^T[\text{var}(x_{1.2})]^{-1}x_{1.2}\right\} &= \mu^T[\text{var}(x_{1.2})]^{-1}\mu + tr\{\text{var}(x_{1.2})]^{-1}\text{var}(x_{1.2})\} \\
&= 0 + tr(I) \\
&= n
\end{aligned}
$$

4. (T14) Suppose that observations $x_1, \ldots, x_n$ have common mean μ, common variance σ^2, and common correlation ρ among pairs, so that $\mu = \mu\mathbf{1}_n$ and $\Sigma = \sigma^2\left[(1-\rho)\mathbf{I}_n + \rho\mathbf{J}_n\right]$ for $\rho \in \left[-\frac{1}{n-1}, 1\right]$. Determine $E(\bar{x})$, $\text{var}(\bar{x})$, and $E(s^2)$.

Solution:

$$
\begin{aligned}
E(\bar{x}) &= E(\frac{\mathbf{1}_n' x}{n}) \\
&= \frac{1}{n} E(\mathbf{1}_n' x) \\
&= \frac{1}{n} (\mathbf{1}_n' \mu \mathbf{1}_n) \\
&= \frac{1}{n} (n\mu) \\
&= \mu
\end{aligned}
$$

$$
\begin{aligned}
var(\bar{x}) &= var(\frac{\mathbf{1}_n' x}{n}) \\
&= \frac{1}{n^2} var(\mathbf{1}_n' x) \\
&= \frac{1}{n^2} (\mathbf{1}_n' var(x) \mathbf{1}_n) \\
&= \frac{1}{n^2} (\mathbf{1}_n' \sigma^2 \left[(1-\rho)\mathbf{I}_n + \rho \mathbf{J}_n\right] \mathbf{1}_n) \\
&= \frac{\sigma^2}{n^2}\, n[1+(n-1)\rho] \\
&= \frac{1+(n-1)\rho}{n} \sigma^2
\end{aligned}
$$

$$
\begin{aligned}
E(S^2) &= \frac{1}{n-1} E[x^T(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n)x] \\
&= \frac{1}{n-1}(\mu\mathbf{1}_n)^T(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n)(\mu\mathbf{1}_n) + \frac{tr[(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n)\Sigma]}{n-1} \\
&= 0 + \frac{tr\{(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n)\sigma^2\,[(1-\rho)\mathbf{I}_n + \rho\mathbf{J}_n]\}}{n-1} \\
&= \frac{\sigma^2}{n-1}\mathrm{tr}\left\{(1-\rho)I_n + \frac{\rho}{n}J_nJ_n - \frac{1-\rho}{n}J_n - \frac{\rho}{n}J_nJ_n\right\} \\
&= \frac{\sigma^2}{n-1} tr\{I_n(1-\rho) - \frac{1-\rho}{n}J_n\} \\
&= \frac{\sigma^2}{n-1}\{n(1-\rho) - (1-\rho)\} \\
&= \sigma^2(1-\rho)
\end{aligned}
$$