PROBABILISTIC ROBOTICS:
ROBOT PERCEPTION

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1

The problem situation is represented in figures 1, 2 and 3. The frame $\mathcal{R} = (0, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ is the reference frame, the frame $\mathcal{R}^{'} = (R, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{k})$ is linked to the robot / image plane of the camera. f is the focal length of the camera, h is the distance between the focal point and the ceiling. The ceiling is the plane defined by equation $z = h - f$. The coordinates in frames $\mathcal{R}$ and $\mathcal{R}^{'}$ are related by equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \\ 0 \end{bmatrix}$$

or using the homogeneous coordinates:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & x_r \\ \sin\theta & \cos\theta & 0 & y_r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

The inverse of this transformation is

$$\begin{aligned} \begin{bmatrix} x^{'} \\ y^{'} \\ z^{'} \\ 1 \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta & 0 & \\ -\sin\theta & \cos\theta & 0 & -R(\theta)\begin{bmatrix} x_r \\ y_r \\ 0 \end{bmatrix} \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta & 0 & -\cos\theta x_r + \sin\theta y_r \\ -\sin\theta & \cos\theta & 0 & -\sin\theta x_r - \cos\theta y_r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \end{aligned}$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Moreover, simple perspective geometry shows that the coordinates$(x_m^{'}, y_m^{'}, \theta_m^{'})$ of the marker in $\mathcal{R}^{'}$ and the coordinates of its image in the camera plane $z = 0$ (x_i, y_i, θ_i) are related by

$$\begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix} = \begin{bmatrix} \frac{f}{h} & 0 & 0 \\ 0 & \frac{f}{h} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_m^{'} \\ y_m^{'} \\ \theta_m^{'} \end{bmatrix}$$

1.1.

$$\begin{bmatrix} x_m \\ y_m \\ \theta_m \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & x_r \\ \sin\theta & \cos\theta & 0 & y_r \\ 0 & 0 & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{h}{f} & 0 & 0 & 0 \\ 0 & \frac{h}{f} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ \theta_i \\ 1 \end{bmatrix}$$

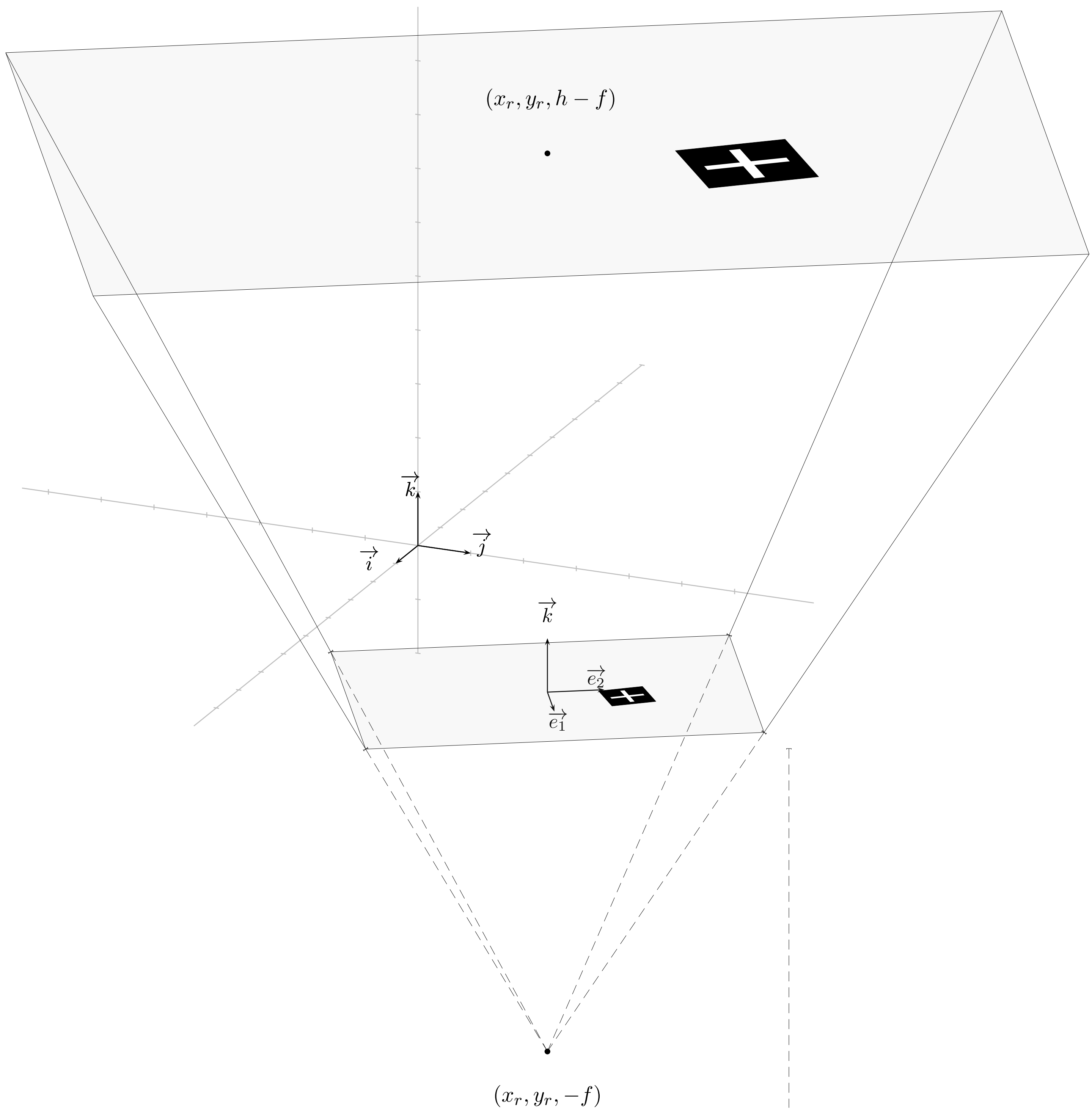

FIGURE 1. Problem setting

or

$$\begin{cases} x_m = \dfrac{h}{f}(x_i \cos\theta - y_i \sin\theta) + x_r \\ y_m = \dfrac{h}{f}(x_i \sin\theta + y_i \cos\theta) + y_r \\ \theta_m = \theta_i + \theta \end{cases}$$

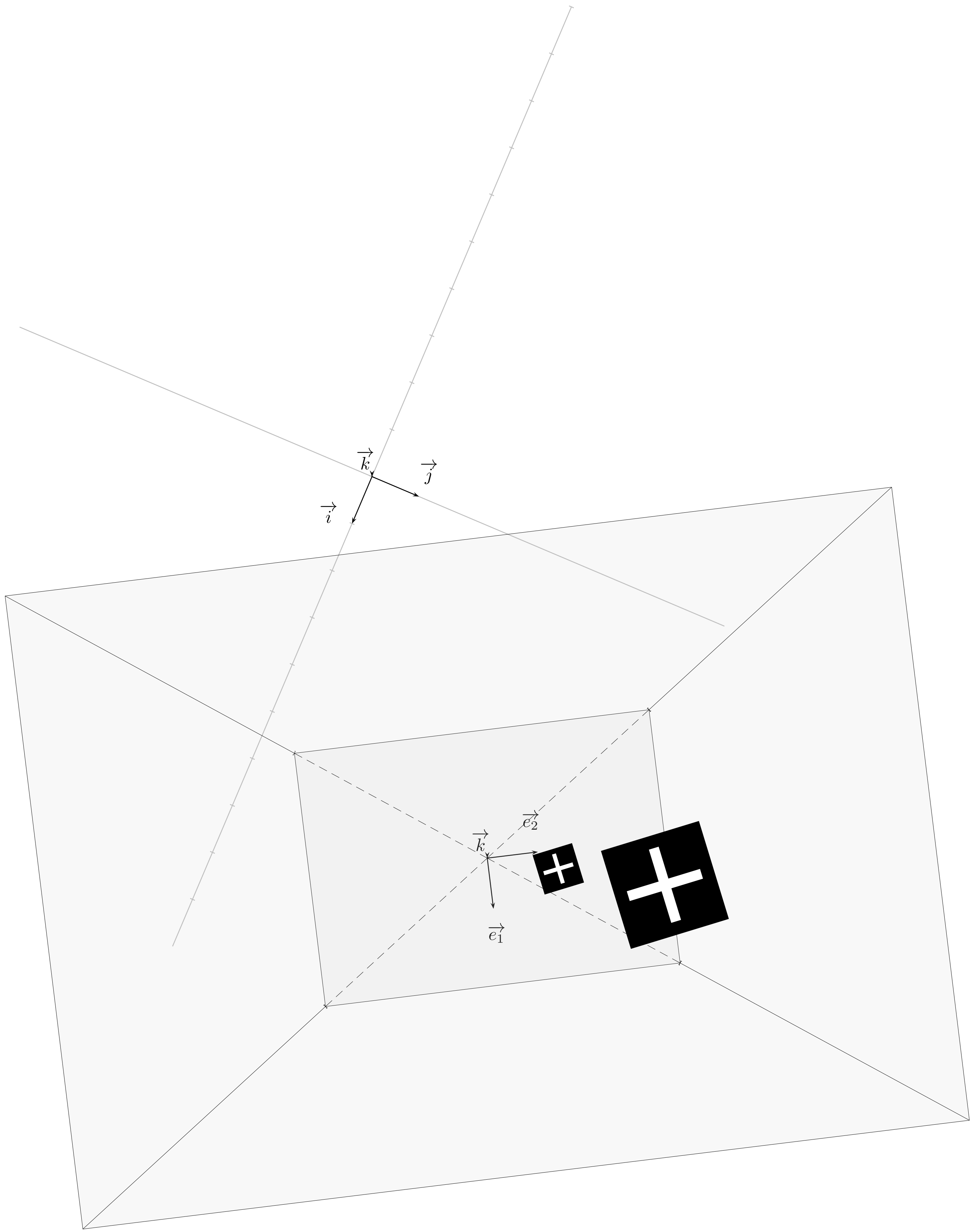

FIGURE 2. Top view

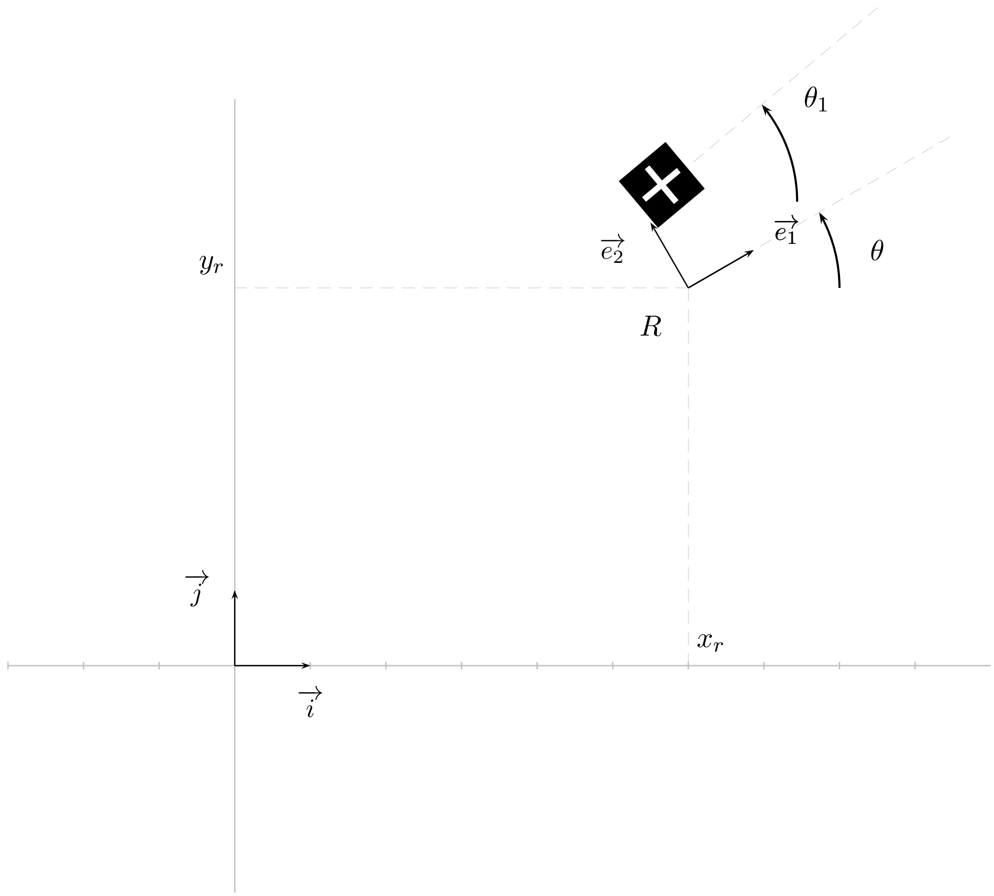

Figure 3. Frames definition

1.2.

$$\begin{bmatrix} x_i \\ y_i \\ \theta_i \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{f}{h} & 0 & 0 & 0 \\ 0 & \frac{f}{h} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & -\cos\theta x_r + \sin\theta y_r \\ -\sin\theta & \cos\theta & 0 & -\sin\theta x_r - \cos\theta y_r \\ 0 & 0 & 1 & -\theta \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ \theta_m \\ 1 \end{bmatrix}$$

$$\begin{cases} x_i = \dfrac{f}{h}(x_m \cos\theta + y_m \sin\theta) - \dfrac{f}{h}(x_r \cos\theta + y_r \sin\theta) \\ y_i = \dfrac{f}{h}(-x_m \sin\theta + y_m \cos\theta) - \dfrac{f}{h}(-x_r \sin\theta + y_r \cos\theta) \\ \theta_i = \theta_m - \theta \end{cases}$$

1.3.

$$\begin{cases} \theta = \theta_m - \theta_i \pmod{\dfrac{\pi}{2}} \\ x_r = x_m - \dfrac{h}{f}(x_i \cos\theta - y_i \sin\theta) & ① \\ y_r = y_m - \dfrac{h}{f}(x_i \sin\theta + y_i \cos\theta) & ② \end{cases}$$

Because of the symetry of the cross, its orientation is defined only modulo $\frac{\pi}{2}$. This means that there is 4 distincts poses of the robot which are compatible with given coordinates of landmark (x_m, y_m, θ_m) and (x_i, y_i, θ_i).

1.4. Suppose we capture 2 indistinguishable crosses on the camera, whose coordinates $(x_{m,j}, y_{m,j}, \theta_{m,j}), \quad j \in \{1, 2\}$ are known (but not the association to respective landmarks). Provided that the relative angle between the two is not 0 or 45° ($\theta_{m,1} - \theta_{m,2} \notin \{k\frac{\pi}{4}, \quad k \in \mathbb{Z}\}$), we can uniquely identify the landmarks by measuring the relative angle since $\theta_{i,1} - \theta_{i,2} = \theta_{m,1} - \theta_{m,2}$. The pose orientation θ is then the unique solution modulo 2π to the system:

$$\begin{cases} x_{m,2} - \dfrac{h}{f}(x_{i,2}\cos\theta - y_{i,2}\sin\theta) = x_{m,1} - \dfrac{h}{f}(x_{i,1}\cos\theta - y_{i,1}\sin\theta) \\ \\ y_{m,2} - \dfrac{h}{f}(x_{i,2}\sin\theta + y_{i,2}\cos\theta) = y_{m,1} - \dfrac{h}{f}(x_{i,1}\sin\theta + y_{i,1}\cos\theta) \end{cases}$$

$$\Leftrightarrow \begin{cases} (x_{i,1} - x_{i,2})\cos\theta + (-y_{i,1} + y_{i,2})\sin\theta = \dfrac{f}{h}(x_{m,1} - x_{m,2}) \\ \\ (y_{i,1} - y_{i,2})\cos\theta + (x_{i,1} - x_{i,2})\sin\theta = \dfrac{f}{h}(y_{m,1} - y_{m,2}) \end{cases}$$

$$\Leftrightarrow \quad \begin{bmatrix} x_{i,1} - x_{i,2} & -y_{i,1} + y_{i,2} \\ y_{i,1} - y_{i,2} & x_{i,1} - x_{i,2} \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \frac{f}{h} \begin{bmatrix} x_{m,1} - x_{m,2} \\ y_{m,1} - y_{m,2} \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \frac{f}{h((x_{i,1} - x_{i,2})^2 + (y_{i,1} - y_{i,2})^2)} \times \begin{bmatrix} x_{i,1} - x_{i,2} & y_{i,1} - y_{i,2} \\ -y_{i,1} + y_{i,2} & x_{i,1} - x_{i,2} \end{bmatrix} \begin{bmatrix} x_{m,1} - x_{m,2} \\ y_{m,1} - y_{m,2} \end{bmatrix}$$

Then we can find the position of the robot using ① and ②.

2

We note ξ the gaussian noise in measurement $\xi \hookrightarrow \mathcal{N}(\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \Sigma)$, where

$$\Sigma = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have

$$\begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix} = \begin{bmatrix} \frac{f}{h} & 0 & 0 \\ 0 & \frac{f}{h} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'_m \\ y'_m \\ \theta'_m \end{bmatrix} + \xi$$

2.1.

$$\begin{bmatrix} x_m \\ y_m \\ \theta_m \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{h}{f} & 0 & 0 \\ 0 & \frac{h}{f} & 0 \\ 0 & 0 & 1 \end{bmatrix} (\begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix} - \xi) + \begin{bmatrix} x_r \\ y_r \\ \theta \end{bmatrix}$$

$$= - \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{h}{f} & 0 & 0 \\ 0 & \frac{h}{f} & 0 \\ 0 & 0 & 1 \end{bmatrix} \xi + \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{h}{f} & 0 & 0 \\ 0 & \frac{h}{f} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \\ \theta \end{bmatrix}$$

This equation shows that conditionned on $\begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix}$ and $\begin{bmatrix} x_r \\ y_r \\ \theta \end{bmatrix}$, the r.v. $\begin{bmatrix} x_m \\ y_m \\ \theta_m \end{bmatrix}$ is gaussian $\begin{bmatrix} x_m \\ y_m \\ \theta_m \end{bmatrix} \hookrightarrow \mathcal{N}(\mu_m, \Sigma_m)$, where

$$
\begin{aligned}
\mu_m &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{h}{f} & 0 & 0 \\ 0 & \frac{h}{f} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \\ \theta \end{bmatrix} \\
\Sigma_m &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{h}{f} & 0 & 0 \\ 0 & \frac{h}{f} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Sigma \begin{bmatrix} \frac{h}{f} & 0 & 0 \\ 0 & \frac{h}{f} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (\frac{h}{f})^2\sigma^2 & 0 & 0 \\ 0 & (\frac{h}{f})^2\sigma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (\frac{h}{f})^2\sigma^2 & 0 & 0 \\ 0 & (\frac{h}{f})^2\sigma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= (\frac{h}{f})^2\Sigma
\end{aligned}
$$

2.2.

$$
\begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix} = \begin{bmatrix} \frac{f}{h} & 0 & 0 \\ 0 & \frac{f}{h} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ \theta_m \end{bmatrix} + \begin{bmatrix} -\cos\theta x_r + \sin\theta y_r \\ -\sin\theta x_r - \cos\theta y_r \\ -\theta \end{bmatrix} \right) + \xi
$$

This equation shows that conditionned on $\begin{bmatrix} x_m \\ y_m \\ \theta_m \end{bmatrix}$ and $\begin{bmatrix} x_r \\ y_r \\ \theta \end{bmatrix}$, the r.v. $\begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix}$ is gaussian $\begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix} \hookrightarrow \mathcal{N}(\mu_i, \Sigma_i)$, where

$$
\begin{aligned}
\mu_i &= \begin{bmatrix} \frac{f}{h} & 0 & 0 \\ 0 & \frac{f}{h} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ \theta_m \end{bmatrix} + \begin{bmatrix} -\cos\theta x_r + \sin\theta y_r \\ -\sin\theta x_r - \cos\theta y_r \\ -\theta \end{bmatrix} \right) \\
\Sigma_i &= \Sigma
\end{aligned}
$$

2.3.

$$
\begin{aligned}
\begin{bmatrix} x_r \\ y_r \\ \theta \end{bmatrix} &= \begin{bmatrix} x_m \\ y_m \\ \theta_m \end{bmatrix} - \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{h}{f} & 0 & 0 \\ 0 & \frac{h}{f} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix} - \xi \right) \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{h}{f} & 0 & 0 \\ 0 & \frac{h}{f} & 0 \\ 0 & 0 & 1 \end{bmatrix} \xi + \begin{bmatrix} x_m \\ y_m \\ \theta_m \end{bmatrix} - \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{h}{f} & 0 & 0 \\ 0 & \frac{h}{f} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix}
\end{aligned}
$$

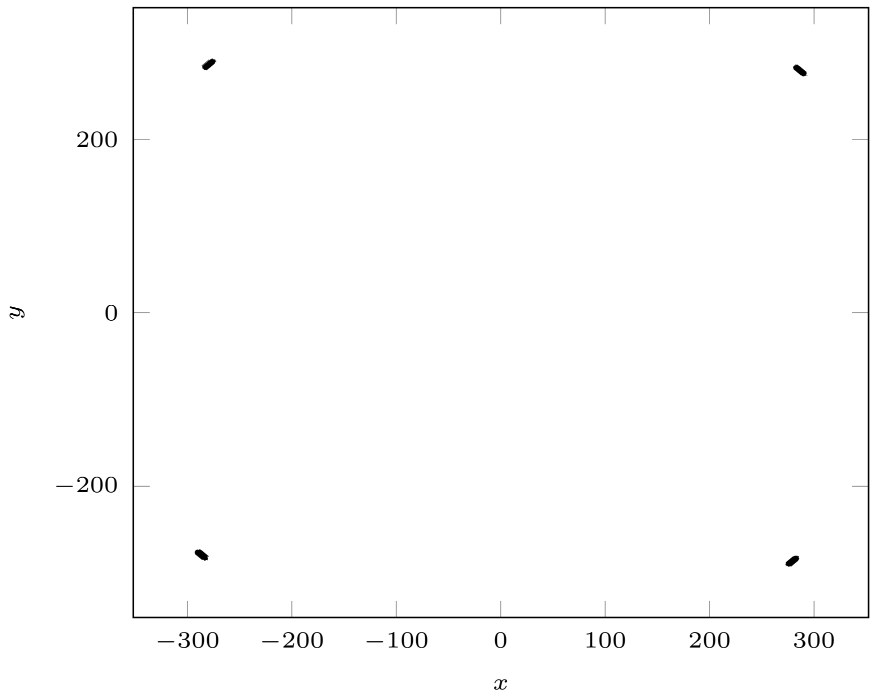

FIGURE 4. 5000 samples of robot pose, $x_m = 0$, $y_m = 0$, $\theta_m = 0$, $x_i = 2\,\text{cm}$, $y_i = 0\,\text{cm}$, $\theta_m = 45°$, $\frac{h}{f} = 200$, $\sigma^2 = 1.0\,\text{cm}^2$.

This equation shows that conditionned on $\begin{bmatrix} x_m \\ y_m \\ \theta_m \end{bmatrix}$ and $\begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix}$, the r.v. $\begin{bmatrix} x_r \\ y_r \\ \theta \end{bmatrix}$ is gaussian $\begin{bmatrix} x_r \\ y_r \\ \theta \end{bmatrix} \hookrightarrow \mathcal{N}(\mu_r, \Sigma_r)$, where

$$\mu_r = \begin{bmatrix} x_m \\ y_m \\ \theta_m \end{bmatrix} - \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{h}{f} & 0 & 0 \\ 0 & \frac{h}{f} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix}$$
$$\Sigma_r = (\frac{h}{f})^2 \Sigma$$

Note that in this exercise the equations between angles are to be understood $\pmod{2\pi}$ since the asymmetry of the landmark allows complete determination of orientation.

3

The equation to be implemented is

$$\begin{cases} \theta = \theta_m - \theta_i \pmod{\frac{\pi}{2}} \\ \begin{bmatrix} x_r \\ y_r \end{bmatrix} = \begin{bmatrix} x_m \\ y_m \end{bmatrix} - \frac{h}{f} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \xi' \end{cases}$$

where $\xi' \hookrightarrow \mathcal{N}(0, \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix})$. Because of the symetry of the landmark, we assume that the r.v. θ is uniformly distributed on $\{\theta_m - \theta_i + k\frac{\pi}{2}, k \in [\![0, 3]\!]\}$. Proposition of implementation in file `robotposespl.m`. I plotted the results of a simulation in figure 4.

4

To do.