MA4L1 Project
Pattern Formation: Turing Instabilities with Schnakenberg Kinetics

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1 Introduction

Morphogenesis is the area of embryonic biological development concerning the development of pattern and form via the reaction and diffusion of chemical species known as morphogens. Mathematical models are an important tool in understanding the mechanics which generate spatial patterns and form. In 1952, Turing first proposed the reaction-diffusion mechanism as a model for pattern formation in the context of biological morphogenesis [1]. His proposal was that two diffusable morphogens, an activator which stimulates production of both substances, and an inhibitor, which slows production, could react and cause spatial patterns due to a spatial instability driven by diffusion. Whilst diffusion usually acts as a steadying mechanism, in Turing's model he suggests that under certain conditions, morphogens exhibiting spatially uniform, stable, steady states in the absence of diffusion, can react and diffuse in such a way that unstable steady-state patterns, known as Turing patterns, are produced. The general form of a reaction-diffusion system is given by:

$$\frac{\partial \mathbf{c}}{\partial t} = D\nabla^2\mathbf{c} + \mathbf{f}(\mathbf{c}). \tag{1}$$

Strictly speaking, this equation implies Fick's law so is only valid for dilute systems, however in this context it is a valid assumption. Where $\mathbf{c}$ is a vector representing morphogen concentrations, ∇^2 is the Laplacian, $\mathbf{f}$ is a function of reaction kinetics, and D is a diagonal matrix representing diffusion coefficients. We can consider (1) as a system of ordinary differential equations. In this project, our main focus will be on systems with two species,

$A = A(\mathbf{x}, t)$ and $B = B(\mathbf{x}, t)$, on a two dimensional domain, $0 < x < 1$, $0 < y < 1$, this gives us [2]:

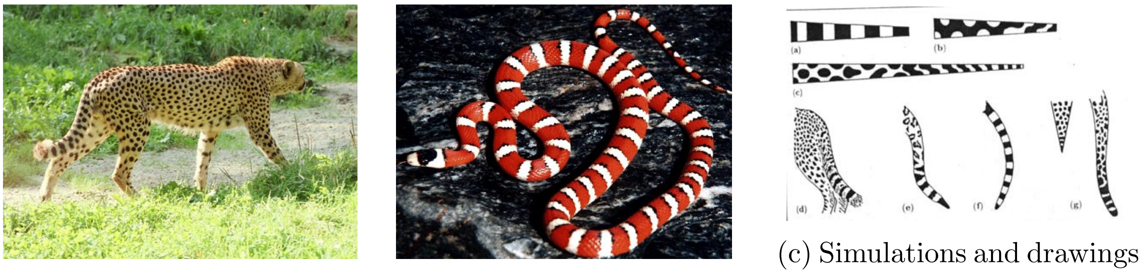
$$\frac{\partial A}{\partial t} = D_A \nabla^2 A + F(A, B), \tag{2}$$

$$\frac{\partial B}{\partial t} = D_B \nabla^2 B + G(A, B), \tag{3}$$

where F and G are the non-linear reaction kinetics.

Without diffusion, $D_A = D_B = 0$ and the morphogens would autocatalytically grow to a linear steady state. Inerestingly, the same result occurs under infinite diffusion, only intermediate levels of diffusion allow inhomogeneity to develop. The presence of diffusion however, gives Turing instabilities drive spatially inhomogeneous solutions which break symmetry and give patterns similar to those observed in nature.

There are a few analogies which have been used in an effort to intuitively describe the phenomenon, most famously is Murray's grasshopper analogy [2]. Another intuitive explanation is based upon predators and prey; an increase in prey causes an increase in predators, which causes a decrease in prey, which causes a decrease in predators, and so on. However, as the prey/predator population increases, the spatial area over which they live also increases, if we assume that the spatial range of a predator increases at a greater rate than the spatial range of the prey, we get pockets of prey between the predators, forming patterns. The shape and size of the domain, relative abundance of activator and inhibitor, and the rate at which the activator and inhibator diffusive, vastly influence the patterns produced. The application to biological morphogenesis is most clearly demonstrated by the abundance of patterns arising on the coats of animals, if this is indeed caused by the activator-inhibator mechanics as Turing suggested, then the size and shape of an animal (i.e. the domain), should have a huge influence on the patterns observed.



(a) Cheetah

(b) Snake

(c) Simulations and drawings of animal patterns [2]

Figure 1: A range of patterns can be observed in nature

2 Schnakenberg Kinetics

In our study, we will focus on one particular set of reaction kinetics, called Schnakenberg kinetics. Together with (3), these kinetics are given by:

$$F(A,B) = k_1 - k_2 A + k_3 A^2 B, \tag{4}$$

$$G(A,B) = k_4 - k_3 A^2 B, \tag{5}$$

where k_1 is a source of chemical species A, $-k_2 A$ represents exponential decay, $k_3 A^2 B$ represents the autocatalysis creating A and using B to facilitate this reaction and k_4 which is a source of B.

It is convenient to make this system dimensionless using a characteristic length L, we set:

$$\mathbf{x}^* = \frac{\mathbf{x}}{L}, \qquad \gamma = \frac{L^2 k_2}{D_A}, \qquad d = \frac{D_B}{D_A}, \qquad u = A\Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}},$$

$$t^* = \frac{D_A t}{L^2}, \qquad a = \frac{k_1}{k_2}\Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}}, \qquad b = \frac{k_4}{k_2}\Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}}, \qquad v = B\Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}},$$

Then we find using the chain rule that:

$$\frac{\partial u}{\partial t^*} = \frac{\partial\Big(A(\frac{k_3}{k_2})^{\frac{1}{2}}\Big)}{\partial t}\frac{\partial t}{\partial t^*} = \frac{L^2}{D_A}\Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}}\frac{\partial A}{\partial t}, \qquad \frac{\partial v}{\partial t^*} = \frac{L^2}{D_A}\Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}}\frac{\partial B}{\partial t}$$

$$\nabla_*^2 u = \Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}}\nabla_*^2 A = \Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}}\Big(\nabla\mathbf{x}^*\Big)^2\nabla^2 A = L^2\Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}}\nabla^2 A$$

$$\nabla_*^2 v = \Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}}\nabla_*^2 B = \Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}}\Big(\nabla\mathbf{x}^*\Big)^2\nabla^2 B = L^2\Big(\frac{k_3}{k_2}\Big)^{\frac{1}{2}}\nabla^2 B$$

We substitute these into (3) and drop the asterisks to derive:

$$\begin{aligned} u_t &= \nabla^2 u + \gamma(a - u + u^2 v) \\ v_t &= d\nabla^2 v + \gamma(b - u^2 v) \end{aligned} \tag{6}$$

Letting $\mathbf{c} = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$, we now impose homogeneous Neumann boundary conditions such that on the boundary of our domain, $\partial\Omega$, we have:

$$(\mathbf{n}\cdot\nabla)\ \mathbf{c}(\mathbf{x},t) = 0 \text{ for } x \in \partial\Omega \tag{7}$$

where n is the outward normal to our domain Ω We take the initial condition:

$$\mathbf{c}(\mathbf{x},0) = \mathbf{c}_0(\mathbf{x}). \tag{8}$$

3 Perturbation Methods

Using the initial condition above where $\mathbf{c}_0$ is a constant, equilibrium solution, and taking:

$$\frac{\partial \mathbf{c}}{\partial t} - D\nabla^2\mathbf{c} = K(\mathbf{c}) \tag{9}$$

where $D = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$ and $K = \left(\begin{smallmatrix} a-u+u^2v & 0 \\ 0 & b-u^2v \end{smallmatrix}\right)$ then since $\mathbf{c}_0$ is a steady state solution we have that:

$$\begin{aligned} \gamma(a-u+u^2v) &= 0 \\ \gamma(b-u^2v) &= 0 \end{aligned} \tag{10}$$

We then introduce a small perturbation $\mathbf{c} = \mathbf{c}_0 + \epsilon\ \mathbf{c}_1$, substituting this in to (9) and Taylor expanding gives [3]:

$$\frac{\partial(\mathbf{c}_0 + \epsilon\ \mathbf{c}_1)}{\partial t} - D\nabla^2(\mathbf{c}_0 + \epsilon\ \mathbf{c}_1) = K(\mathbf{c}_0 + \epsilon\ \mathbf{c}_1) \tag{11}$$

$$\frac{\partial(\mathbf{c}_0)}{\partial t} - D\nabla^2(\mathbf{c}_0) + \frac{\partial(\epsilon\ \mathbf{c}_1)}{\partial t} - D\nabla^2(\epsilon\ \mathbf{c}_1) = K(\mathbf{c}_0) + \epsilon\ K'(\mathbf{c}_0)\mathbf{c}_1 + \epsilon^2\ \frac{K''(\mathbf{c}_0)}{2}\mathbf{c}_1^2 + ... \tag{12}$$

$$\frac{\epsilon\ \partial(\mathbf{c}_1)}{\partial t} - \epsilon\ D\nabla^2(\mathbf{c}_1) = \epsilon\ K'(\mathbf{c}_0)\mathbf{c}_1 + \mathcal{O}(\epsilon^2) \tag{13}$$

$$\frac{\partial(\mathbf{c}_1)}{\partial t} - D\nabla^2(\mathbf{c}_1) \approx K'(\mathbf{c}_0)\mathbf{c}_1 \tag{14}$$

4 Stability Analysis

4.1 Homogeneous Steady State

We mentioned that in the absence of diffusion we expect a stable, homogeneous steady state solution. To find a steady state solution for (6) we require

$$a - u_0 + u_0^2 v_0 = b - u_0^2 v_0 = 0 \tag{15}$$

solving this gives:

$$u_0 = a + b \qquad v_0 = \frac{b}{(a+b)^2} \tag{16}$$

Starting with the steady state solution $\mathbf{c}_0 = \left(\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right) \implies K = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, with no diffusion, this state must be linearly stable and satisfy [4]:

$$u_t = \gamma(a - u + u^2 v) \tag{17}$$

$$v_t = \gamma(b - u^2 v) \tag{18}$$

Letting

$$\mathbf{w} = \mathbf{c} - \mathbf{c}_0 = \left(\begin{smallmatrix} u - u_0 \\ v - v_0 \end{smallmatrix}\right) \tag{19}$$

and linearising as above with $D = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we get:

$$\mathbf{w}_t = \gamma K'(\mathbf{c}_0) \cdot \mathbf{w}$$

$$\implies \frac{d\mathbf{w}}{dt} = \gamma \begin{pmatrix} 2u_0 v_0 - 1 & -2u_0 v_0 \\ u_0^2 & -u_0^2 \end{pmatrix} \cdot \mathbf{w} \tag{20}$$

We then look for solutions of the form $\mathbf{w}(t) \propto e^{\lambda t}$ where λ is an eigenvalue. In order for our steady state to be linearly stable we need $\mathcal{R}(\lambda) < 0$ [3] which means the solution will exponentially decay as $t \to \infty$. Substituting $\mathbf{w}(t) \propto e^{\lambda t}$ into (20) gives the following eigenvalue equation:

$$|\gamma K'(\mathbf{c}_0) - \lambda I| = \begin{vmatrix} \gamma(2u_0 v_0 - 1) - \lambda & -2\gamma u_0 v_0 \\ \gamma u_0^2 & -\gamma u_0^2 - \lambda \end{vmatrix} = 0 \tag{21}$$

$$\text{letting } A = K'(\mathbf{c}_0)$$

$$\implies \lambda^2 - \gamma tr(A)\lambda + \gamma^2 det(A) = 0 \tag{22}$$

$$\implies \lambda_\pm = \frac{\gamma tr(A)}{2} \pm \sqrt{\frac{\gamma^2 (tr(A))^2}{4} - det(A)} \tag{23}$$

Writing A in terms of a and b gives:

$$A = \begin{pmatrix} \frac{2b}{(a+b)} - 1 & -\frac{2b}{(a+b)} \\ (a+b)^2 & -(a+b)^2 \end{pmatrix} \tag{24}$$

For linear stability, we require that $det(A) = (\lambda_+ \lambda_-) > 0$, and $tr(A) = (\lambda_+ + \lambda_-) < 0$. The first condition means that the real parts of both eigenvalues have the same sign (if we have complex eigenvalues, they must be conjugate pairs and so will have equal real parts); the second condition implies that sign is negative, and hence the system is linearly stable [5]. Solving this gives the following requirements:

$$(a+b)^2 > 0 \tag{25}$$

$$(a+b)^3 > (b-a). \tag{26}$$

It is worth noting that this section is related to the laws of thermodynamics. In particular, that energy is conserved. If the real part of the eigenvalues were not negative we would have exponentially growing oscillations which equates to perpetual motion. Although, there is in effect an "energy" input from A and B being continuously added to the system byt the k_1 and k_4 terms. This must be a requirement of "energy" being taken out of the system by the first order decay term, and we must have that k_2A at least matches that put in my k_1 and k_4.

4.2 Diffusion Driven Instability

Considering (6) in 2D, in the presence of diffusion, which we now write as:

$$\frac{d\mathbf{w}}{dt} = \gamma A\mathbf{w} + D\nabla^2\mathbf{w} \text{ with } D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \tag{27}$$

We look for solutions of the form:

$$\mathbf{w}(\mathbf{x}, t) = c\mathbf{W}(\mathbf{x})e^{\lambda t} \tag{28}$$

where c is a constant, then:

$$c\lambda I\mathbf{W}(\mathbf{x})e^{\lambda t} = c\gamma A\mathbf{W}(\mathbf{x})e^{\lambda t} + cD\nabla^2\mathbf{W}(\mathbf{x})e^{\lambda t} \tag{29}$$

$$\implies \lambda I\mathbf{W}(\mathbf{x}) = \gamma A\mathbf{W}(\mathbf{x}) + D\nabla^2\mathbf{W}(\mathbf{x}). \tag{30}$$

Letting $\mathbf{W}(\mathbf{x}) = \mathbf{C}X(x)Y(y)$, we now find $\mathbf{W}(\mathbf{x})$, subject to Neumann boundary conditions, such that:

$$D\nabla^2\mathbf{W}(\mathbf{x}) = -k^2\mathbf{W} \tag{31}$$

$$X''(x)Y(y) + X(x)Y''(y) = -k^2X(x)Y(y) \tag{32}$$

Letting the eigenvalue of the spatial problem (the wavenumber), $k^2 = k_1^2 + k_2^2$, we find $X(x) = \mathbf{C}_1 cos(m\pi x)$ and $Y(y) = \mathbf{C}_2 cos(n\pi y)$, where $\mathbf{C}_1, \mathbf{C}_2$ are constant vectors and m, n are integers, therefore:

$$\mathbf{W}(\mathbf{x}) = \mathbf{C}cos(m\pi x)cos(n\pi y) \text{ where } \mathbf{C} \text{ is constant} \tag{33}$$

We now look to solve:

$$0 = \lambda I \mathbf{W}(\mathbf{x}) - \gamma A \mathbf{W}(\mathbf{x}) + Dk^2 \mathbf{W}(\mathbf{x}) \tag{34}$$

$$= \left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \gamma \begin{pmatrix} \frac{2b}{(a+b)} - 1 & -\frac{2b}{(a+b)} \\ (a+b)^2 & -(a+b)^2 \end{pmatrix} + k^2 \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \right] \cdot \mathbf{W}(\mathbf{x}) \tag{35}$$

Computing the determinant of (35) gives:

$$\lambda^2 - tr(\gamma A - k^2 D)\lambda + det(\gamma A - k^2 D) = 0 \tag{36}$$

We have that $tr(\gamma A - k^2 D) = \gamma tr(A) - k^2(1+d)$, since $k^2(1+d) > 0$, and from our linear stability requirements in the absence of diffusion we know that $tr(A) < 0$, therefore $tr(\gamma A - k^2 D) < 0$. In order for instabilities to be present we need to satisfy $\mathcal{R}(\lambda) > 0$ for at least one eigenvalue.

$$\lambda_{\pm} = \frac{tr(\gamma A - k^2 D) \pm \sqrt{(tr(\gamma A - k^2 D))^2 - 4det(\gamma A - k^2 D)}}{2} \tag{37}$$

If $det(\gamma A - k^2 D) < 0$ then $\mathcal{R}(\lambda_+) > 0$.

$$det(k^2 D - \gamma A) = dk^4 + \left[\gamma(a+b)^2 - \frac{2b\gamma d}{a+b} + \gamma d \right] k^2 + \gamma(a+b)^2 \tag{38}$$

We note that since $dk^4 > 0$ and $\gamma(a+b)^2 > 0$ so we have the condition:

$$\gamma(a+b)^2 - \frac{2b\gamma d}{a+b} + \gamma d < 0$$

$$\implies (a+b)^3 < d(b-a)$$

$$\text{Using (26) gives } d > 1 \tag{39}$$

We now minimise (38) by differentiating with respect to k^2 and setting this equal to 0 to find that the minimum occurs at $k^2 = \frac{b\gamma}{a+b} - \frac{\gamma}{2} - \frac{(a+b)^2\gamma}{2d}$. Substituting this in to (38) gives the condition:

$$\gamma^2 \left(-\frac{b^2 d}{(a+b)^2} + \frac{bd}{(a+b)} - \frac{(a+b)^2}{4} - \frac{(a+b)^4}{4d} - b(a+b) - \frac{d}{4} \right) < 0 \tag{40}$$

Rearranging we complete our set of conditions:

$$0 < (b-a) < (a+b)^3 < d(b-a) \tag{41}$$

$$\left(d(b-a) - (a+b)^3 \right)^2 > 4d(a+b)^4 \tag{42}$$

Together, these conditions define a space called the Turing space, when these conditions hold, the system will be linearly unstable. We find bounds on this range by taking the points where $det(\gamma A - k^2 D) = 0$, which occur at:

$$k_{\pm}^2 = \gamma \frac{[d(b-a) - (a+b)^3] \pm \sqrt{[d(b-a) - (a+b)^3]^2 - 4d(a+b)^4}}{2d(a+b)} \tag{43}$$

For $k_-^2 < k^2 < k_+^2$, whenever $k^2 = (m\pi)^2 + (n\pi)^2$, solutions of the form:

$$\mathbf{w}(\mathbf{x}, t) = \mathbf{C}_{m,n} cos(m\pi x) cos(n\pi y) e^{\lambda(k^2)t} \tag{44}$$

will be linearly unstable. We can take linear combinations of these solutions which will also be linearly unstable.

Equation (43) implies that the domain over which the problem exists will influence whether we observe Turing patterns or not. Since the values of k for which we observe instabilities are discrete, in order to observe patterns we require that our domain is large enough for such a value of k to occur. The larger the domain size is, the more patterns we observe. It has been suggested that this is why smaller animals are usually uniform in colour, this corresponds to γ being small. Medium sized animals such as zebra, big cats, have many spatial patterns because γ is neither too large, nor too small, and very large animals, corresponding to large γ may have patterns but the structure is very fine so they are less noticeable [2].

5 Computational Model

We wish to simulate (6) over our domain, using a finite difference method, discretizing over space and time. We assume that at $t = t_0$ our initial time, the concentrations of the morphogens u and v are independent and randomly distributed over our grid (i.e. there is some noise in the system), and for simplicity we take our space steps $\Delta x = \Delta y = h = 0.04$. Our finite difference discretization of t with time step Δt is:

$$u_t \approx \frac{u(\Delta t(n+1)) - u(\Delta t n)}{\Delta t} \tag{45}$$

$$= u_t + \mathcal{O}(\Delta t) \tag{46}$$

We take Neumann boundary conditions as before. We use a five point stencil, finite difference method to approximate the 2D Laplacian, this is defined by:

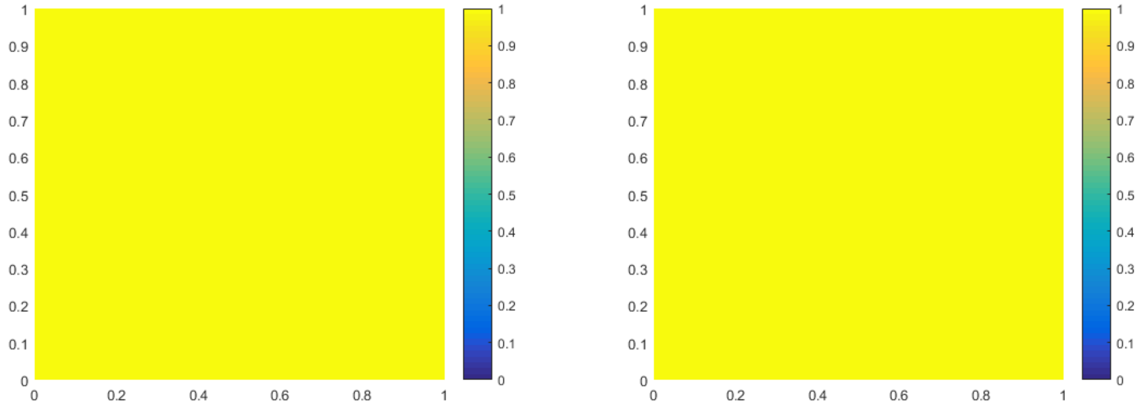
$$\nabla^2 u(x,y) \approx \frac{u(x+h,y) + u(x-hx,y) + u(x,y+h) + u(x,y-h) - 4u(x,y)}{h^2} \tag{47}$$

$$= \nabla^2 u(x,y) + \mathcal{O}(h^2) \tag{48}$$

To ensure convergence we take $\Delta t \leq \frac{h^2}{2}$ and then experimentally determine a value of Δt for which we observe a steady state. We adapt a MATLAB [6] script from [7] which is included in the appendix. We use an $50x50$ grid to begin with and take $\Delta t = 0.00016$ which is small enough for the scheme to be stable.

6 Results and Discussion

Our simulation includes four variables, we set $s = 0.0021$ which is an experimentally determined value which gives clear patterns, and $\gamma = 1$, we are then free choose variables (a, b, d) satisfying (41) and (42) so that we can observe the instabilities. First we wish to observe the steady state solution in the absence of diffusion, we do this by setting $d = 0$ and we set $a = 0.1$ and $b = 0.9$ which in satisfy the stability requirements. As we expect, in Figure 2a we observe no pattern. We next include diffusion, but choose parameters which don't satisfy (41) and (42), again as expected, we observe in Figure 2b, the solution reaching a stable, equilibrium concentration with no patterns observed.



(a) $a = 0.1$, $b = 0.9$, $d = 0$

(b) $a = 0.9$, $b = 0.1$, $d = 100$

Figure 2: Plots of concentration of u at $t = 40.0$

We now look to generate some diffusion driven patterns to back up the theory, choosing conditions satisfying (41) and (42), and as expect we see a diffusion driven pattern emerge (Figure 3).

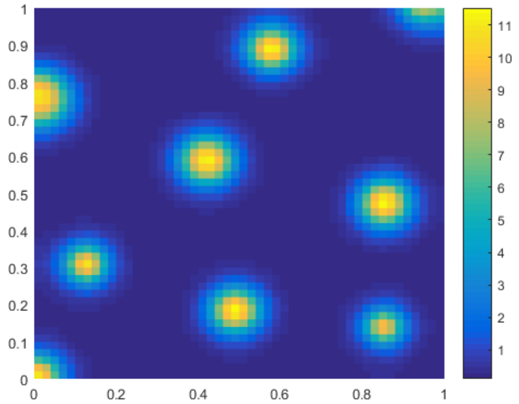
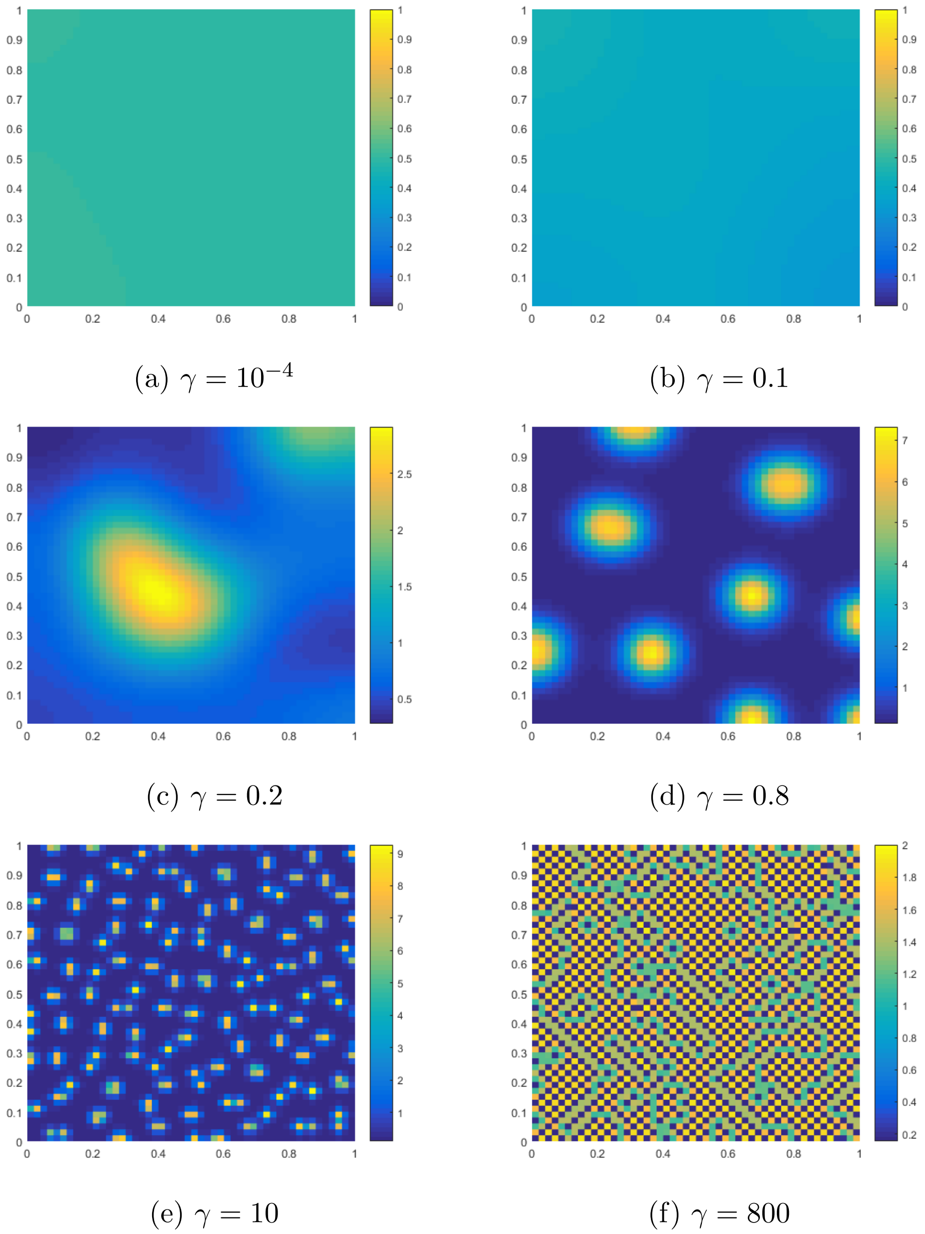


Figure 3: Plot of concentration of u with $a = 0.1$, $b = 0.9$, $d = 250$, at $t = 20.0$



(a) $\gamma = 10^{-4}$

(b) $\gamma = 0.1$

(c) $\gamma = 0.2$

(d) $\gamma = 0.8$

(e) $\gamma = 10$

(f) $\gamma = 800$

Figure 4: Plots of concentration of u with $a = 0.1$, $b = 0.9$, $d = 100$, at t=40.0

We now investigate the effect of varying γ, this corresponds to varying the scale of the domain area since $\gamma \sim L^{1/2}$, we plot the results in Figure 4. For $\gamma = 10^{-4}$ we see no influence from the reaction kinetics and the result is an even mix of both morphogens, this corresponds to the domain being too small for an allowed mode to fit into it. For $\gamma = 0.1$, we start to see an influence from the kinetics, and although the domain is still small enough that we don't observe any patterns, we see that the concentration of species u is lower since we have a high term b. When we increase to $\gamma = 0.2$ we begin to see a pattern forming but the domain is still quite small so only one mode occurs, appearing more as a mark than a pattern, we also seem the maximum value of u over the domain increase. When $\gamma = 0.8$ we see a large scale, pattern of spots corresponding to several allowed value of k fitting within our domain. For $\gamma = 10$ we see a much finer scale pattern as the domain becomes large in relation to the size of the pattern. By the time we get to $\gamma = 800$, many modes are introduced and the pattern develops to be much more complicated, here we are really limited by the scale of our discretisation meaning our simulation gives a 'checkerboard' pattern rather than an identifiable continuous one. The plots do have the expected qualitative behaviour as we vary γ, however due to the limitation of the simulation speed, we have been unable to identify a relation between γ and the number of nodes observed.

Whilst carrying out our simulations we have grown to understand the complexity and intricacy of reaction-diffusion systems. We found that for a sufficiently large domain, as long as conditions (41) and (42) are met, we observe instability, and that the mode of the instabilities depends on the size of the domain. We have seen how the Schnakenberg model can lead to interesting pattern formations similar to those found in nature. However while we have seen general trends and qualitative behaviour we have not defined specific regions wherein each kind of pattern exists. Further avenues to explore could include further analysis in the patterns on a square domain. Furthermore, we could expand the study to different domain geometries and even to 3D patterns. One limiting factor of this research has been how relatively slow the simulation is, taking a couple of minutes to generate each plot, this has meant that we were unable to carry out an investigation over a large sample of parameter values to try to derive relations between them and the plots generated.

References

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Appendices

MATLAB Script for Finite Difference Approximation

```
%Solve a Turing model system of equations in 2-D space over time. Apply
%Euler's Method to a semi-discretized Reaction-Diffusion system.
%clear all
%Grid size
Tf=1000000;
a=0; % Lower boundary
b=1; % Upper boundary
```

```matlab
M=50; % M is the number of spaces between points a and b.
dx=0.04; %(b-a)/M; % dx is delta x
dy=0.04; %(b-a)/M;

x=linspace(a,b,M+1); % M+1 equally spaced x vectors including a and b.
y=linspace(a,b,M+1);

%Time stepping
dt=0.00016; %100*(dx^2)/2; % dt is delta t the time step
N=Tf/dt; % N is the number of time steps in the interval [0,1]

%Constant Values
s=0.0021 %Scaling
D=20; % D is the Diffusion coefficient Du/Dv
g=0.0021; % sizes the domain for particular wavelengths
alpha=0.09; % a is alpha, a coefficient in f and g (-a is gamma)
beta=0.91; % b is beta, another coefficient in f and g

%pre-allocation
unp1=zeros(M+3,M+3);
vnp1=zeros(M+3,M+3);

%Initial Conditions
un=rand(M+3,M+3); %Begin with a random point between [0,1]
vn=rand(M+3,M+3);
for n=1:N
    for i=2:M+2
        un(i,1)=un(i,3); %Boundary conditions on left flux is zero
        un(i,M+3)=un(i,M+1); %Boundary conditions on right
        vn(i,1)=vn(i,3);
        vn(i,M+3)=vn(i,M+1);
    end

    for j=2:M+2
        un(1,j)=un(3,j); %Boundary conditions on left
        un(M+3,j)=un(M+1,j); %Boundary conditions on right
        vn(1,j)=vn(3,j);
        vn(M+3,j)=vn(M+1,j);
    end

    for i=2:M+2
        for j=2:M+2
            %Source function for u and v
            srcu=alpha-un(i,j)+vn(i,j)*(un(i,j))^2;
            srcv=beta-vn(i,j)*(un(i,j))^2;
            uxx=(un(i-1,j)-2*un(i,j)+un(i+1,j))/dx^2; %Laplacian u
            vxx=(vn(i-1,j)-2*vn(i,j)+vn(i+1,j))/dx^2; %Laplacian v
            uyy=(un(i,j-1)-2*un(i,j)+un(i,j+1))/dy^2; %Laplacian u
            vyy=(vn(i,j-1)-2*vn(i,j)+vn(i,j+1))/dy^2; %Laplacian v
            Lapu=uxx+uyy;
            Lapv=vxx+vyy;
            unp1(i,j)=un(i,j)+dt*(s*Lapu+g*srcu);
            vnp1(i,j)=vn(i,j)+dt*(D*s*Lapv+g*srcv);
        end
    end

    un=unp1;
    vn=vnp1;

    % Graphing
    if mod(n,6250)==0
        %subplot(2,1,2)
        hdl = surf(x,y,un(2:M+2,2:M+2));
        set(hdl,'edgecolor','none');
        axis([ -1, 1,-1,1]);
        %caxis([-10,15]);
        view(2);
        colorbar;
        fprintf('Time t = %f\n',n*dt);
        ch = input('Hit enter to continue :','s');

        if (strcmp(ch,'k') == 1)
            keyboard;
        end
    end
end
```