Assignment 2: Production Scheduling

Transportation and Scheduling (FEM21043) – January 2022

January 13, 2022

We consider a production facility that produces one type of product. For a planning horizon of T periods, the demand for each period is exactly known. Demand in a period can be met by production in that period, or from inventory. If production takes place in a period, a capacity on the production quantity must be satisfied. Both the costs for production and those for holding inventory are *pseudolinear*, as defined below. The aim of this exercise is to find a production schedule that meets all demands and respects the production capacities, against minimal costs.

The Production Scheduling Problem (PSP) can be formulated as an Integer Linear Program (ILP). In order to do so, we first introduce some notation. Denote the time horizon as $\mathcal{T} = \{1, \ldots, T\}$. We define d_t as the demand to be met in period $t \in \mathcal{T}$ and c_t as the production capacity in period $t \in \mathcal{T}$. The costs for producing x items in period $t \in \mathcal{T}$ are given by

$$P_t(x) = \begin{cases} 0 & \text{if } x = 0, \\ P + px & \text{otherwise.} \end{cases}$$

Here, $P \in \mathbb{N}$ is a positive integer, and $p \in \mathbb{N}$ is a non-negative integer. Similarly, the costs for holding I items in inventory at the end of period $t \in \mathcal{T}$ are given by

$$H_t(I) = \begin{cases} 0 & \text{if } I = 0, \\ H + hI & \text{otherwise.} \end{cases}$$

Again, $H \in \mathbb{N}$ is a positive integer, and $h \in \mathbb{N}$ is a non-negative integer.

In order to formulate the PSP as an ILP, we introduce non-negative integer decision variables x_t and I_t for every period $t \in \mathcal{T}$, that represent the production quantity in period $t \in \mathcal{T}$ and the number of products in inventory at the end of period $t \in \mathcal{T}$, respectively. We also introduce binary decision variables y_t and z_t, which are 1 if x_t and I_t are strictly positive in period $t \in \mathcal{T}$, respectively. The PSP can be formulated as the following ILP.

$$\text{(PSP)} \quad \min \sum_{t=1}^{T} (Py_t + px_t + Hz_t + hI_t)$$

such that

$$\begin{aligned} I_{t-1} + x_t &= d_t + I_t, && \forall t \in \mathcal{T}, \\ x_t &\leq c_t y_t, && \forall t \in \mathcal{T}, \\ I_t &\leq D_{t+1,T} z_t, && \forall t \in \mathcal{T}, \\ y_t, z_t &\in \{0,1\}, && \forall t \in \mathcal{T}, \\ x_t, I_t &\in \mathbb{N}, && \forall t \in \mathcal{T}, \end{aligned}$$

with $D_{t_1,t_2} = \sum_{t=t_1}^{t_2} d_t$.

PSP can also be solved using a dynamic programming (DP) algorithm. Let B be any upper bound on the optimal solution value of PSP and let $F \in \mathbb{N}$ be a factor such that $1 \leq F \leq B$. The upper bound B can, for example, be obtained using a heuristic. We define $B' = (\lfloor B/F \rfloor + T)F$. We consider a set of budgets $\mathcal{B} = \{0, F, 2F, \ldots, B'\}$. For $b \in \mathcal{B}$ and $t \in \mathcal{T}$, we define $F_t(b)$ as the maximal inventory that can be achieved at the end of period $t \in \mathcal{T}$, if the total costs for the first t periods must be below b and the budgets allocated to all periods are a multiple of F. The value b can be seen as the budget that is available for the first t periods. We can derive a recurrence relation for $F_t(b)$, for $t \in \mathcal{T}$ with $t \geq 2$, by dividing the budget b in a budget a for the first $t-1$ periods, and a budget $b-a$ for the t-th period. We first define, for $a, b \in \mathcal{B}$ and $t \in \mathcal{T}$ with $t \geq 2$, the functions

$$F_t^1(a,b) = \max_{0 \leq x_t \leq c_t} \{F_{t-1}(a) + x_t - d_t : P_t(x_t) + H_t(F_{t-1}(a) + x_t - d_t) \leq b - a\}$$

and

$$F_t^2(a,b) = \max_{0 \leq I_t < F_{t-1}(a) - d_t} \{I_t : H_t(I_t) \leq b - a\}.$$

It then holds for $b \in \mathcal{B}$ and $t \in \mathcal{T}$ with $t \geq 2$ that

$$F_t(b) = \max_{a \in \mathcal{B}, 0 \leq a \leq b} \left\{ \max\{F_t^1(a,b), F_t^2(a,b)\} \right\}.$$

For general factors F, the smallest budget $b \in \mathcal{B}$ such that $F_T(b) \geq 0$ provides an approximation of the optimal solution value: Such a budget b is at most TF above the optimal solution value. For $F = 1$, this procedure provides the optimal solution value.

On Canvas, you can find 5 instances of the PSP. The aim of this assignment is to compare the performance of the ILP to that of the DP.

a. Try to solve each instance using the ILP formulation with a time limit of 5 minutes and allowing the solver to use at most 1 thread. If the instance can be solved to optimality, report the computation time and the optimal solution value. For the instances that cannot be solved to optimality, report the best lower and upper bound. Also report the programming language and the solver (and version) you have used. Finally, provide the solution of Instance 3, in the format specified below.

b. Develop a heuristic for the PSP. Describe the heuristic and report the solution values and the computation times for all instances. Again, report the solution of Instance 3.

c. Implement the dynamic programming algorithm (with a quadratic running time). For each instance, select a factor F such that the computation time is below 5 seconds. Report the lower bound, upper bound, and computation time that are obtained for this factor. Iteratively halve the factor F and run the dynamic program, as long as the computation time is below 5 minutes. Report the solution of Instance 3.

d. Implement the dynamic programming algorithm (with a linear running time). For each instance, use the DP to find the optimal solution value. Report the computation time and the optimal objective value for each instance. (You need not report any solutions for this subquestion.)

Your answers have to be handed in before Friday January 21, 2022, at 18:00 via Canvas. You should hand in the following files.

- 1 PDF-file of at most 5 pages with the answers to the above questions.
- 1 zip-file containing all the code that you have used.
- 3 text files, each with a solution for Instance 3. Each line in these text files should describe the decision variables x_t and I_t in a period $t \in \mathcal{T}$. Thus, each line is of the following form:

  ```
  280 115
  ```

  The following files should be submitted.

  – `solution_ILP.txt`
  – `solution_heur.txt`
  – `solution_DP.txt`