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Lecture Note 9.2: CDOs and Default Correlation

Last time we described several types of asset backed securities, and introduced CDOs based on traded corporate bonds. Today's topic is to think about valuing this class of securities.

Understanding the pricing and the riskiness of CDOs presents some unique challenges. We will see why this is. We will also see some shortcuts that were frequently used by practitioners and discuss conceptual problems with them.

Outline:

I. The pricing problem.

II. What is default correlation?

III. High-dimensional models.

IV. Summary

I. The valuation problem.

- CDOs are an example of a security whose cash-flows (payoffs) at each date depend on the survival of a set of reference entities $\{X^{(i)}\}_{i=1}^{N}$ with $N > 1$.
  - ▶ Let us assume the cash-flows from the security at time t can be written as some function C of the form

$$C(t) = C(1_{\left\{\tau^{(1)} \le t\right\}}, 1_{\left\{\tau^{(2)} \le t\right\}}, \ldots, 1_{\left\{\tau^{(N)} \le t\right\}})$$

    where $\tau^{(i)}$ is the (random) default time of entity i and $1_{\{A\}}$ is the indicator function of event A.

  This class of securities is sometimes called *credit basket* products.

- Implicitly our discussion is going to be concerned with CDOs that are not "large" in a statistical sense.
  - ▶ Although there may be several dozen bonds in the pool, we will not assume that we can model the default rate by a single average, as we discussed last time.

  Rather, today, we are concerned with exactly replicating/hedging the effect of each of the individual defaults.

- For CDOs, we will assume we can trade in the underlying assets (bonds and/or CDS).
  - ▶ If we can show that an appropriate dynamic strategy can exactly replicate the cash-flows for each tranche, it will follow that we can value them as the discounted expectation of their cash-flows under the risk-neutral measure.

- ▶ Recall from last time that this is not true for all types of asset-backed securities.
- ▶ And for today's CDOs, we will have to think very carefully about how we identify the risk-neutral measure.

- Let me illustrate the nature of the issues that arise in pricing CDOs by means of an idealized example where the basket is really small: just 2 entities.

- This CDO is defined by:

  The asset side consisting of exactly two underlying bonds of risky entities, X and Y, each with notional value 100.

  The liability side consists of just two CDO tranches, J (for junior) and S (senior), whose "attachment point" is 100, i.e. the two tranches have equal face value.

- Assume all cash-flows are at the end (T) and that if either X or Y defaults its bonds have zero recovery.

- Then at T:
  - ▶ Everybody gets their principal back if nobody defaults.
  - ▶ J gets par unless *either* X or Y defaults.
  - ▶ S gets par unless *both* X and Y default.

- Clearly tranche J has more risk than S, and so will trade at a lower price. **But how much?**

- The key observation is that there is some extra risk here that *is not totally described by the individual default probabilities* of X and Y.

- Suppose the X and Y each individually have 10% chance of dying by T.

- Then we could describe this situation in (at least) two very different ways, according to the following tables.

Scenario One Probabilities:

	Y dead	Y alive	total prob:
X dead	1	9	10
X alive	9	81	90
total prob:	10	90	

Scenario Two Probabilities:

	Y dead	Y alive	total prob:
X dead	9	1	10
X alive	1	89	90
total prob:	10	90	

- J prefers the second scenario. But this case is much <u>worse</u> for S: the chance of losing are NINE TIMES higher.

- An important thing to understand about valuing CDOs is that the guys at the bottom (the lowest tranche) typically get wiped out *unless all the reference entities survive*. Whereas the guys at the top ONLY get hurt *if all the reference entities die*.

- In other words, investors in basket products are exposed to <u>default correlation risk</u>. But the sign of the exposure differs for different degrees of seniority.
  - ▶ Junior tranches are said to be "long correlation", and senior tranches are "short correlation."
  - ▶ Remember: *correlated default equals correlated survival.*

- Each tranche holder is speculating the same way (they are long) on *overall* credit risk of the basket. But they may be exposed quite differently to the risk of <u>correlated defaults.</u>

- Now consider a second example: counterparty risk in CDS.

- Last week, we computed how the default risk affected the CDS fee under some particular default models.
  - ▶ Now I just want to draw an analogy with the simple CDO above.

- Suppose you are interested in buying default protection on X <u>from</u> Y.

- Let's say we are just buying a simple binary credit put, with zero recovery in default except that it has the additional exposure to Y.

- Then its payoffs at T are:

Risky Counterparty Payoffs:

	Y dead	Y alive
X dead	0	100
X alive	0	0

which looks very much like our junior tranche above,

Junior Tranche Payoffs:

	Y dead	Y alive
X dead	0	0
X alive	0	100

- Clearly the counterparty risk is going to cause this put to be priced at a discount to riskless ones that we discussed earlier. But again we can see that the amount of this discount is going to depend on the probability of *joint* default.

- Relative to the CDO example, the payoff of our put occurs in an "off-diagonal" cell of the matrix. Intuitively, this tells us we are going to <u>dislike</u> correlation.

- We can write the payoffs of both our example securities with the indicator functions introduced earlier.

$$P^{JuniorCDO}(T) = 1_{\left\{\tau^{(X)}>T\right\}} \cdot 1_{\left\{\tau^{(Y)}>T\right\}}$$

and

$$P^{RiskyCDS}(T) = 1_{\left\{\tau^{(X)}\leq T\right\}} \cdot 1_{\left\{\tau^{(Y)}>T\right\}}.$$

- To compute expectations that give us the prices, we need to have the risk-neutral <u>joint</u> distribution of the two survival variables.

- Let us define:

$$H^{XY}(t_1, t_2; t) \equiv \text{Prob}_t^Q\{\tau^{(X)} \leq t_1 \text{ and } \tau^{(Y)} \leq t_2\}$$

or, equivalently, the joint density $h^{XY}(t_1, t_2; t)$ via

$$H^{XY}(t_1, t_2; t) = \int_0^{t_1} \int_0^{t_2} h^{XY}(u, v; t)\, dudv.$$

- Then the CDO tranche value can be written

$$B_{t,T} \int_T^{\infty} \int_T^{\infty} h^{XY}(u, v; t)\, dudv.$$

and the credit put with risky counterparty is

$$B_{t,T} \int_T^{\infty} \int_0^{T} h^{XY}(u, v; t)\, dudv.$$

- The trouble is *where do we get this joint distribution?*

- We have seen earlier in the semester that we can extract the individual – or marginal – risk-neutral default distributions from the prices of traded CDSs.
  - ▶ Recall, if we have an entire term structure of CDS fees for each entity, then there is a straightforward iterative system we can solve to get $h^{(i)}(T;t)$ for each T.

- But, as the example above makes clear, the two marginal distributions are <u>not</u> sufficient to determine the joint distribution.

- While there is a great deal of information in the traded prices of default insurance for each entity, the CDS market does not supply enough information to value our securities.
  - ▶ We need to supply extra information about *default correlation*.
  - ▶ Building, estimating, and testing such a specification adds significant complexity to valuing credit basket products.

II. What is default correlation?

- What do we really mean by default correlation?
- In fact, there are distinct ways of quantifying the intuitive idea of clustering in time, and they do not always represent exactly the same thing.
- Mathematically, for any two random variables, the standard correlation coefficient is defined to be the covariance divided by the product of the standard deviations.
- So for any two entities, we can define the correlation between their survival indicator variables.

$$r_{XY}(T) \equiv$$

$$\frac{\mathrm{E}[1_{\{\tau^{(X)}>T\}}\,1_{\{\tau^{(Y)}>T\}}] - \mathrm{E}[1_{\{\tau^{(X)}>T\}}]\mathrm{E}[1_{\{\tau^{(Y)}>T\}}]}{\sqrt{\mathrm{E}[(1_{\{\tau^{(X)}>T\}})^2] - (\mathrm{E}[1_{\{\tau^{(X)}>T\}}])^2}\sqrt{\mathrm{E}[(1_{\{\tau^{(Y)}>T\}})^2] - (\mathrm{E}[1_{\{\tau^{(Y)}>T\}}])^2}}$$

  - Or, more simply

$$\frac{p_{XY} - p_X p_Y}{\sqrt{p_X(1-p_X)}\,\sqrt{p_Y(1-p_Y)}}$$

    where p_X stands for the risk-neutral survival probability of X, and p_{XY} stands for the risk-neutral probability of joint survival to T.

  - This formula gives one possible definition of default correlation.

- A second type of correlation we could define is the correlation between the default times themselves.

$$\tilde{r}_{XY} \equiv \frac{\mathrm{E}[\tau^{(X)}\ \tau^{(Y)}] - \mathrm{E}[\tau^{(X)}]\mathrm{E}[\tau^{(Y)}]}{\sqrt{\mathrm{E}[(\tau^{(X)})^2] - (\mathrm{E}[\tau^{(X)}])^2}\sqrt{\mathrm{E}[(\tau^{(Y)})^2] - (\mathrm{E}[\tau^{(Y)}])^2}}$$

- This quantity is nice in that it is not defined with respect to a particular time horizon.
  - ▶ Notice that it is not the same thing as $r_{XY}(T)$.

- Statistically, how could one estimate default correlations for two companies given that they're both still alive!?
  - ▶ Practitioners typically rely on historical default rates for companies in particular categories.
    - ∗ For example, for Apple and Samsung, we could compare joint survival probabilities for pairs of consumer electronics firms in different countries.
  - ▶ I'm sure you can imagine more sophisticated ways of doing things like this.

- Last week, we discussed several modelling assumptions that can link defaults across entities.

(A) Correlation between asset value or stock price processes in a structural model.

(B) Correlations between default intensity processes.

(C) Common jumps into default.

(D) Or even common jumps in intensities.

- Specifying a model with one or more of these features (and appropriate prices of risk) will identify the risk neutral joint survival distribution $H^{XY}(t_1, t_2; t)$.

- Again, it is worth pointing out that the different types of correlation here are not identical.
  - ▶ Default intensity correlation will not be equal to default-time correlation, and so on.
  - ▶ Different modeling aproaches will dictate different estimation strategies. For example, correlations in CDS spreads could be used to estimate correlations in default intensity processes.

III. High-dimensional models.

- As we discussed last week, to value a CDO tranche we could simulate histories of default experience by simulating correlated paths of each entity's (risk neutral) default intensity.
  - ▶ And then, conditional on each intensity at each date, we simulate the default outcomes separately.

- However there is a practical issue with this approach: with a large basket it can involve a lot of computer power.
  - ▶ Many CDOs have more than 100 names in their asset portfolio. So, with stochastic intensities, you'd have to simulate, e.g., 100 correlated λs for perhaps 1000 time steps (e.g., 250 step/year times 4 years).
  - ▶ Then, to get accurate values for the low-risk tranches (which don't experience default often), we might need 100000 paths.
  - ▶ This would be very slow, especially if we wanted to try lots of different input values to see how the answer changed.
  - ▶ And this is assuming that there aren't other sources of risk that we also want in the model.
    - ∗ We might also want to relax the assumption that the recovery rates, $R^{(i)}$ are known in advance and the riskless rate is fixed.

- In fact, all the work described above generates information we don't really need, namely, the entire path of each firm's experience. *All we really need to know are the <u>default dates</u> of each name.*
  - ▶ Let me re-emphasize this point. If we are trying to compute
$$\mathrm{E}_{\mathrm{t}}^{\mathrm{Q}}[\int_t^T e^{-r(u-t)}\Gamma^{UL}(u)du]$$
where Γ^{UL} are the cash flows to a given tranche, then the amount of those cash-flows is solely determined by which of the underlying reference entities are alive at each date.
  - ▶ Thus, conditional on the realized default <u>times</u> of all of our entities (call them $\tau^{(1)}, \tau^{(2)}, \ldots \tau^{(N)}$), our valuation is just a deterministic exercise in mapping the cash-flows to the tranches.

- An efficient model, then, would be one that told us directly the distribution of the default dates.
  - ▶ Then we could just draw one random number for each entity from that distribution for each simulated outcome, instead of an entire path.

- We have seen that we can infer the risk-neutral default distribution from market prices of single-name credit protection (assuming known recovery rates).

- If $H(s)$ is the implied probability that $\tau \leq s$, then we can draw random numbers from this distribution by the so-called *inverse cdf* method.

- Here's how it works:

  (i) Draw u from a uniform random number generator.

  (ii) Compute $H^{-1}(u)$, i.e. find the value, t^* on the horizontal axis such that $H(t^*) = u$.

  (iii) Repeat.

  By construction t^* has the right distribution:

  $\mathrm{Prob}(t^* \leq s) = \mathrm{Prob}(H^{-1}(u) \leq s) = \mathrm{Prob}(u \leq H(s)) = H(s)$

  since $u \sim U([0, 1])$.

- Sometimes the distributions extracted from CDS prices can be approximated by a simple exponential function:

$$H(s, t) \doteq 1 - e^{-\hat{\lambda}^Q (s-t)}.$$

  ▶ If that holds, then $H^{-1}(u) = -\log(1 - u)/\hat{\lambda}^Q$.

  ▶ Or, since $1 - u$ also $\sim U([0, 1])$, we can just put $t^* = -\log(u)/\hat{\lambda}^Q$.

- We have seen that a model with constant jump intensity λ would in fact lead to exactly this type of cdf. BUT, if you want to model the intensity as stochastic, the cdf won't be exponential. And the cdf itself will change over time.

- Returning to our multivariate world, what we really want is to simulate default times from an N-dimensional process which has specified *marginal* default intensities.
  - ▶ For this we need some specification for how the marginals are composed into the joint distribution.

- As we have already emphasized, the CDS market tells us <u>nothing</u> about this.

- If we just apply our inverse cdf method to each of the marginals independently, then that corresponds to the very strong assumption that the default events are independent – which may not be realistic.

- It is not obvious, even for the case of constant-intensity exponential random variables, how one should even <u>define</u> correlated defaults.

- One way of constructing correlated exponential outcomes is to do the following.

  (i) Draw one normally distributed vector $Z \sim \mathsf{N}(0, \mathcal{R})$ using some correlation matrix $\mathcal{R}$.

  (ii) Turn these variables into uniform variates (on $[0, 1]$) by applying the one-dimensional normal CDF: $\nu^{(n)} = \mathcal{N}(Z^{(n)})$.

  (iii) Turn these into exponentially distributed draws via $\tau^{(n)} = -\log(\nu^{(n)})/\hat{\lambda}^{Q,(n)}$

  This is called the "normal Gaussian copula" method (NGC).

- If the normal draws for two entities are highly correlated, then the default times (scaled by their respective intensities) will be highly correlated.

- Notice the last step is just the inverse cdf method. The second step might be called the inverse-inverse cdf method for generating uniform numbers.

- The first step is the one we really have to pay attention to.

- There is no particular reason to use the multivariate normal distribution in the first step. It is just a distribution for which correlated variates are easy to simulate. But so is the Student-t distribution, for example.
  - ▶ In other words, the use of <u>normal</u> copula is not a consequence of any model about normality of asset prices or CDS fees, or a required condition for no-arbitrage pricing. It is purely for convenience.

- It's not true, for example, that a default component model like the one last week (where every entity is exposed to a common systematic default event) will yield a joint distribution that corresponds to a Gaussian copula.
  - ▶ Moreover, if the $\lambda^{Q,(i)}$ are stochastic, then the default times won't be exponentially distributed.

- It is not at all obvious that <u>any</u> reasonable model model of risk factors and their market prices of risk can be combined to yield this type of risk-neutral distribution.
  - As far as I know, there is also no *empirical* support for the hypothesis that the *true* joint default distribution is well described via a Gaussian copula.

- To believe in a given specification for a "pricing measure," it must be the case that you have empirical support for a true model, and can identify the parameters required for the risk-adjustment via the returns of traded assets.

- Financial engineers who decided they were free to specify an arbitrary pricing process to suit their computational needs seem to have lost touch with the underlying foundations of the theory.

- But....

- Notice how easy the NGC approach makes life!
  - One single draw from an N-dimensional normal gives us all the information we need for an entire default history of all our N names.
  - So evaluating expected outcomes by averaging over $M = 100000$ histories can be done in microseconds.
  - I'll illustrate this below.

- There is an even further simplification one can do that still preserves the marginal distributions.
  - ▶ Assume that the normal variate Z is the sum of two normal terms, one systematic and one idiosyncratic :

$$Z^{(n)} = \sqrt{\rho}\; Z^{syst} + \sqrt{1-\rho}\; Z^{idio,(n)},$$

    where the idiosyncratic components are independent across names.

  - ▶ This means every two firms have the same pair-wise copula correlation ρ.

- This version of the NGC used to be called the "market standard model".

- Notice that, like the Black-Scholes model, the market standard model now only depends on one unobservable parameter, ρ.
  - ▶ And, like the Black-Scholes formula, one can invert the model to find an *implied correlation*.
    - ∗ One tries different ρ until the implied model price matches the observed market price of a given CDO tranche.
    - ∗ It is remarkable that the simulation step is fast enough to permit a such a search computation.
  - ▶ But note that this parameter has a murky meaning, even under the model.
    - ∗ It is <u>not</u>, for example, equal to the correlation in firms' firm values or bond prices.

* It is not even the correlation between their intensity processes.

So one must be extremely careful in interpreting statements about this "correlation."

- ▶ Like implied vol, however, one can always transform market prices into these units and use them to talk to other traders – even if no one accepts or believes the model.
- ▶ In reality, it was usually the case that different tranches of *the same CDO* traded at different implied correlations.
  * This is direct evidence that the model does not capture the market's beliefs about the risk-neutral default distribution.

- However, for many years the "market standard model" was used to price CDOs.

- It cannot be over-emphasized that there is no economic underpinning at all to this model. It is merely a conjecture about what the risk neutral joint distribution might look like – given the marginals.

- While the dangers and inconsistencies of using this model for pricing are now widely understood, it can still be useful, at least as a benchmark.
  - It gives us a really easy tool for quantifying the distribution (true or risk neutral) of tranche loss outcomes.

- If I have an `M`-vector `lambdaQ` of marginal default probabilities with copula correlation `rho`, and I want to simulate a number `pools` of pool histories, it's three lines of MATLAB code:

```
%---------------------------------------------------------------
M=125;  pools=100000;

systematic    = repmat( randn(1,pools), M, 1);
ncop          = normcdf( sqrt(1-rho)*randn(M,pools) + sqrt(rho)*systematic );
default_times =-log( ncop )./lambdaQ;

%---------------------------------------------------------------
```

- That's it! Now you have one matrix of 125 X 100k default times.

- The next piece of code shows how to mechanically map these random times into cashflows for one particular tranche, defined by a lower and upper attachment point.
  - This part would be the same for any model, given the default times.
  - It is not specific to the NGC model.

  (The code is quite wasteful of RAM since stores several large matrices and it never clears any memory.)

```
%------------------------------------------------------------------

T = 5;                  % CDO maturity

recovery = 0.40;        % assumption

lower_pt  =  7 ;        % attachment points for a particular tranche
upper_pt  =  20 ;        % expressed as percentage.

tr_width  = upper_pt-lower_pt;

loss_per_def = (100/M)*(1-recovery);
recov_per_def  = (100/M)*recovery;


% We already got the default time matrix from copula step above.
% Turn them into an increasing sequence.

default_times = sort(default_times);       % sort() operates columnwise

before_T      =  (default_times<=T);   % indicator for defaults before maturity
                                       % (right side is a boolean operator)


% Build matrices of entire pool experiences at each default time.

pool_loss       = loss_per_def.*before_T;
pool_recov      = recov_per_def.*before_T;

% Our tranche will only feel events that happen after total pool losses reach
% the lower attachment point. So keep track of running cumulative sum.

% We will also need to keep track of cumulative recoveries

CPL = cumsum( pool_loss );
CPR = cumsum( pool_recov );

% (these sums only count the events before maturity)
```

```
% Build matrix of tranche cash-flow experiences at each default time

tr_losses =  max(0,  min(CPL,upper_pt) - max(CPL-pool_loss,lower_pt) );

% To see how this works, if M=100, T=5, recovery=1/2, L=2, U=4,  then
%  one column of these matrices might look like:

%       default times   [ 0.4  0.9  1.2  1.7  2.1  2.8  3.3  3.5  4.7  5.1....]

%                  CPL= [ 0.5  1.0  1.5  2.0  2.5  3.0  3.5  4.0  4.5  4.5....]
%             min term  [ 0.5  1.0  1.5  2.0  2.5  3.0  3.5  4.0  4.0  4.0....]
%   rightmost max term  [ 2     2    2    2    2   2.5  3.0  3.5  4.0  4.0....]
%  max(0, difference )  [ 0.0  0.0  0.0  0.0  0.5  0.5  0.5  0.5  0.0  0.0....]


tr_loss_pct       = tr_losses./tr_width;

%  Total (undiscounted) default losses of the sample is then the vector sum(tr_los
figure(1), hist(sum(tr_loss_pct));


% Tranche may also experience lowering of principal from notional recovery

tr_recovery  = max(0, min(CPR, 100-lower_pt) - max(CPR-pool_recov, 100-upper_pt) )

tr_recovery_pct   = tr_recovery./tr_width;


% Outstanding notional is face value minus losses minus recoveries

tr_principal = 1.00 - cumsum(tr_loss_pct) - cumsum(tr_recovery_pct);

%----------------------------------------------------------------------
```

- Now if you believe your sampling algorithm represents the risk-neutral distribution, it is straightforward to value the tranche via discounting the cash-flows experienced along every path,
  - The code will assume this is a synthetic (unfunded) CDO so there is no repayment of principle.

- The tranche pays a fixed fee (like a coupon rate) which is paid on as a percentage of the remaining principle balance.

    * Just like for a CDS, we can compute the fair fee by setting the PV of the coupon payment side equal to the PV of the credit protection side.

(Again, this part of the code is not specific to the NGC model.)

```
%-----------------------------------------------------------------
r=0.01;       % riskless rate assumption


pv_tranche_pct_loss_pct  = sum( exp(-r*default_times) .*  tr_loss_pct );


% for coupon PV, need time intervals between defaults

delta_t =default_times - [zeros(1,pools) ; default_times(1:end-1,:)];

% integrate e^{-r(s-t)} over each interval and scale by amount of principal

pv_coupon_multiplier = ((exp(-r*(default_times-delta_t))-exp(-r*default_times))/r)
                          .* tr_principal.*before_T;

pv_coupon_pct = sum(pv_coupon_multiplier);

% done!

fair_fee =  mean(pv_tranche_pct_loss_pct)/mean(pv_coupon_pct)

%-----------------------------------------------------------------------
```

- Next time, I will use this model to show how the loss distributions change as some of the input parameters change.
  - However, just as we cannot assert that this model can tell us the *price* of the CDO tranche, we can also not assert that varying the parameters tells us the *hedge ratios*.
  - Even if we believed the model, it assumes its inputs (the marginal intensities and correlations) are constants, and hence the sensitivities are inconsistent measures of risk within a given pool.

IV. Summary.

- Modeling credit *correlation* is crucial for pricing and hedging of multi-name credit products.

- Marginal default probabilities (credit spreads) do not nail down joint probabilities.

- So we can't price basket products in a completely "model free' way just using CDS data.

- Historical data can tell us correlations between, e.g., stock prices or bond credit spreads for two entities (that are still alive).
  - ▶ We may also need to use historical information on correlation between actual default events among similar entities, e.g., as measured by industry or geography or size.

- Ultimately there must be a feasible hedging strategy for joint-default risk in order for the logic of no-arbitrage theory to apply.

- While the computational simplicity of valuation via the direct simulation of correlated default times is appealing, this approach is not directly grounded in a no-arbitrage model.