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Lecture Note 7.1: Unpredictable Volatility

The Black-Scholes-Merton methodology views the world as subject to a single source of uncertainty: dS. In most applied financial engineering problems, however, there are many sources of uncertainty. The next item on our agenda is to generalize our models to be able to try to incorporate more risks in our valuation framework.

To motivate this undertaking, we start by examining the assumption of constant volatility. *What if that is wrong?* How much does it matter? And how does it affect pricing and hedging decisions? With any model, we always need to question the assumptions.

To address these issues, we have to do two things: (1) quantify how volatility changes, and then (2) figure out how these changes affect the no-arbitrage argument, and thus how they change the solutions for valuations and hedge ratios.

Outline:

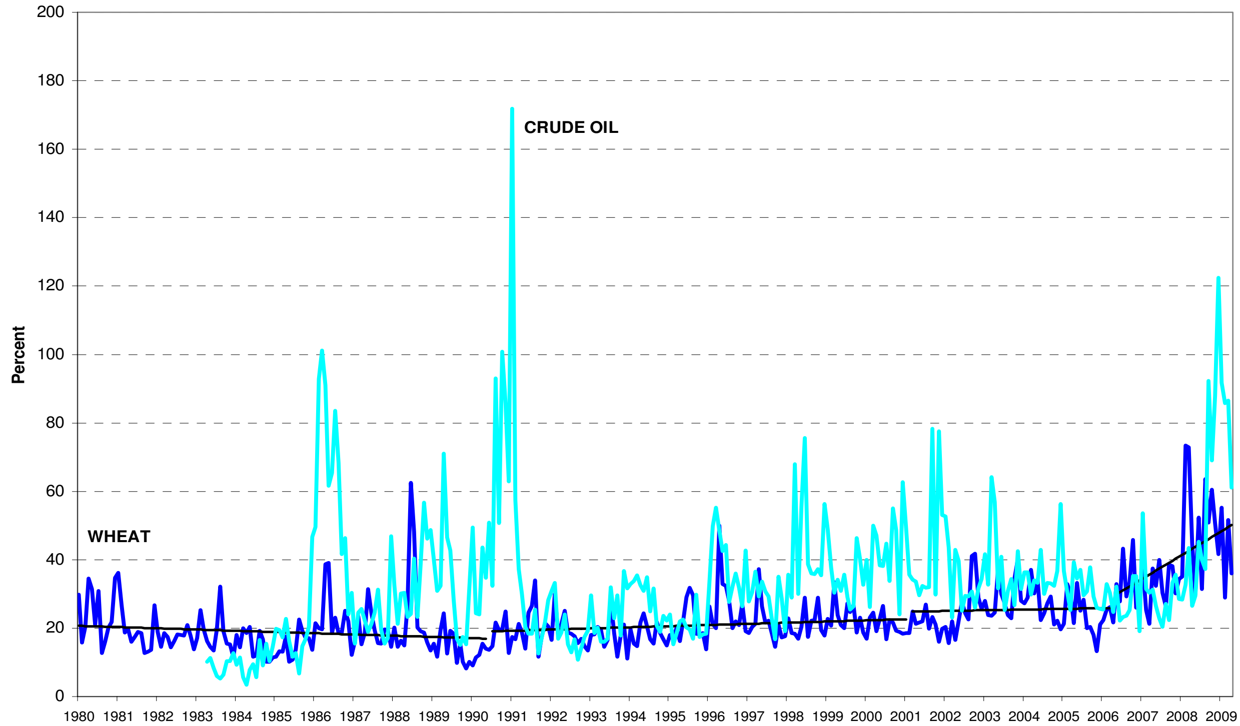
I. The Behavior of Realized Volatility
II. The Behavior of Implied Volatility
III. Modeling Unpredictable Volatility
IV. The No-Arbitrage Argument
V. Interpretation
VI. Results: Implications for Options Prices.
VII. Summary

I. The Behavior of Realized Volatility

- As soon as we start to apply the Black-Scholes formula to any real valuation, we immediately confront the difficult task of deciding what σ is.

- In terms of the model, there is no ambiguity: σ is the standard deviation of returns to the underlying.

- The trouble comes when we look at actual returns.



Data source: CME Group

- In looking at **realized** volatilities, there appear to be large differences across different samples of returns for the same underlying instrument.

- However we now need to think carefully about three questions:

(A) Does **true** volatility really change?

  - Just because some periods have higher *estimated* volatility than others, doesn't mean the underlying process changed.
  - You would expect to observe that anyway, just by chance.

(B) Are volatility changes due to dependence upon S and t?

  - Remember that we can already handle volatilities that change predictably through time, or with the level of the underlying (as in the constant elasticity of variance model).
  - Perhaps such *deterministic volatility* models are good enough.

(C) Does it matter?

  - Perhaps the variations in volatility aren't big enough to have much **economic** impact on options prices.

- Let's consider each of these questions.

(A) Do volatilities really change?

- Even if volatility really <u>were</u> constant, looking at any given historical period we would find times of high and low <u>realized</u> volatility.

  ▶ How can we distinguish between the hypotheses that *(i)* <u>true</u> volatility changed, and *(ii)* it just looked that way?

- The natural place to start is to divide up time into distinct samples, and compare realized volatilities.

- To tell whether volatility is changing from period to period, we need to be able to accurately estimate what it <u>is</u> in each period. Then we can decide if any changes we found were just due to chance.

- The standard estimator for the volatility of a sample of returns is

$$\hat{\sigma} = \sqrt{\frac{1}{N}\sum_{t=t_1}^{t_N} r_t^2} \qquad \text{(Not annualized).}$$

  ▶ Here $r_{t_i} = \log((P_{t_i} + D_{t_i})/P_{t_{i-1}})$ where P is price and D is any dividends paid.

  ▶ Note that we're not subtracting the sample mean from the rs before squaring them. **Why?**

  ▶ In general zero is a much better estimator of the **expected** return than whatever the sample average turned out to be in any particular data set.

- ▶ We would typically annualize our estimate by multiplying by
$$\sqrt{\text{number of periods of length } \Delta t \text{ per year}}$$

- You can build better, fancier estimators that incorporate more information (volume, high, low, etc). But the simple one works well *if volatility really is constant.*

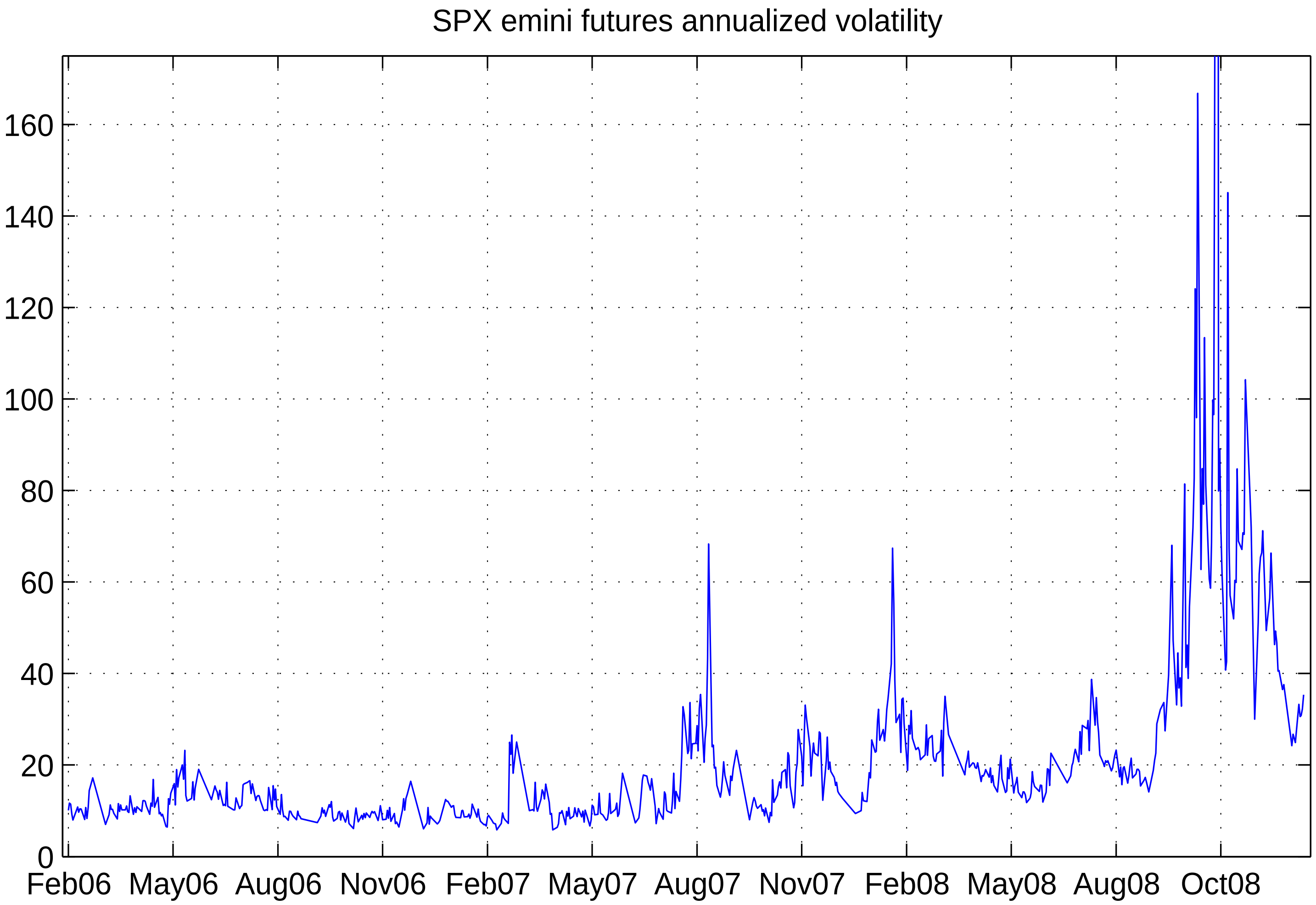
  - ▶ The margin-of-error in the estimate is approximately
$$\frac{\sigma}{\sqrt{2N}}$$
    which is already small with as few as 20-30 data points.

  - ▶ With much bigger samples, this sampling error goes to zero very fast: **volatilities are easy to estimate when they really are constant.**

- There are around 400 1-minute returns per day for exchange traded futures. So the estimation error is about 0.035 σ.

- With this data density, it is effectively possible to observe actual volatility at different, (non-overlapping) intervals in time.

- The plots on the next page show what you get when you look at the S&P 500 futures at this time-scale.



- Here the error in the individual days' estimates is tiny.
  - Statistically, there is <u>no chance</u> that the fluctuations could have occured without <u>actual</u> changes in volatility.
- So, yes, we can conclude volatility changes over time.
  - There is nothing special about these futures either. All asset prices show a large degree of *heteroskedasticity.*
- It is not just a question of volatility *looking like* it changes.

(B) Are the changes predictable functions of S and t?

- We have already discussed some sorts of volatility changes.
  - ▶ There are quiet periods and busy periods, from the perspective of information arrival (e.g. weekends vs. earnings announcements).
  - ▶ Companies get less volatile as they grow.
  - ▶ Commodity volatilities can be seasonal.
  - ▶ Bond prices fluctuate less as they approach maturity.
- These are all examples of time-varying, but predictable, volatility changes that we can model as $\sigma(S, t)$.
  - ▶ We know that the Black-Scholes PDE and also binomial grids can be modified to take into account time- and level-dependency.
  - ▶ Another name for a specification like this is a *deterministic volatility function* (DVF).
- The question is: Can we build such a function so as to account for the variation we observe in realized vol?
- Answer: No. At least not completely.
  - ▶ Although models like this can definitely improve volatility forecasts, they cannot explain all of the changes in realized volatility.

(C) Economic Significance

- So volatility does in fact represent a second source of randomness.
- Does it matter?
- Really, what we want to know is: when is volatility *constant enough* for pricing and hedging derivatives?
  - ▶ Perhaps the surprises that occur in volatility, while statistically significant, wouldn't cause big hedging/replication errors for someone using a constant volatility model.
- This can be the case. The answer depends upon:

  1. How large are the possible volatility surprises?

  2. Will they average-out over time?

  3. How sensitive is a derivative's value to volatility?

  The first two issues are empirical and depend on the asset.

- Referring to the graphs above , it is not uncommon to see swings of plus and minus 50% in the level of vol, and these can persist for periods of several months.

- For instance, if today's instantaneous σ for wheat is 16%, it could easily turn out to be 24% or 8% over the next 6-months.

- Overall, the *volatility of volatility* seems to be 50-100% per year, and the *half-life* of shocks seems to be a few months, usually.
  - These numbers are typical of most asset types, although every asset is somewhat different.
  - While most volatility series do look *mean reverting* (or stationary) there can also be *structural breaks* (permanent shocks).
    - For example when a company fundamentally alters its assets.

- The third issue raised above was whether the typical changes in vol *matter* for derivatives.
- Perhaps not. Some derivatives, like short-dated options far out-of-the-money, aren't very sensitive to volatility.
  - Likewise, long dated options might not be affected by short-term transitory movements (if the vol mean-reverts).

- BUT we have seen many examples of options whose value can definitely change *a lot* when, for example. we plug in two σs that differ by 50% and evaluate the Black-Scholes formula.

- However – be careful!

  ▶ If volatility can change, then *the Black-Scholes formula is wrong.*

    * So the sensitivity of the formula to σ doesn't really tell us what we need to know.

  ▶ That is, the Black-Scholes "vega" ($\partial c^{BS}/\partial\sigma$) is an **inconsistent** measure of volatility risk.

  ▶ Indeed, it is conceptually wrong to even use it in this fashion.

- When we want to quantify how much an option's value (or a hedging strategy's profit) would change as the result of volatility evolution, we need to use *an options pricing model which takes into account such changes.*

  ▶ So, while we know intuitively that, yes, volatility is economically important, to be more precise we will have to return to this topic later in the lecture.

- Before we go further, let's see what the options markets themselves can teach us about σ.

II. The Behavior of Implied Volatility

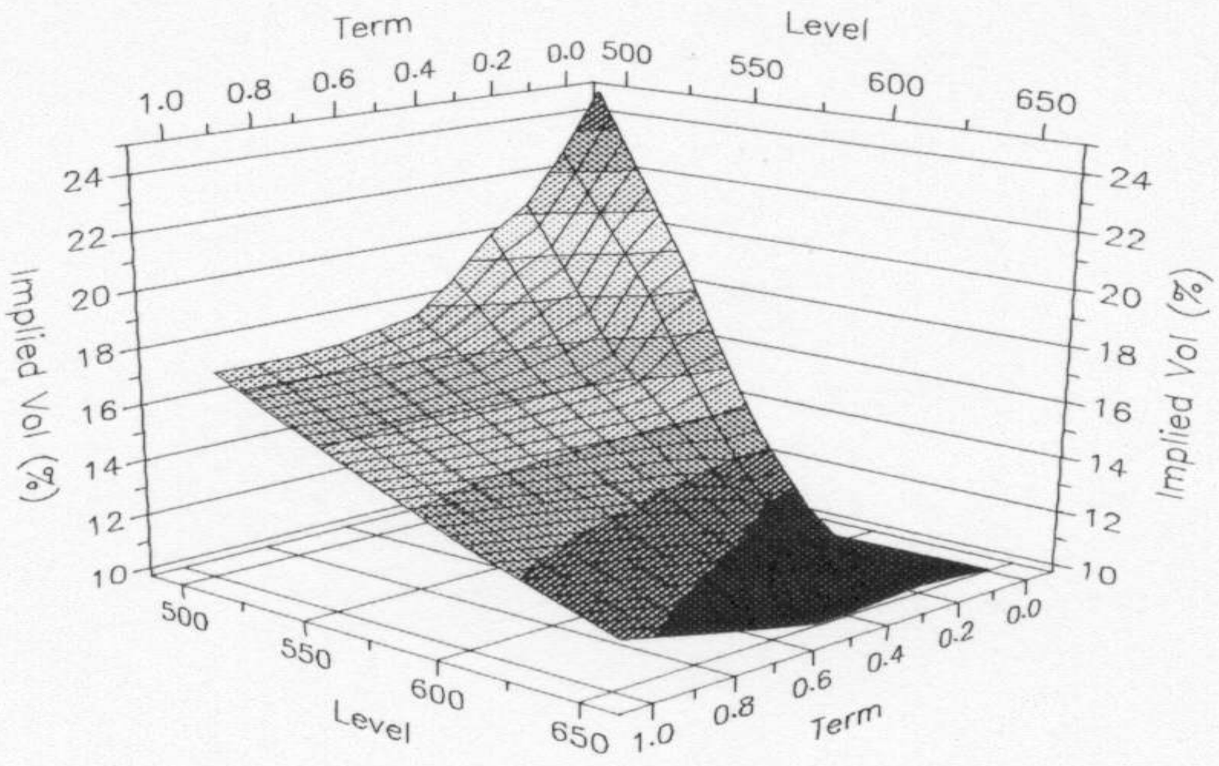
- If the Black-Scholes model were true, then, on any given day, if we looked at the prices of traded options on S for different strikes and different maturities, we would <u>have</u> to observe that

(A) they all had the same Black-Scholes implied volatility, and

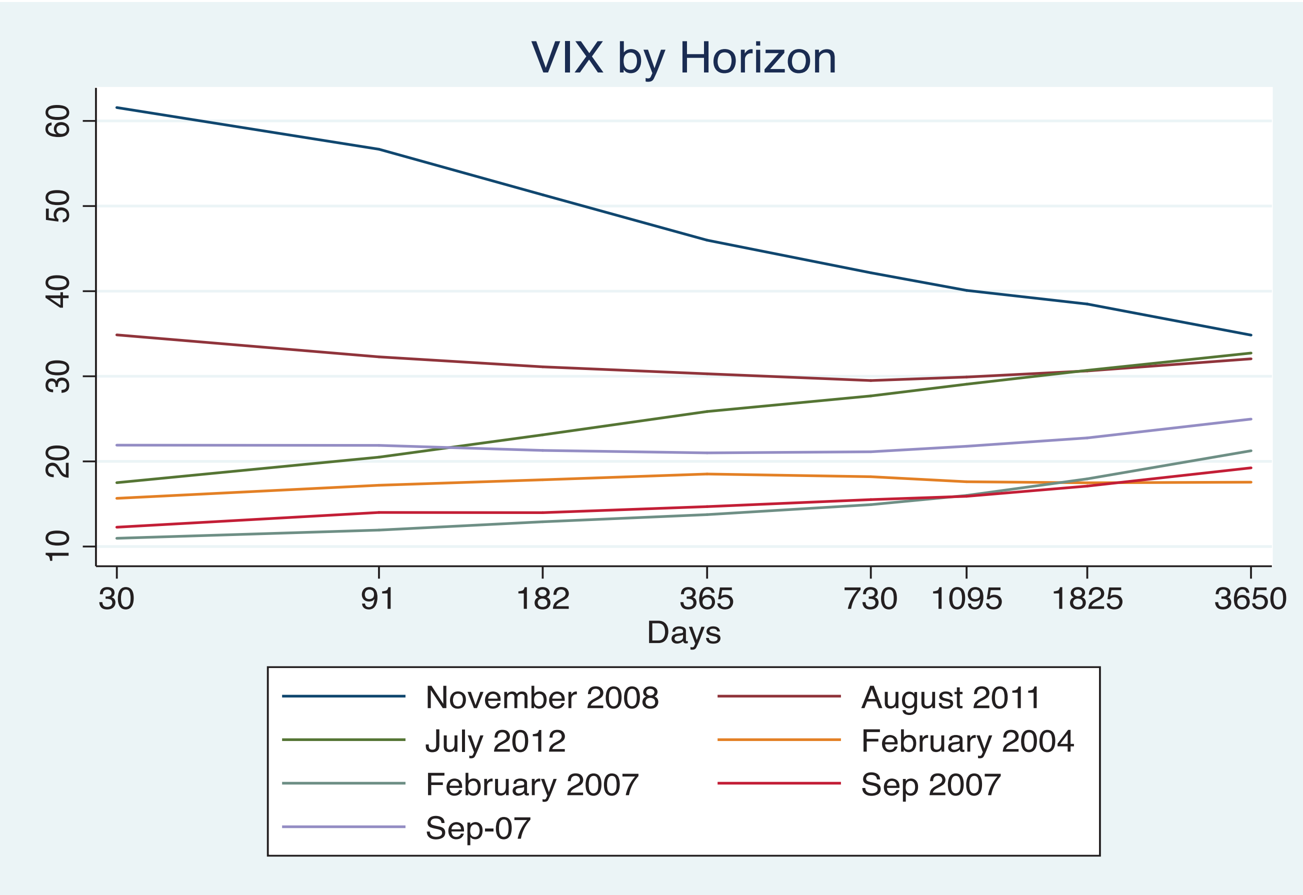
(B) that implied number was the true statistical volatility of S's returns, both in the past and in the future.

- In other words, if Black-Scholes is right, σ is constant, in the past and in the future.

- This is not what options markets tell us – at all.



- In reality, we **never** observe constant implied volatilities in **any** market.
  - Instead, we observe a complex implied volatility *surface*, as a function of K (or K/S) and T.
  - The plot of implied vols as a function of expiration (with K fixed) is called the *term structure.*
  - The plot of implieds vs K/S is called the *smile.*
  - (Remember, even if we don't believe Black-Scholes we can still convert prices into BSIV units for ease of comparison.)
- The plot shows the implied surface for the S&P 500 index on a typical day.
  - The smiles are usually skewed toward the lower strikes.
  - The term structure at-the-money usually slopes up
  - The surface gets smoother for farther expirations.
- Even worse (for Black-Scholes), these surfaces move around a lot over time.
  - Here are some term-structures of implied vols for the index as observed on different dates.



Notes: Average VIX for the indicated month by horizon. Source: Goldman Sachs.

- So clearly options markets don't believe the Black-Scholes specification of constant volatility.
  - And if the assumption is wrong, then so is the formula.

- This implies that implied volatilities computed from the model are inconsistent estimators of beliefs about future return volatility.

- Is there a way we can translate options prices consistently back into a model of σ which we could then interpret as forecasts of the actual future variablility of S?

- One thing you can do is to back out deterministic volatility functions of the type we thought about above: $\sigma(S, t)$.
  - In other words, choose a DVF to fit observed options prices.
  - Equivalent to picking a different u and d at **every node** in a binomial tree to MAKE the grid agree with the market.

- We already learned how to use Merton's deterministic volatility rule to back out a term structure of implied forward volatilities.
  - One can use the same idea to get a separate term-structure for each strike price, and thus build an entire forward volatility surface, or DVF.

- Once we back out this DVF, then we can see if future actual volatility really does behave the way the surface predicts.

- But now there is a different test as well.
  - We can ask whether the DVFs predict future *implieds*.
  - If the functions are correct, then, for example, next month's one-month smile must equal the implicit one-month smile in today's two-month smile.

- Empirically they turn out to do a very <u>poor</u> job of forecasting changes in realized volatility AND changes in implied volatility.

- Dumas, Fleming and Whaley (1998) attempted to fit deterministic vol models for S&P 500 options for a six year sample.
  - Their prediction errors one week later are mostly greater than just constant volatility models.
  - The implied hedging policies also perform poorly.
  - They conclude:

    > *...there may be no economic significance to the deterministic volatility functions implied by options prices.*

- Does this tell us that options markets are bad at forecasting future volatility?
- **NO.**
- It tells us that imputing deterministic volatilty *models* is the wrong way of interpreting options prices.
  - These models cannot jointly account for options prices and the actual volatility of the underlying returns.
  - But we don't have any reason to think the markets actually <u>believe</u> the DVFs.
- What, then, is the alternative to deterministic models of σ?

III. Stochastic volatility.

- Ok, so it's worthwhile to see what random volatilities do to derivative prices. To do this, we have to replace our old constant σ (or $\sigma(S,t)$) with some new σ_t stochastic process.

- In other words, we have to have a model for volatility which allows for random changes.

- In our continuous-time setting, we do that by specifying a stochastic differential for σ, just like we did when we modeled the underlying price. For example

$$d\sigma_t = m(\cdot)\,dt + s(\cdot)\,dW_t^\sigma.$$

  - ▶ The left side is the instantaneous change in the instantaneous vol.

  - ▶ The right side is just like our stock model: a deterministic term plus a random normal innovation. Each of these multiplied by a coefficient function.

  - ▶ We are assuming we can observe this process today, but we cannot predict exactly where it will go in the future.

- With such a model, we can ask:

  If volatility changes this way, how much will it cost us to hedge that additional risk?

- This will then give us a new theory for pricing options.

- To have a complete model we now need to specify:
  (a) What is the drift function $m(\cdot)$?

  (b) What is the diffusion function $s(\cdot)$?

  (c) How is the Gaussian innovation, dW^σ, correlated with the dW^S that drives the underlying price?

  ▶ We'll call that correlation ρ.

- For the drift, we want to capture the idea that σ is mean-reverting. We can do this by ensuring that the drift is down when σ is too high, and up when it's low. For instance

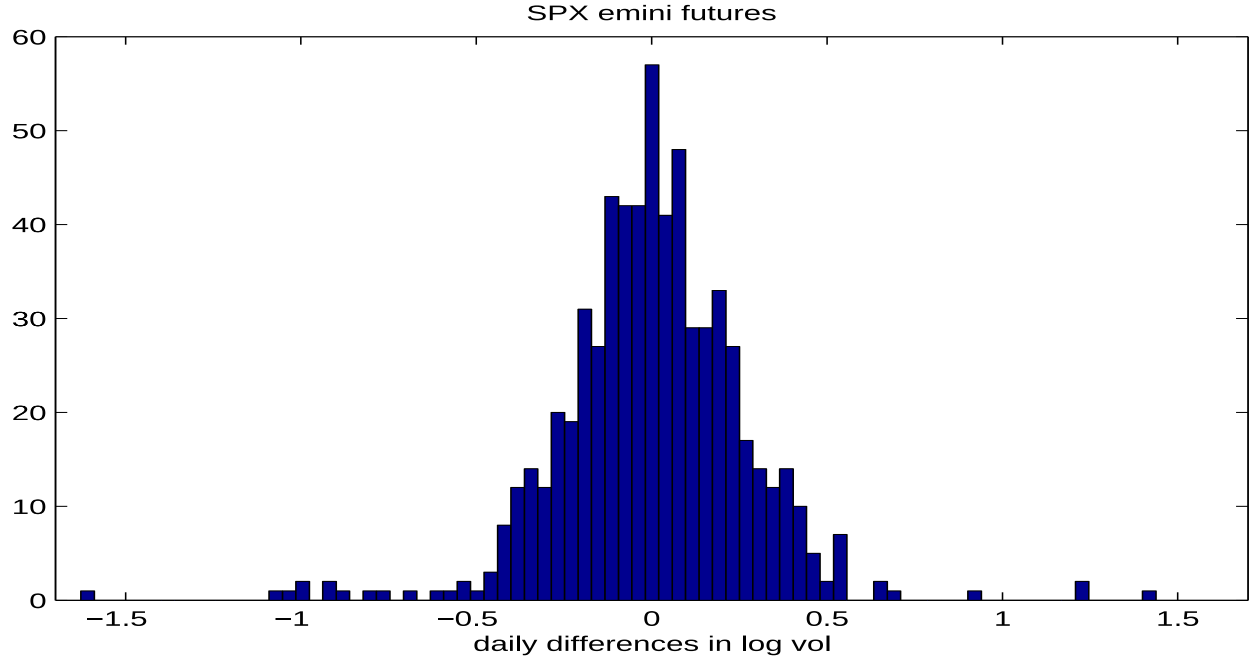
$$m = \kappa(\sigma_0 - \sigma_t).$$

- Here σ_0 is the level σ reverts to, and κ is a constant determining how fast it reverts.
  - ▶ A big κ means that volatility will never stay far from its long-run average for very long. If $\kappa = 0$ then volatility will be like a random walk.
  - ▶ A typical process might have a **half-life** of six-months, which corresponds to about $\kappa = 1.4$.

- Simple linear mean-reversion is not the only possible specification of course. But let's adopt it for now.

- For the diffusion coefficient $s(\ )$, the simplest choices are either (a) a constant; or (b) a constant times σ_t itself.

- Notice that choice (b) means that we are assuming log vol is effectively a geometric Brownian motion: its *percentage* changes are normally distributed.
  - ▶ This looks to be a good model for many assets.



- Finally, we'll assume ρ is a constant, although it could, in principle, also be a function of S.

- Note that these choices (or any others) must actually be tested and verified in actual data for each application.
  - ▶ Just like for our dS model, we have to estimate the different constants that appear in our specification, and then also see whether those estimated values do a good job of describing the return dynamics out-of-sample.
  - ▶ Doing the **econometrics** now becomes a significant additional burden in using these models.

- To be concrete, for the rest of the lecture I'm going to assume this model:

$$d\sigma_t = \kappa(\sigma_0 - \sigma_t)\, dt + s\, \sigma_t\, dW_t^\sigma.$$

  And I'm going to assume we have fixed values of κ, σ_0, s, and ρ that we're comfortable with.

  ▶ But keep in mind that this particular specification is motivated mainly by convenience – not necessarily reality.

- Indeed, there is a vast literature devoted to estimating and testing different types of time-series models for volatility.

- **Example:** What does stochastic volatility look like?
  - ▶ Suppose we have a share whose price is 100 and its initial volatility is 40% (and $\mu = 0$).
  - ▶ Suppose it's long-run volatility is 25% and the speed of reversion (κ) is 0.75.
  - ▶ Now let the volatility of volatility be 50%. This is s.
  - ▶ Last, put $\rho = 0$.
  - ▶ Here's what a few weeks ($dt = 1/52$) might look like:

(A)	**(B)**	**(C)**	**(D)**	**(E)**	**(F)**	**(G)**
Starting Stock Price S_t	Starting Volatility σ_t	Random S change = normal(0,1) $\times\sqrt{dt}$	New S = (A)+ (A)·(B)·(C)	Random vol change = normal(0,1) $\times\sqrt{dt}$	Drift in volatility= $0.75\times$ $(0.25-$(B)$)dt$	New volatility= (B)+(F)+ 0.50·(B)·(E)
100.00	0.4000	-0.1401	94.39	-0.2224	-0.0022	0.3533
94.39	0.3533	0.0852	97.24	0.0357	-0.0015	0.3582
97.24	0.3582	0.7040	99.69	-0.1465	-0.0016	0.3304
99.69	0.3304	0.2347	107.42	0.1962	-0.0012	0.3616
107.42	0.3616	0.0820	110.60	-0.1116	-0.0016	0.3398
110.60	0.3398	-0.0893	107.25	0.0733	-0.0013	0.3510
107.25	0.3510	0.0527	109.23	0.0304	-0.0015	0.3549
109.23	0.3549	-0.1399	103.81	-0.1278	-0.0015	0.3307
103.81	0.3307	-0.0027	103.72	-0.3010	-0.0012	0.2797
103.72	0.2797	-0.0067	103.52	-0.0082	-0.0004	0.2782

- Notice that, compared to the Black-Scholes world, we've considerably complicated things for ourselves already.

  We used to have only 1 constant we had to specify and now we have 4.

  And this is for a simple model.

- What's more, you might think that we may have started down a "slippery slope" here.
  - ▶ Should we believe that these 4 new parameters are any more likely to be constant than our original σ was?
  - ▶ *Is there any economic theory or law of nature that implies that they should be fixed?*

- There really isn't. We are on a slippery slope. These parameters might be unstable over time too.

- However, it will usually turn out that small changes to $\kappa, \sigma_0, s,$ and ρ will make *much smaller* changes to options prices than similar changes in σ make to Black-Scholes.
  - ▶ In that sense, the model is now more *robust*.

- But it's right to be thinking along these lines, because we don't know in advance that it will work out like this.

- And we do need to check our valuation results in each case to see if they are especially sensitive to some of the parameters.
  - ▶ This can greatly enhance our understanding of the risks of particular derivatives.

IV. The No-Arbitrage Argument

- Now what? We have a volatility model. What does it mean for options prices?

- Where, in the Black-Scholes argument did we *use* the constant volatility assumption?

- Recall that what we did was to imagine holding an option and holding hedge positions that made the whole portfolio instantaneously riskless.

- The key step was that we figured out exactly what the random component of the option's price change was going to be over the next instant of time (using Itô's lemma).
  - Once we quantified those random bits, we knew how to knock them out.

- Where we used the constant volatility assumption was in saying that the only random thing that made the option value change was the stock (or the underlying asset).

- Mathematically, when we found the differential dC of the function $C(\,)$, we only had to differentiate with respect to one random thing: dS

- Now we have another random thing contributing to the change in $C(\,)$. So we need to use a two-dimensional version of Itô's chain rule.

- Here's what it says. Suppose we have two random things X and Y and that their correlation is ρ:

$$\begin{aligned} dX &= a_X(\,)\,dt + b_X(\,)\,dW^X \\ dY &= a_Y(\,)\,dt + b_Y(\,)\,dW^Y. \end{aligned}$$

Now suppose that C depends on both of these (and time). Then

$$\begin{aligned} dC = \quad & \frac{\partial C}{\partial X}\,dX + \frac{\partial C}{\partial t}\,dt + \frac{1}{2}b_X^2\frac{\partial^2 C}{\partial X^2}\,dt \\ + \quad & \frac{\partial C}{\partial Y}\,dY + \frac{1}{2}b_Y^2\frac{\partial^2 C}{\partial Y^2}\,dt \\ + \quad & \rho b_X b_Y\frac{\partial^2 C}{\partial X \partial Y}\,dt \end{aligned} \tag{1}$$

- This is pretty awful. But it's really just a straightforward extension of what we had before.
  - ▶ Now we have the old terms plus similar ones for Y.
  - ▶ plus an extra term that corrects for X and Y being correlated.

- For our case, we don't have X and Y. We have S and σ.

$$\begin{aligned} dS &= \mu S\,dt + \sigma S\,dW^S \\ d\sigma &= \kappa(\sigma_0 - \sigma)\,dt + s\,\sigma\,dW^\sigma. \end{aligned}$$

- So, fine, all we have to do is plug these particular as and bs into the Itô formula and we get dC.

- Now before writing down a giant ugly formula, think about what's going to happen.

- The ugly formula is going to have one term that multiplies dW^S which is the random part of next instant's stock change.
- We're trying to figure out how to form a riskless portfolio. So to get rid of that term we will hold a stock position of exactly the right size to offset that.
- But then we're not done.
- There will still be a part of dC that depends on dW^σ that we won't have gotten rid of.
- What do we do?

- If we're going to offset the σ risk, there has to be something else out there we can use to offset it with. **The stock itself can't do the job**.
- So we're stuck – unless we assume there is already another option or derivative available for us to trade, whose price also depends on σ.
- **For no arbitrage arguments to work, there must be (at least) as many securities available to trade as there are sources of risk in the economy.**
  - And, of course, we need to be able to trade these
    - which includes short selling them.

- So let's select one **other** derivative, P, say, which also depends on S and σ.
- In order to hedge C's volatility risk with P we have to know how P varies with σ.
  - In other words, we *also need to solve for* dP.
  - That makes the problem a bit harder.
- Now we have to go to the ugly equations.
- Three pages and it'll be over!

(A) The full expression for dC is

$$\begin{aligned} C_S \mu S\, dt + C_S \sigma S\, dW^S + C_\sigma \kappa(\sigma_0 - \sigma)\, dt + C_\sigma s\, \sigma\, dW^\sigma \\ + \;\; \frac{1}{2}\sigma^2 S^2 C_{SS}\, dt + \frac{1}{2} s^2\, \sigma^2 C_{\sigma\sigma}\, dt \\ + \;\; \rho s \sigma^2 S C_{\sigma S}\, dt + C_t\, dt \end{aligned}$$

where I'm using subscripts to denote partial derivatives.

(B) Now I'm going to define a portfolio Π^C to be long one unit of C and short C_S shares of stock so that I can forget about the stock exposure. And

$$\Pi^C = C - C_S\, S - L^C.$$

As in the Black-Scholes argument we also borrow whatever amount L^C is necessary to make the strategy self-financing. So all profits come from capital gains. In other words, the gains from holding Π^C are

$$
\begin{aligned}
dC - C_S\, dS - rL^C\, dt = [&\frac{1}{2}\sigma^2 S^2 C_{SS} + \frac{1}{2}s^2\, \sigma^2 C_{\sigma\sigma} \\
&+\rho s \sigma^2 S C_{\sigma S} + \ C_\sigma \kappa(\sigma_0 - \sigma) + C_t - rL^C]\, dt \\
&+ C_\sigma\, s\sigma\, dW^\sigma
\end{aligned}
$$

which I'm going to write as

$$
d\Pi^C = a^C \Pi^C\, dt \ + \ b^C \Pi^C\, dW^\sigma
$$

where

$$
\begin{aligned}
a^C \ &\equiv \ \frac{1}{\Pi^C}[\frac{1}{2}\sigma^2 S^2 C_{SS} + \frac{1}{2}s^2 \sigma^2 C_{\sigma\sigma} + \rho s \sigma^2 S C_{\sigma S} \\
&\qquad + C_\sigma \kappa(\sigma_0 - \sigma) + C_t - rL^C] \qquad (2) \\
b^C \ &\equiv \ \frac{1}{\Pi^C}\, [C_\sigma\, s\, \sigma] \qquad (3)
\end{aligned}
$$

(C) Next, I go through the exact same steps with my second option P.

- define Π^P as the value of a portfolio holding P but with the stock risk hedged away.
- So, just changing C into P,

$$
d\Pi^P = a^P \Pi^P\, dt \ + \ b^P \Pi^P\, dW^\sigma
$$

(D) Now the key step: we can get rid of the dW^{σ} part by combining Π^C and Π^P.

- Hold one unit of Π^C and short $(b^C\Pi^C)/(b^P\Pi^P)$ units of Π^P.
- That gives us the riskless position we were after.
- Do you see why I picked that hedge ratio?
- Since both the P and C portfolios are self financing, so is this combination.

(E) Now we use our no-arbitrage condition and assert that **the expected return of that portfolio of portfolios must be $\mathbf{r}$:**

$$\left(a^C\Pi^C - \frac{b^C\Pi^C}{b^P\Pi^P}\,a^P\Pi^P\right) = r \cdot \left(\Pi^C - \frac{b^C\Pi^C}{b^P\Pi^P}\,\Pi^P\right)$$

- What assumption about payouts am I making here?

(F) Simplify this by cancelling out all the Πs:

$$a^C - \frac{b^C}{b^P}\,a^P = r - r\,\frac{b^C}{b^P} \quad \text{or,}$$

$$a^C - r = \frac{b^C}{b^P}\,(a^P - r) \quad \text{or,}$$

$$\frac{(a^C - r)}{b^C} = \frac{(a^P - r)}{b^P} \equiv \lambda. \tag{4}$$

where I am defining λ to be this ratio.

(G) Now take the equation $(a^C - r)/b^C = \lambda$ and put back in what a^C and b^C were. Specifically, write

$$a^C \Pi^C - r\Pi^C = \lambda\, b^C \Pi^C$$

Then use

$$r\Pi^C = r[C - C_S \cdot S - L^C], \quad b^C \Pi^C = s\,\sigma\, C_\sigma$$

and

$$a^C \Pi^C = \frac{1}{2}\sigma^2 S^2 C_{SS} + \frac{1}{2} s^2 \sigma^2 C_{\sigma\sigma} + \rho s \sigma^2 S C_{\sigma S} + C_\sigma \kappa(\sigma_0 - \sigma) + C_t - rL^C$$

When the dust all settles, what you get is

$$\frac{1}{2} s^2 \sigma^2 C_{\sigma\sigma} + \rho s \sigma^2 S C_{\sigma S} + \frac{1}{2}\sigma^2 S^2 C_{SS} + rSC_S$$
$$-rC + C_t + [\kappa(\sigma_0 - \sigma) - \lambda s \sigma] C_\sigma = 0 \qquad (5)$$

- That is the partial differential equation we were after.

V. What does it all mean?

- Having got our PDE, we simply give it to a mathematician and tell him to solve it for us. And, in a minute, we'll look at what he would come up with. But first, all that algebra concealed some really important lessons that I didn't stop to point out.

(A) The Market Price of Risk

- The key equation in the whole mess was
$$\frac{(a^C - r)}{b^C} = \frac{(a^P - r)}{b^P} \equiv \lambda.$$
This says a number of things.

- What we managed to do was get all the stuff related to our first option C separated from all the stuff involving the second one P. What this means is that **this ratio**, which I labeled λ, **has nothing to do with any particular derivative**. It must be an inherent property of the economy.

- Once we got here, we could forget about P. Nothing relating to it appears in our final PDE.

- But what is λ? Well what are those two ratios?

- The way I defined the as, each one is the expected percentage change on a particular portfolio which was exposed **only** to σ risk. Likewise, each of the bs was the volatility of one of those portfolios.
$$d\Pi^P/\Pi^P = a^P\,dt \;+\; b^P\,dW^\sigma$$

- So $(a - r)$ is just an *excess expected return*. And the ratio is a risk-reward measure, also known as a Sharpe ratio. The equation just says:

  every portfolio exposed only to dW^{σ} must have the same Sharpe ratio.

- That means λ is **the expected excess return you get for every unit of exposure you have to volatility risk.**

- In other words it's the market price of volatility risk.
  - ▶ We can similarly define a market price of any type of risk.
  - ▶ The market price of **stock market** risk, for example, is just the Sharpe ratio for holding a long position in the market portfolio.
  - ▶ The market price of **interest rate** risk, would be the Sharpe ratio for a portfolio *with a long exposure* to r risk.
    - ∗ So this could be estimated from the the Sharpe ratio for holding a short position in Treasury bonds.

- All of these numbers bould be different from each other, and all of them could change over time.
  - ▶ We should also note that there are, in principle, different market prices of volatility risk for different underlying assets.
    - ∗ For example, the market price of S&P 500 volatility risk is not the same as the market price of crude oil volatility risk.

- That's the economics of lambda. But when we hand our PDE to our mathematicians to solve, they're going to want to know a bit more than that.

  What <u>number</u> is it? What do we plug in to the computer?

- You can't look lambdas up in the newspaper. So what can you do?

- Just like estimating volatilities when using Black-Scholes, there are really only two things you can do:

  1. Estimate it from historical data; or

  2. Infer a value from other derivatives.

- To estimate λ_σ from historical data you would need to:

  (a) find historical returns on portfolios exposed only to σ risk (such as options portfolios);

  (b) compute their realized excess returns and volatilities;

  (c) forecast future Sharpe ratios from the historical average.

- Empirical studies have typically concluded that there is a significantly negative market price of volatility risk, at least for stock index volatility.

  ▶ How should we interpret that?

- Our second choice was to find implied lambdas from observed options prices.

- Here we would need to solve the PDE for a whole bunch of different options with a whole bunch of different λs and then pick the one that "best" fits the prices in the market today.

- There are some serious difficulties with this too.
  - ▶ What we are doing in the first place is trying to figure out what the option is *worth*. If we are willing to take the market prices of options as given, then what's the point of the whole exercise?
  - ▶ Also, the problem of backing-out a value from current derivatives prices isn't simple because we <u>also</u> have a bunch of other unobservable parameters (ρ, s, etc). So it's not as easy as getting an implied volatility from a call.

- Still, in some situations it does make more sense to single out particular derivatives, take their prices as given and impute λ from them.
  - ▶ For example, in pricing a fixed-income derivative, it would be natural to regard treasury bonds as primary assets.

- We'll talk more about extracting these λs next time.

- What I want you to remember is that, when you see a market price of risk in a derivatives model, you must ask what assumptions are being made about it, and why, and whether there is any evidence for them.

(B) The Replicating Portfolio

- One other thing to keep track of in the derivation of the PDE is the positions needed to replicate an option with stochastic volatility.
  - ▶ Even though P doesn't appear in the valuation equation, we still can't ignore it. Since we can't trade σ directly, we have to use P (continuously!) to hedge the volatility risk in C.
  - ▶ And the right hedge ratio was the expression

$$\frac{b_C \Pi_C}{b_P \Pi_P} \text{ which is } C_\sigma / P_\sigma.$$

  - ▶ So to figure out what this is, we will ultimately have to solve the PDE twice (to get $P(S, \sigma)$ as well as $C(S, \sigma)$) so that we can compute each rate-of-change.
  - ▶ These are the **correct vegas** of the two options.
- As in our original no-arbitrage argument, the replicating portfolio also has the usual stock and risk-free bond positions too.
  - ▶ But these quantities also will be slightly different from what they would be for the same option under the Black-Scholes model.
    - ∗ For example, δ isn't $\mathcal{N}(d_1)$ anymore.

(C) New Terms in the PDE

- Now let's have another look at that PDE we want to solve:

$$\frac{1}{2}s^2\sigma^2C_{\sigma\sigma}+\rho s\sigma^2SC_{\sigma S}+\frac{1}{2}\sigma^2S^2C_{SS}+rSC_S-rC+C_t+[\kappa(\sigma_0-\sigma)-\lambda s\sigma]C_\sigma = 0.$$

- Notice a few things
  - ▶ It's basically the Black-Scholes PDE with some more stuff.
  - ▶ If we had assumed different functional forms for the mean and variance of the σ process, these would just get dropped in in the obvious places here. We don't have to go through the whole argument again.
  - ▶ Likewise, we could have had ρ and λ be any functions of S, σ, and t and it wouldn't change the argument. You'd just plug them in here.

- The most important thing that should strike you, though, is the fact that all the parameters of the σ process now appear.

  Unlike the Black-Scholes case, it is no longer true that we don't care about the true drifts of the processes.

- Actually we still don't care about the drift of the stock.

- What accounts for this asymmetry? There are two underlying sources of randomness here, but they aren't being treated equally?

- Suppose that you could actually trade units of σ directly. How would the derivation have been changed?

- ▶ Instead of that complicated business with selling some amount of another option, we would just have "shorted" C_σ units of volatility directly and got

$$dC - C_S\,dS - C_\sigma\,d\sigma = [\frac{1}{2}\sigma^2 S^2 C_{SS} + \frac{1}{2}s^2\sigma^2 C_{\sigma\sigma} + \rho s\sigma^2 S C_{\sigma S} + C_t]\,dt$$

  which is a riskless position.

- ▶ So it would have drift $r[C - C_S S - C_\sigma \sigma]$, which would give us the PDE

$$\frac{1}{2}s^2\sigma^2 C_{\sigma\sigma} + \rho s\sigma^2 S C_{\sigma S} + \frac{1}{2}\sigma^2 S^2 C_{SS} + rSC_S - rC + C_t + r\sigma C_\sigma = 0.$$

- ▶ Now <u>this</u> makes *no reference to the drift term <u>or</u> to λ.*

- So we have learned that *there is a fundamental difference between traded and non-traded sources of randomness when it comes to no-arbitrage pricing.*

- **For non-traded risks, we care about both the market price of those risks and about their true expected changes**.

- There is one last thing we need to think about before we can hand the PDE off to our mathematician.

(D) Boundary Conditions

- When we add another source of uncertainty, our PDE adds a dimension.
- Mathematically, this means, we need to specify additional information about what happens to our derivative at the extreme ranges of the new variable.
- Economically, it means we can now incorporate any features of the contract that specify cash-flows that depend on the new variable.
  - ▶ So, we could use our PDE to value a *variance swap* that pays σ_t^2 every day.
    - ∗ Or, in practice, it would pay the daily realized variance $\hat{\sigma}_t^2$.
  - ▶ Would this then make volatility a "traded asset"?
  - ▶ How would this differ from a contract on implied volatility?
- Once we have specified appropriate boundary conditions, we can (finally) turn to the solution of our 2-dimensional problem.

VI. Results.

(A) Pricing

- In the next lecture we'll talk about general methods for solving our new two-dimensional PDE.

- For now, let's think about the intuition behind the model.
  - ▶ How should random volatility affect the shape of the probability distribution of S_T?

- Think about simulating future paths of the underlying.

- When there's some chance that volatility can be really big, that means you *might* get paths along which really extreme things start to happen.
  - ▶ That will occur if there happen to be a lot positive random shocks to σ_t

- But the other side of the coin is that you might *instead* get a lot of negative volatility shocks.
  - ▶ That will make for very boring stock price paths.

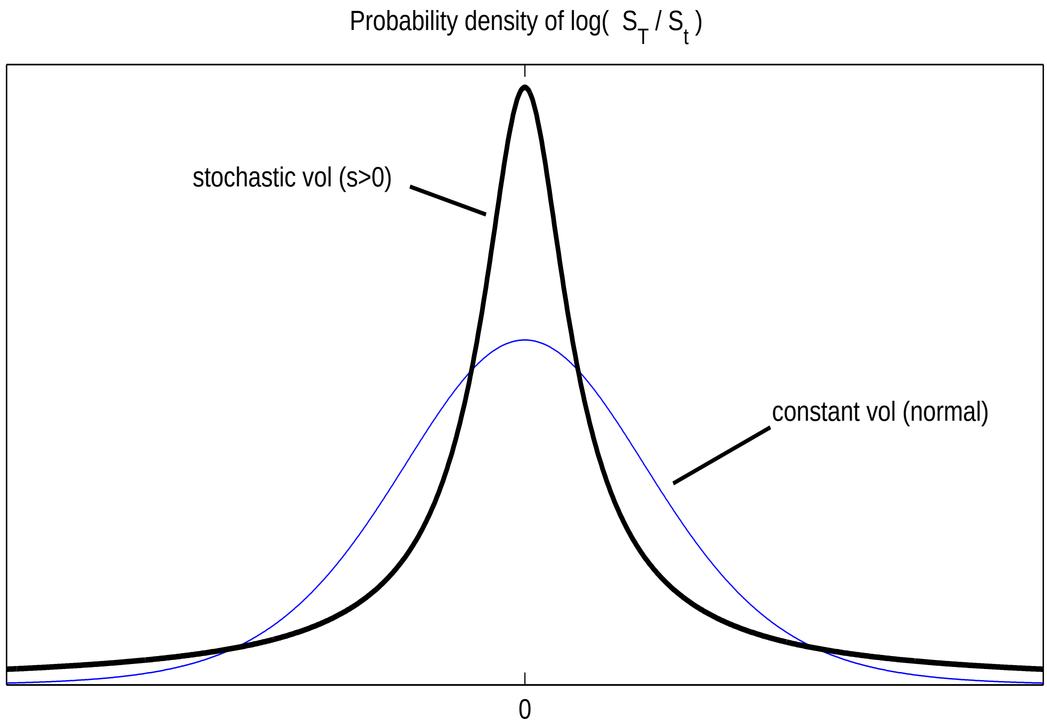
- So, compared to a constant volatility model (with the same average level of volatility) stochastic volatility means

(**A**) *fatter tails of the return distribution* AND

(**B**) *more mass near zero* AND

(**C**) *thinner "shoulders".*

- Here's an (exaggerated) graph comparing the two distributions.



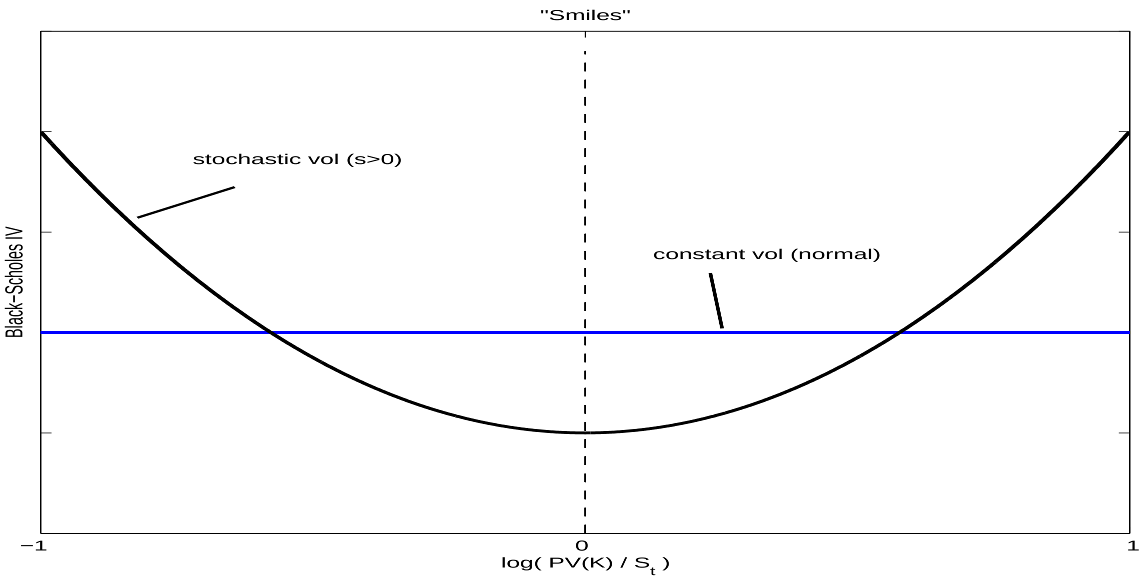
- The picture immediately suggests how the SV model will differ from Black-Scholes when it comes to pricing options.

  Far out-of-the money puts and calls become relatively MORE valuable.

  Near-the money puts and calls become relatively LESS valuable.

- It's easiest to compare prices in terms of their Black-Scholes implied volatilities.

- Here's a plots of the "smiles" for the two models for a fixed maturity:



- Of course if we use the constant volatility model (Black-Scholes) to price options and then convert back to BSIV's we get a constant.
  - ▶ This doesn't happen for stochastic volatility.

- Which parameters in the model do you think influence the steepness of the smile?

- For options close to maturity, the vol of vol doesn't matter much.
  - ▶ In fact, the BSIV will actually equal σ_t as $T \to t$.

- Now let's think intuitively about what happens to the return distribution as we vary ρ.

▶ If ρ is positive then stock paths along which S *starts* to go down are likely to also have σ_t *start* to go down.

▶ So *after an initial move down*, S will be LESS volatile.

▶ So further down moves become unlikely.

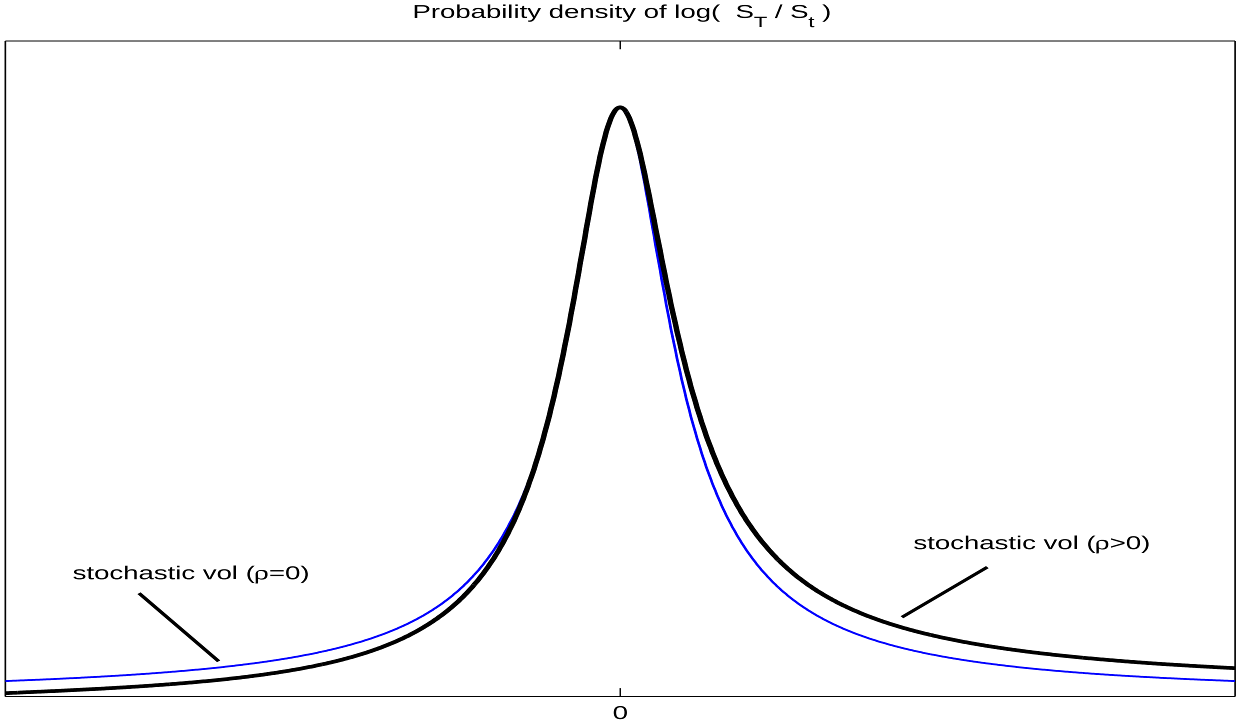
∗ Conclusion: crashes are less likely.

▶ The same logic works in reverse if ρ is positive and the path of S starts to go UP.

∗ Now *further* big up-moves become relatively MORE likely (since σ_t is likely to be higher).

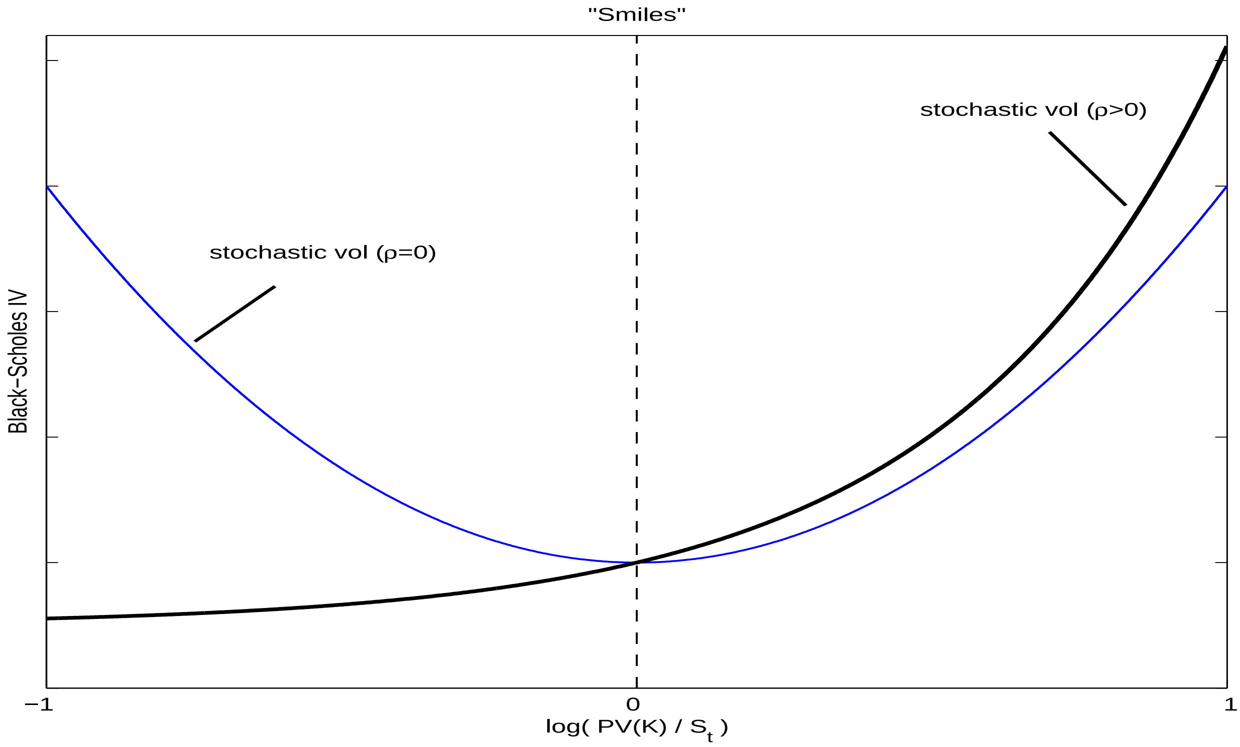
∗ Hence upward price explosions are more likely.

• Here's another exaggerated picture of the overall impact on the time-T return distribution:



• Now again, think about what this means for options prices.

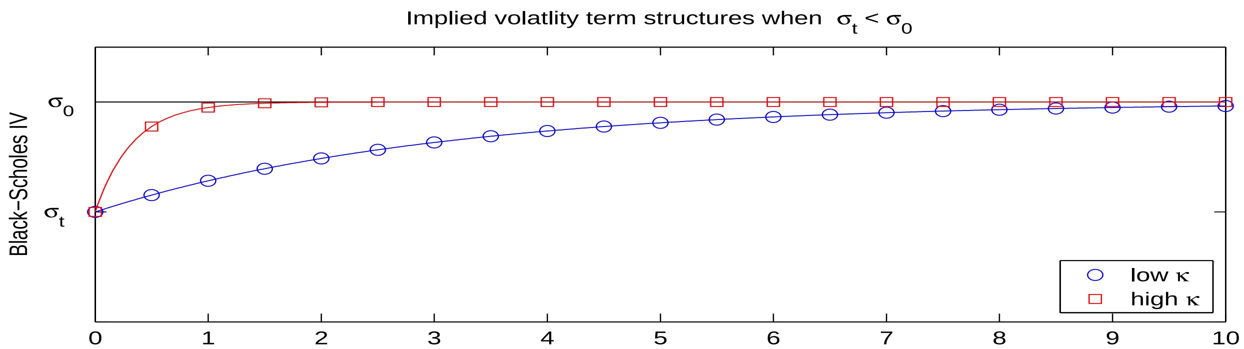
• Positive ρ makes it relatively more likely that out-of-the-money calls will pay off (than under a symmetric distribution) and less likely that out-of-the-money puts will. (And, of course, put-call-parity means that out-of-the-money puts have to have the same BSIV as in-the-money calls.)

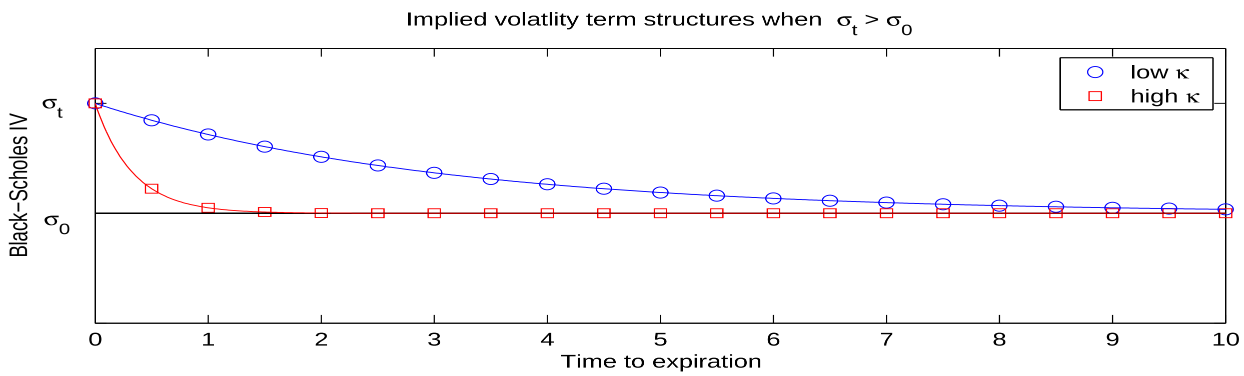


• The model can generate skewed smiles too!

• For negative rho, the pictures (and the intuition) are just reversed.

- Finally, stochastic volatility models can also generate realistic <u>term-structures</u> of implied vols.
  - When $\sigma_t > \sigma_0$, the long-run level, the term structure slopes down. It slopes up in the opposite case.
  - The steepness of the slope is governed by κ, the speed of mean-reversion.





- To summarize, stochastic volatility models BOTH fit the data of actual return distributions better than deterministic vol models AND correspond to what options markets tell us people really do expect about the future.

(B) Hedging

- We know that if we have the wrong model, we will necessarily have the wrong replication/hedging strategy for any given derivative.
- The question now is *how wrong?*
- It's possible that the model gives hedge ratios that, while different from Black-Scholes, don't yield very different net cash-flows when averaged over the life of an option.
- Melino & Turnbull (1995) investigated this.
  - ▶ They compared realized hedging errors for different maturity options held to expiration under three strategies

    (i) Doing no volatility hedging;

    (ii) Eliminating the Black-Scholes "vega" using periodically updated implied vols ;

    (iii) Hedging volatility using the correct SV volatility exposures.
  - ▶ All of the strategies were kept delta neutral.
  - ▶ They did a lot of simulations and tabulated:

    1. The average cost minus the cost of the option.

    2. The variablility (root-mean-square) of this replication error across simulations.

- They find that the stochastic vol model does deliver dramatically reduced hedging errors for options of over 90 days life.

- Trying to use Black-Scholes to do **vega** hedging (as well as delta hedging) generates the worst results of all.
  - Shows the danger of using a model in an inconsistent fashion.

- In terms of the replicating portfolios, the deltas are not too different between the models. The big differences are in the volatility hedges.
  - In general, the stochastic volatility model tells you you have less volatility risk than Black-Scholes would make you think.
  - As a practical matter, when trading is costly, overhedging doesn't just raise the riskiness of the payoff. It also makes the whole strategy more expensive.
  - The effect is bigger for longer term options since volatility shocks tend to reverse over time.

- What about in real-life situations?

- An extensive study by Bakshi, Cao, and Chen (1997) compared SV hedging results to those delivered by more complicated models (as well as BS).
  - They use actual options prices and returns for the S& P500. With hundreds of options and daily data, they have over 15,000 observations.
    - Note that most of these options have less than 90 days to maturity.
  - The stochastic-vol model lowers average absolute errors the most.
  - It also delivers more accurate deltas and leads to hedging errors which are less sensitive to the re-balancing frequency.
- To conclude, it appears that the economic importance of modeling changing volatility can be large, both in terms of accurate pricing and in managing risk or hedging derivatives positions.

VII. Summary

- This lecture showed how to relax the assumption that volatility is constant and still price derivatives by no-arbitrage.

- We observed that deterministic vol models are inadequate at describing both actual returns and the volatility expectations embedded in options prices.

- We then asked: when do we <u>need</u> to use more complicated models?

- Answer: it may be dangerous to trust Black-Scholes if:
  - ▶ Your underlying has high volatility of volatility, or that shocks to volatility are persistent; or

  - ▶ Your derivatives have a long time till expiration.

- In these cases, you need a model which incorporates the costs of hedging volatility exposure into the derivatives price.

- Besides this topic, our development has shown that we can incorporate **multiple sources of risk** into the no-arbitrage framework.

- We saw that there has to be another security available to hedge the new source of risk, and you have to be able to trade it continuously and costlessly.

Lecture Note 7.1: Summary of Notation

SYMBOL	PAGE	MEANING
r_{t_i}	p4	*return from one particular date t_{i-1} to another date t_{i-1}*
$\hat{\sigma}$	p4	*estimated volatility using the sample of N returns r_{t_i}*
$\sigma(S,t)$	p7	*deterministic (non-random) volatility function*
$\frac{\partial c_{BS}}{\partial \sigma}$	p11	*Black-Scholes vega*
σ_t	p16	*instantaneous volatility of underlying returns at time t*
$m(\ ), s(\ )$	p16	*generic drift and diffusion functions for a continuous-time process, in particular, a volatility process*
dW_t^σ	p16	*instantaneous random normal innovation in volatility at time t of mean zero and variance dt*
κ, σ_0	p17	*speed of mean-reversion and long-run level of volatility*
ρ	p17	*correlation between underlying returns and volatility changes*
s	p17	*instantaneous volatility of volatility (a constant)*
P, C	p25	*prices of two derivatives which depend on S and σ*
Π^C	p25	*value of a delta-hedged portfolio holding one unit of C*
L^C	p25	*amount of riskless borrowing or lending in portfolio Π^C*
a_C, b_C	p26	*instantaneous expected change and volatility of portfolio Π^C*
a_P, b_P	p26	*analogous to the above for delta-hedged portfolio Π^P which holds one unit of P*
λ	p27	*market price of volatility risk = Sharpe ratio for portfolio exposed only to σ risk*