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Lecture Note 4.2: Applications of the Binomial Model

Introduction:

In this note we will extend the binomial model to price a variety of complex derivatives. This isn't just an exercise. For some of these options, the binomial model is actually *the best* method of deriving no-arbitrage prices and hedge ratios, and is widely used in practice.

The procedure – just iterating backwards from the terminal payoffs – gives us an amazingly simple and general recipe for valuing all kinds of derivatives.

Outline:

I. Adding Payouts to the Underlying

II. American Options

III. Some Exotic Options

IV. Issues of Convergence

V. Summary

I. Payouts to the Underlying

(A) Continuous Yield

It's simple to modify the basic binomial argument for the case of continuous proportional payouts on the underlying. The most important application is for currency options.

- Consider a one-period call on the U.S. Dollar from the perspective of a Japanese investor.

- Let:

$$\begin{aligned} u &= 1 + \text{rate of return if Yen price of Dollar goes up} \\ d &= 1 + \text{rate of return if ¥/\$ goes down} \\ \bar{r}_d &= 1 + \text{interest rate for borrowing and lending Yen} \\ \bar{r}_f &= 1 + \text{interest rate for borrowing and lending Dollars} \end{aligned}$$

- As before, we are somewhere on our grid and we know the two possible values, C_u and C_d, that the call could have next period.

- Now if we hold δ Dollars, we put these in U.S. risk-free deposits and they each become $\delta\bar{r}_f$ Dollars (worth $\delta uS\bar{r}_f$ or $\delta dS\bar{r}_f$) next period. So the two replication conditions become:

$$\delta u\bar{r}_f S - \bar{r}_d L = C_u \quad (1)$$

$$\delta d\bar{r}_f S - \bar{r}_d L = C_d \quad (2)$$

- We do the exact same algebra as last time. There are just two steps:

  1. Solve these two equations for δ and L.

  2. Plug these into the no-arbitrage equation

  $$C = \delta S - L$$

  and re-arrange.

- I'll let you fill in the details. In the end we get the same relation:

$$\mathbf{C} \;=\; \frac{\bar{\mathbf{q}}\,\mathbf{C_u} + (\mathbf{1}-\bar{\mathbf{q}})\,\mathbf{C_d}}{\bar{\mathbf{r}}_\mathbf{d}}$$

- Only now we have

$$\bar{q} \equiv \frac{(\bar{r}_d/\bar{r}_f) - d}{u - d} \quad \text{and} \quad (1 - \bar{q}) \equiv \frac{u - (\bar{r}_d/\bar{r}_f)}{u - d}.$$

- Note that if we have continuously compounded domestic and foreign interest rates r_d and r_f (instead of simple one-period rates) then

$$(\bar{r}_d/\bar{r}_f) \approx e^{(r_d - r_f)\Delta t}.$$

- In a nutshell, we just have to modify the pseudo-probability q to reflect the yield on the underlying but otherwise things are the *exact same*.

  ▶ We still just discount next period's possible values by $B\bar{q}$ or $B(1 - \bar{q})$ to get the current value.

- This analysis applies equally well to models with convenience yields, borrowing fees, storage costs etc.
- Recall the key assumption here: *we have to know these rates in advance.*
  - ▶ But the method **doesn't** require that they are constant over all of the grid!
  - ▶ In the more general case, you would just compute a different $\bar{q}$ at each node.

(B) Discrete Payouts (Dividends)

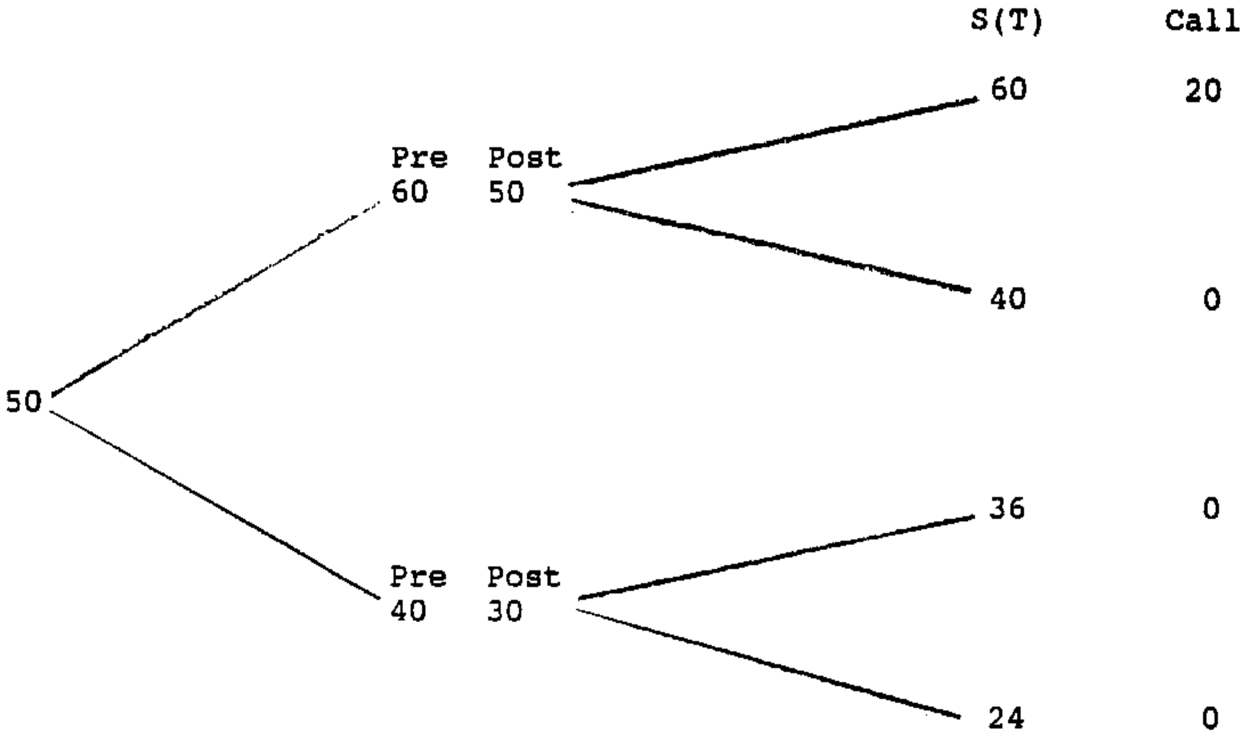
- We can include dividends in the binomial model as long as:

  (a) We know the timing and dollar amount of each dividend to be paid between t and T. *or,*

  (b) We know the timing and proportion of the stock price that will be paid from t to T.
- For the European case, all we do is modify the stock price tree to account for it dropping by the amount of the dividend at the ex date.
  - ▶ We'll do the American case in few minutes.

- **Example:** Current stock price is \$50 and company will pay a \$10 dividend in one month. If,
  - ▶ It returns ± 20 % each month (simple).
  - ▶ The monthly (simple) interest rate is 10%.

  What is the value of a European call option with a strike price of 40 if it expires in two months?



- Here $\bar{r} = 1.1, u = 1.2, d = 0.8$, so $q = \frac{(1.1-0.8)}{(1.2-0.8)} = 0.75$.
- Then $c_u = 13.636$ and $c_d = 0$.
- In setting up the replicating portfolio at time zero, does anything change because of the dividend?

- No.
- The stock return is the same at each node whether we use the pre- or post-div prices. So u and d are the same. So just use the same q.
  - ▶ Conclude $C = 0.75 * 13.636/1.1 = 9.30.$

- Actually the assumption of constant monetary dividends is not very sensible in practice. There are a few problems:
  - ▶ The tree no-longer recombines after the dividend with a fixed u and d. This means we have an increasingly messy tree to keep track of.
  - ▶ The grid can go negative!

  Also, for a long horizon, it can't be sensible to assume a riskless dividend stream from a risky stock!

- A solution to all these problems is to model the firm as paying out a fixed *percent of its share price* on the dividend dates.
  - ▶ It causes the grid to recombine. (Check!)
  - ▶ We won't be modelling a company as potentially paying out more than it is worth! (Grid stays positive.)
  - ▶ It makes the dividend stream as risky as the share.

  We'll do an example below.

II. American Options

- American style options are substantially harder to price in most models.
  - ▶ The no-arbitrage value of the option is the value it has under the smartest possible exercise strategy.
  - ▶ The holder of the option essentially has a *second* option at each instant to exchange his original option for the underlying asset.
  - ▶ We saw that early exercise of a call is never optimal *if there are no payouts to the underlying.*
  - ▶ But if there *are* such payouts, holding a call that is deep in-the-money may be foregoing more in yield than the time-value of the option is worth.
  - ▶ So computing the optimal exercise point involves weighing the interest lost against the time decay of the option premium.
  - ▶ The solution varies throughout the life of the option, and is very difficult to describe analytically.
- In the binomial model, though, it's trivial:
  - ▶ Just work backwards as usual, but replace the model's value with the option's intrinsic value at every node where the latter is higher.
- Return to our currency option example.

- Parameters: $S = 100¥/\$$, $K = 100¥/\$, r_d = 1\%$, $r_f = 6\%$ (quarterly compounded annual rates). $T = 1$ year.
- Consider a 4-step tree.
- To construct the tree
  - ∗ Choose u, d to model a 12% annual volatility for the spot rate.
  - ∗ As we have seen, one way of doing this is $u = 1/d = \exp(\sigma\sqrt{\Delta t}) = 1.0618$.
  - ∗ Using our formula for the psuedo-probability:

$$\bar{q} = \frac{(\bar{r}_d/\bar{r}_f) - d}{u - d} = 0.382.$$

- Here's what the European tree looks like:

Spot ¥/$				127.12	**European**	**call**			27.12
			119.72					18.20	
		112.75		112.75			9.94		12.75
	106.18		106.16			4.93		4.86	
100.00		100.00		100.00	2.32		1.86		0.00
	94.18		94.18			0.71		0.00	
		88.69		88.69			0.00		0.00
			83.53					0.00	
				78.66					0.00

- For an American call, proceed as before to time step $T-1$. Then notice that we should exercise at both in-the-money nodes.

	27.12
$\mathbf{18.20 < 19.72}$	
	12.75
$\mathbf{4.86 < 6.18}$	
	0.00
0.00	
	0.00
0.00	
	0.00

		27.12
	19.72	
$\mathbf{11.32 < 12.75}$		12.75
	6.18	
2.36		0.00
	0.00	
		0.00
	0.00	
		0.00

- Then do the same for the next step. Here only the most in-the-money node should be exercised.

- Then at the last two steps, the time-value of the option is big enough that we shouldn't exercise early. Final tree:

American call				27.12
			19.72	
		12.75		12.75
	6.32		6.18	
2.96		2.36		0.00
	0.90		0.00	
		0.00		0.00
			0.00	
				0.00

- Notes about the solution:
  - ▶ The American call is quite a lot more valuable here.
    - $*$ The difference is greatest for high S/K or low r_d/r_f.
  - ▶ Tree gives us *optimal* exercise strategy, as well as the corrected hedge ratios.

- With <u>discrete</u> payouts, do the same node-by-node checking for early exercise, using the grid-shifting (described above) for the time-step when the payment is made.

- **Example:** Consider this stock tree:

						86.4
					72	
		72	$\searrow$	60		57.6
	60				48	
50		48	$\searrow$	40		38.4
	40				32	
		32	$\searrow$	26.7		25.6
					21.3	
						17.1

  - ▶ Here the company will pay the dividend to people that own the stock by the third period.
  - ▶ If you exercise a <u>call</u> on that date you get the money. So, for a call, use the cum-div prices (to the left of the arrows) to figure out exercise value.
    - $*$ For a <u>put</u> you would always wait until the stock goes ex. So use those prices to evaluate the put exercise strategy.

- As above, we compare the "dead" value to the "live" value – what the option will be worth if we don't exercise.
- Suppose $q = 0.55, \bar{r} = 1.02, K = 40$. Then unexercised options will be worth

		46.4
	32.8	
21.9		17.6
	9.5	
5.2		0
	0	
0		0
	0	
		0

- So, comparing these to what we could get by exercising <u>prior to</u> the dividend, we learn that it *is optimal* to exercise at the top two nodes.
- Replace the values there with intrinsic, and finish the tree:

						46.4
					32.8	
		32	$\nwarrow$	21.9		17.6
	20.8				9.5	
13.05		8	$\nwarrow$	5.2		0
	4.2				0	
		0	$\nwarrow$	0		0
					0	
						0

- Notice I skipped the step of checking for exercise at all the steps other than the third. Why?

- “Bermudan” options are a type of exotic that allows early exercise *at some times but not at others*.
  - Examples:
    - Executive stock option that can't be exercised for first 5 years.
    - Cash-settled index options that can only be exercised at each day's close.
  - Simple to handle with binomial tree:
    - Just enforce the early-exercise inequality at time steps for which it is permitted.

III. Exotic Options

- Over the years, investment banks have dreamed up a variety of odd payoff structures, which – despite their complications – are true derivatives in the sense that they can be perfectly replicated.

- Supposedly "tailored to suit the client's special needs", these securities are often just tools for confusing people into overpaying.

- In other words, your initial intuition about what they are worth might be very wrong!

- For many of them, the binomial model provides a fast, easy-to-program valuation method (as well as hedge strategy).

- Let's consider some examples.

(A) Variable-strike American Options

- Some options have exercises prices that change over their lives, e.g. $K = K(t)$.

- Corporate bonds often contain embedded options in the form of call features (the right of the company to buy them back – or redeem them – before maturity). Typically the prices at which they can do this, the strike prices, vary over the life of the bond.

- If you own the bond, you have implicitly sold a call option (on the non-callable equivalent) whose strike price depends on time.

- Value this just like an ordinary American option, except adjust the early-exercise payoff function at each time-step to reflect the changing strike.

(B) Compound Options

- A compound option is an option on an option, like the right to buy a put. (This might be a "cheap" form of insurance.)

- Say the underlying option expires at T_1. The option on the option could expire at any $T_2 \leq T_1$.

- Value it in two steps:

  1. Value the underlying option on a binomial tree for *its* underlying asset extending out to T_1, saving the values at each node.

  2. Then re-use the grid (out to T_2) treating *those* values as the underlying for the second option.

- In other words, step 1 converts the stock grid to a grid of e.g. ordinary put values. Then in step 2 we can value any derivative on it using the same q!

- No reason to stop there: we can now do options on options on options on ...

(C) Barrier Options

- Barrier options are options that turn into something else when the price of the underlying asset crosses some pre-determined level (or levels).

- Classic examples:

  Knock-out options which immediately become worthless when either $S \leq S^\star$ ("down-and-out") or $S \geq S^\star$ ("up-and-out").

  Knock-in options which are not activated *until* either $S \leq S^\star$ ("down-and-in") or $S \geq S^\star$ ("up-and-in").

- More exotic variants:
  - ▶ Binary (or "touch") options.
  - ▶ Time-varying barriers.
  - ▶ Multiple barriers.

- Valuation is straightforward in a binomial world.

- For knock-outs, just work backwards and put zeros at every node across the barrier.

- For knock-ins, do a first pass (as with compound options) to compute the value of the ordinary option that you get if the barrier is crossed.

  - ▶ Then do a second pass, putting those values on the nodes for which the barrier is crossed.

(D) An Exotic Example.

- You want to offer a client a type of protection on a non-dividend paying stock whose price today is \$ 100. Assume that, the stock will move either up by $33\frac{1}{3}$% or down by 25% in each of the next three years. The risk-free rate is 6% per year.

- Using annual time steps, value a one-year at-the-money American put *with the option to extend* for a second and third year.

- The client will pay an initial fee, f, for the first year, and at the end of the year, he can either terminate the option (either exercising or not) or else elect to pay you the same fee again to keep the option alive for the next year. The third year works the same.

- If we assume a particular value for f, we can value the contract by working back from the end of the grid. Then, if the value of the contract at time zero is more or less than the initial fee the contract is not fairly priced.

- For example, assume the fee is $f = 11$. Then (with $u = 1.333, d = 0.75, \bar{r} = 1.06$, and $q = 0.53143$) at the end of year-2 the valuation looks like this:

stock:				**continuation value:**			
			237.037				0
		177.777				.	
	133.333		133.333		.		0
100		100		.		11.051	
	75		75		.		25
		56.25				38.089	
			42.1875				57.8125

- The client will choose to pay 11 at the middle node, but at the bottom node he will not, because $38.089 - 11 < K - S = 43.75$. (Or course, in the top node, he would not pay 11 for a claim worth zero.)

- Next, stepping back one-period, we have

	0			0				0
0			0				0	
	0			0		0.465		0
11.051		$\longrightarrow$	0.051		$\longrightarrow$		0.051	
	25			25		19.76		25
38.089			43.75				43.75	
	57.8125			57.8125				57.8125

- This time, we find that it is not optimal to extend at <u>either</u> of the $T = 1$ nodes. In the up node, the continuation value is only 0.465, which is not worth paying 11 for. And in the down node, 19.76 is less than intrinsic which is 25. So we replace these values by 0 and 25 .

- Now in going back the last step, the payoffs look just the same as at the ud node at time 2. So we know its value today is 11.051, or after the initial fee, 0.051.

- From this analyis, we can see that the fair fee cannot be less than the value of a one-period put, which is 11.051. Nor can it be more: if we repeat the analysis with $f = 11.1$, for example, we will find a negative value today.
  - This is not a general result: it just happens to be true in this example because we only used one time step per year.
- The important thing is to notice how, at each node, we just had to evaluate the continuation decision faced by the option owner.
- Also, remember, whatever the payoffs are, we always use the same q to move back through the tree. *Never use the true probability for anything!*

IV. Notes about Convergence

- An obvious feature of the binomial model is that it gives you different values for your derivative with different choices of the number of steps, N, in the tree, even when you pick u, d and π to keep the mean and variance of the underlying the same for all N.

- But if you plot the values you get for different N, you see that they are always converging as N goes up.

- This is because the underlying binomial processes are converging to a unique Brownian motion process. So the derivative prices are converging to their value in a continuous-time world.

- In practice, people naturally think of their binomial scheme not as a serious representation of the real world, but instead as a computational device for getting at the continuous-time values.

- This makes sense:
  - ▶ The Brownian model of dynamics seems more realistic.
  - ▶ The alternative is to attribute economic meaning to some particular N.

- So the practical question then arises: *How large must we make* $\mathbf{N}$ *to get close to the limit?*

- Answer: not large for ordinary European options (N=10 is usually plenty), but sometimes *huge* e.g. for exotics.

- A lot of technical literature is devoted to figuring out how to get binomial routines to give the continuous-time answer in the shortest time (i.e. with the smallest N).
- But, before turning our backs on the binomial prices themselves, it is worth remembering that sometimes time and prices really *do* move discretely instead of continuously.
  - ▶ Markets close
  - ▶ Liquidity can dry up
  - ▶ Prices can gap up or down
- In some situations the binomial model may be more realistic.
- When valuing barrier options, for example, it might be very important to take account of the fact that prices can jump <u>through</u> the barrier without hitting it exactly.
- In short, it isn't necessarily the case that the discrete model is <u>more false</u> than the continuous-time one, or merely an approximation to it.
- Remember, all our models are only approximate descriptions of real markets.

V. Summary.

- The binomial tree framework gives us a way to derive no-arbitrage prices for a lot of very complicated derivatives.

- You now know how to handle
  - ▶ Underlyings with payouts;
  - ▶ American options;
  - ▶ Several types of exotic options.

- The method doesn't give closed-form solutions, but it is fast and easy to program.

- Although we did not pursue it, the method can also handle different shaped (asymmetric or time-dependent) grids, and payouts that differ at different nodes.

- As N gets large, binomial prices converge to those that hold in a continuous-time Brownian world, which is the one Black and Scholes used.

Lecture Notes 4.1 and 4.2: Summary of Notation

Symbol	Page	Meaning
π	*L4.1, p2*	*probability that security price goes from* S *to* S^+
u, d	*L4.1, p4*	*multiplicative changes* $= S^+/S$ *and* S^-/S *respectively*
$\mathrm{E}[\,], \mathrm{Var}[\,]$	*L4.1, p5*	*expectation and variance of a random variable*
T, N	*L4.1, p5*	*length of time and number of time steps in binomial tree*
$N!$	*L4.1, p6*	$= N \times N-1 \times N-2 \times \cdots \times 2 \times 1$
μ_N, σ_N^2	*L4.1, p7*	*mean and var.* N*-period continuously-compounded return*
Δt	*L4.1, p9*	*length of time in years between nodes* $= T/N$
$\mathcal{N}(\mu, \sigma^2)$	*L4.1, p11*	*normal distribution with mean* μ *and var.* σ^2
dS	*L4.1, p11*	*change in S over a time interval of length* dt
dW	*L4.1, p11*	*random normal component of* dS *of mean 0 and variance* d
δ, L	*L4.1, p14*	*amount of underlying asset and borrowing held in*
		replicating portfolio for call
$\bar{r}$	*L4.1, p17*	*one plus the simple, per-period riskless interest rate*
δ_u, L_u	*L4.1, p17*	*amount of replicating asset and borrowing when at node* uS
C_{uu}, C_{ud}	*L4.1, p17*	*call value when stock is at nodes* u^2S *and* udS *respectively*
q	*L4.1, p22*	$\frac{\bar{r}-d}{u-d}$ $=$ *pseudo-probability of an up-move on the grid*
$C_T(S_T(j))$	*L4.1, p27*	*Payoff at* T *of any European derivative*
		if the underlying ends up at $S_T(j) = S_t\, u^j d^{N-j}$
C_t	*L4.1, p27*	*value today of that derivative*
$b(x)$	*L4.1, p28*	*price of a butterfly paying 1 if* $S_T = x$
$\bar{\mu}$	*L4.1, p29*	*one-plus expected simple per-period return of underlying*
$\bar{r}_d, \bar{r}_f$	*L4.2, p2*	*one plus the simple per-time-step riskless interest rate*
		in the domestic and foreign currencies respectively
B	*L4.2, p3*	*price of a one-period riskless 0-coupon bond* $= 1/\bar{r}_d$
$\bar{q}$	*L4.2, p3*	$\frac{(\bar{r}_d/\bar{r}_f)-d}{u-d}$ $=$ *pseudo-probability of an up-move on the grid*
		when the underlying pays a continuous yield
$S^\star$	*L4.2, p15*	*knock-in/knock-out price for barrier options*