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Lecture Note 5.1: The Black-Scholes Option Pricing Formula

- In 1973, Black and Scholes derived a closed-form solution for the price of a European call on an asset whose price is a geometric random walk in continuous time. **Why should we care?**

- Afterall, we already know we can get the same answer – and can price much more interesting derivatives – just by using the binomial model with a large N. There are three reasons.

- First, closed-form solutions are useful for analyzing properties of options prices *in general*, rather than just for a specific tree.

- Second, they can sometimes speed-up calculations.

- But really the main reason to study the Black-Scholes model is to understand how the no-arbitrage argument works in continuous time. This will show us incredibly powerful ways to approach *all* derivatives.

Outline:

I. Some continuous-time math
II. The Black-Scholes-Merton argument
III. A general derivatives equation
IV. Recognizing all the assumptions
V. The Black-Scholes solution for a European call
VI. Properties of the solution
VII. Summary

I. Stochastic Calculus

- Last week, I introduced the geometric Brownian motion model as the continuous-time limit of binomial processes. We describe such processes via a stochastic differential equation like:

$$\begin{aligned}\frac{dS}{S} &= \mu\,dt \;+\; \sigma\,dW \quad \text{or} \\ dS &= \mu S dt \;+\; \sigma S dW \quad \text{or, in general,} \\ dS &= a(S,t)dt \;+\; b(S,t)dW.\end{aligned}$$

- Recall, the left side is the instantaneous change in S (S is the price of the underlying asset), and the right side has two components.

  The deterministic part. The dt term says part of the change in S over a little time interval is due to things we know at the start of the interval. The coefficient $a(\,)$ is called the <u>drift</u> of dS.

  The random part. dW is a Normally distributed random variable with zero mean and variance dt. It is independent of the past and totally unpredictable at time t. Here $b(\,)$ is called the <u>diffusion coefficient</u> of dS. The whole dW term has variance b^2dt.

- If we approximate dt by the discrete time step $\Delta t = t_2 - t_1$, then we can approximate our new model by

$$\Delta S \;(= S_{t_2} - S_{t_1}) \;\; = a()\,\Delta t + b()\,\Delta W_{t_2}$$

which we can view as a recipe for how S evolves:

$$S_{t_2} = S_{t_1} + a()\,\Delta t + b()\,\Delta W_{t_2}$$

- Now suppose the value of a call is a function, $C(S,t)$, of the asset price. Then its changes over time are also driven by the same source of randomness W that changes S. Then suppose we knew that

$$dC = \alpha(S,t)dt \; + \; \beta(S,t)dW.$$

- Then we'd know how to hedge C: combine it with S in the proportion β/b. **Why?** Because that would get rid of all the randomness.

$$\begin{array}{rlcl} & \alpha(S,t)dt & + & \beta(S,t)dW \\ - & \frac{\beta}{b}\cdot[a(S,t)dt & + & b(S,t)dW] \\ = & \text{something}\, dt & + & \text{nothing}\, dW \end{array}$$

- So what we need is a "chain rule" for random processes that tells us how to find the differential of a function of S in terms of the differential of S itself.
  - ▶ We need a recipe to get $dC(S,t)$. That is, we want to know

$$\Delta C \approx C(S+\Delta S, t+\Delta t) - C(S,t)$$

  - ▶ This will give us $\alpha(\ )$ and $\beta(\ )$.

- The solution was found in 1951 by a mathematician named Itô. It says that, if C is a smooth enough function to have partial derivatives, then

$$dC(S,t) = \frac{\partial C}{\partial S}\, dS \; + \; \frac{\partial C}{\partial t}\, dt \; + \; \frac{1}{2} b^2 \frac{\partial^2 C}{\partial S^2}\, dt \qquad (1)$$

- Notice: this is just like the chain rule from ordinary calculus except for the last term:

$$df(X,Y) = \frac{\partial f}{\partial X}\, dX \; + \; \frac{\partial f}{\partial Y}\, dY$$

- Sometimes we will use subscripts to denote partial derivatives, so that equation (1) becomes

$$dC = C_S\, dS \ + \ C_t\, dt \ + \ \frac{1}{2} b^2 C_{SS}\, dt$$

or

$$dC = (C_S\, a(S,t) + C_t + \frac{1}{2}\, b(S,t)^2\, C_{SS})\, dt + C_S\, b(S,t)\, dW$$

- Recall that partial derivatives are just *rates of change* of C when either t or S changes by a unit amount.
  - ▶ This is the same thing as the slope of a graph of C against either S or t.
  - ▶ And then second derivatives are just rates of change of rates of change – or slopes of slopes.
- The formula then tells us that $\alpha = (C_t + C_S\ a + \frac{1}{2} b^2 C_{SS})$ and $\beta = C_S\ b$.
- We will not need to compute them for any specific function. But we do need to be able to manipulate them.
- Armed with **Itô's lemma**, we can readily understand the Black-Scholes argument.

Historical Note: The argument Black & Scholes actually used is much more convoluted than the one we are going to see, which was developed by Robert Merton.

Merton didn't get his name on the formula, but he did share the Nobel Prize for it.

II. The Black-Scholes Argument

- Unlike the binomial case, the continuous-time no arbitrage argument isn't going to give us the value of an option directly as the value of a replicating portfolio. Instead, it's going to give us a *relationship* that characterizes the option's value.

- Here's the argument. You need to understand each of the following steps.

  Step 1. Assume the price of the underlying asset follows

  $$\frac{dS}{S} = \mu\, dt \ + \ \sigma\, dW. \tag{2}$$

  Assume it pays no dividends. Let r be the continuously compounded, instantaneous risk-free interest rate. You can borrow or lend at this rate, so interest accrues at rate $r\,dt$.

  Step 2. Suppose that the price of a call is ONLY a function of S and t. That is, **there are no other sources of randomness affecting the value of the option.** Also assume the function $C(S, t)$ is smooth (so that we can apply Itô's lemma).

  Step 3. Construct a portfolio that is long the call and short $\partial C/\partial S$ shares of stock. Invest an amount I in the risk-free asset. (We will determine what these quantities are later in the argument.) Write the position value as

  $$\Pi \equiv C - C_S \cdot S + I.$$

Step 4. At all times, increase/decrease your risk-free investment by the cost of any changes in the stock position. With this self-financing policy, there is no cash in-flow or out-flow to your position. So its changes in value over dt come only from the positions you start the period with:

$$d\Pi = dC - C_S dS + rIdt.$$

This is a restriction on dI. Without it, in general, $d\Pi$ *also* would reflect changes due to $I(S)$ and $C_S(S)$ changing.

Step 5. Use Itô's lemma to rewrite the dC term:

$$\begin{aligned} dC - C_S dS + rIdt &= C_s\, dS \ + \ C_t\, dt + \ \frac{1}{2}\sigma^2 S^2 C_{SS}\, dt - C_s\, dS + rIdt \\ &= C_t\, dt \ + \ \frac{1}{2}\sigma^2 S^2 C_{SS}\, dt + rIdt \qquad (3) \end{aligned}$$

This is the instantaneous profit from our position.

Step 6. Deduce that, since this gain is riskless and has no cash-flows, **if it were not also equal to $\mathbf{r\,\Pi\,dt}$ there would be an arbitrage opportunity.** This is the key step.

Step 7. Conclude,

$$r(C - C_S \cdot S + I)\, dt = C_t\, dt \ + \ \frac{1}{2}\sigma^2 S^2 C_{SS}\, dt + rIdt$$

Hence:

$$\mathbf{\frac{1}{2}\sigma^2 S^2 C_{SS} + C_t - rC + rSC_S = 0}.$$

- That's it. That's seven steps to a Nobel Prize.

III. A General Derivatives Equation.

- Perhaps you are wondering why

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - rC + rS\frac{\partial C}{\partial S} = 0 \tag{4}$$

  is worth so much excitement. It doesn't tell us the price of a call, it doesn't look like it tells us the price of anything.

- The reason is that the argument is incredibly general.

- If you re-do the steps assuming the derivative has payouts $\Gamma(t)$ per unit time and the underlying has yield d, the derivation is the same and the equation just says

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - rC + (r-d)S\frac{\partial C}{\partial S} + \Gamma = 0 \tag{5}$$

- *Any* derivative on S must satisfy such an equation. We said C was supposed to be the price of a call. But we never referred to this. We never used any special features of calls in the argument.

- This means that we can price all possible derivatives under this model by solving (4) or (5).
  - ▶ And, in fact, it is a very familiar (to mathematicians) type of partial differential equation (PDE), whose solution techniquess are well understood.

- In other words, this gives us everything but the final answer.

- Different securities impose different boundary conditions on the solution to (4), which means they impose different payoff functions at maturity, or along barriers etc.

- **Example of boundary conditions:** Consider an "up and out" knock-out call. If the strike is K and the knock-out happens at price X, then to get the value of the call while it is alive, we would solve the equation with the additional conditions:

$$\begin{aligned} C(T, S_T) &= \max[S_T - K, 0] \text{ for } S_T < X \\ C(t, S_t) &= 0 \text{ for all } t \text{ and all } S_t \geq X \\ C(t, 0) &= 0 \text{ for all } t \end{aligned}$$

- Once we have boundary conditions, the theory of partial differential equations also guarantees that
  1. **a solution exists,** which is lucky, and moreover,
  2. **only one solution exists.**

- That tells us that, if we find a particular solution to the equation, it must be **the** price of our derivative.
  - ▶ Otherwise, we wouldn't know that there wasn't some other one out there, and we wouldn't know which one ought to be the price.

- Our PDE also gives us some interesting insights about what the different terms mean. Consider.

Interpretation in terms of hedging profit & loss.

Imagine we are hedging a position. We buy the derivative, sell $\delta = C_S$ shares, and finance the whole thing by lending. How does our profit and loss behave over time?

- Assuming our theory holds, we know that we will make zero profit *over each instant.*

- But now we can say something about the components of our gains/losses.

- Since we are hedged, our random gains and losses from the stock and the call exactly offset.

- Our interest costs (per unit of time) are going to be: $r(SC_S - C)$.

- The option has time-decay of $C_t \equiv \frac{\partial C}{\partial t}$ (regardless of what the stock price does).

- But now look at our PDE again:

$$\text{hedge book profit} = 0 = C_t + r(SC_S - C) + \frac{1}{2}\sigma^2 S^2 C_{SS}.$$

- That third term must also be part of our profit/losses or else things won't add up. But what is it?

- Where else can P&L come from?

- It must be that the gains or losses per unit time from our hedge adjustment are

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \equiv \frac{1}{2}\sigma^2 S^2 \ \gamma.$$

- In particular, this will be positive or negative depending on the sign of "gamma", the curvature of the derivative price with respect to the stock.

Interpretation of absence of μ.

- Perhaps the most important point to notice about the Black-Scholes-Merton PDE: the underlying stock's expected rate of return μ appears *nowhere*.

  * **An option on a stock expected to go up 30% a year is worth the same as one on a stock expected to go down 30% a year.**

    - all other things (σ, S, K, T, r) being equal.

  * We noticed this in the binomial model's solution. But here we see it *even before we ever solve the equation.*

  * And it applies to all derivatives, not just options.

IV. Recognizing the Assumptions

The Black-Scholes model makes a lot of "strong" assumptions. In discussing these, I want to distinguish between three different types of assumption.

(A) Old Assumptions.

- Just like the binomial model, and, indeed, *all* our no-arbitrage models, the Black-Scholes model uses the assumption of perfect-markets

  - ▶ No transations costs.
  - ▶ Riskless borrowing and lending and short-selling are possible.
  - ▶ No taxes.
  - ▶ No counterparty/legal risk.

- We don't want to brush these aside. (In fact, we will talk a lot about them later.) But first, I want to focus your attention on the **new** assumptions we had to make.

(B) New Assumptions

- There are really only two. But they matter.

  1. Continuous trading is possible.

  2. Prices cannot jump.

- Obviously these two are related. If either is not satisfied, we cannot construct instantaneously riskless portfolios.

- In practice, in the largest currency and fixed-income markets, continuity is not a bad approximation.
- Still, that doesn't tell us anything. **We don't know that a little violation means a little change to the model's results.**
- We will have to investigate that.
- The last category of assumptions are those concerning the interest rate and volatility processes. We need to clarify exactly what is old here and what is new.

(C) Interest Rate and Volatility Assumptions.

- First, we did assume there were no payouts, borrowing fees, etc. when we derived equation (4). But that was purely to slim down the argument. If there is a continous percentage payout y to the underlying (such as a continous dividend yield, or a foreign interest rate), it is not hard to modify Steps 1-7 to get

$$\frac{1}{2}\sigma^2 S^2 C_{SS} + C_t - rC + (r - y)SC_S = 0. \qquad (6)$$

- When Black,Scholes and Merton solved the PDE (6), they did so under the additional assumptions that
  1. The volatility σ is a constant.
  2. The risk-free rate r is constant.
  3. y is zero.
- But what did we actually have to assume about about r, y, and σ?

- Really **they can be any functions of S and t.**

- So, for example, you could even have random interest rates in a currency model as long as the exchange rate determined the interest rates.

- The key point is that **there is only allowed to be one source of randomness in the economy** which affects the derivative. We express this by saying that the Black-Scholes model is a one state variable model.

- Actually, we had precisely the same level of generality in the binomial model. We were allowed to vary r across different nodes of the tree, and could even have varied the tree spacing (u and d) at different times and levels. It increased computational complexity, but it didn't affect the no-arbitrage argument.

- Now let's have a look at what happens when we do make the additional assumptions

  1. The volatility σ is a constant.

  2. The risk-free rate r is constant.

  3. y is zero.

V. The Black-Scholes Solution

(A) Theorem (Black and Scholes (1973), Merton (1973)): The solution to (4) when r and σ are constant, and subject to the boundary condition

$$c(S_T, T) \;=\; \max[S_T - K, 0]$$

is given by

$$c(S, K, T, r, \sigma) = S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) \qquad (7)$$

where $\mathcal{N}(\cdot)$ is the **cumulative normal distribution function**, and where

$$d_1 \;\equiv\; \frac{\ln\left(\frac{S}{Ke^{-r(T-t)}}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \quad \text{and} \qquad (8)$$

$$d_2 \;\equiv\; \frac{\ln\left(\frac{S}{Ke^{-r(T-t)}}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} = d_1 - \sigma\sqrt{(T-t)}. \qquad (9)$$

Example: Price a 10 year European call struck at 400 on a stock trading at 100 when the risk-free rate is 6% and the stock volatility is 25%.

$$
\begin{aligned}
Ke^{-r(T-t)} &= 400 \cdot e^{-0.06 \cdot 10} = 219.53 \\
\ln\left(\frac{S}{Ke^{-r(T-t)}}\right) &= \ln(100/219.52) = -0.786 \\
\sigma^2(T-t) &= 0.625 \\
d_1 &= (-0.786 + 0.5 \cdot 0.625)/\sqrt{0.625} = -0.599 \\
d_2 &= (-0.786 - 0.5 \cdot 0.625)/\sqrt{0.625} = -1.390 \\
\mathcal{N}(d_1) &= 0.2745 \ \text{ (with the help of a computer)} \\
\mathcal{N}(d_2) &= 0.0823
\end{aligned}
$$

Conclusion:

$$c = 100 \cdot 0.2745 - 219.53 \cdot 0.0823 = 9.385$$

(B) Proof?

- A sufficient proof of the theorem, is just to plug the answer in to the PDE and verify that it works (and satisfies the boundary conditions). Then, because we know PDE solutions are unique, there can't be any other one.
- There is absolutely nothing interesting about all the calculus involved in solving the PDE "from scratch".
- We'll show later that you can also prove it from our general risk-neutral expectations formula.

VI. Properties of the solution

(A) Interpretation of the $\mathcal{N}$s.

- We can look at the Black-Scholes formula as another example of a formula we saw last time:

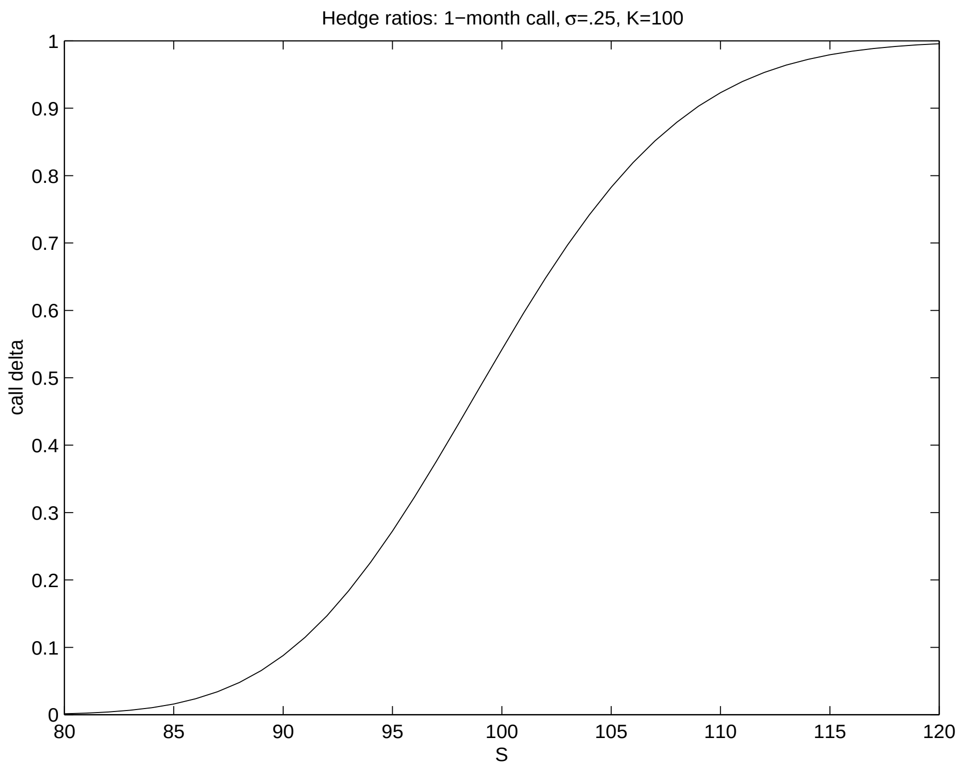
$$c = \delta S - L$$

- With a little calculus, we can differentiate the Black-Scholes formula and prove that

$$\frac{\partial c}{\partial S} = \mathcal{N}(d_1).$$

(**Hint:** use Leibnitz' Rule and the fact that $\mathcal{N}(x) = 1 - \mathcal{N}(-x)$.)

- This gives us a nice interpretation of $\mathcal{N}(d_1)$. It is just **the stock hedge ratio under the Black-Scholes model.**



- Next, from the Black-Scholes formula, we see that

$$c - \mathcal{N}(d_1)S = -Ke^{-r(T-t)}\mathcal{N}(d_2)$$

- But also

$$c - \mathcal{N}(d_1)S = c - \frac{\partial C}{\partial S}S = c - \delta S$$

- And we know that $c - \delta S = -L$ is the quantity of risk-free bonds in the replicating portfolio.

- Hence

$$L = Ke^{-r(T-t)}\mathcal{N}(d_2).$$

- This gives us an interpretation of $\mathcal{N}(d_2)$. It is **the percentage of the present value of the strike price we need to borrow to form the replicating portfolio.**

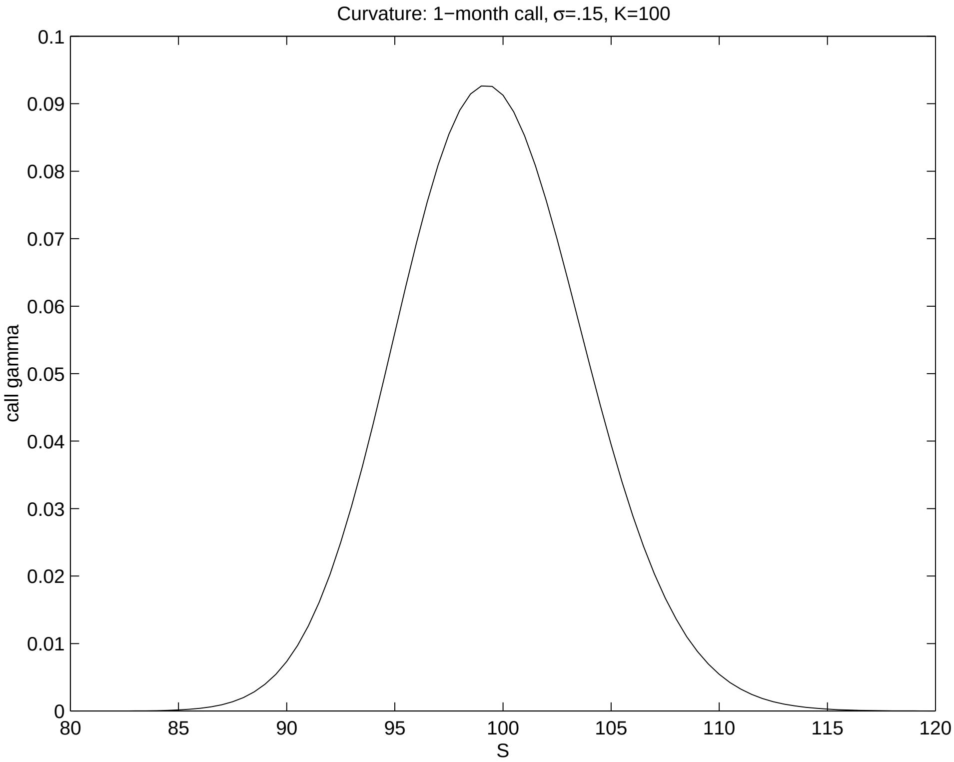
- As an exercise (which I will leave to you), one can now go back and verify that putting $I = -L$ does indeed make the replicating strategy self-financing.

(B) Rates-of-change of the solution

- Now that we have $c()$ explicitly, we can go back and compute those partial derivatives that appeared in the PDE to get a feel for what they look like.

- We already saw $\frac{\partial c}{\partial S}$. Now here's <u>its</u> slope

$$\frac{\partial^2 c}{\partial S^2} = \frac{\partial}{\partial S}\left[\frac{\partial c}{\partial S}\right]$$



(Note that to get the exact formula for this requires a lot more messy calculus.)

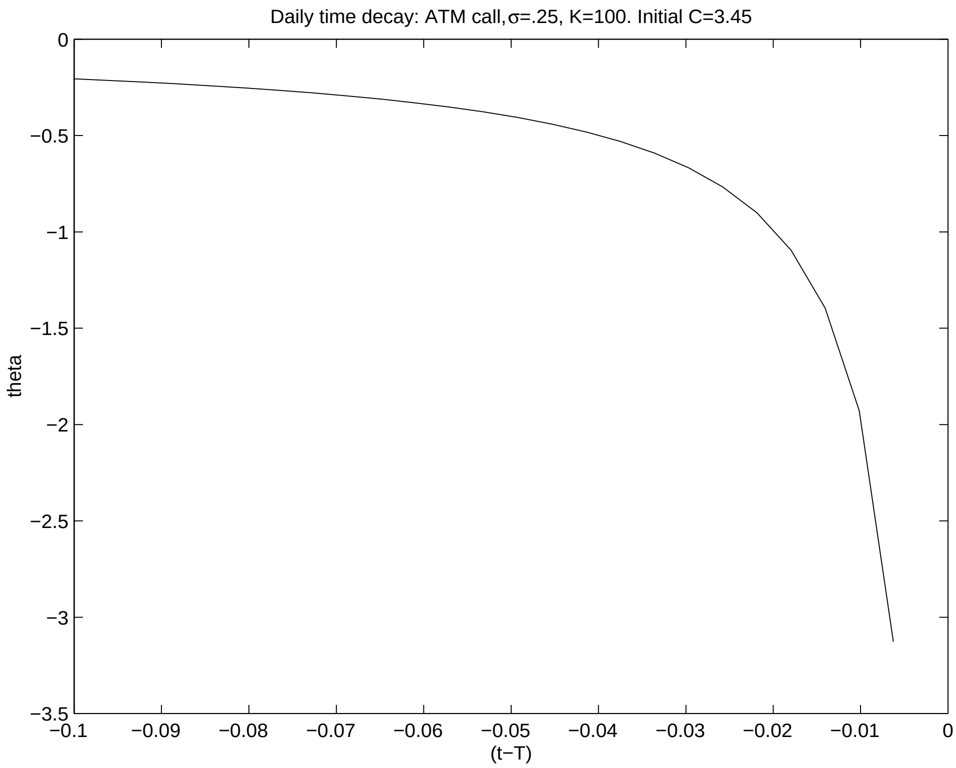
- As we deduced above, when this curve is high it means we have to be doing a lot of (costly) adjustment to the replicating portfolio.

- **Q:** How do you think this graph changes as we get closer to expiration?

- Next, let's look at $\frac{\partial c}{\partial t}$.

  ▶ What curve will this be the slope of?

  ▶ What should <u>that</u> curve look like?



- Surprisingly, nearly all of the time-decay in the option's premium is concentrated at the very end of its life.

- **Q:** Does this graph show that calls have negative expected return?

(**C**) Expected Returns.

- In fact, let's consider what we know about the expected return (or risk premium) of any derivative in the Black-Scholes-Merton world.

- Suppose we have an arbitrary claim $Q(S)$ on a non-dividend paying asset S that obeys $dS = \mu S\ dt + \sigma S\ dW$.

- The instantaneous expected (excess) return to Q is the predictable part of dQ/Q minus the riskless rate. (I'm assuming there are no payouts to holding Q itself.)

- Using Ito's lemma, we can write

$$dQ = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Q}{\partial S^2} + \mu S \frac{\partial Q}{\partial S} + \frac{\partial Q}{\partial t}\right) dt + \text{something}\ \ dW.$$

- But the Black-Scholes PDE says Q must also satisfy:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Q}{\partial S^2} + \frac{\partial Q}{\partial t} = rQ - rS\frac{\partial Q}{\partial S}.$$

- So the dt terms of dQ are

$$\mu S \frac{\partial Q}{\partial S} - rS\frac{\partial Q}{\partial S} + rQ.$$

- Dividing by Q to get expected *percentage* change, and subtracting r we get the expression for Q's risk premium:

$$\pi_Q = \frac{\partial Q}{\partial S}\ \frac{S}{Q}\ (\mu - r).$$

- This shows that the expected excess return is the product of the delta, the *gearing*, and the stock's risk premium.
  - Assuming the latter is positive, Q's risk premium is positive or negative depending upon the sign of delta.

- **Q:** Does this mean puts on the market have negative expected return?

(D) Risk-neutral expectations.

- **Question:** How would the Black-Scholes PDE we derived have been different if the stock price model had been:

$$\frac{dS}{S} = r\,dt \;+\; \sigma\,dW \tag{10}$$

instead of

$$\frac{dS}{S} = \mu\,dt \;+\; \sigma\,dW? \tag{11}$$

- **Answer:** not at all.

- We already noticed that it doesn't matter what μ is. So it might as well have been r as anything else, for all we care.

- But that first stock model would be awfully peculiar.

  It would say the stock's expected return over dt is the same as that of a perfectly riskless deposit.

- That would mean investor's were getting no compensation for the fact that the stock is risky.

- Such a model could <u>only</u> hold **if everybody were risk-neutral.**

  ▶ Or, at least, neutral with respect to S-risk.

- They aren't. And that model <u>doesn't</u> hold.

- But if it did, then how would people value options?

- ▶ Well, if they are neutral with respect to the risks in S, they must be neutral with respect to the risks in *derivatives that depend only on* S.

- ▶ Then, **risk-neutral people value assets as the expected value of their future cash-flows**, discounted at the risk free rate.

- So we know how they would compute the value of any European derivative on S. They'd:

  (1) Use their instantaneous stock model to figure out the probability distribution of all the possible time-T stock prices.

  (2) From that, deduce the distribution of all possible final call values.

  (3) From that, calculate their expected payoff from holding the call.

  (4) Then discount that number by multiplying by $B_{0,T}$.

- Of course, if they preferred, they could get it the same way we have to: by solving an ugly partial differential equation, subject to particular boundary conditions.

  - ▶ But both methods would have to give them the same answer.

- Actually, if they priced the option the second way, as we already observed, they would have to solve the exact same PDE, with the same boundary conditions that we would.
- Which means....THEY MUST GET THE SAME ANSWER THAT WE DO.
- So what?
- This tells us that we must also be allowed to use their other method.
  - ▶ We could have gotten our solution, by changing *our* stock model to *theirs*,
  - ▶ Just **switch** μ **to** $\mathbf{r}$ and then do steps (1)-(4) above.
  - ▶ That doesn't mean we believe their model. It's just a computational trick.
- Is it **always true** – with ANY model of stock price behavior – that you can value ANY derivative with this method?
- No. Look back at the logic we used, and you will see that it is rather specialized to this case.
  - ▶ We needed to know that our derivative solved a particular PDE which didn't involve the true μ.
  - ▶ Not all derivatives do in all models.
- So we can't apply the technique blindly. But when we can, it is often the best way to value derivatives.

(E) Implied Volatility

- An important feature you may have noticed about the Black-Scholes solution is that it contains only one input that is not directly observable: the volatility σ.

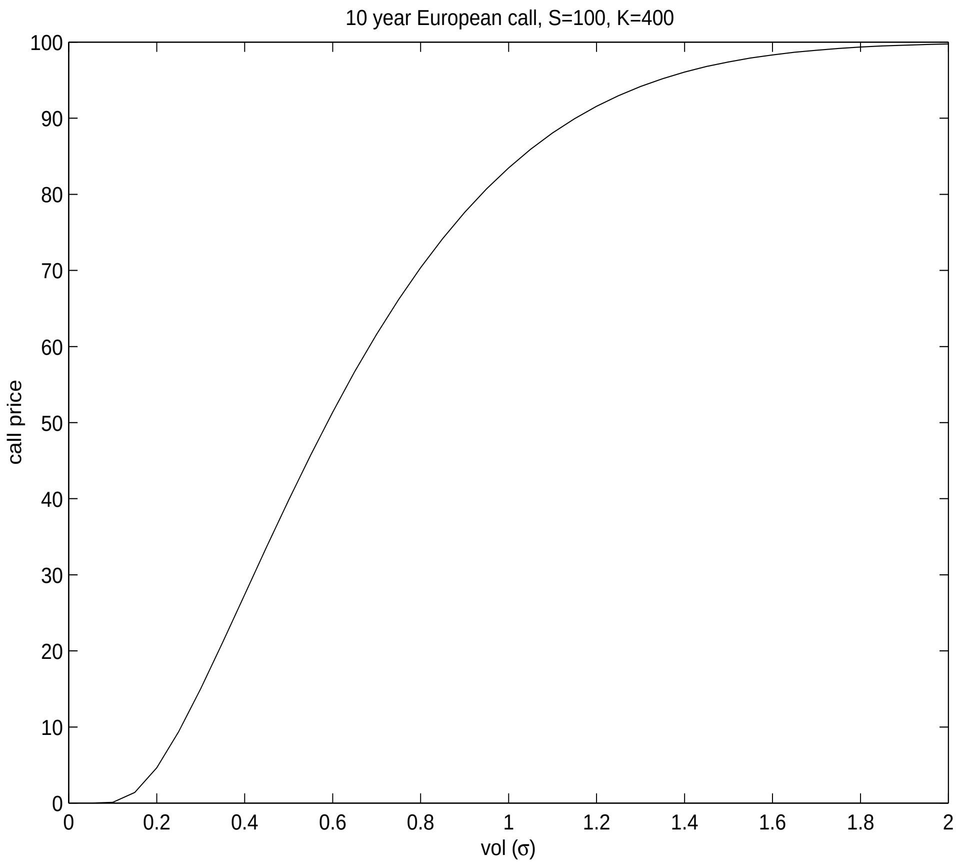
- But another way of looking at it is as a formula **for a volatility** given the price of a call.

- If we plot the call value against σ for any particular option we notice three things:

  1. The curve is always upward sloping.

  2. For $\sigma = 0$ it just gives the intrinsic value of the option.

  3. For $\sigma = \infty$ it just gives S.



- Mathematically, these facts imply that the pricing function has a unique inverse. For every possible call price (within the bounds allowed by static arbitrage) there is one, and only one σ which, plugged into the formula, would give that price.
- That particular σ is called the option's Black-Scholes **implied volatility** for the price. Some people drop the Black-Scholes part and just refer to implied volatilities (IVs) or occasionaly implied standard deviations (ISDs).
- **Don't be confused:**
  - Talking about an option's price in terms of these units doesn't imply that we believe the model that produced the formula.
  - Nor is there any reason to equate "true" volatilities to IVs, since we may not completely believe the model.
- A perfect analogy is with the yield-to-maturity of a bond.
  - It's just a handy way of normalizing prices across different maturities and coupons (or strikes).
- We will spend a lot more time thinking about volatilities. Clearly they are key to pricing options.
- The Black-Scholes formula is the first time we have seen that precisely quantified.

VII. Summary

- We derived a general partial differential equation for a any derivative in an model whose uncertainty is generated by a single Brownian motion price process.
- The derivation of this equation was the foundation for modern financial engineering.
- The main <u>new</u> assumptions to get the PDE were:
  1. Continuous trading is possible.
  2. Prices cannot jump.
- When volatility and interest rates are constant, the equation happens to have a closed-form solution for the case of a European call. That solution was discovered by Merton and Black and Scholes.
- Closed-form solutions are nice for analysis, but for more interesting cases, the PDE has to be solved by numerical techniques.
- Taking discounted risk-neutral expectations is one such technique, which we will study in detail in several applications.

Lecture Note 5.1: Summary of Notation

Symbol	Page	Meaning
dS	p2	*change in price of asset whose price is S over time interval dt*
dW	p2	*random normal variable of mean zero and variance dt*
μ, σ	p2	*expected percent change (continuously compounded) and standard deviation of percent changes in S*
$a(\,), b(\,)$	p2	*generic functions decribing expected change and standard deviation of changes in S*
dC	p3	*change in price of derivative whose price C is known to depend only on S and t*
$\alpha(\,), \beta(\,)$	p3	*functions to be determined decribing expected change and std. dev. of changes in C*
C_t	p4	$\frac{\partial C}{\partial t}$ *the rate of change of C with respect to t*
C_S	p4	$\frac{\partial C}{\partial S}$ *the rate of change of C with respect to S*
C_{SS}	p4	$\frac{\partial^2 C}{\partial S^2}$ *the rate of change of C_S with respect to S*
Π	p5	*value of portfolio long one unit of derivative, short C_S units of the underlying, and with I invested in the risk-free asset*
r	p5	*instantaneous (e.g. overnight) riskless rate, assumed constant*
X	p8	*knock-in/knock-out price in exotic example*
δ	p9	*"delta" $= \frac{\partial C}{\partial S}$, the hedge ratio of C*
γ	p10	*"gamma" $= \frac{\partial^2 C}{\partial S^2}$, the convexity of C*
y	p12	*continuous payout on underlying asset*
$\mathcal{N}(x)$	p14	*cumulative normal distribution function evaluated at x*
d_1, d_2	p14	*arguments of $\mathcal{N}$ appearing in the Black-Scholes formula*
$Q(S)$	p20	*price of any derivative on S*
IV	p26	*Black-Scholes implied volatility*