FIN 513: Homework #7

Due on Tuesday, April 10, 2018

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Problem 1

(a) **Disagree.** Under CAPM, market price of risk for an arbitrary asset is defined as $\lambda_X = \rho_{X,M}\lambda_M$, where X is an arbitrary asset, and M denotes market. Let X denotes temperature, and if temperature increases are associated with lower stock prices, it means that correlation between temperature and stock price, $\rho_{X,M}$, is negative. Since market price of risk associated with stock price is positive, which associated with global warming will be negative, not positive.

(b) **Agree.** From the equation $\lambda_X = \rho_{X,M}\lambda_M$, market price of risk for an arbitrary asset is calculated as multiple of correlation coefficient between the asset and market and Sharpe ratio for the stock market. Since correlation coefficient is between -1 and 1, its absolute value must be smaller than the Sharpe ratio for the stock market.

Problem 2

(a) Since the market price of volatility risk is given as $\lambda(v) = \lambda_0 + \lambda_1 v$, risk-neutralized process of v is derived as follows.

$$\begin{aligned} dv_t &= \kappa(\bar{v} - v_t)dt - \lambda(v)s_v dt + s_v dW_t^v \\ &= (\kappa\bar{v} - \lambda_0 s_v - (\kappa + \lambda_1 s_v)v_t)dt + s_v dW_t^v \\ &= k^*(\bar{v}^* - v_t)dt + s_v dW_t^v \end{aligned}$$

where $\kappa^* = \kappa + \lambda_1 s_v$, and $\bar{v}^* = \frac{\kappa\bar{v} - \lambda_0 s_v}{\kappa^*}$. Since κ^* and $\bar{v}^*$ are constant, v_t follows Ornstein-Uhlenbeck process under risk-neutral measure.

(b) Let F denote forward price for the contract. If there is no arbitrage opportunity, F should satisfy the following equation.

$$\begin{aligned} \mathrm{E}^*[B_{0,T}(\sigma_T - F)] &= 0 \\ \Rightarrow F &= \mathrm{E}^*[\sigma_T] \end{aligned}$$

Where $B_{0,T}$ denotes discount factor from current date to the maturity and E^* denotes expectation operator under risk-neutral measure. It is known that a random variable which follows Ornstein-Uhlenbeck process is normally distributed, so $v_t = \log\sigma_t^2$ is normally distributed. Therefore, there is a closed form of $\mathrm{E}^*[\sigma_T]$. From the analogy of distribution in lecture note 7.2, v_T is distributed as $N(v_0 e^{-\kappa^* T} + \bar{v}^*(1 - e^{\kappa^* T}), \frac{s_v^2}{2\kappa^*}(1 - e^{-2\kappa^* T}))$. Since $\mathrm{E}^*[\sigma_T] = \mathrm{E}^*[e^{\frac{1}{2}v_T}]$, and $\frac{1}{2}V_T$ is distributed as $N(\frac{1}{2}[v_0 e^{-\kappa^* T} + \bar{v}^*(1 - e^{\kappa^* T})], \frac{s_v^2}{8\kappa^*}(1 - e^{-2\kappa^* T}))$, no arbitrage forward price is derived as follows.

$$F = \mathrm{E}^*[\sigma_T] = e^{\frac{1}{2}[v_0 e^{-\kappa^* T} + \bar{v}^*(1 - e^{\kappa^* T})] + \frac{s_v^2}{16\kappa^*}(1 - e^{-2\kappa^* T})}$$

(c) Letting $T \to 0$, since $e^{\kappa^* T} \to 1$, forward price F converges to $e^{\frac{1}{2}v_0}$, which is equal to σ_0.

Problem 3

Since $G_T = \frac{\partial Y_T}{\partial Y_t} = 1 + \int_t^T -\kappa \frac{\partial Y_u}{\partial Y_t} du + \int_t^T s_Y \frac{\partial Y_u}{\partial Y_t} dW_u = 1 - \kappa \int_t^T G_u du + \int_t^T s_Y G_u dW_u$ (assuming some regularity conditions hold), stochastic process G_T follows $dG_T = -\kappa G_T dT + s_Y G_T dW_T$, and its initial value G_t is equal to 1. Therefore, we can evaluate delta of the claim within simulation. It can be implemented by choosing the same random number from simulating Y_T, and applying the number to G_T process, and choose the initial number as 1. Furthermore, since both Y process(similar to Vasicek model) and G process(geometric brownian motion) have closed-form solution, it is possible to generate Y_T and G_T directly.

Problem 4

Setting $r_0 = 3.675\%$, which is 3-month maturity yield, using the formula from the lecture note, G_0 and G_1 were calculated first. By using calculated G_0 and G_1, and changing r^*, bond prices and zero coupon rates were calculated by using the following formula.

$$
\begin{aligned}
B_{0,T} &= e^{G_0(0,T) - r_0 G_1(0,T)} \\
G_0(0,T) &= \frac{1}{\kappa^2}(G_1(0,T) - T)(r^* \kappa^2 - \frac{b^2}{2}) - \frac{(bG_1(0,T))^2}{4\kappa} \\
G_1(0,T) &= \frac{1}{\kappa}(1 - e^{-\kappa T}) \\
r_T &= -\frac{\log B_{0,T}}{T}
\end{aligned}
$$

Using excel solver, the optimal r^* which minimizes average absolute error between market rate and model rate was calculated. It was calculated as about 4.614%. Average absolute error was calculated as about 0.0003. Figure 1 shows plot of fitted curve and actual curve. Fitted curve has monotonically increasing feature. Actual curve, in contrast, has a humped shape. Table 1 shows the optimization result.

Problem 5

Using simulation technique mentioned on the problem set, 10,000 sample paths were created and swap value is calculated. Value is calculated for spread-payer. Parameters were used from answer of problem 4. Table 2 shows summary statistics of the simulation. The average value was calculated as 14.50(million), and standard deviation is calculated as 0.498. Therefore, it seems that the swap is profitable for spread payer since the present value of swap is positive. Of course, it is profitable for spread payer “on average”, however, there is a possibility that spread payer gets lots of loss. Figure 2 shows histogram of swap values from each path. As you can see, there is a fat tail in the distribution. From the table 2, the lowest value of the swap is -878.68, which is enormously high loss. Among 10,000 sample paths, the number of paths which consequences to negative value is 1,270, larger than 10% of total number of paths. It is because when interest rate goes out

Maturity(Year)	Yield(Market)	G_0	G_1	Yield(Model)	Absolute Error
0.25	3.68%	-0.0001	0.2469	3.69%	0.0001
0.5	3.79%	-0.0006	0.4877	3.70%	0.0009
0.75	3.84%	-0.0013	0.7226	3.71%	0.0013
1	3.86%	-0.0022	0.9516	3.72%	0.0014
2	3.85%	-0.0086	1.8127	3.76%	0.0009
3	3.82%	-0.0187	2.5918	3.80%	0.0002
4	3.81%	-0.0321	3.2968	3.83%	0.0003
5	3.81%	-0.0485	3.9347	3.86%	0.0005
6	3.83%	-0.0677	4.5119	3.89%	0.0007
7	3.85%	-0.0892	5.0341	3.92%	0.0007
8	3.88%	-0.1130	5.5067	3.94%	0.0007
9	3.91%	-0.1387	5.9343	3.96%	0.0006
10	3.93%	-0.1662	6.3212	3.99%	0.0005
11	3.96%	-0.1953	6.6713	4.00%	0.0004
12	3.99%	-0.2258	6.9881	4.02%	0.0004
13	4.01%	-0.2577	7.2747	4.04%	0.0003
14	4.03%	-0.2907	7.5340	4.05%	0.0002
15	4.05%	-0.3248	7.7687	4.07%	0.0002
16	4.07%	-0.3598	7.9810	4.08%	0.0001
17	4.09%	-0.3957	8.1732	4.09%	0.0001
18	4.10%	-0.4324	8.3470	4.11%	0.0001
19	4.12%	-0.4698	8.5043	4.12%	0.0000
20	4.13%	-0.5078	8.6466	4.13%	0.0000
21	4.14%	-0.5464	8.7754	4.14%	0.0000
22	4.15%	-0.5855	8.8920	4.15%	0.0000
23	4.16%	-0.6251	8.9974	4.16%	0.0000
24	4.17%	-0.6651	9.0928	4.16%	0.0001
25	4.18%	-0.7055	9.1792	4.17%	0.0001
26	4.18%	-0.7463	9.2573	4.18%	0.0001
27	4.19%	-0.7873	9.3279	4.19%	0.0001
28	4.20%	-0.8287	9.3919	4.19%	0.0001
29	4.21%	-0.8702	9.4498	4.20%	0.0001
30	4.21%	-0.9121	9.5021	4.20%	0.0001

Table 1: Optimization Result: $r^* = 4.614\%$, average absolute error: 0.0003

Problem 5 continued on next page. . .

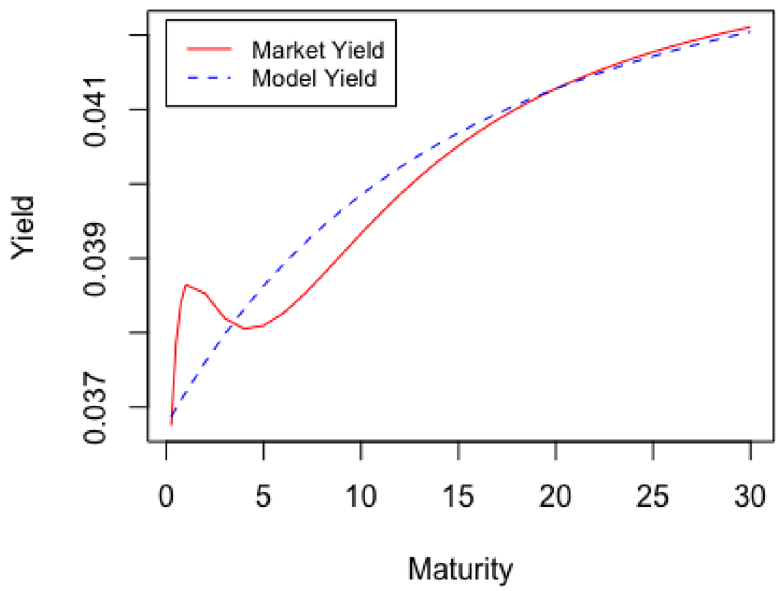

Figure 1: Market yield and model yield

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	St.Dev(Simulation)
-878.68	24.56	29.43	14.50	30.56	33.05	0.498

Table 2: Summary statistics from Monte-Carlo simulation

of the range, amount of spread accumulates, therefore the amount of money has to be paid increases sharply.

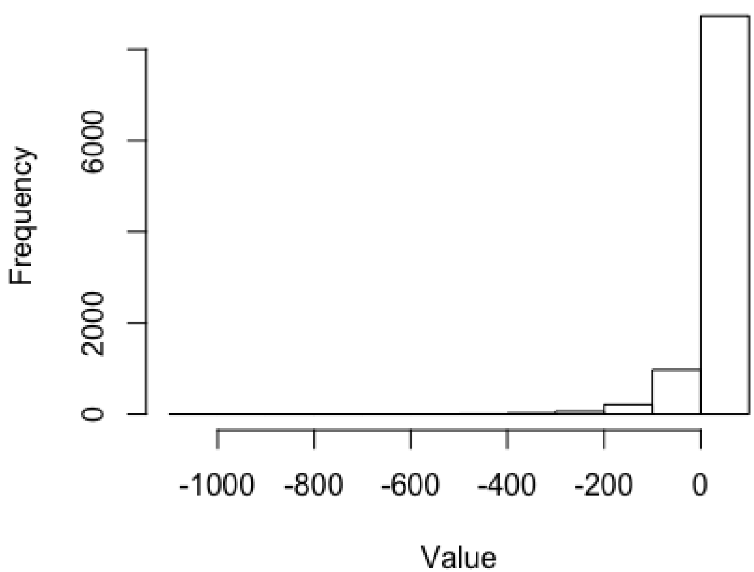

Figure 2: Histogram of swap values