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Lecture Note 6.1: Structural Models of Credit Risk

Introduction:

This lecture explores the insight we can gain through applying the Black-Scholes continuous-time methodology to the problem of valuing different parts of a firm's capital structure. As financial engineers, we need models like this in order to design securities for firms, as well as to trade and hedge them.

The models we will see build on the premise that all a firm's liabilities – bonds, stock, warrants, leases, or anything else –can be viewed as derivatives whose underlying is the assets to which they are claims.

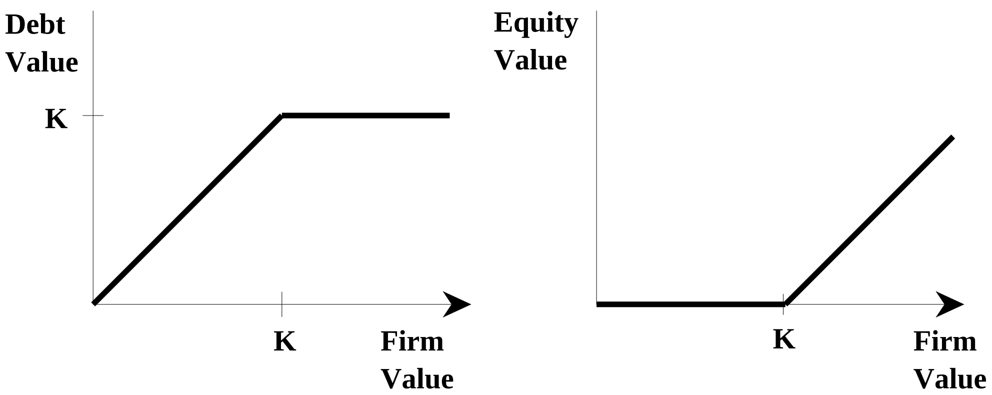
This gives us a powerful method for approaching credit risk. Even though the models we will see are simplifications, they are still very useful in practice. Moreover, this is a very active area of research, in which better models are being continually developed.

Outline:

I. Structural Models of Credit Risk
II. Equity-based Models
III. Application: Optimal Capital Structure
IV. Improving the Models
V. Summary

I. Structural Models of Credit Risk

- Early on in finance I'm sure you learned that one can think of a firm's equity as a call option on the value of its assets. This picture probably looks familiar:



- The idea is: the firm has debts of face value K, and at their maturity, T, stock holders get $\max[V - K, 0]$ where V is the value of the underlying assets. Bond holders get the rest.

- This is called the **Merton model** of risky debt.

- As simple as the set-up is, it gives us the power to actually give specific, quantitative answers to questions about the co-determination of bond and stock prices for such an idealized firm.

- For example: it is well known that increasing V's volatility transfers value from bond holders to stock holders.
  - ▶ The model tells us <u>how much</u>.

- Of course, we wouldn't expect the predictions to apply exactly to any real firm. But the model suggests how such relationships work across firms generally.
  - ▶ It is also the starting point for learning how to handle more complex capital structures.

- I want to focus now on the specific topic of credit spreads, i.e. what determines the relative value of corporate bonds. The credit spread is just defined as the difference between a risky bond's yield-to-maturity and that of an equivalent riskless one.

- Let's set the basic assumptions we are going to use throughout the lecture. We are in a Black-Scholes world with **perfect markets** and **continuous trading.** But some of the details now need to be clarified.

  1. It is now the total value of the firm's assets that evolves according to the standard geometric Brownian motion

  $$\frac{dV}{V} = (\mu - \Pi)\, dt \ + \ \sigma\, dW.$$

  We are taking this specification as given. In particular:

  - ▶ We know the volatility of the underlying asset value, *and it won't change.*

  - ▶ Also we assume we know the total payout percentage to all holders of claims to V – denoted $\Pi\ dt$ here. This is the sum of all coupons, dividends, and net new financings that the company will engage in – modeled as a constant yield.

- ▶ Note that V is assumed to be a market price – not an accounting value. While the no-arbitrage argument formally assumes that units of V can be traded (and sold short), it is actually sufficient to imagine that there is some other traded asset that is perfectly correlated with V.

2. There are no taxes or reorganization costs. Clearly this is unrealistic, and it can easily be relaxed. But when it holds it has an important consequence: **The Modigliani-Miller Theorem holds.**

Recap: *The market value of the firm is independent of its capital structure and depends only on the cash-flows from the underlying assets.*

The Argument: If there were any particular <u>best</u> capital structure, and the firm didn't have it, somebody would just buy up all its securities and finance it with the better structure – thus making arbitrage profits. Since anybody could do this, the market prices of the existing claims will be bid up to the point where arbitrage is impossible, which means the value of the firm with the <u>original</u> structure is just the same as with the <u>best</u> one.

As a consequence, V is just the sum of the prices of all the firm's claims.

3. The firm won't do any unanticipated financing between now and T. We need this because if they issued more claims it could change how the firm's value is divided up at T.

- ▶ Also note that this assumption means we know the firm's payout policy won't/can't change. The Merton model in fact assumes no dividends.

4. Interest rates are constant. This looks odd when we are trying to build bond pricing models. It is. But all of the models we'll see can be (and have been) extended to random interest rate economies.

- ▶ It turns out that the generalization doesn't affect any of the conclusions – at least for the kinds of interest rate processes we have seen in the developed world in our lifetimes.
- ▶ So, for our purposes today, fixing r is not a big deal.

5. All contracts are enforcable and cannot be renegotiated. This didn't sound so bad for exchange traded options. It matters here, because it rules out, for example, strategic bankruptcy filings.

- Now, with all those assumptions, suppose we are given an arbitrary claim, F, whose value depends only on V. Let's let F have a payout per unit time (a coupon rate) of Γ.

- When we re-do the continuous-time no-arbitrage argument with these payouts, we find that F obeys this version of the fundamental PDE:

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2} + \frac{\partial F}{\partial t} - rF + (r - \Pi)V\frac{\partial F}{\partial V} + \Gamma = 0. \quad (1)$$

- Now all we have to do is specify the boundary conditions that characterize any specific security, and then solve (1) subject to them.

- Let's look at some specific examples.

(A) Credit spreads in the Merton model.

- Formally, in terms of the framework we just set up, Π and Γ are both zero, because the bonds are zero-coupon and the firm doesn't pay dividends. The debt has face value $F^\star$ and maturity T and the boundary conditions are

$$\begin{aligned} F_T(V) &= \min[V, F^\star] \\ F_t(V = 0) &= 0 \quad \text{for all } t < T \\ F_t(V = \infty) &= F^\star \cdot e^{-r(T-t)} \quad \text{for all } t < T \end{aligned}$$

- For this model, we don't need to re-solve the PDE because we already know what the solution is.

- If the equity value is $c(V)$, then $F = V - c$. Since c is just given by the Black-Scholes formula with V being the underlying and $F^\star$ being the strike.

- Rerranging the terms slightly shows that $V - c$ is

$$F^\star e^{-r(T-t)} \left\{ \mathcal{N}(d_2) + \frac{1}{d}\mathcal{N}(-d_1) \right\}$$

where $d \equiv F^\star e^{-r(T-t)}/V$ is the "leverage" of the firm (and has nothing to do with d_1, d_2).

- The continuously compounded **yield-to-maturity**, y_τ, (where $\tau \equiv T - t$) is always defined for a zero coupon bond by the equation

$$e^{-y_\tau\ \tau} = \frac{F}{F^\star} \quad \text{hence} \quad y_\tau = -\frac{1}{\tau}\log(\frac{F}{F^\star}).$$

Example: A two year discount bond whose price is 87 (per 100 face value) has y-t-m

$$-(1/2)\log(0.87) = -0.5 \times -.139 = 0.0696 \approx 7\% \text{ per year.}$$

- With our formulas, we can calculate the yield spread in this model

$$y_\tau - r = -\frac{1}{\tau}\log[\mathcal{N}(d_2) + \frac{1}{d}\mathcal{N}(-d_1)].$$

Example: Find the credit spread on a five-year bond on a firm whose assets have 20% volatility, when the risk-free rate is 6% and the firm's assets are worth twice the face value of the debt.

- First, notice that we don't care what V and $F^\star$ are, We only need their ratio.

$$d = \frac{F^\star}{V} \cdot e^{-0.06 \cdot 5} = 0.5 \cdot e^{-0.06 \cdot 5} = 0.370$$

because we are given that $V = 2F^\star$.

- Let's suppose $F^\star = 100$. This will give us the answer as a percent of the bond's face value, which is how bonds are really quoted.

- We just use the regular formulas for d_1 and d_2.

$$d_1 = \frac{\log\left(\frac{V}{F^\star e^{-r(T-t)}}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}$$

$$= \frac{\log \frac{1}{0.37} + 0.5\,(0.2^2)\,5}{0.2\,\sqrt{5}} = 2.44$$

Likewise

$$d_2 = \frac{\log \frac{1}{0.37} - 0.5\,(0.2^2)\,5}{0.2\,\sqrt{5}} = 2.00$$

- So the bond is worth

$$100e^{-0.06 \cdot 5}(\mathcal{N}(2.00) + (1/.37)\mathcal{N}(-2.44)) = 73.84\%$$

- This gives a credit spread of

$-(1/5) \cdot \log(.7384) - 0.06 = .0607 - .06 = 0.0007 =$ 7 basis points

- These are from Merton's 1974 paper

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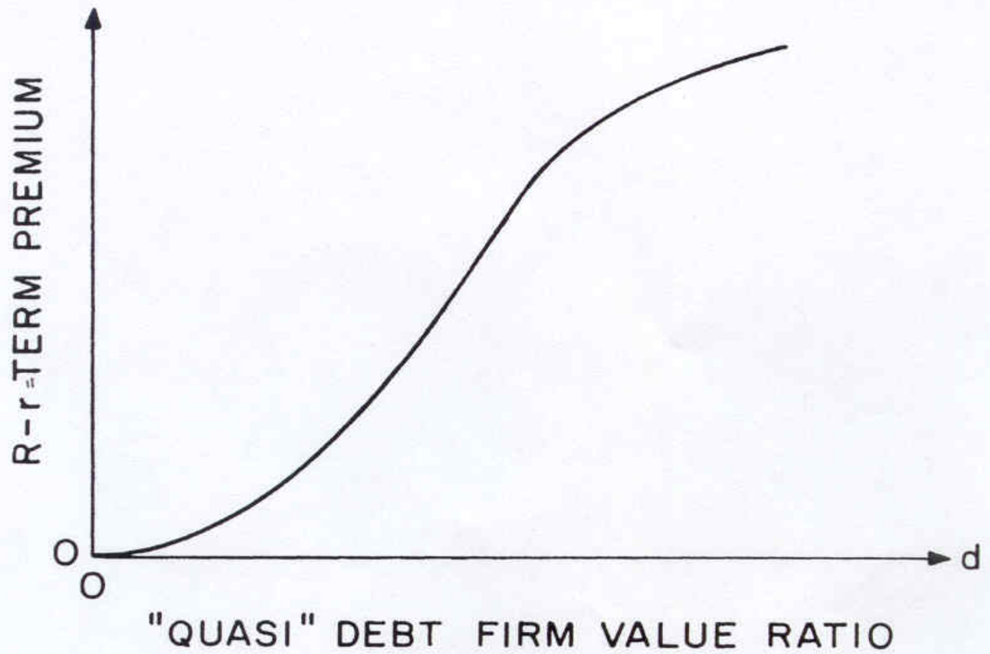


FIGURE 1

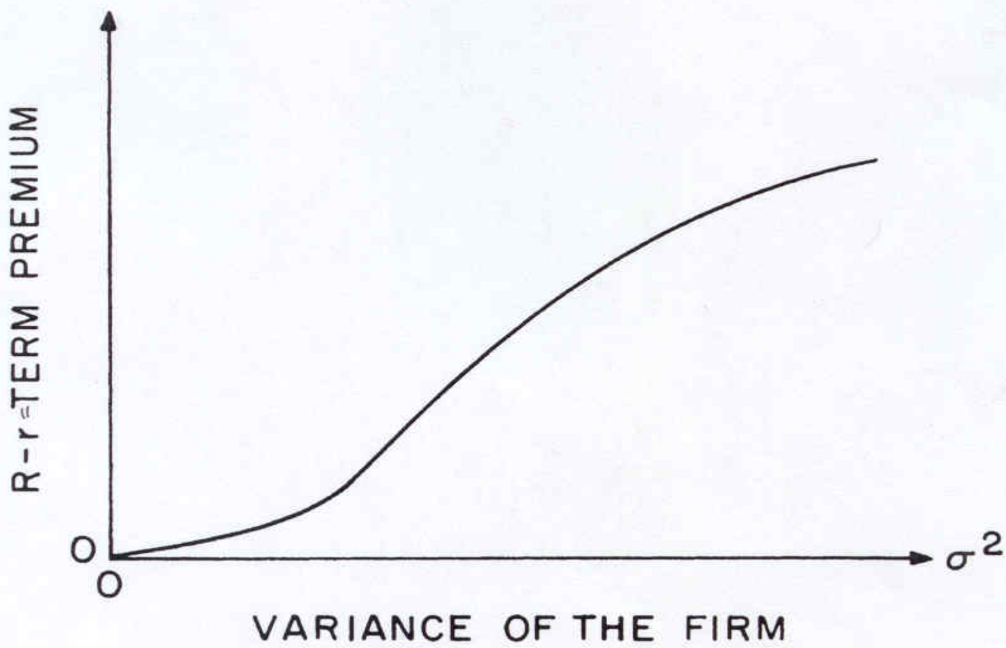


FIGURE 2

The Pricing of Corporate Debt 131

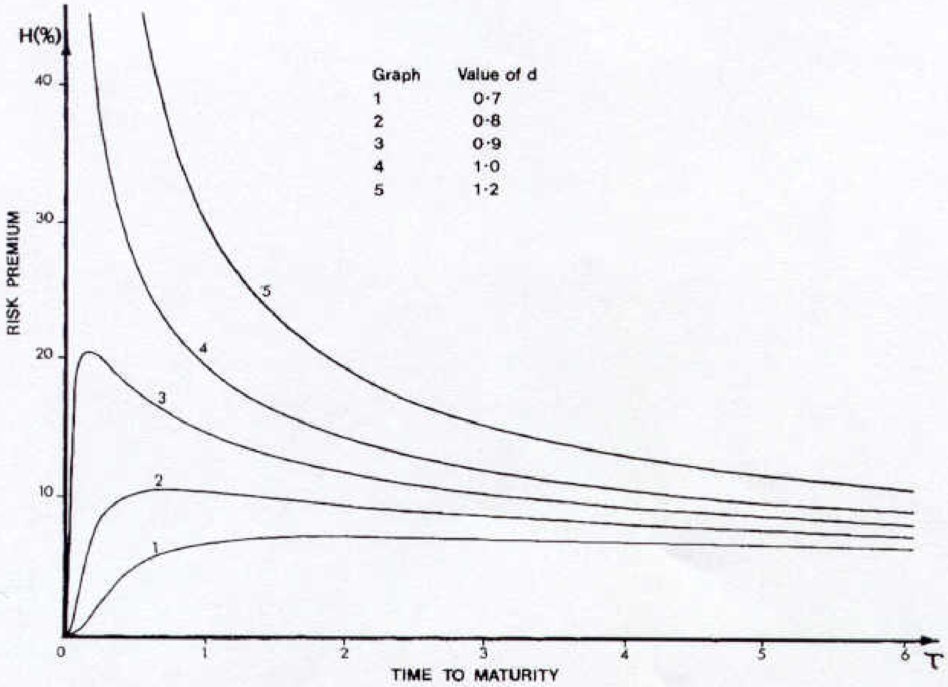


Figure 1. Risk premium as a function of time to maturity.

- This graph, which shows the <u>term structure of credit spreads</u>, is the most interesting one. It tells us that there are two very distinct cases in this model.
  - ▶ For highly levered firms ($d \geq 1$), the costs of short-term debt are enormous, but decline rapidly with maturity.
  - ▶ For firms with not much debt ($d < 1$), long-term debt is *more* expensive.

- The Merton model introduces a very fundamental idea that other researchers have built on:

  We can model credit risk with no-arbitrage methods.

- This approach to credit is called **structural modeling.**
  - ▶ It is "structural" in the sense that attempts to model default in terms of the structure of each firm's assets and liabilities.
  - ▶ It is often contrasted with "reduced form" models, which we will meet later on.

- Notice that one can use the Merton model in practice despite the fact that the two most important important inputs to the model, V_t and σ_V, are unobservable. **How?**
  - ▶ From the firm's stock price, we can immediately compute the market value of equity, S.
  - ▶ Moreover <u>its</u> volatility, σ_S, can be estimated from historical stock returns or option implied volatilities.
  - ▶ We also have formulas for what both of these quantities are, in terms of the model's other inputs.

$$
\begin{aligned}
S(V, \sigma_V) &= V\, \mathcal{N}(d_1) - e^{-r(T-t)}\, F^{\star}\, \mathcal{N}(d_2) && (2) \\
\sigma_S(V, \sigma_V) &= \frac{\partial S(V)}{\partial V}\, \frac{V}{S(V)}\, \sigma_V && (3)
\end{aligned}
$$

▶ The second equation just follows from applying Ito's lemma to the first equation. (And the first term on the right in the second equation is just the Black-Scholes delta, of course.)

▶ Given values for the quantities on the left, we have two equations in two unknowns.

▶ Although they are highly nonlinear, (remember V appears inside d_1 and d_2!) we can always solve them for V and σ_V with the help of a computer.

(B) Other Bankruptcy Assumptions.

- A wide range of complex structural models have been developed with the goal of trying to relax as many as possible of the unrealistic assumptions in the Merton model.

- It turns out the most important part of a structural specification is how it models <u>default events</u>.

  How is default triggered?

  How much is recovered upon default?

- In the Merton model, creditors recover all of V, and default can only happen at the one bond's maturity, T.

- This is not a very realistic depiction of bankruptcy.
  - ▶ It means the firm can become massively insolvent $(V < F^\star)$ and the creditors can't do anything. Management gets to stay in charge, and the equity holders get to keep their call option on V.
  - ▶ Conversely, if there were any coupon interest on the debt, owners might *want* to declare bankruptcy if asset values deteriorated a lot.

- Improving these simplistic assumptions was an early priority in structural modeling.

- An alternative assumption is that at some low level $V = V_B$ the firm is put out of its misery and the creditors (bond holders) take control and divide the remaining assets.
  - ▶ Black and Cox (1976) modified the basic Merton model to include such a *bankruptcy barrier*.
  - ▶ A similar, more flexible version was developed by **Longstaff and Schwartz** in 1995.

- Now, the idea is bond holders enforce a protective covenant, and liquidate the firm before T if assets lose too much value.

- Notice that now the bankruptcy event is not defined by reference to any one specific debt issue.
  - ▶ This makes it possible to incorporate more complex capital structures than allowed by Merton, including coupon debt.

- Formally, we solve our PDE subject to the new boundary conditions.

$$
\begin{aligned}
F(T,V) &= F^{\star}, \quad V > V_B(T) \\
F(t,V_B(t)) &= (1-W)F^{\star}, \quad t \leq T
\end{aligned}
$$

- Here W is the "writedown" due to the costs of bankruptcy.
  - ▶ Different bonds of the same issuer can have different W, reflecting their seniority.
  - ▶ Even though the law stipulates "absolute priority" of creditor classes, in practice it rarely happens that way.

- A simple choice is just to have V_B be a constant.

- <u>Unlike</u> Merton, assets will never be liquidated to pay off maturing bonds. So the implicit assumption is that the firm can re-finance maturing issues as long as $V > V_B$.

- Here's what credit spreads look like as a function of
  $X = V_t/V_B =$ assets as a fraction of the lower bound (i.e. how much they can still lose before defaulting), and
  $W =$ the percent of the face value that will be lost in bankruptcy.

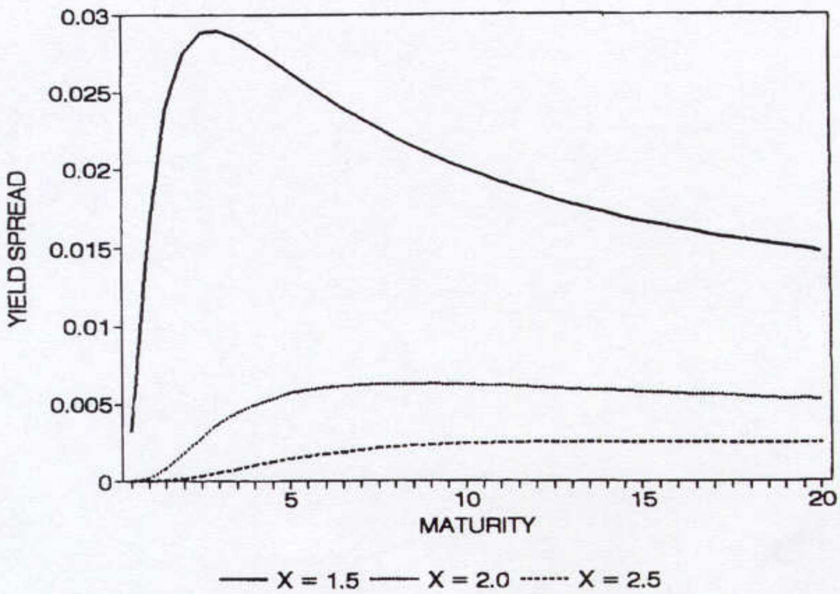


Figure 2. Credit spreads for an 8 percent bond for different values of X. The parameter values used are $r = 0.04$, $w = 0.5$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\alpha = 0.06$, $\beta = 1.00$, and $\eta^2 = 0.001$.

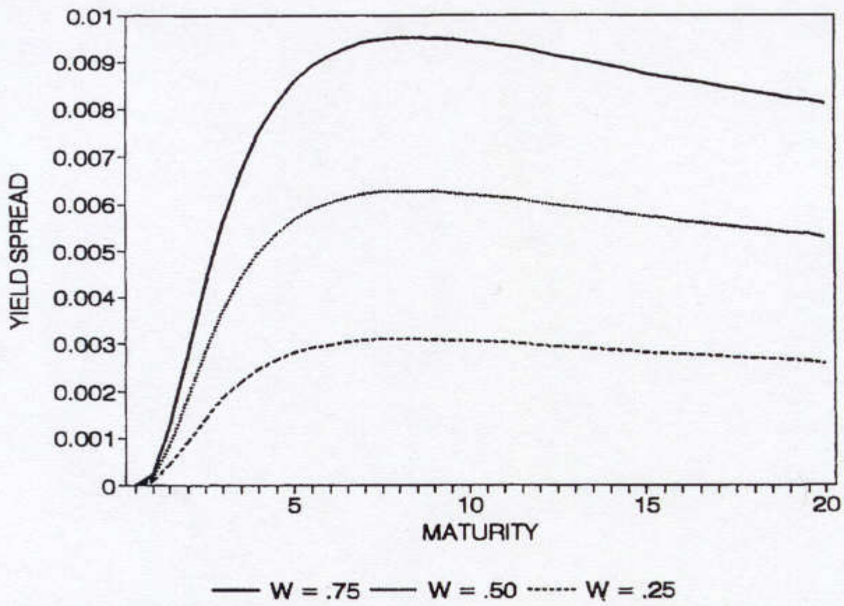


Figure 3. Credit spreads for an 8 percent bond for different values of w. The parameter values used are $X = 2.0$, $r = 0.04$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\alpha = 0.06$, $\beta = 1.00$, and $\eta^2 = 0.001$.

- The credit spreads the model predicts actually do agree pretty well with observed market levels.
  - As a typical practice, one might calibrate V_B to match the true default probability to a particular horizon, as proxied by the firm's credit rating
  - The asset value, its volatility, and its expected growth rate can then be extracted from observed values for the same quantities for the firm's equity, as with the Merton model.
- These types of model are really used by big banks and hedge funds. You can actually model real companies' capital structure unlike the Merton model.
  - Bonds can have coupons.
  - The firm can have many different outstanding debt issues.
  - It's not hard to add random interest rates. And r can be correlated with V.
- Structural models for the most part cannot be solved in closed form. But we know we can always solve the PDE that characterizes each claim by our probabilistic technique.
  - Simulate V after changing its drift to r (or $(r - \Pi)$). Then value by discounted expectations.
  - Notice that under the adjusted model, bankruptcy will typically occur more frequently than its true probability.
    - Why? Think about the simulation paths....

- ▶ Can you see why no-arbitrage credit spreads are predicted to be lower when r goes up? There is some evidence that this is actually true empirically.

- Notice what the technique is doing: simulating forward V_t from $t = 0$ to $t = T$ – but *stopping when V_t hits V_B* – if that happens before T.

- So we can represent the value of an ordinary coupon bond (with principal$= 1$, coupon$= \Gamma$) in the LS model as:

$$F(V_0) = B_{0,T}\ \mathrm{E}_{\mathrm{t}}^{\mathrm{Q}}[1_{\{\tau>T\}}] + \Gamma \sum_{T_n} B_{0,T_n}\ \mathrm{E}_{\mathrm{t}}^{\mathrm{Q}}[1_{\{\tau>T_n\}}]$$
$$+(1-W) \int_0^T B_{0,t}\ \mathrm{E}_{\mathrm{t}}^{\mathrm{Q}}[1_{\{\tau\in dt\}}]dt.$$

where τ denotes the random time at which V_B first hits (crosses) V_B. (The sum in the second term is over all the coupon payment dates.)

- We can actually evaluate this in closed form using the *first-passage-time density* for a geometric Brownian motion:

$$\text{Prob}^Q[\tau \in dt] = \mathrm{E}_{\mathrm{t}}^{\mathrm{Q}}[1_{\{\tau\in dt\}}]$$
$$= \frac{\log\left(\frac{V}{V_B}\right)}{\sqrt{2\pi\sigma^2 t^3}} \exp\left\{-\frac{1}{2}\frac{(\log[V/V_B] + (r - \frac{1}{2}\sigma^2)t)^2}{\sigma^2 t}\right\} dt.$$

- You can use this density to do the integrals above analytically. But it's not clear this is any easier than just doing simulations!

II. Equity-Based Models

- Now I want to show you another way of approaching risky bonds that can be much more practical for hedging.

- I mentioned that the Longstaff-Schwartz type models tell you how to hedge risky debt, but a problem is they tell you how to do it *in terms of units of V* – the whole firm.
  - ▶ To actually do that you'd have to be trading in little baskets of the firm's entire capital structure, which isn't very realistic.

- A clever trader could get around that by reasoning that, if a firm has two bonds, and we know the sensitivity of each to V then we can just use one to hedge the other. <u>True.</u>

- An even better way to do it would be to substitute the stock for one of the bonds, since stocks are more widely traded.

- So we would need to know what the model said the sensitivity of S to V was.

- But to price stocks using that model you have to first model every single other claim in the firm's capital structure, and then view equity as the residual
  - ▶ This is way too cumbersome to be practical.
  - ▶ And it might not be a very good model of stock prices. (They could turn out to be negative!)

- Why not try a more direct approach?

- If we know F (a given bond's price) is a function of V, and S is a function of V, why not just cut V out of the model all together and model F as a function of S?

- In short, go back to

$$\frac{dS}{S} = \mu_S \, dt \;+\; \sigma_S(S,t) \, dW$$

- Before we go this route, we had better think carefully about one issue though:

  Does it make sense to write down a model of the share price before we look at the firm's capital structure?

- In the Merton model, for example, the stock's volatility is an *endogenous* function of the firm's capital structure.

- If, instead, we specify an (exogenous) volatility for dS in advance, we have to put some thought into writing down a specification that realistically describes how a (levered!) firm's share price will behave.

- We do run in to one problem right away if, for example, we want to just re-use the Black-Scholes stock model.

  How do we capture bankruptcy?

- A natural answer is: bankruptcy is when the stock hits zero.

- But the log-normal process can't hit zero. **Why?**

- So what to do?

- One solution: specify some small positive stock price (e.g. 0.01) which corresponds to "bankruptcy".

  Problem: Might get very different answers depending on if you choose 0.01 or 0.02, etc.

- Better solution: **use a different model**.
  - I already argued that the **constant elasticity of variance** (CEV) model, for example, is a more realistic depiction of volatility dynamics.
  - Recall it says:

$$dS = \mu\, S\, dt \ + \ \omega \sqrt{S}\, dW \quad \text{or}$$

$$\frac{dS}{S} = \mu\, dt \ + \ \frac{\omega}{\sqrt{S}}\, dW.$$

  - For our purposes today, an even more important thing that's good about the CEV model is that it allows S to hit zero. This takes some math to prove. But, as you would expect (and want), it turns out that the chances of hitting zero are higher when μ is lower and/or σ is higher.
  - One thing you *can* see from $dS = \mu S\, dt + \omega\sqrt{S}\, dW$ is that, once S does hit zero, it stops. It really does die.
  - I simulated 1000 paths of the model for 10 years using daily steps and a starting volatility of 30% and an expected return of 6%.
    - About 3% went bankrupt – about what you might find for A or Baa companies.

- With this process, it's easy to go through our PDE argument again (I'll let you do this for yourself), and wind up with

$$\frac{1}{2}\omega^2 S \frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial t} - rF + S(r-d)\frac{\partial F}{\partial S} + \Gamma = 0. \quad (4)$$

- Now another advantage of thinking of S as the underlying asset is that some bonds have boundary conditions that are explicitly given in terms of the stock price. The most common ones are **convertible bonds.**

- These allow the holder the right to exchange them (at any time, usually) for a certain fixed number of shares.
  - ▶ Usually the indenture specifies a *conversion price*, S_C, which is related to the number of shares per bond, N, by

    $$N = F^{\star}/S_C$$

    where $F^{\star}$, as before, is the face value of the bond.

- This would be a nightmare to try to implement via Longstaff & Schwartz. But it's simple now. It just requires that we use a terminal payoff of $\max[F^{\star}, NS_T]$ and impose an early-exercise constraint.

- So, all together, the set of boundary conditions is now

$$\begin{aligned} F_T(S) &= \max[F^{\star}, NS_T] \\ F_t(S_t) &\geq NS_t \text{ for all } t < T \\ F_t(S=0) &= 0 \text{ for all } t < T \end{aligned}$$

- Here, the second condition is the early-exercise condition; the third is bankruptcy.

- By the way, if you thought there was some reason to believe you would recover more than zero, just change the third one to something like

$$F(S=0) = (1-W)F^{\star} \text{ for all } t < T.$$

  In practice most convertibles are very junior, unsecured debts. So it's rare that they get anything in bankruptcy.

- Another feature of real-world bonds is that the issuers usually have the right to redeem them early at a **call price** F^C.

- To translate that into boundary conditions, we need to know when redemption will actually occur.

- When should manager's exercise their right to call bonds?

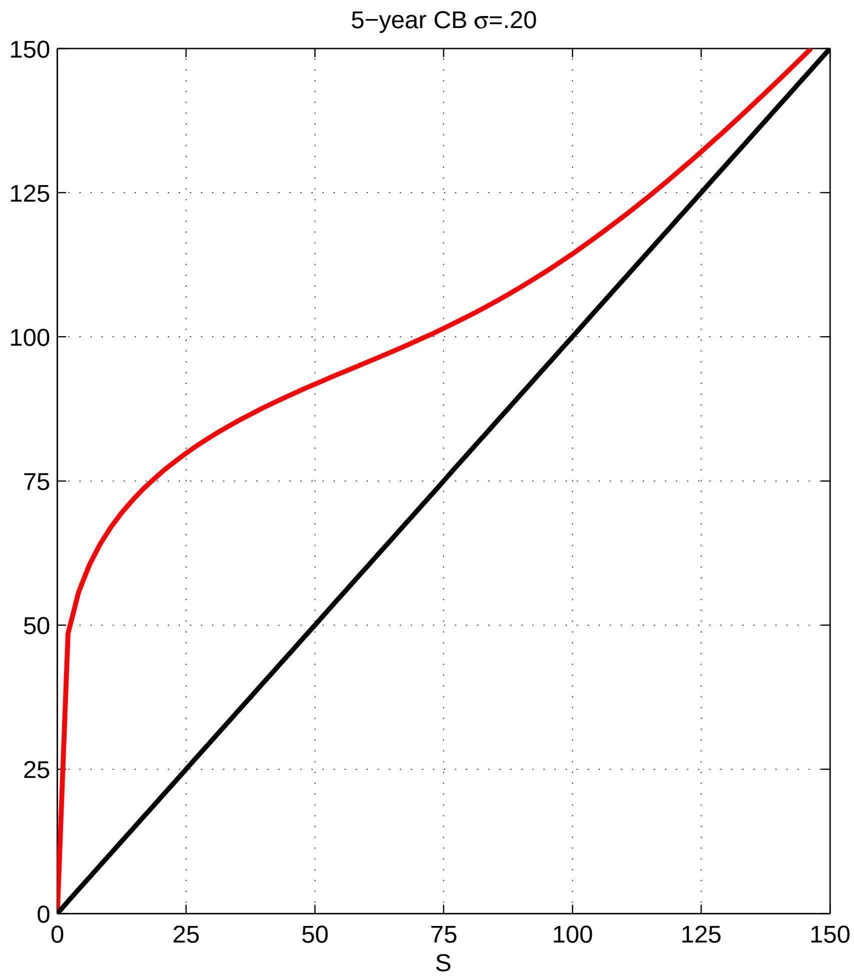
- Easy: they should do it whenever the bonds would be worth more than F^C if they didn't call them. *Why?*
  - Because their job is to maximize the stock holder's value, which, in this case, is the same as minimizing bond holder's value.
  - Imagine managers found their bonds were worth $F^C + \$1$. Then they could always make their share holders \$1 by calling the bonds for F^C and issuing new bonds with a lower coupon which was worth only F^C.

- So, for our valuation problem, handling call features is simple: we just assume the firm's managers are rational and hence *treat the call price as the ceiling on the bond's value.*

- This means we can add a fourth boundary condition:

$$F_t(S_t) \leq F^C(t) \text{ for all } t < T.$$

- Now we just solve the PDE (numerically). Here's what the solutions look like for the non-callable case.



- Based on this plot, do CB holders prefer more or less σ?

- Notice that a convertible bond is **not at all** like just a (riskless) bond plus a call.
  - ▶ We are explicitly taking into account the credit risk of the principal re-payment.

- These models work extremely well in practice, both for valuation and hedging.

- It's also easy to play "what if" games, and figure out how changing or adding different features changes the price function. For example:
  - ▶ We could make the bonds *puttable* by enforcing a different lower boundary condition.
  - ▶ We could add sinking-funds/ammortizing principal.
  - ▶ We could see how much we would need to raise the coupon if we wanted to lower the number of shares and keep price constant.
  - ▶ And so on.

  This could be very useful to a potential issuer.

- Finally, we can return to the problem we started with and just value an ordinary risky bond by letting the number of shares you get go to zero.

III. Optimal Firm Policies.

- An important and interesting generalization of the Merton-class of structural models are ones in which the firm's decisions are *endogenized*.

- In other words, we would like the models to take into account the fact that managers make financing decisions that determine credit risk in some sort of a rational manner.
  - ▶ Even if managers never change the asset side of their balance sheet (i.e., V is exogenous), we may have a much more accurate depiction of the payouts different claims will experience when we consider their actions on the liability side.
  - ▶ Ideally, we would develop a complete picture of *all future financing activity* such as new debt issuance, dividend changes, SEOs, share repurchases, etc.

- Besides helping to price risky debt, this objective of modelling optimal managerial behavior is <u>the</u> central problem in **corporate finance**.
  - ▶ As we will see, financial engineering techniques can offer important insights – not just to traders – but to CFOs of ordinary companies.

- To illustrate this line of research, let me show you one model that sheds important light on the question of *where the bankrupcy barrier might be.*

- If our objective is to model optimal financing decisions, then the first thing we have to do is *explain why the M&M theorem doesn't hold.*
  - ▶ If it does, all financing decisions are irrelevent!

- So let's make add back two important considerations that Modigliani and Miller assumed away:
  1. Taxes. Payments to bondholders confer a tax shield (in most countries) on the company's earnings that dividends do not. Debt might be good.

  2. Bankruptcy costs. There are direct costs (lawyers) as well as indirect ones (loss of employees, disruption of production), none of which accrues to security holders. Debt might be bad.

- Now put these in the setting we used in the earlier part of the lecture, where the firm's asset values V are taken as given and modelled as

$$dV = \mu V\, dt \ + \ \sigma V\, dW.$$

  As before, the firm will be liquidated when V hits V_B. Also there is only one debt issue outstanding, and there are no dividends.

- What can we say about the firm's total value?

- **It's not $\mathbf{V}$.**

- We have to include the effects of taxes and bankruptcy costs.
- The neat idea here is that we can treat these by *thinking of them as two new securities.*
  1. A claim to the bankruptcy costs, which is just like a credit put – it pays nothing unless and until default, when it pays, say $\alpha \cdot V_B$. Call its value F^{BC}.
    - Note that we are assuming α is known and constant.

  2. A claim to the tax shield. The tax shield is just a fraction of the coupon payment. If the marginal effective tax rate is Ω, this claim generates cash-flows of $\Omega \cdot \Gamma$ as long as the firm is alive. (Remember Γ is the coupon.) Call the value of this F^{TS}.
    - Note that we are assuming the tax shields accrue to the firm's owners regardless of whether they have taxable income.
- The whole firm is then worth

$$v = V + F^{TS} - F^{BC}.$$

- The plan: We will value the separate pieces of the firm and see how they vary *as a function of the things managers choose:* the amount of debt , and the default point.
  - Then we can figure out the **best** levels for these things from the stand point of maximizing manager's objective function.

- There is a neat special case when we can get some closed-form results.
  - ▶ This is the case where all the firm's debt is perpetual.
  - ▶ This is the **Leland (1995) model** of capital structure.
- We know that all the claims satisfy our no-arbitrage PDE.
- Now, without knowing anything about the solutions, we can already say one thing about them for sure: they don't vary with time!
  - ▶ This follows from the assumption that debt is perpetual.
  - ▶ Mathematically, this means that $\frac{\partial F}{\partial t} = 0$, so our PDE has one less term.
- All we need to do is figure out the particular boundary conditions the three claims satisfy. A little bit of thought yields these:

(A) The bond must satisfy

$$
\begin{aligned}
F(V_B) &= (1-\alpha)V_B \\
F(\infty) &= \Gamma/r \quad = \text{value of a risk-free perpetuity.}
\end{aligned}
$$

(B) The tax shield is almost the same.

$$
\begin{aligned}
F^{TS}(V_B) &= 0 \quad \text{(it ceases when the firm dies)} \\
F^{TS}(\infty) &= \Omega\Gamma/r
\end{aligned}
$$

(C) The bankruptcy cost claim must satisfy

$$\begin{aligned} F^{BC}(V_B) &= \alpha V_B \qquad = \text{what the lawyers get} \\ F^{BC}(\infty) &= 0 \end{aligned}$$

(D) And the stock is just the leftover

$$S = V - (F - F^{TS} + F^{BC}).$$

- Remarkably, the PDE has closed-form solutions for all these cases!
- Let $X \equiv 2r/\sigma^2$. Then it is easy to verify that these work:

$$F = \frac{\Gamma}{r} + \left[(1-\alpha)V_B - \frac{\Gamma}{r}\right]\left[\frac{V_B}{V}\right]^X \tag{5}$$

$$F^{TS} = \Omega\frac{\Gamma}{r}\left[1 - \left(\frac{V_B}{V}\right)^X\right] \tag{6}$$

$$F^{BC} = \alpha V_B \left[\frac{V_B}{V}\right]^X = \frac{\alpha}{V^X} \cdot V_B^{1+X} \tag{7}$$

- Now we come to our first policy question:
  How is V_B determined?
- Leland's answer is simple: managers choose the default point to maximize the stock price value.
  - ▶ Notice that this may *not* be the same as maximizing total firm value.

- ▶ The assumption is that, once the firm issues its debt, debt holders cannot stop equity holders from abandoning their claim (declaring bankruptcy) whenever it is best for them to do so.
- ▶ Remember, equity holders have to pay the coupon interest here. (We assumed no net payouts, Π to holders of V.)
- ▶ The model thus takes into account an important feature of real-world firms: *agency conflicts*.

- Mathematically, we can readily expand the equity value expression above – taking Γ as fixed – and then pick V_B to maximize it via the usual method:
  - ▶ Differentiate with respect to V_B, then
  - ▶ Set that equal to zero, then
  - ▶ Solve for V_B

- **Result:**

$$V_B = (1 - \Omega)\frac{X}{1+X}\frac{\Gamma}{r}.$$

  - ▶ For later, let me call this nP where $P = \Gamma/r$ and n is the factor multiplying it. (And note that $0 < n < 1$.)
  - ▶ This gives the optimal abandonment point for equity holders as a fraction of the present value of the perpetual debt obligation.

- Now we can address the second question:
  How much debt should the firm have?

- Leland's analysis is again straightforward.
  - ▶ Ask what value of P would maximize the whole firm's value.
  - ▶ Here we take as given that, once we fix P, manager's will implement the V_B policy we just found.

- The crucial thing to remember now is that the whole firm's worth **isn't $\mathbf{V}$**.

- Instead it's $\quad V + F^{TS} - F^{BC}$.

- So mathematically, we set the derivative of this with respect to P equal to zero, which is the same as

$$\frac{d}{dP} F^{TS} = \frac{d}{dP} F^{BC}$$

- The things we have to differentiate are just powers of P though. So it's not hard.

- I will let you verify that solving for P gives

$$P^* = \frac{V_0}{n} \left[1 + X + \alpha X \frac{1 - \Omega}{\Omega} \right]^{-1/X}$$

  - ▶ Note that the solution still depend on where V is at the time the decision is being made. I denoted that V_0.

- Ideally, the firm would want to adjust its policies as its asset value changes. But Leland assumes that frictions (i.e. investment banker fees) are so large that debt issuance or retirement is never worthwhile.

- Now we have completely solved the model. We go back and plug in this P in to our expression for the optimal V_B and then substitute both back into equation (5) to get our bond's value and its credit spread.
  - Then we can analyze how these credit spreads evolve with V_t. But we have to remember that the V_B and P do not themselves evolve after time zero.

- Here's what Leland gets when he plots total firm value against leverage using different levels of asset volatility.
  - Note: the horizontal axis is (market value of debt)/(market value of firm)

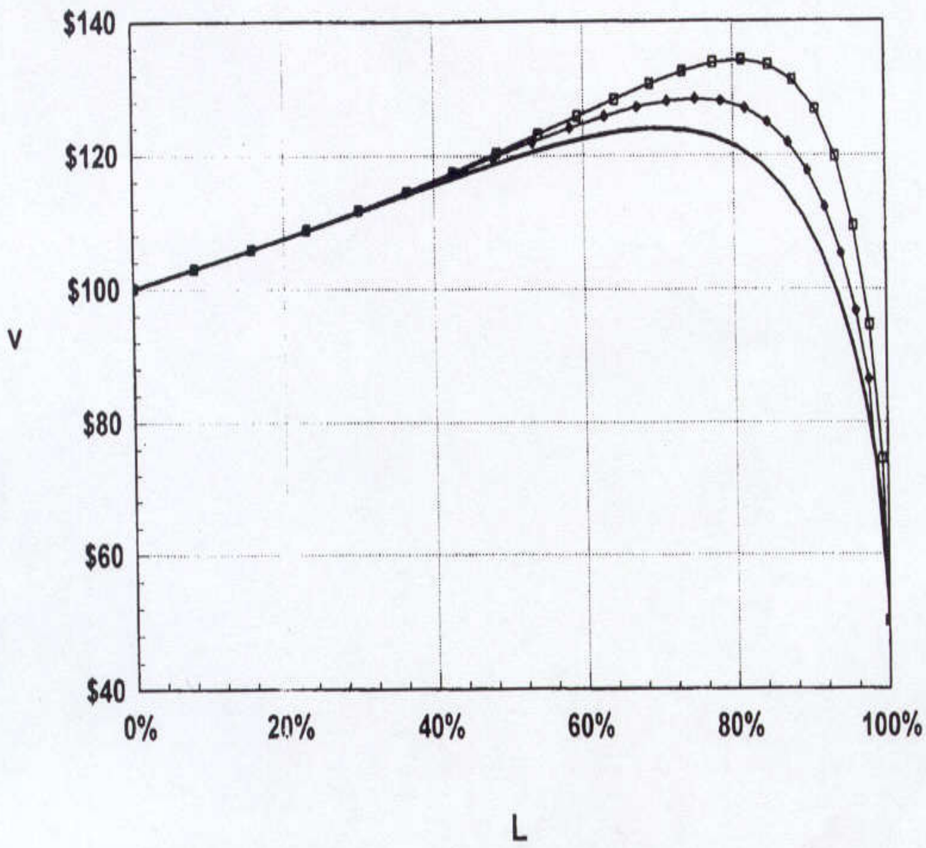


Figure 7. Total firm value as function of the leverage, when debt is unprotected. The lines plot total firm value, v, at varying levels of leverage L, for three levels of asset volatility, σ: 15 percent (*open square*), 20 percent (*filled diamond*), and 25 percent (*solid line*). It is assumed that the risk-free interest rate r = 6.0 percent, bankruptcy costs are 50 percent ($\alpha = 0.5$), and the corporate tax rate is 35 percent ($\tau = 0.35$).

- Look at the size of the value creation here. The graphs fix $V = 100$. So the model is saying that financing decisions can have huge benefits to the firm, adding up to 35% or so of the intrinsic value of the assets' cash-flow if you get the leverage right. And there are large penalties for over-doing it too.

  Capital structure matters!

- By quantifying the optimal debt and bankruptcy policies, the Leland model allows us to analyze all kinds of important issues.
  - ▶ How does the optimal debt level change with tax rates?
  - ▶ How does asset volatility affect the liquidation decision?
  - ▶ How do bankrupcty costs raise the cost of debt?
  - ▶ How does r influence the value of a credit put?
  - ▶ And so on.
- Moreover, returning to our initial motivation, we can directly check to see the impact of endogenous decision making by seeing how credit spreads change when we pick a non-optimal V_B.
  - ▶ For example, a calculation that I will leave as an exercise is to see how the credit spread changes if we assume the managers pick P at time zero to maximize *equity* value.
    - ∗ The answer would show us how important it is to take into account the agency conflict that influences managers' decisions.
- Of course, like the Merton model, the Leland model is just a first step towards richer depictions of the full financial policy problem.

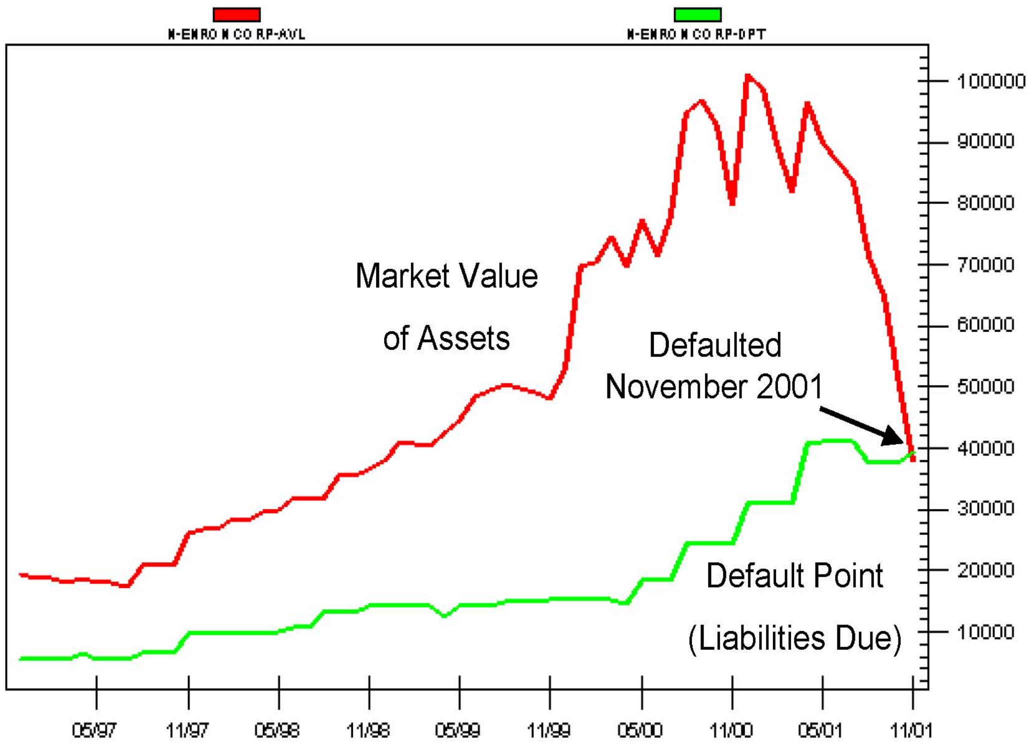
IV. Further Enhancements

- Firm credit risk models have been enhanced in recent years to try to capture progressively more realistic features of debt contracts.

- Here is a brief list of some of the directions in which progress is being made.

Surprises. The models we looked at are based on continuous state variables. In particular, this means *bankruptcy is never a surprise.* This is unrealistic. Indeed, traditional structural models have a hard time explaining non-zero credit spreads for highly rated companies and short term debt for the this reason. The natural fix is to incorporate some form of jumps into the models. Models that include discontinuous shocks to asset values perform much better – but they have to depart from the pure no-arbitrage framework.

The default process. We have used an imaginary bankruptcy threshhold, V_B, to capture default events of all type. In reality there is no single trigger that determines default. Instead it is largely determined by the degree of creditor leniency. We might be able to better model the determinants of this through the use of *time-varying default threshholds.* (Recent research suggests there are systematic components here – and perhaps discontinuous jumps too). Another important feature of credit distress is the potential for *strategic renegotiation* outside of bankruptcy.



Future financing policies. It is very important to try to capture not just the value of a firm's claims today, but what *changes* are likely to occur. Predicting what claims will the firm sell (or buy) in the future allows us to more accurately assess the ways in which assets will be divided in the future, and how risky those assets are likely to be. Lots of money has been <u>lost</u> by people using static models which failed to account for the shock to credit that can occur when, for example, a firm suddenly increases leverage a lot (e.g., for an LBO or a recapitalization.) Models of *dynamic capital structure* attempt to capture the optimization process of firm managers as they adjust their debt to capture tax benefits and hedge against bankruptcy risk.

Risky debt recovery. A big assumption is that we know in advance what every bond will be worth in bankruptcy. That's ok if it's a small number anyway (i.e. for very junior debt). But real-world investors face *recovery risk*. Debt recovery will be affected by, among other things, industry and macroeconomic conditions. So there is *systematic* risk here, which may be unhedgeable. There are timing uncertainites here as well because bankruptcy work-outs can take years.

Financial constraints. Most models assume that firms can raise new capital to meet *cash-flow* needs – either from new equity or new debt – as long as it is solvent. More realistic models take into account that external finance may be difficult or impossible to raise in bad climates (i.e., when the firm has shortfalls of cash). Thus *liquidity*, as well as solvency, can be a trigger for credit events.

- This is a very lively field. Models that address the concerns listed above are being developed and proposed right now.

V. Summary

- We looked at how the Black-Scholes methodology could be applied to the hard problem of valuing securities subject to default risk.

- Even the simplified models we have seen today are enjoying a lot of attention right now from traders, risk managers, rating agencies, and regulators.

- We examined two classes of models.

(A) The first took the firm's *asset value process* as given and tried to quantify how risky all of the securities were, relative to each other.

(B) The second took the firm's *stock price process* as given and gave us a useful way to price and hedge real-world bonds.

- In these models there is the common implication that *bankruptcy risk is hedgable* and as a consequence **the price of credit risk should not depend on the true probability of default, but on the "risk-neutral" probability.**

- There generally aren't closed-form solutions to these models.
  - ▶ But we can always solve the PDE by taking discounted risk-neutral expectations of the payoffs to each claim.

Lecture Note 6.1: Summary of Notation

Symbol	Page	Meaning
V	p2	*value of a firm's assets*
μ, σ	p3	*expected growth rate and volatility of* V
Π	p3	*total yield per unit time to all firm's claim holders*
F	p6	*value of some particular claim on the firm*
Γ	p6	*payout per unit time to holders of claim* F
$F^\star$	p6	*face value of zero coupon bond in Merton model*
c	p7	*value of equity in Merton model*
d	p7	$PV(F^\star)/V$ *in Merton model*
y_τ	p7	*yield to maturity of 0-coupon bond maturing in* τ *years*
$S(V,t), \sigma_S(V,t)$	p12	*equity value and volatility in Merton model*
V_B	p13	*asset value which triggers bankruptcy*
X	p14	V/V_B *in the Longstaff-Schwartz model*
W	p14	*percent of face value lost in bankruptcy*
τ	p17	*first (random) time* V_t *falls to level* V_B
$E^Q[\ \ ]$	p17	*expectation computed using RN probabilities*
$\mu_S, \sigma_S(S,t)$	p19	*expected growth rate and volatility of share price* S
ω	p20	*volatility when* $S = 1$ *in CEV model*
N,S_C	p22	$N =$ *number of shares received on conversion of bond with face value* $F^\star$ *and conversion price* S_C
F_C	p23	*call price in effect at time* t

Lecture Note 6.1: Notation – continued

Symbol	Page	Meaning
v	*p27*	*total value of all firm's claims in the Leland model*
F^{TS}, F^{BC}	*p27*	*value of claim tax shield and bankruptcy costs*
α	*p27*	*percent of* v *lost in bankruptcy in Leland model*
Ω	*p28*	*marginal effective corporate tax rate*
X	*p29*	$2 \cdot r/\sigma^2$
$P^{\star}$	*p31*	*optimal quantity of debt*