FIN 513: Homework #5

Due on Tuesday, March 6, 2018

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Problem 1

Let V and S denote sum of values of options and stock price respectively, then we can denote portfolio of the market maker as $\Pi = V - \Delta S$. By Ito's lemma, the following equation follows.

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 - \Delta dS$$

Since the portfolio has zero delta, $d\Pi = \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 = \frac{\partial V}{\partial t}dt + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2}dt$ holds. Since the portfolio became riskless, by no arbitrage principle, its return must be equal to risk-free rate as follows.

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2}dt = r\Pi dt$$

Since parameters are given as $\frac{\partial^2 V}{\partial S^2} = -1.725$, $S = 143$, $r = 0.05$, $\sigma = 0.7$, plugging them into the equation above, we can obtain the following result.

$$\frac{\partial V}{\partial t} + \frac{1}{2} \times (143)^2 \times (0.7)^2 \times (-1.725) = 0.05 \times 30,000,000$$
$$\Rightarrow \frac{\partial V}{\partial t} = 1,508,642.26$$

Assuming a year is equal to 365 days, if the stock price is unchanged, the expected value of the positions is approximately $\Pi + d\Pi = \Pi + \frac{\partial V}{\partial t}dt = 30,000,000 + 1,508,642.26 \times \frac{1}{365} = 30,004,133.3$. The value might not be exact since the whole procedure was implemented on continuous time framework which is not exactly consistent to this problem. However, since $dt = \frac{1}{365}$ is small enough, errors can be ignored.

Problem 2

Since it is assumed that Black-Scholes assumptions hold, put option price is calculated as follows.

$$p = B_{t,T}(KN(-d_2) - FN(-d_1))$$
$$d_1 = \frac{\log(F/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sqrt{T-t}$$

Let p_1, p_2, p_3, p_4 denote price of put options with time to maturity from 2 months to 5 months, respectively. Since all parameters need for valuing option are given except volatility, and it is assumed that all of options have same implied volatility, it is possible to calculate volatility by using the formula inversely. In other words, we can obtain implied volatility by searching σ which equates sum of option prices to 2 millions. (find σ such that $\sum_{i=1}^{4} p_i = 2,000,000$) By taking some numerical procedures, the implied volatility is calculated as about 9.63%.

Problem 3

(a) Under Black-Scholes economy, call option price is calculated as $c = e^{-r(T-t)}\mathrm{E}_t^Q[\max(S_T - K, 0)]$, where E_t^Q is an expectation operator under risk neutral measure conditioning at time t. Since we already know that $c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$ and $p = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$, where $d_1 = \frac{\log(S/Ke^{-r(T-t)})+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$, expected payoff under risk neutral measure can be derived as follows.

$$\mathrm{E}_t^Q[\max(S_T - K, 0)] = Se^{r(T-t)}N(d_1) - KN(d_2)$$
$$\mathrm{E}_t^Q[\max(K - S_T, 0)] = KN(-d_2) - Se^{r(T-t)}N(-d_1)$$

Under risk neutral measure, expected return of physical measure μ is converted into risk free rate, r. Therefore, by converting all r into μ, we can obtain true expected payoff. Therefore, true expected payoff of options are derived as follows.

$$\mathrm{E}_t[\max(S_T - K, 0)] = Se^{\mu(T-t)}N(d_1) - KN(d_2)$$
$$\mathrm{E}_t[\max(K - S_T, 0)] = KN(-d_2) - Se^{\mu(T-t)}N(-d_1)$$
$$d_1 = \frac{\log(S/Ke^{-\mu(T-t)}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sqrt{T-t}$$

By using the formula above, expected payoff of straddle is calculated as follows. It is well-known that a

Volatility	Expected Payoff
0.2	17.506
0.3	25.376
0.4	33.348
0.5	41.306

Table 1: Expected payoff under physical measure

straddle is more worth when volatility is larger. From the result, we can find expected payoff of straddle increases as volatility increases, so the result is consistent with that fact.

(b) By using the Algorithm 1 mentioned below, it is possible to implement dynamic replication strategy. Table 2 shows the result from simulation.

Volatility	0.2	0.3	0.4	0.5
Payoff from analytic solution	17.506	25.376	33.348	41.306
Payoff from dynamic replication	1.457	1.242	1.691	1.860
Standard deviation of simulation	0.146	0.226	0.306	0.407

Table 2: Simulation result

(c) As shown in Table 2, there is a large difference between analytic payoff and value of replication strategy.

Algorithm 1: Monte-Carlo Simulation for Dynamic Replication

Input: S_0, K, μ, r, d, σ, T, N, M

`//` N `and` M `denote number of time steps and number of paths, respectively.`

Output: Average of payoff, Standard Deviation

Set P as an array `// P: array of portfolio value at maturity for each sample path`

Set $\Delta t = T/N$

for $i \leftarrow 1$ ***to*** M **do**

  Initialize S, D, C as an array `// S: stock price path, D: delta path, C: cash path`

  Set $S_1 = S_0$, $D_1 = N(d_1; S_1) - N(-d_1; S_1)$, $C_1 = -D_1 S_1$.

  for $j \leftarrow 1$ ***to*** N **do**

    Generate random number Z which follows $N(0,1)$.

    Calculate $S_{j+1} = S_j e^{(\mu - d - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} Z}$

    if $j \neq N$ **then**

      `// Delta of straddle is equal to sum of deltas of call and put.`

      Calculate $D_{j+1} = N(d_1; S_{j+1}) - N(-d_1; S_{j+1})$

      `// The rest of cash after rebalancing`

      Calculate $C_{j+1} = C_j e^{r\Delta t} - (D_{j+1} - D_j) S_{j+1}$.

    end

    else

      `//` $j = N$ `case`

      Calculate $P_i = D_N S_{N+1} + C_N e^{r\Delta t}$. `// Value at maturity at` i^{th} `path`

    end

  end

end

return $\bar{P}$, $\sigma_P/\sqrt{M}$ `//` $\bar{P}$`: average of` P`,` σ_P`: standard deviation of` P.