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Lecture Note 2.1: Swaps

In the next few lectures we will examine *swap contracts*.

Swaps are an amazingly broad class of derivatives. We will describe the major types of swaps and indicate some of the different financial engineering problems that swaps are designed to solve.

Our main goal is to try to understand the valuation principles behind <u>all</u> swaps. We begin by analyzing swaps that are a lot like forwards contracts. As with forwards, we will figure out the *value* of an existing swap, and then deduce the no-arbitrage *rate* for a new swap (so that its initial value is zero). We start by solving these problems in a world of perfect markets.

Outline:

I. What are swaps?

II. Commodity swaps.

III. Currency swaps.

IV. Transactions costs.

V. Marking-to-market and margining.

VI. Summary.

I. Definition of a swap.

- A swap contract is an agreement between two parties to exchange two *streams* of some good or cash flows, over several dates until some terminal horizon T.

- The simplest swap can be viewed as the exchange of

  (A) A fixed quantity of a good, for

  (B) A fixed amount of money
  (or fixed monetary amount of the good)

  at a <u>series</u> of pre-specified dates.

- It is a somewhat surprising fact that all the swaps we will consider can be built up from simple swaps like this.

- Besides the good, the swap contract also specifies:
  - ▶ the dates of exchange.
  - ▶ delivery instructions for both parties (including netting).
  - ▶ conditions for assignability or cancellation;
  - ▶ remedies in the case of default.

- Swaps have a lot of similarities to forwards.
  - ▶ Swaps originally traded only in OTC markets.
  - ▶ The terms are customizable.
  - ▶ There is often no marking-to-market (although this is changing).
  - ▶ Neither side pays the other to enter a swap.

- Most contracts follow the legal template of the ISDA Master Agreement, and are enforceable under U.S. or English law.

II. Swap pricing: Commodity swaps.

- We now examine the no-arbitrage valuation of several types of swaps. For this part of the lecture, we assume **markets are perfect**. The most important aspects of this assumption are

  Riskless borrowing and lending. In particular, we imagine that there is a term structure of riskless zero coupon bonds of price $B_{t,T}$ for any maturity you want *and* you can buy them (lend) or costlessly short sell them (borrow).

  Reliable counterparties. Unless otherwise indicated, we will assume that both sides to any swap contract are certain to perform their obligations over the life of the swap.

- A commodity swap is identical to a long-term fixed-price contract to deliver the good.

- The reason for their existence is obvious: both producers and consumers may wish to lock in the revenue or cost of some physical product that they know they will need repeatedly.
  - ▶ Energy products (heating oil, jet fuel, etc) are prominent examples.

Example: What is this contract worth?

- Consider the two sides of a two-year, semi-annual swap on 100 Oz of gold for \$1500. Value each side separately.

  Side 1 Delivery of 100 Oz of gold in 6 months, 100 Oz of gold in 12 months, 100 Oz of gold in 18 months, and 100 Oz of gold in 24 months.

  - ▶ A gold "annuity".
  - ▶ Can convert to a \$ payment stream via gold forward sales.

  Side 2 Delivery of \$150,000 at same dates.

  - ▶ Equivalently \$150,000 worth of gold at the then-prevailing spot price.
    - ∗ Possibly simpler to implement, since then just exchange the net amount of gold.
  - ▶ Either way, value as just a certain payout stream of cash.

Example (continued) So suppose interest rates and forward prices are as shown below:

Date	Bond	Fwd	Side 1 delivers		Side 2 delivers	
$t - t_0$	$e^{-r_t(t-t_0)}$	$F_{t_0,t}$	Gold or	t_0 dollars	Gold or	t_0 dollars
0	1.0	1400 (spt)	-	-	-	-
0.5	0.970	1435	100	139196	$1500/S_{t_1}$	145500
1.0	0.941	1485	100	139740	$1500/S_{t_2}$	141150
1.5	0.912	1540	100	140450	$1500/S_{t_3}$	136800
2.0	0.883	1565	100	138190	$1500/S_{t_4}$	132450
Total present value				-557575		-555900
Net value to Side 2						**+1675**

- No-arbitrage pricing at its simplest, once you take as given:
  - forward prices of gold to each delivery date; and
  - prices of $ to each date.

- Replicating portfolios:

  Side 1: buy forwards to each date (price=0) and buy enough T-bills to fund the forward purchases.

  Side 2: buy T-bills.

- **Conclusion:** Commodity swaps can be viewed as derivatives whose underlyings are riskless bonds and forwards on the commodity itself.

- Let's express the swap value in a formula. Say today is time 0.

- Value of the contract to gold receiver (side 2) at $t = 0$:

$$V_0 = \sum_{t=\frac{1}{2}:\frac{1}{2}}^{2} e^{-r_t t} \; (100 \; F_{0,t}) - \sum_{t=\frac{1}{2}:\frac{1}{2}}^{2} e^{-r_t t} \; (\$150,000)$$

$$= (100) \sum_{t=\frac{1}{2}:\frac{1}{2}}^{2} B_{0,t} \; F_{0,t} - (100 \; X) \sum_{t=\frac{1}{2}:\frac{1}{2}}^{2} B_{0,t}$$

where X is the swap price in the contract (here 1500).

- What \$/oz. swap price must prevail today ($t = 0$) for a fair swap? Just set $V = 0$, and solve for X.

$$X_0 = \frac{\sum_t^T B_{0,t} \; F_{0,t}}{\sum_t^T B_{0,t}}.$$

- This swap "rate" tells you the fair price to use for a T-year fixed-price contract.
  - ▶ Not today's spot price.
  - ▶ A weighted average of forward prices.
    - ∗ What do the weights look like as a function of time-horizon?

Example (continued)

- What is the fair price today for a *new* gold swap like the one we valued above?

$$\frac{1391.96 + 1397.40 + 1404.50 + 1381.90}{0.970 + 0.941 + 0.912 + 0.883} = 1504.50$$

Q: How could we define the hedge ratio for the swap above?

- Recall (from last week) that we can view forward contracts as themselves derivatives of the spot market.
  - So we could also view *swaps* as derivatives of the underlying spot market.
  - But, recall, to price forwards by no-arbitrage we had to assume **known carry costs**:
  - If this condition is met, then we can replace all the Fs in the above formula by the function of S that we derived last week.

$$X_0 = \frac{\sum_t^T B_{0,t}\ S_0\ e^{(r-y+u)t}}{\sum_t^T B_{0,t}} = S_0\ \frac{\sum_t^T e^{(-y+u)t}}{\sum_t^T e^{-rt}}.$$

- So, we can still price swaps by no-arbitrage, even if a forward market *doesn't* exist (if the underlying is a tradeable asset with predictable carry cost).

- But it is important to understand that we can value the swap by no-arbitrage even if we *can't* value the forwards by no-arbitrage – as long as there is a forward market.
  - In that case, the valuation formula *does* still apply, even for things that cannot be transported into the future.
    - For example, the swap formula works for a swap contract that promises delivery of fresh (unfrozen!) fish to a sushi restaurant at the start of every week of the year... as long as there are traded forwards on fish.
    - Another example is electricity, which is not storable (yet). Electricity swaps can be replicated by trading in electricity forward markets – which do exist.

- A simple variation on a standard swap is a **forward swap**. This just means both parties agree to begin the exchanges at a date sometime in the future.

<u>Example</u> (continued)

Q: What is the fair price today for our gold swap if we agree to start it in one year (i.e. skip the first two exchanges)?

A: This one-year-forward one-year swap must again have a price that makes the net value today of both sides zero.

$$0 = \sum_{t=1.5}^{2} B_{0,t}\ (100\ F_{0,t}) - \sum_{t=1.5}^{2} B_{0,t}\ (100\ X_{0,1,1}).$$

Or, with our numbers,

$$X_{0,1,1} = \frac{1404.50 + 1381.90}{0.912 + 0.883} = 1552.30$$

III. Currency swaps.

- Currency swaps are identical to commodity swaps where the "commodity" is the foreign currency (euros, say).

- Same formulas for value and fair rate, given forward prices.

- Actually, those formulas are for just the most basic currency swap – an **interest only** swap.

- In practice, many currency swaps originate as the exchange of cash-flows from two bonds. So, *unlike* commodity swaps, there is then an additional exchange of the principal amounts of the bonds at the start and end dates.
  - ▶ Still simple to value (assuming riskless payments) once forward prices are given.
  - ▶ Introduces another free parameter in the contract: the relative principal amounts.
    - ∗ The periodic payments, being coupons, are then expressed as a % of these amounts.

Example

- 7 year maturity, 1.25% coupon bond on ¥1b to be "swapped into dollars." (Assume annual coupons.) *How do we set the other side so as to make the swap fair?*

- ▶ Could either fix dollar principal amount and solve for fair dollar interest payment, *or*
- ▶ Fix dollar interest payment (e.g. 7-year treasury rate) and solve for principal amount.

- What would be the fair dollar interest rate today if dollar principal is \$10m?
- Just equate values of both currency flows and solve for X:

$$¥1,000 \cdot (1 - 0.0125 \sum_{t=1}^{7} e^{-r_t^{¥} t} - e^{-r_7^{¥} \cdot 7})$$
$$= S_0 \cdot \$10 \cdot (1 - \frac{X}{100} \sum_{t=1}^{7} e^{-r_t^{\$} t} - e^{-r_7^{\$} \cdot 7}).$$

where S_0 is today's spot yen-per-dollar rate.

- The first line is the value of the yen cash-flows promised by the side who issued the yen bond.

(A) He gets the principal from the issue at the start (hence the $+1$ in parantheses).

(B) Then he pays the interest payments (middle) and final principal (last term).

The formula just equates the value of this side to the value of the dollar flows to be received in the swap.

- Currency swaps are really easy to value and price (as long as we ignore counterparty default risk).

IV. Transactions costs.

- We will talk a lot about the assumptions behind the no-arbitrage valuation formula in the next lectures.

- I have already called your attention to the assumptions about counterparty risk, and the existence of riskless borrowing/lending.

- What about the effect of ignoring <u>transactions costs</u>?

- Here the key observation is that, as with forwards, the replicating positions for swaps are **static**.
  - ▶ As soon as we initiate the position, we can undertake all the replicating transactions instantly, and then never change them.

- So, if those initial transactions are costly (due to fees, bid/ask spreads, etc.) we can just build those costs into the value of the derivative.
  - ▶ In other words, as with forward prices, our theory will again give us upper and lower no-arbitrage bounds for the swap rate, instead of a single price.

  - ▶ For example, if we are asked by a customer to quote a swap price at which we would sell a commodity, and we want to take into account the bid/ask spreads that we face in the forward markets and bond markets, we would quote:

$$X_0^{ask} = \frac{\sum_t^T B_{0,t}^{ask} \ F_{0,t}^{ask}}{\sum_t^T B_{0,t}^{bid}}.$$

  * If we sell at a price lower than that, we will lose money if we do the hedging transactions.

  * But *anyone* would be making risk free money if they were able to sell at a higher price. So (as long as there is competition) we can't charge any more than this.

- Note that the formula is affected by many spreads (more so than forwards) because there are so many replicating transactions.
  - This may mean that the resulting bounds $[X_0^{bid}, X_0^{ask}]$ are quite far apart.

- **Remember:** our theory has nothing to say about what price should prevail for a derivative inside its no-arbitrage bounds.
  - We cannot assert, for example, that the market price "should be" close to the prefect-markets price or the mid-point of the band.

  - The price will be determined by supply and demand. That's all.

V. Mark-to-Market and Margining Swaps.

- We deduced above the value (at time 0, say) for an old commodity swap with swap price X (to the long side)

$$V(X) = \sum_t^T B_{0,t}\ F_{0,t} - X\ \sum_t^T B_{0,t},$$

and the fair price X^* for a new swap:

$$X^* = \frac{\sum_t^T B_{0,t}\ F_{0,t}}{\sum_t^T B_{0,t}}.$$

Hence

$$X^*\ \sum_t^T B_{0,t} = \sum_t^T B_{0,t} F_{0,t}.$$

- Combining these, the value of the old swap can also be expressed simply as

$$V(X, X^*) = (X^* - X)\ \sum_t^T B_{0,t}.$$

- This is a pretty obvious result: If you offset your old swap with a new one, you lock-in the certain payments $X^* - X$ for the life of the swap.
    - ▶ The formula just says yor position is worth the present value of these payments.
    - ▶ This is also called the mark-to-market formula for the swap.
- As we have noted, many swap market participants have recently become subject to the requirement that they *will* be marked-to-market on a daily basis.

- In the case of centrally cleared swaps, the clearinghouse (CCP) will perform the calculation, and collect the amount V – also called the *variation margin* – from the loser and pay it to the winner.
- For each non-cleared swap, the counterparties will agree on a third-party custodial agent to perform the same margin calculation and collection.

And, after the cash is paid, the swap price is re-set every day to the new price, X^*.

- An important question is whether the marking-to-market changes the fair price of the swap.
- The answer is no.
- Here is a simple argument to show why:

(A) Suppose you were long a non-margined swap with swap price X_0.

(B) Suppose every day, t, you sell <u>and</u> buy new swaps at X_t (with the same terms and dates as the original swap, and no counterparty risk).

- Clearly doing this has creates zero value, now and in the future.

(C) If, also, every day (t_2) you package your old long position (from t_1) with your new short position, that pair of positions is worth $V(X_{t_2}, X_{t_1})$.

(D) You should therefore be indifferent between retaining these two trades and selling them for this price.

- But selling each day's offset positions and retaining the new long swap with rate X_{t_2} is exactly the same thing as having the position marked-to-market.

- The point to remember here is that marking a position to market every day is economically identical to unwinding your position each day and then establishing the same position all over again.
  - Since doing this has no economic costs or benefits, it does not effect the value of the position.
  - Thus a swap with this feature has no advantage or disadvantage compared to one without it.
  - Hence the two should have the same fair price.

- But remember! We are ignoring counterparty risk.

- The real reason to have margining is to minimize <u>that</u> risk.

- The point we are making now, is that *aside from counterparty risk mitigation*, the simple act of realizing the gains and losses continuously (instead of all at the end) does not change the valuation of the contract.

VI. Summary.

- The swaps we looked at today are just like baskets of forward contracts.

- So it is not surprising that the no-arbitrage swap price should be a weighted average of the underlying forward prices.
  - ▶ You **do** need to know that formula!

- The replicating/hedging positions for each side of a swap are just positions in forwards and zero coupon bonds whose quantities are equal to the quantity of the promised payments.

- Transactions costs mean that arbitrage can enforce upper and lower bounds on the swap price.

- Absent counterparty risk, marking swaps to market does not change their valuation.

Lecture Note 2.1: Summary of Notation

Symbol	Page	Meaning
$S_{t_0}, F_{t_0,t}$	*p5*	*spot and forward-to-t price of an asset at time t_0*
$B_{t_0,t}$	*p5*	*riskless zero coupon bond maturing at t*
r_t	*p5*	*continuously-compounded riskless rate until time t*
$V_0(S_0)$	*p6*	*value at $t=0$ of an existing swap when spot is at S_0*
$\sum_{t=\frac{1}{2}:\frac{1}{2}}^{2}$	*p6*	*indicates that the expression following is to be*
		evaluated with $t = \frac{1}{2}, 1, \frac{3}{2}, 2$ and the terms added up
y, u	*p7*	*borrowing rate and storage cost (c.c.) for a commodity*
X_{0,t_1,t_2}	*p9*	*fair rate at $t=0$ for a forward swap running from t_1 to t_2*
S_0	*p11*	*spot yen/dollar exchange rate*
$[X_0^{bid}, X_0^{ask}]$	*p13*	*lower and upper no-arbitrage bands for swap rate*
		when there are transactions costs
$V(X, X^*)$	*p14*	*value an existing swap with swap price X when the current*
		fair swap price is X^*