Homework 7

1. (*CAPM prices of risk.*) Agree or disagree with the following statements concerning the classic Capital Asset Pricing Model (CAPM). Explain your reasoning.

   (a) *If the CAPM is correct, then the market price of risk associated with global warming will be positive if temperature increases are associated with lower stock prices.*

   (b) *The CAPM predicts that the market price of volatility risk must be smaller in absolute value than the Sharpe ratio for the stock market.*

2. (*Nonconstant price of volatility risk.*) Consider a model for $v = \log \sigma^2$, the log variance for some asset. Suppose v obeys the Ornstein-Uhlenbeck process

$$dv_t = \kappa\,(\bar{v} - v_t)\,dt + s_v\,dW_t^v$$

   where $\kappa, \bar{v}$, and s_v are constants.

   (a) Show that, if the market price of volatility risk is $\lambda(v) = \lambda_0 + \lambda_1 v$, then the risk-neutralized version of v is still a Ornstein-Uhlenbeck process, with a re-definition of the constants.

   (b) What is the no arbitrage forward price for a contract paying σ_T at time T under this model? *(Use the fact that, for a normally distributed random variable $X \sim N(\mu,\ \Sigma)$ we have $E[e^X] = e^{\mu + \frac{1}{2}\Sigma}$ and refer to Lecture Note 7.2 for the mean and variance of v_T. The answer is a mess, but simplify it as best you can.)*

   (c) What is the limit of the formula as $T \to 0$?

3. (*Deltas from simulation.*) In most problems, it is not easy to compute deltas directly from Monte Carlo output. But for some applications we can get them via simulating an auxillary process.

   Suppose we have a European claim paying $f(Y_T)$ at T, where Y is a diffusion and the riskless rate, r, is constant. We want to compute (under the risk-neutral measure):

$$\delta/B = \frac{\partial}{\partial Y_t}\ \mathrm{E_t}\left[f(Y_T)\right] = \mathrm{E_t}\left[\frac{\partial f(Y_T)}{\partial Y_t}\right] = \mathrm{E_t}\left[f'(Y_T)\ \frac{\partial Y_T}{\partial Y_t}\right].$$

   (Here the middle inequality requires some regularity conditions so that the expectation of the partial derivative is well defined.) From the right-hand expression, it is apparent that we can compute $f'(Y_T)$ from simulation of the Y process. So we could compute the whole thing if we knew $G_T = \frac{\partial Y_T}{\partial Y_t}$.

   Suppose $Y_T = Y_t + \int_t^T \kappa(\bar{Y} - Y_u)\ du + \int_t^T s_Y\ Y_u\ dW_u$. What stochastic process does dG_T obey? (Be careful: we want its evolution as T changes, while holding t fixed.) What is its initial value G_t? Explain how you could evaluate G_T within a routine that simulates Y_T.

4. (*Implicit prices of risk.*) Use the Euro area government yield curve from March 13 2007 on the Compass site to find a value for the market price of risk that best fits the curve using the Vasicek model. Assume $\kappa = 0.10$ and $b = 0.0065$.

   - Set r_0 to the shortest maturity yield on the curve. Then, for a range of $r^\star$ values, evaluate the functions $G_0(0,T)$ and $G_1(0,T)$ in Lecture Note 7.2 out to a maturity of $T = 30$ years. Convert the bond prices into (zero-coupon) rates for each T.
   - Compute the absolute average error in fitting the rates for each $r^\star$. Then report the value that gives you the lowest error. Note that $r^\star$ is all you need. It's not necessary to convert it into a λ.

   Show a plot of your fitted curve and the actual yield curve.

5. *(Snowball Swap.)* In 2007, a Portuguese transportation company Metro do Porto (MdP) entered into an unusual swap with Banco Santander, a Spanish bank, (BST). Later on, this swap went very badly for MdP, and they sued BST in court, claiming that the swap was so unfair at the start that *they themselves must have been too unsophisticated to understand it!* – and therefore Santander had committed fraud in selling it to them. Your task is to value the swap at its start date via Monte Carlo simulation.

   Every quarter, starting on March 13 2007, BST paid MdP a fixed fee of 0.0300 times the notional amount. (All rates are annual, so this is 75 basis points each quarter). Starting on March 13 2009, MdP paid a value called the "spread", which was updated one quarter in advance. (So the March 13 2009 spread payment was determined on December 13 2008.) The spread was updated according to the following rules:

   (a) If the 3-month rate, r_t, is between 0.02% and 0.06%, then the new spread is the old spread minus 0.0050 (50 bp), *except that* the new spread cannot be below zero.
   (b) If r_t is below 0.0200, then the new spread is the old spread <u>plus</u> 2 times $(0.0200 - r_t)$.
   (c) If r_t is above 0.0600, then the new spread is the old spread <u>plus</u> 2 times $(r_t - 0.0600)$.

   The spread is also an annualized rate. So the payment each quarter is 0.25 times spread times notional. On December 13 2008, the old spread is zero. The swap terminates on December 13, 2022. The swap notional amount is 89 million EUR. What is its net value to MdP on March 13 2007? (Also report your numerical standard error.)

   Using the fitted risk-neutralized model for r_t from the previous question, draw 10000 paths from 2007 to 2022 using daily time-steps. (You can assume the spot rate, not the 3-month rate, is used in the spread computation.) Along each path, compute the net cash-flows from the swap to/from MdP, cumulate their discounted sum, and then average the results. Some things to remember:

   - Start the simulations from the r_0 value you used in fitting the yield curve.
   - At each time-step, t, where there is no cash payment, just increment r_t by adding dr and increment the discount factor $DF_{0,t} = e^{-\int_0^t r_u\,du}$ by multiplying by $e^{-r_t\,\Delta t}$.
   - At time-steps corresponding to cash flows, $t = 0, 0.25, 0.5, ...$, and using the *spread* value from the previous quarter figure out the net cash-flow due at that date; multiply it by the discount factor, and add it to the total cash-flow of the path.
   - Then use r_t to update the *spread* amount to be paid at the end of the quarter.
   - When you get to the end (2022), save the discounted cash-flow sum for that path. Average these over all your simulations.

   Plot a histogram of the distribution across paths of dcf outcomes.