FIN 513: Homework #3

Due on Tuessday, February 13, 2018

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Problem 1

The put-call parity of American options on a currency is $e^{-r_f(T-t)}S - K \le C - P \le S - e^{-r_d(T-t)}K$ or $B^f S - K \le C - P \le S - B^d K$, where S and K denotes spot exchange rate and strike price as usual, r_d and r_f denotes domestic risk-free rate and foreign risk-free rate, respectively.

(Proof) Suppose not.

1. Suppose $C - P > S - e^{-r_d(T-t)}K$. Then an arbitrage opportunity exists by constructing the following portfolio.

   (a) Buy a put option and a unit of foreign currency.

   (b) Write a call option and borrow $e^{-r_d(T-t)}K$ amount of domestic currency on risk-free rate.

   All possible situations can be separated by two cases: early exercise of written option occurs and no early exercise of written option. Assuming that early exercise occurs at $t^* < T$, payoff of the portfolio is as follows.

   Case 1. Early exercise at $t^* < T$ occurs.

   (a) $e^{r_f(t^*-t)}S_{t^*}$

   (b) $-(S_{t^*} - K) - e^{-r_d(T-t^*)}K$

   The sum of payoff from two strategies is $(e^{r_f(t^*-t)}-1)S_{t^*}+(1-e^{-r_d(T-t^*)})K$, which is positive.

   Case 2. No early exercise.

   i. $S_T > K$

   (a) $0 + e^{r_f(T-t)}S_T$

   (b) $-(S_T - K) - K$

   ii. $S_T \le K$

   (a) $(K - S_T) + e^{r_f(T-t)}S_T$

   (b) $0 - K$

   In this case, the portfolio also has a positive payoff regardless of spot exchange rate at maturity.

   Since the portfolio has a positive payoff at all possible situations, there must be a cost for implementing the strategies if there is no arbitrage opportunity. However, by the assumption, the initial cost for constructing portfolio is negative, so a contradiction occurs. Therefore, $C - P \le S - e^{-r_d(T-t)}K$ must hold.

2. Suppose $C - P < e^{-r_f(T-t)}S - K$. Then there is also an arbitrage opportunity exists considering the following portfolio.

(a) Buy a call option and invest K amount of domestic currencies on domestic risk-free rate.

(b) Write a put option and sell short a foreign risk-free zero coupon bond.

By using similar procedure above, existence of arbitrage can be derived.

Case 1. Early exercise at $t^* < T$ occurs.

(a) $e^{rt^*}K$

(b) $-(K - S_{t^*}) - e^{-r_f(T-t^*)}S_{t^*}$

In this case, the sum of payoff from two strategies above is $S_{t^*}(1 - e^{-r_f(T-t^*)}) - (e^{rt^*} - 1)K$, which is positive.

Case 2. No early exercise.

i. $S_T > K$

(a) $(S_T - K) + e^{r(T-t)}K$

(b) $0 - S_T$

ii. $S_T \leq K$

(a) $0 + e^{r(T-t)}K$

(b) $-(K - S_T) - S_T$

At the maturity, the portfolio has a positive value regardless of spot exchange rate at T.

Since the portfolio has a positive payoff at all possible situations, there is an arbitrage opportunity since we assumed that there is a negative initial cost for constructing this portfolio. Therefore, $C - P \geq e^{-r_f(T-t)}S - K$ must hold.

Combining all results above, the put-call parity mentioned above must hold. Otherwise, there would be an arbitrage opportunity.

Problem 2

(a) **False.** Even if $c(K,T) > S_t$, in order to lock arbitrage profit in, it needs to trade underlying asset. However, in this case, underlying asset is not tradable, constructing arbitrage portfolio is impossible.

(b) **True.** Suppose $p(K,T) > B_{t,T} \cdot K$. Then by selling short a put option with strike price K and maturity T, and buying a zero coupon bond with face value K and maturity T.

(c)

(d)

(e)

Problem 2 continued on next page...

(f)

(g)

(h)

Problem 3

(a) Figure 1 represents plots of function f, g, h over the range -1 to 2.

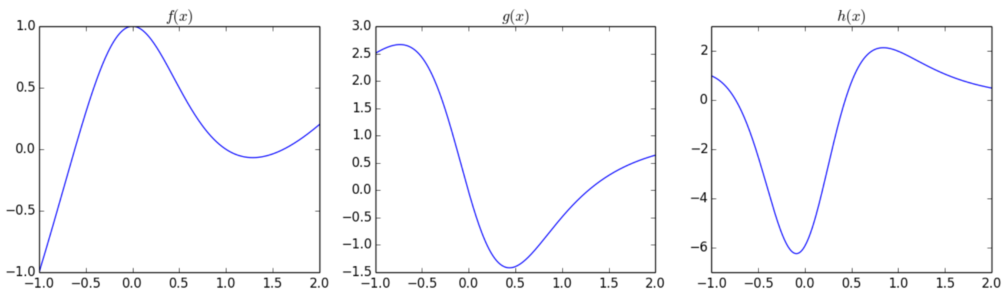

Figure 1: $f(x)$, $g(x)$, $h(x)$ (from left to right)

(b) From the figure 1, it seems that from $x = 0.5$(approximately) the function f is convex since its slope starts increase at this point. Actually, using some calculus, since $h(x) = -\frac{2(x^3-9x^2-3x+3)}{(x^2+1)^3}$, $f(x)$ is convex from $x = 0.4422$.

(c) It appears that from $x = -0.1$ to 0.8(approximately) the function looks convex. It is because function $h(x)$ starts to increase at -0.1 and stops increasing at 0.8.

(d) Visually, it seems that from $x = -0.5$ to 0.4 $h(x)$ looks convex.

Problem 4

(a) There is an arbitrage opportunity. Since the probabilities mentioned are exact, there is just two possibilities that stock price moves up to \$120 or down to \$90, so if someone sell short a call option with strike price 120, then there would be an arbitrage because the price of call option is positive and there is no amount pay at maturity with certainty.

Problem 4 continued on next page. . .

(b) As long as the statistical probabilities are true, using the idea of binomial tree, option price should be equal to followings.

$$
\begin{aligned}
c(100) &= e^{-0.06\times\frac{1}{12}}(0.4\times 20 + 0.6\times 0)\\
&= 7.96\\
c(110) &= e^{-0.06\times\frac{1}{12}}(0.4\times 10 + 0.6\times 0)\\
&= 3.9\\
c(120) &= e^{-0.06\times\frac{1}{12}}(0.4\times 0 + 0.6\times 0)\\
&= 0
\end{aligned}
$$

However, it is completely different from the price of options traded in the market. There are two possibilities that makes deviation between theoretical prices and market prices. First, there would be differences between statistical probabilities and market-implied probabilities. One main reason why $c(100)$ and $c(110)$ using statistical probabilities is higher than market prices is because relatively higher (statistical) probability is allocated to the state that stock prices moves up to 120. In other words, if a probability less than 0.4 is allocated to the state in which stock price goes up to 120, $c(100)$ and $c(110)$ would be closer to the market price. Second, there would be more states rather than two states. Regarding the statistical probabilities, there are just two states, and under those states $c(120)$ is always equal to zero whatever probabilities are allocated to. It means that market prices of option imply that there are more states than statistical view, especially states that stocks price will be larger than 120. Summing up, statistical probabilities assumes more simple states than market implied view, and allocate more probabilities to states that stock prices are larger than 100.