FIN 513: Homework #5

Due on Tuesday, March 6, 2018

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Problem 1

Let V and S denote sum of values of options and stock price respectively, then we can denote portfolio of the market maker as $\Pi = V - \Delta S$. By Ito's lemma, the following equation follows.

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 - \Delta dS$$

Since the portfolio has zero delta, $d\Pi = \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 = \frac{\partial V}{\partial t}dt + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2}dt$ holds. Since the portfolio became riskless, by no arbitrage principle, its return must be equal to risk-free rate as follows.

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2}dt = r\Pi dt$$

Since parameters are given as $\frac{\partial^2 V}{\partial S^2} = -1,725$, $S = 143$, $r = 0.05$, $\sigma = 0.7$, plugging them into the equation above, we can obtain the following result.

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2} \times (143)^2 \times (0.7)^2 \times (-1725) = 0.05 \times 30,000,000 \\
&\Rightarrow \frac{\partial V}{\partial t} = 10,142,258.6
\end{aligned}$$

Assuming a year is equal to 365 days, if the stock price is unchanged, the expected value of the positions is approximately $\Pi + d\Pi = \Pi + \frac{\partial V}{\partial t}dt = 30,000,000 + 10,142,258.6 \times \frac{1}{365} = 30,027,787.01$. The value might not be exact since the whole procedure was implemented on continuous time framework which is not exactly consistent to this problem. However, since $dt = \frac{1}{365}$ is small enough, errors can be ignored.

Problem 2

Since it is assumed that Black-Scholes assumptions hold, put option price is calculated as follows.

$$\begin{aligned}
p &= B_{t,T}(KN(-d_2) - FN(-d_1)) \\
d_1 &= \frac{\log(F/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 - \sqrt{T-t}
\end{aligned}$$

Let p_1, p_2, p_3, p_4 denote price of put options with time to maturity from 2 months to 5 months, respectively. Since all parameters need for valuing option are given except volatility, and it is assumed that all of options have same implied volatility, it is possible to calculate volatility by using the formula inversely. In other words, we can obtain implied volatility by searching σ which equates sum of option prices to 2 millions. (find σ such that $\sum_{i=1}^{4} p_i = 2,000,000$) By taking some numerical procedures, the implied volatility is calculated as about 9.63%. Prices of each option are calculated as Table 1.

Problem 3

(a) Under Black-Scholes economy, call option price is calculated as $c = e^{-r(T-t)}\mathrm{E}_t^Q[\max(S_T - K, 0)]$, where E_t^Q is an expectation operator under risk neutral measure conditioning at time t. Since we

Maturity	Price
2 months	351,710
3 months	464,836
4 months	555,220
5 months	628,235

Table 1: Price of options

already know that $c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$ and $p = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$, where $d_1 = \frac{\log(S/Ke^{-r(T-t)})+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$, expected payoff under risk neutral measure can be derived as follows.

$$\mathrm{E}_t^Q[\max(S_T - K, 0)] = Se^{r(T-t)}N(d_1) - KN(d_2)$$
$$\mathrm{E}_t^Q[\max(K - S_T, 0)] = KN(-d_2) - Se^{r(T-t)}N(-d_1)$$

Under risk neutral measure, expected return of physical measure μ is converted into risk free rate, r. Therefore, by converting all r into μ, we can obtain true expected payoff. Therefore, true expected payoff of options are derived as follows.

$$\mathrm{E}_t[\max(S_T - K, 0)] = Se^{\mu(T-t)}N(d_1) - KN(d_2)$$
$$\mathrm{E}_t[\max(K - S_T, 0)] = KN(-d_2) - Se^{\mu(T-t)}N(-d_1)$$
$$d_1 = \frac{\log(S/Ke^{-\mu(T-t)}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sqrt{T-t}$$

By using the formula above, expected payoff of straddle is calculated as follows. It is well-known that a

Volatility	Expected Payoff
0.2	17.506
0.3	25.376
0.4	33.348
0.5	41.306

Table 2: Expected payoff under physical measure

straddle is more worth when volatility is larger. From the result, we can find expected payoff of straddle increases as volatility increases, so the result is consistent with that fact.

(b) By using the Algorithm 1 mentioned below(See the last page), it is possible to implement dynamic replication strategy. Table 3 shows the result from simulation. Simulation was performed with 10,000 sample paths and 500 time steps.

Volatility	0.2	0.3	0.4	0.5
Payoff from analytic solution	17.506	25.376	33.348	41.306
Payoff from dynamic replication	1.457	1.242	1.691	1.860
Standard deviation of simulation	0.146	0.226	0.306	0.407

Table 3: Simulation result

(c) As shown in Table 3, there is a large difference between analytic payoff and value of replication strategy. It is because the replication strategy cannot reflect curvature of the straddle with respect to spot price. It just replicated straddle price with respect to “direction” of spot price: going up or down of price. However, value of straddle is actually independent to direction of spot price. What actually matters is the amount of change in price. Payoff of straddle gets larger when spot price at maturity is far from the strike price, so volatility is the key feature. Since delta of a straddle increases as spot price increases, the position requires more underlyings if spot price gets higher and vice versa. It means the position has “buy-high, sell-low” strategy. The loss from “buy-high, sell-low” is compensated through treating curvature of straddle such as including options in replicating portfolio. However, in this portfolio, there is no such securities, therefore it cannot replicate payoff of straddle. Of course, it is possible to think that this delta-hedged portfolio should have zero value at maturity. However, it is not because the expected return of spot price is not zero. Since delta-hedging portfolio is a portfolio replicating payoff of derivatives from “directional” changes of underlying asset price, the portfolio has positive value because a straddle has positive value if underlying asset price goes up. If expected return of spot price is zero, expected payoff of replicating portfolio would be zero. Table 4 shows expected payoff of replicating portfolio when $\mu = 0$. Simulation was performed with 10,000 sample paths and 500 time steps. Unlike

Volatility	Expected Payoff	Standard Deviation
0.2	-0.011	0.124
0.3	0.116	0.193
0.4	-0.002	0.273
0.5	0.007	0.359

Table 4: Expected payoff($\mu = 0$)

results in Table 3, expected payoffs are closed to zero, and all expected payoffs are within $0 \pm \sigma$ where σ is standard deviation of simulation. It means that we cannot say expected payoffs are different to zero, statistically. Therefore, we can conclude that the replicating portfolio should not be zero unless expected return of spot price is not zero.

Problem 4

(a) By using similar procedure with Algorithm 1, only changing D_{j+1} as $b(S_{j+1}-S_0)$, the expected cash-flow and standard deviation can be obtained. Table 5 shows expected cash flow and standard deviation with respect to volatilities. Simulation was performed with 50,000 sample paths and 500 time steps. From the

Volatility	Expected Cash Flow	Standard Deviation
0.2	4.23	0.19
0.3	6.08	0.47
0.4	8.31	0.92
0.5	13.51	1.85

Table 5: Expected cash flow of the strategy

table, the expected cash flow increases as volatility increases. However, we cannot say "long volatility" strategy because the positive return is caused by positive expected return, not volatility. Table 6 shows the expected cash flow where expected return of spot price is zero. From the table, although volatility

Volatility	Expected Cash Flow	Standard Deviation
0.2	0.13	0.14
0.3	-0.49	0.36
0.4	0.60	0.82
0.5	0.87	1.49

Table 6: Expected cash flow of the strategy($\mu = 0$)

is same as Table 5, we can find that expected cash flow is quite low. There is even a negative expected cash flow. Almost all cash flows are statistically indifferent to zero. Therefore, we cannot conclude that the strategy is "long volatility" strategy. Instead, the strategy invests in positive expected return of spot price.

(b) By using similar simulation procedure, expected return and Sharpe ratio was calculated as Table 7. Sharpe ratio was calculated as (Expected return / Standard deviation) since we assumed that risk free rate is equal to zero.

From the result, it can be find that expected return increases as volatility of spot price increase, however Sharpe ratio decreases since standard deviation increases more rapidly than expected return. Therefore, from the perspective of mean-variance, increase in volatility does not seem attractive since it increases risks more than expected return.

Volatility	Expected Return	Standard Deviation	Sharpe Ratio
0.2	1.76%	0.08%	22.63
0.3	2.30%	0.19%	12.33
0.4	3.21%	0.38%	8.49
0.5	6.06%	0.90%	6.72

Table 7: Expected return and Sharpe ratio

Algorithm 1: Monte-Carlo Simulation for Dynamic Replication

Input: S_0, K, μ, r, d, σ, T, N, M

// N and M denote number of time steps and number of paths, respectively.

Output: Average of payoff, Standard Deviation

Set P as an array // P: array of portfolio value at maturity for each sample path

Set $\Delta t = T/N$

for $i \leftarrow 1$ ***to*** M **do**

    Initialize S, D, C as an array // S: stock price path, D: delta path, C: cash path

    Set $S_1 = S_0$, $D_1 = N(d_1; S_1) - N(-d_1; S_1)$, $C_1 = -D_1 S_1$.

    for $j \leftarrow 1$ ***to*** N **do**

        Generate random number Z which follows $N(0,1)$.

        Calculate $S_{j+1} = S_j e^{(\mu - d - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} Z}$

        if $j \neq N$ **then**

            // Delta of straddle is equal to sum of deltas of call and put.

            Calculate $D_{j+1} = N(d_1; S_{j+1}) - N(-d_1; S_{j+1})$

            // The rest of cash after rebalancing

            Calculate $C_{j+1} = C_j e^{r\Delta t} - (D_{j+1} - D_j) S_{j+1}$.

        end

        else

            // $j = N$ case

            Calculate $P_i = D_N S_{N+1} + C_N e^{r\Delta t}$. // Value at maturity at i^{th} path

        end

    end

end

return $\bar{P}$, $\sigma_P/\sqrt{M}$ // $\bar{P}$: average of P, σ_P: standard deviation of P.