University of Illinois, Urbana-Champaign *Professor Tim Johnson*
FIN 513 - Spring 2018 *tcj@illinois.edu*

Lecture Note 1.1: Forward Contracts

This lecture note examines the pricing and hedging of forward contracts. For some of them, no arbitrage pricing is easy. But not always!

You be already familiar with these instruments. They are very important in finance. Our goal is to make sure you know how the markets work in practice and see precisely how and where the assumptions of the theory may be violated.

Outline:

I. Definitions and Contractual Features

II. Pricing Forward Contracts

(**A**) currencies

(**B**) stock indexes

III. An Example

IV. More on Short selling.

V. Commodity Forwards

VI. Summary

I. Contract Definitions.

- A forward contract is an agreement between two parties to exchange something at a prespecified price on a prespecified date.
  - ▶ The person who agrees to buy is said to be **long**, and the person who agrees to sell is said to be **short**.

- Contract specifications must include:

(A) Amount and quality of good to be delivered; mechanics of delivery (when/where).

(B) Price (F) and currency of denomination;

(C) Time of delivery (T) also called settlement or maturity date;

(D) Remedies in the event of default; collateralization (if any).

- Detailed legal terms must be negotiated between trading counterparties in advance.

- Actually, forward transactions are not intrinsically different from "spot" trades. In fact, most "spot" trades of assets *do* specify a few days ahead for settlement.

- The idea of locking in contract terms in advance of physical settlement has been around for at least 700 years!

- Forward contracts are almost all traded in over-the-counter (OTC) markets, which means that there is no centralized exchange.
  - Instead, participants interact through telephone, web, and other communications systems, on a bilateral basis.

- Properties of Forward contracts:
  - Customizable;
  - No reporting of trades;
  - No public prices;
  - May not be assignable;
  - No realization of gains/losses prior to T (unless margined);
  - **Credit risk.**

- In most forward markets, there is no cental counterparty (CCP) who guarantees contracts, nor any mechanism to collateralize gains and losses.
  - The analysis below will assume this is the case.
  - However new rules on margin for non-cleared derivatives are now starting to apply to some forward positions. (More on this later.)

II. Forward Prices

- By convention, forward contracts (and futures) are entered into without either side paying the other anything up-front. Our task now is to describe mathematically the forward price, $F(\cdot)$, at which both sides will agree the contract is worth zero today.

- In other words, if soybeans today cost \$6.21 a bushel, what is the correct price for June-delivery soybeans?

- Let's answer this by looking at foreign exchange forwards first.

(A) Pricing currency forwards.

- Example (perfect markets):

  ▶ Spot ¥/\$ exchange rate is 120.00.
  One year (annually compounded) ¥interest rate is 0.15%.
  One year (annually compounded) \$ interest rate is 0.30%.

  ▶ ¥120.00 today = ¥120.18 in 1 year.
  ¥120.00 today = \$ 1.00 today
  \$ 1.00 today = \$ 1.0030 in 1 year.

  ▶ Notice that – even without a forward contract – we can already enter into trades that *transform future-dollars to future yen* at a guaranteed rate.

  1. Borrow dollars;
  2. Sell them for yen;
  3. Invest the yen.

  ▶ Here's the picture:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\yen 120.00 \arrow[r, shift left] \arrow[d, shift right] & \yen 120.18 \arrow[l, shift left, "0.15\%"] \arrow[d, shift right, "?"'] \\
\$\ 1.00 \arrow[r, shift left, "0.3\%"] \arrow[u, shift right, "120.00\ \yen/\$"'] & \$\ 1.0030 \arrow[l, shift left] \arrow[u, shift right]
\end{tikzcd}
$$

Now **1 Year from now**

- If there is a forward contract, there would then be another *completely equivalent* way of converting future-dollars into future-yen.

- So it better be the case that forward rate ensures :
$ 1.00 in 1 year = 120.00 X 1.0015/1.003 = ¥119.82 in 1 year.

  ∗ The two methods better give the same result or something is seriously wrong.

- In math,

$$F_{t,T}(S_t, R^{\yen}, R^{\$}) = S_t \times \left[\frac{1 + R^{\yen}}{1 + R^{\$}}\right].$$

* Using continuously compounded interest rates, this expresssion is

$$F_{t,T}(S_t, r^{¥}, r^{\$}) = S_t \times e^{(r^{¥} - r^{\$})(T-t)}$$

where $(T-t) = 1$ in the example.

* This formula assumes both S and F are quoted in yen-per-dollar. If we want to view FX rates like prices of any other commodity, this convention means we are viewing the dollar as the "good" and using yen as our numeraire.

▶ Note that using continuously compounded rates does not mean we are assuming the interest is actually *paid* continuously.

* The r s are the rates that apply for borrowing/lending to time T with all cash flows at the end.

* Also Note that we do not have to assume a flat term structure here. Sometimes I will distinguish rates at different maturities with notation like $r_{t,T}$

▶ We can also write the formulas above in terms of the prices of riskless zero coupon bonds maturing at T:

$$F_{t,T}(S_t, B^{¥}_{t,T}, B^{\$}_{t,T}) = S_t \cdot B^{\$}_{t,T} / B^{¥}_{t,T}$$

since, in either currency, $B_{t,T} = e^{-r_{t,T}\ (T-t)}$.

▶ Now, what did we mean precisely when we said the formula "better hold....or something is wrong"?

- The no-arbitrage argument formalized.
  - Suppose the forward price were *anything other* than 119.82. Call it 119.00. As long as this case, we can get free money. How?
    - ∗ Buy the thing that is too cheap (forward dollars).
    - ∗ Sell the thing that's too rich (spot dollars).
    - ∗ Put the yen in the bank.
    - ∗ Borrow the dollars to deliver. (Use yen as collateral).
    - ∗ Then . . . do nothing!
  - Arbitrage Table:

Transaction	Cashflow (at t)	Cashflow (at T)
Buy 1.00 dollars forward	0	-119.00 yen + 1 dollar
Lend 119/1.0015 yen	-118.82 yen	+119.00 yen
Borrow 1/1.003 dollars	+.9970 dollars	-1 dollar
Sell .9970 dollars spot @ 120.00	+119.64 yen - 0.9970 dollars	0
Total	+0.82 yen	0 yen , 0 dollars

  - Do this times 1,000,000,000,000,000,000,000!
  - If instead of 119.00, the forward price were 121.00, we'd reverse these trades and again make free money.
  - We could now formally **prove** our pricing law by similarly showing that ANY number other than $S\ [\frac{1+R^{¥}}{1+R^{\$}}] = 119.82$ likewise implies arbitrage profits.

- The strategy implemented in our table above is called a "cash and carry" arbitrage trade.
  - We replicate a forward position with a "cash" position, and then just "carry" it into the future.
- Notice that the replicating strategy for the forward:

  involves a spot position and two bond positions. The government bonds (or riskless bank deposits) and the foreign currency are the underlying securities.

  is perfect. The cashflows from the underlying assets *exactly* off-set those from the forward *regardless of where spot is at settlement.*

  is available. Nothing stops us from doing it.

  is static. No intermediate transactions required.

  has known price today. We're taking spot exchange rate and both interest rates as given.

- Problems with the argument?
  - Are there any reasons why we might ever **not** observe

$$\text{Forward} = \text{Spot} \times e^{(r_{t,T}^{domestic} - r_{t,T}^{foreign})(T-t)}$$

    in real life?
  - Certainly!

▶ The biggest problem: Transactions costs.

In real life, there are bid/ask spreads in both the spot market and in the borrowing/lending markets.

Q: What do they do to the argument?

A: The (unique) no-arbitrage price instead becomes a pair of prices (upper and lower) defining a no-arbitrage *region*.

* The upper bound is the cost of replicating a long forward position, i.e., the buy-and-hold cost *including all transaction charges*.
  - If you could ever sell a forward above that price, you would be happy to!
* The lower bound is the cost of replicating a short forward, i.e., the borrow-and-sell cost including all charges.
  - If you could ever buy a forward below that price...

Q: Whose costs determine the lower and upper bounds?

A: If markets are competitive, the players with the *lowest costs* will enforce the *tightest* no-arbitrage bounds.

Last time we asserted that the basic paradigm of financial engineering is that financial instruments are worth whatever it costs to replicate them.

Now we can amend that statement to be more precise:

> An instrument is worth *less* than the <u>least</u> it costs to replicate it, and is worth *more* than the <u>least</u> it costs to hedge it.

(Remember, hedging a long position is the same as replicating a short position)

- Some other big assumptions in the cash-and-carry argument:
  - $*$ No counterparty risk.
  - $*$ The existence of riskless borrowing/lending markets.
  - $*$ Enforcibility of contracts.

  We will talk a lot more about these as we go along.

(**B)** Forward Price of a Stock Index.

- How would we determine the forward price for a basket of stocks that make up an index?
  - ▶ Let's suppose the forward is a physical delivery contract, and all the stocks are freely tradeable and liquid.
  - ▶ Then the basic argument is the same as for currencies: "cash and carry".
- This time, let's consider the upper and lower arbitrage limits separately.
- First we'll we figure out how much it will cost to buy and hold the basket till delivery
  - ▶ The cost of doing that must be the upper bound on F. (Call it $\overline{F}$.)
    - ∗ If someone ever wanted to pay more than $\overline{F}$, we would agree to go short to them, then buy all the shares and hold them, and make riskless profit.
- As usual, we start out neglecting transactions costs.

- **Buy-and-hold cost:**
  - ▶ The cost of buying a unit of the index in the spot market is assumed known: S_t.
    - ∗ I'm using S to denote the sum of all the individual stock prices times their weight in the index.

- We borrow to pay for it at r using the shares as collateral – just as we did with yen.
  - Net cash-flow today: 0.
  - Amount owed at T: $S_t \cdot e^{r(T-t)}$.
  - As in the currency example, r is being expressed in continuous compounded form for mathematical convenience but we are assuming the interest is payable at T.
    - So r here means the time-T rate we can lock in today.
    - We are not assuming this rate is constant over time, or the same for all T.
- We also get any dividends paid between now and T.
  - For now, **approximate** the stream of dividends we get from all the individual stocks as a continuous **percentage** flow at the rate d per unit time.
  - This is a bit crude. We will relax it later to think about discrete payouts.
  - For large baskets of stocks, like the S&P 500, the dividends are fairly smoothly spread out over time.
  - With this approximation d is just like the foreign interest rate: the continuous yield on holding a unit of the underlying good.
  - Note that we are also assuming d is known today.

- This is a reasonable assumption for U.S. companies up to about a year ahead.

▶ As with the currency case, we can use the yield we will get from the "good" to reduce the *amount* of it that we need to hold today.

▶ Instead of buying 1.0 units of the index, buy $N_t = 1/e^{d(T-t)}$ units and reinvest in shares as the dividends are received.

∗ This ensures that we will have exactly 1.0 unit at T.

∗ You can verify for yourself that this strategy is *self-financing:*

- The cost of the additional shares you need to purchase each day is exactly equal to the dividend income you will receive.

- (It is exact in continuous time. When the trading interval is $\Delta t = 1$ day it is a close approximation.)

∗ And buying fewer shares today reduces the amount of money we need to borrow.

∗ So the net amount owed at T is $N_t\, S_t\, e^{r(T-t)} = S_t\, e^{(r-d)(T-t)}$.

▶ **Conclude** F cannot be *more than* the cost of transporting shares into the future:

$$\overline{F_{t,T}} = S_t\ e^{(r-d)(T-t)}$$

- Now what about a lower bound?
  - ▶ For that, we have to figure out how much it will cost to sell and *borrow* one share till delivery.
  - ▶ That (in future dollars) must be the lower bound on F. (Call it $\underline{F}$.)
  - ▶ Notice that we are now considering a position that involves **short selling** the stocks: selling something we do not already own.
  - ▶ We assumed above that we could borrow cash using shares as collateral. And borrowing shares using money as cash is just the same transaction from the other side.

- **Sell-and-borrow cost:**
  - ▶ Find someone who owns the index shares and wants to finance their long position at rate r.
  - ▶ Borrow shares from them today; sell shares in the market; No net cash flow.
    - ∗ Use the money received, S_t, to collateralize the stock loan. (The lender will still pay us interest on it.)
    - ∗ So the cash we have at T is $S_t e^{r(T-t)}$.
  - ▶ However, we also owe the stock lender any dividends we receive on his shares.

* As above, we can initally borrow fewer shares, $1/e^{d(T-t)}$, and then, borrow and sell more shares at rate d per unit time to generate the cash that we owe the lender.
* This is just our reinvestment step above, but in reverse.
* When time T arrives we will have borrowed exactly 1.0 units of the index.

- Again, selling fewer shares initially (and earning less interest on the money) reduces the amount we will have at T to $S_t \ e^{(r-d)(T-t)}$.
- **Conclude:** F cannot be *less than* the lower bound:

$$\underline{F_{t,T}} = S_t \ e^{(r-d)(T-t)}$$

* If anyone were willing to sell us the index forward below this price, we would go long, and take the short-spot position in the individual stocks above.
* At date T, we take delivery of the basket of shares from our forward purchase and return it to the lender.

- Of course, since we neglected all transactions costs, upper and lower bound are the same number.

$$F_{t,T} = S_t e^{(r-d)(T-t)}$$

III. An Important Example

- Here are some closing prices for Dow index forwards from one day earlier this winter:

	Date		$(T_n - t)$	Closing	$R_{S/A}$%
t	Nov 08	2017	–	23430	–
T_1	Dec	'17	0.069	23417	1.08
T_2	Jan	'18	0.153	23414	1.18
T_3	Feb	'18	0.238	23403	1.27
T_4	Mar	'18	0.315	23382	1.33
T_5	Jun	'18	0.567	23370	1.47
T_6	Sep	'18	0.819	23380	1.54
T_7	Dec	'18	1.069	23376	1.61
T_8	Jun	'19	1.567	23369	1.67
T_9	Dec	'19	2.069	23357	1.75

- The "Closing" column shows the end-of-day prices of forwards to different settlement horizons, T_n. (The first row is the index itself, or "spot").

- The right column shows the risk-free interest rate (semi-annually compounded) to each date.

- What can we learn from these prices?
  - ▶ Do people expect the Dow to fall over the next two years?
  - ▶ Does that represent "the market's forecast"?

- Here's another table:

	Date		r_{cc}	$\frac{1}{T_n - t}\log(\frac{F}{S})$	diff
T_1	Dec	'17	0.0108	-0.0081	0.0189
T_2	Jan	'18	0.0118	-0.0042	0.0160
T_3	Feb	'18	0.0127	-0.0049	0.0175
T_4	Mar	'18	0.0133	-0.0066	0.0198
T_5	Jun	'18	0.0146	-0.0045	0.0192
T_6	Sep	'18	0.0153	-0.0026	0.0180
T_7	Dec	'18	0.0160	-0.0021	0.0182
T_8	Jun	'19	0.0166	-0.0017	0.0183
T_9	Dec	'19	0.0174	-0.0015	0.0189

- The right-hand column is the difference between the third and fourth columns.
- What do those numbers mean?
  - These are very highly forcastable because the stocks in the index are large companies with very stable payout policies.
- The key point to realize is that there are **3 term-structures** in this data: the futures; the interest rates; and the underlying yield (dividends).
  - *Using the law of no-arbitrage, any two of them must imply the third.*
  - So, given two, **the third contains no information.**
  - In particular, holding interest-rates and total carry costs fixed, futures prices tell us <u>nothing</u> about "expected" spot prices in the future.

IV. More on Shorting Stocks.

- Selling short stocks is a key step in many arbitrage strategies that we will encounter.

- So it is worthwhile to take closer look at how it actually works in real life.

- For now, I want to call your attention to two features of the "stock loan" market.

1. Borrowing fees.

▶ Sometimes it is difficult (or even impossible) to find a stock holder willing to lend shares that you might want to short. In this case, you may be willing to pay an extra fee for the privilege.

  ∗ Usually, that fee is expressed as a continuously compounded percentage of the value of the stock. Call it w.

  ∗ This is then just like an extra dividend stream that the owner of the stock can generate by loaning the shares out.

▶ In practice, we often see very high w values – sometimes over 100%! – for stocks that a lot of people want to short, like "hot" IPOs in the tech sector.

  ∗ Sometimes this accounts for people's apparent willingness to hold stocks that "everybody knows are overvalued"

▶ But the reverse can happen too. There may be shares that no one really wants to short or to hold as collateral. Then we can observe *negative* fees.

  ∗ In that case, w is more like a storage cost for a commodity.

▶ As far as forward prices go, the modification to the formula is straightforward:

$$F_{t,T} = S_t \cdot e^{(r-(d+w))(T-t)}.$$

▶ Like our assumption about the dividend stream, there is an assumption here that you can lock in the rate w for the full period until T.

  ∗ The common practice in the markets, however, is that either borrower or lender can rengotiate w – or even cancel the transaction – at any time.

2. Discrete dividends.

▶ I noted above that when stocks pay dividends the borrower has to pay the dividend to the lender.

  ∗ The lender no longer gets it from the company because, once the borrower sells the stock in the market, the company sees the new buyer as the legal owner of those shares.

▶ For most companies, dividends are paid 1, 2 or 4 times a year. In the U.S., quarterly dividends are the most common.

- For a single stock, then, we should not assume the payments are continuous. How does that affect the forward pricing formula?

- Let's forget about borrowing fees now, and continue to assume the amount and timing of the dividends between t and T is known.

- Consider the net buy-and-hold forward cost $\overline{F}$.

  * Without the dividend, our cost at T is $S_t\ e^{r(T-t)}$ because we have to borrow S_t dollars at t.

  * But if we know the dividends with certainty, we can sell short zero-coupon bonds to each dividend date and that will generate some extra cash today.

  * To be specific, suppose there are N dividends of amount D_n to be paid at dates u_n between now and T.

  * Let $PV(D)$ stand for $\Sigma_{n=1}^{N}\ B_{t,u_n}\ D_n$.

  * This is the amount we would get today from shorting the zero coupon bonds, i.e., borrowing against the future dividends.

  * Then we pay off these short positions with the dividends at each time u_n.

  * This amount reduces the cash we have to borrow to go long the share, and hence the total cost of carrying the position.

- Of course, since we are still ignoring transactions costs, the argument works in reverse for the sell-and-hold cost.
- Pricing result:

$$F_{t,T} = [S_t - PV(D)]\, e^{r(T-t)}$$

- The argument works just as well for other assets with lumpy payouts – like coupon bonds.
- I will leave it for you to think about the impact on the argument of *uncertainty* about the amount and timing of dividends.
  - Sometimes there are surprises in real life.

V. Pricing Commodity Forwards.

- Now let's return to soybeans.
- Can we use our "cash and carry" argument once again?
- At this point we can guess what it would say:

$$F_{t,T} = S_t \; e^{\left\{\left(\begin{array}{l}\text{yield to}\\ \text{holding cash}\end{array}\right) - \left(\begin{array}{l}\text{yield to holding}\\ \text{commodity}\end{array}\right)\right\}(T-t)}$$

  since this is what we derived for currencies and stock indexes.
  - ▶ The yield for holding cash is just r.
  - ▶ The yield for holding the underlying good was its payout rate (assuming it to be continuous, for simplicity).
  - ▶ The difference between them is the <u>net cost of carry</u>.

- And, in fact, this formula <u>sometimes</u> remains true, just with different names for the yield terms.
  - ▶ Dividend yield corresponds to (negative) **storage costs.**
    - ∗ These include cost of spoilage, warehousing, etc.
    - ∗ If we assume that these are (1) proportional to the value of S, and (2) is incurred continuously at rate u, and (3) incurred by the owner (not the borrower), then these are just like an interest rate.

- ▶ Holders of the commodity will also benefit if they can lend it out.

- ▶ Lending fee often called **convenience yield.**

  - ∗ Also referred to as the lease or rental rate.

  - ∗ Again express this cost as a proportional yield, y.

- Then the pricing formula says:

$$F_{t,T} = S_t \cdot e^{(r-y+u)(T-t)}.$$

Q: Why did I say this only holds "sometimes"? What has to be true about the commodity to enforce the arbitrage?

- There are two reasons why the transactions in the cash-and-carry replication may not be possible for a commodity.

- First, to enforce the **lower** no-arbitrage bound, an arbitrageur would need to be able to hold a short position in the commodity until delivery.

  - ▶ In practice it is impossible to borrow many commodities – at any price.

  - ▶ More subtle: units of the commodity need to be perfectly interchangeable.

- Second, to enforce the **upper** no-arbitrage bound, an arbitrageur would need to be able to hold a long position in the commodity until delivery.

- ▶ Not all goods are storable!
  - ∗ Fresh orange juice.
  - ∗ Electricity.

- Our analysis of commodities highlights the limitations of the cash-and-carry argument.

  Arbitrage can enforce the forward price formula if and only if the underlying good can be transported in to the future at a cost that is known today.

- Do you think the argument works for:
  - ▶ Real estate?
  - ▶ Beaujoulais nouveau?

- Just because we cannot price something by no-arbitrage doesn't mean people can't trade it!

VI. Summary

- Cash-and-Carry replication yields arbitrage pricing of forwards, given the spot price, and the relevant payouts and/or interest rates.

- Main assumption: carry costs are known in advance, or term borrowing/lending of cash and the underlying to settlement date are possible.

- Transactions costs turn the no-arbitrage *price* into upper and lower no-arbitrage *bounds*.

- IF no-arbitrage pricing applies, then there is *no extra information* in forward prices about where spot will be in the future.

- Short-selling to execute arbitrage trades requires another step: borrowing the underlying asset.
  - For some underlyings, this may be impossible or illegal.
  - Or it may involve a separate fee, like a rental rate.
  - If borrowing an asset is costly, then owners of it can make extra money by lending it.

- When the no-arbitrage argument fails, forwards are *no longer derivatives*. Instead they become primary securities that help complete the market.

Lecture Note 1.1: Summary of Notation

Symbol	Page	Meaning
F	p2	*delivery price of a forward contract (or "forward price")*
T	p2	*delivery date (or maturity) of contract*
S_T	p5	*spot market price of underlying asset at* T
$F_{t,T}$	p5	*forward price at time* t *for delivery time* T
R	p5	*simple annual risk-free rate*
r	p6	*continuously-compounded risk-free rate per annum*
$r_{t,T}$	p6	*continuously-compounded rate per annum*
		at time t *for a risk-free payment at* T
$B_{t,T}$	p6	*price at* t *of a risk-free zero-coupon*
		bond maturing at T
$\overline{F}, \underline{F}$	p11,14	*upper and lower no-arbitrage forward price bounds*
d	p12	*dividend yield on a stock index*
$R_{S/A}$	p16	*annual riskless rate (semiannually compounded form)*
w	p18	*percentage fee paid to borrow stock*
D_n	p20	*lump-sum dividend to be paid at time* u_n
$PV(D)$	p20	*present value of all remaining dividends between* t *and* T
u	p21	*storage cost for physical commodity*
y	p22	*borrowing fee or "convenience yield"*