FIN 513: Homework #4

Due on Tuesday, February 20, 2018

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Problem 1

(a) By using the formula $u = e^{\sigma\sqrt{\Delta t}}$ and $d = 1/u$, the binomial tree is constructed and by using backward induction, price of call option was calculated as 0.0561506, price of put option was calculated as 0.0659037. Since a straddle is a portfolio of a call and put for long position both, price of straddle is calculated as $0.0561506 + 0.0659037 = 0.122054$.

(b) Using the same tree constructed in (a), straddle price is calculated by same method with each maturity, and plotted as Figure 1. It seems that $\frac{\partial \Pi}{\partial t}$ has negative value since price of straddle is decreasing.

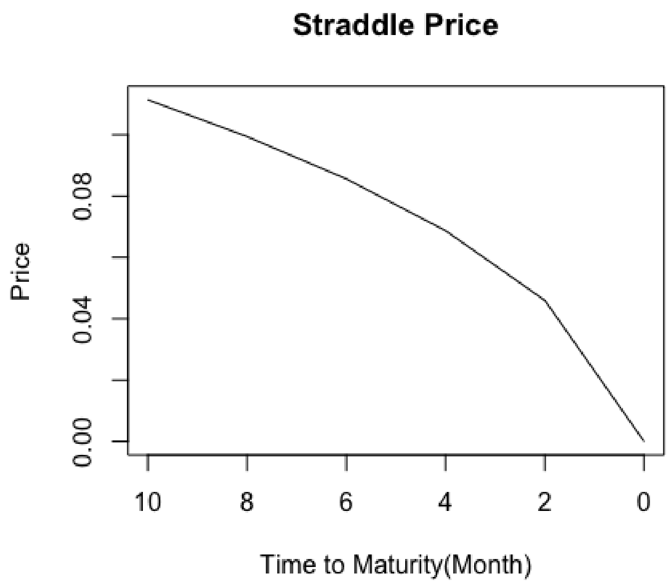


Figure 1: Straddle price

(c) Even though the price of straddle is decreasing on the plot, it does not mean buying a straddle has a negative expected return. It is because the values of straddle corresponding maturity were calculated assuming others are equal. Since straddle has positive payoff unless underlying price at maturity is exactly same as price at initial date, it is likely to make a profit if variation of stock price is high enough. However, since Figure 1 ignores variation of underlying prices, it seems that holding a straddle makes negative return even if it is not true.

Problem 2

Since the payoff at one year depends on payoff at 6 months, in order to value this contract, it is necessary to separate cases with respect to payoff at 6 months. Considering the currency option example used in the lecture notes, the possible profits at 6 months are {20, 10, -2.62}.

i. $C = 20$ *at 6 month:* This case corresponds to exchange rate which becomes 112.75 at $t = 0.5$. Given this, the possible variation of exchange rate at $t = 1$ has just three states. Figure 2 represents the possible states of exchange rate given the rate is 112.75 at 6 months. In this case, since C is already 20,

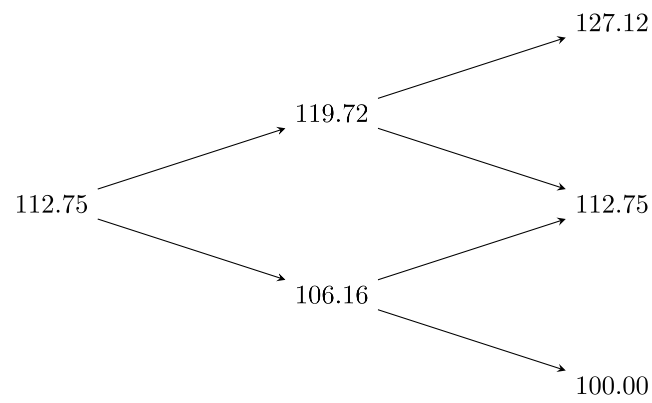


Figure 2: Exchange rate tree given rate 112.75 at $t = 0.5$

and the possible state is always greater than reference level, payoff at $t = 1$ will be zero. Therefore the value of contract at $t = 0.5$ will be automatically 20 given the exchange rate is 112.75.

ii. $C = 10$ *at 6 month:* This case corresponds to exchange rate which becomes 100 at $t = 0.5$. Using the same procedure, it is possible to calculate final payoff, and calculate the value of contract given the rate is 100 at $t = 0.5$. As shown in Figure 3, it was calculated as 15.16.

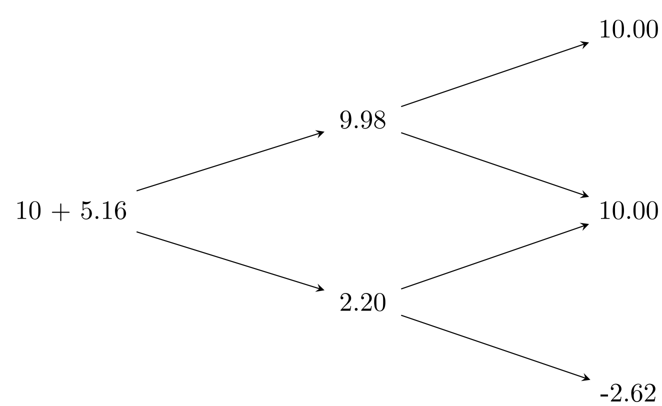


Figure 3: Contract value calculation using tree: given rate is 100 at $t = 0.5$

iii. $C = -2.62$ *at 6 month:* This corresponds to exchange rate which becomes 88.69 at $t = 0.5$. Using the same procedure, contract value can be calculated as Figure 4.

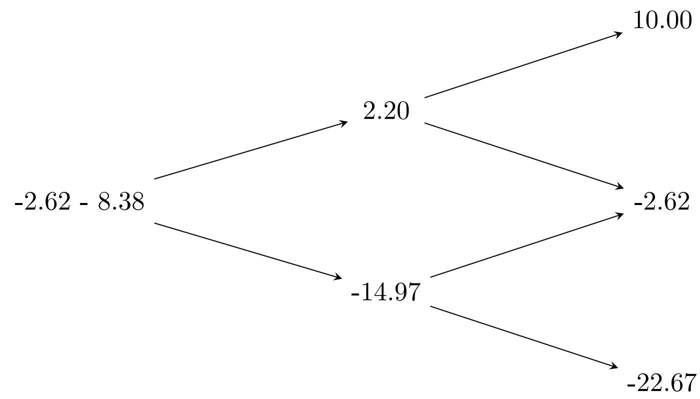


Figure 4: Contract value calculation using tree: given rate is 100 at $t = 0.5$

Combining results above, the value of contract at $t = 0.5$ for all states can be known. Therefore, by constructing another binomial tree, the value of contract at initial date can be calculated by backward induction. Figure 5 shows binomial tree used for calculating value of the contract. The value was calculated as 5.86.

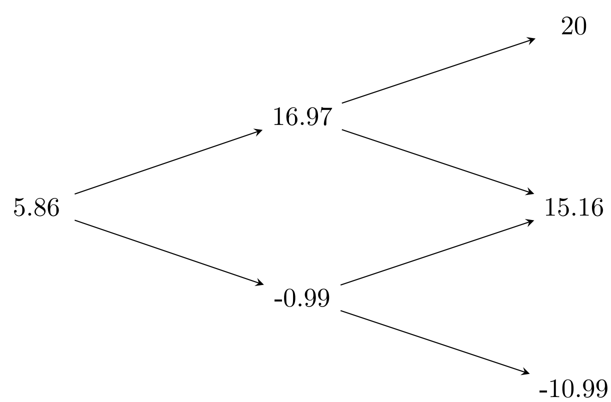


Figure 5: Contract value tree

Problem 3

False. It is not necessary to assume the options are not marked to market. The only assumption to derive options prices under binomial model is that underlying asset and riskless bond are tradable(both long and short) at each steps. Regardless of marking-to-market, it is possible to price option only if we can replicate payoff of options by trading underlying asset and riskless bond at each node.

Problem 4

a. Let $\bar{r}^d$ and $\bar{r}^f$ denote gross return of domestic riskless bond and foreign riskless bond, respectively. Then the risk-neutral probability $\bar{q}$ is calculated as $\bar{q} = \frac{\bar{r}^d/\bar{r}^f - d}{u-d}$.

i. *Assume* $\bar{q} > 1$. Since $u - d > 0$, $\bar{q} = \frac{\bar{r}^d/\bar{r}^f - d}{u-d} > 1$ is equivalent to $\frac{\bar{r}^d}{\bar{r}^f} - d > u - d \Rightarrow \frac{\bar{r}^d}{\bar{r}^f} > u > d$. Using this inequality, it is possible to make an arbitrage profit using the following strategies.

(1) Borrow an foreign currency and exchange to domestic currency.

(2) Invest domestic currency from exchange of foreign currency to riskless bond.

After one period, assume that the exchange rate becomes higher to $S_t \times u$. Then the investor have to pay $S_t u \bar{r}^f$ to the lender, and gets $S_t \bar{r}^d$ from the investment of domestic riskless bond. Therefore, the final payoff of the strategy is equal to $S_t \bar{r}^d - S_t u \bar{r}^f = S_t(\bar{r}^d/\bar{r}^f - u)$. Because we assumed that $\bar{r}^d/\bar{r}^f - u > 0$, and there is no initial cost to make the portfolio, the investor can make an arbitrage profit. Case in which exchange rate becomes $S_t \times d$ is analogous.

ii. *Assume* $\bar{q} < 0$. $\bar{q} = \frac{\bar{r}^d/\bar{r}^f - d}{u-d} < 0$ is equivalent to $\frac{\bar{r}^d}{\bar{r}^f} - d < 0 \Rightarrow \frac{\bar{r}^d}{\bar{r}^f} < d < u$. In this case, it is also possible to make an arbitrage profit using the following strategies.

(1) Borrow S_t amount of domestic currency and exchange into a foreign currency.

(2) Invest a foreign currency to riskless bond.

Then after one period, assuming the exchange rate becomes to $S_t \times d$, the investor will get $S_t d \bar{r}^f$ and have an obligation to pay $S_t \bar{r}^d$. Therefore, the final payoff the strategy is equal to $S_t d \bar{r}^f - S_t \bar{r}^d = S_t(d - \bar{r}^d/\bar{r}^f)$, which is positive. Therefore, the investor can make an arbitrage profit since there is no initial amount of investment. Case in which exchange rate becomes $S_t \times u$ is analogous.

b. First, let us assume that exchange rate y becomes $y \times u$ or $y \times d$ after one month. Assuming the expect return of investing in Afghani is zero, in order to match the expected return, the following equation should hold.

$$y = 0.5 \times yu + 0.5 \times yd$$

Cancelling y out on both sides and rearranging the terms, equation $d = 2 - u$ is obtained. Using the result, in order to match standard deviation, we can construct the following equation.

$$0.5 \times (u-1)^2 + 0.5 \times (d-1)^2 = 0.5 \times (u-1)^2 + 0.5 \times (1-u)^2 = (\frac{0.1}{12})^2$$
$$\Rightarrow (u-1)^2 = (\frac{1}{120})^2$$
$$\Rightarrow u = 1 \pm \frac{1}{120}$$

Since $u > d$, $u = 1 + \frac{1}{120} = 1.0083$ and $d = 2 - u = 0.9917$. By using the given interest rate to calculate risk-neutral probability $\bar{q}$, it is calculated as $\bar{q} = \frac{1.02/1.08 - 0.9917}{1.0083 - 0.9917} = -2.8467$, which is not between 0 and 1, does not satisfy the condition derived in (a). Therefore, if the value of u and d calculated above are used for constructing tree, arbitrage occurs.

Problem 5

a. Using the formula $u = e^{\sigma \Delta t}$, $d = 1/u$, a tree is constructed(Figure 6). However, since volatility changes at each nodes, u and d changes(as shown on Figure 7), it does no longer recombines, consequently. Since

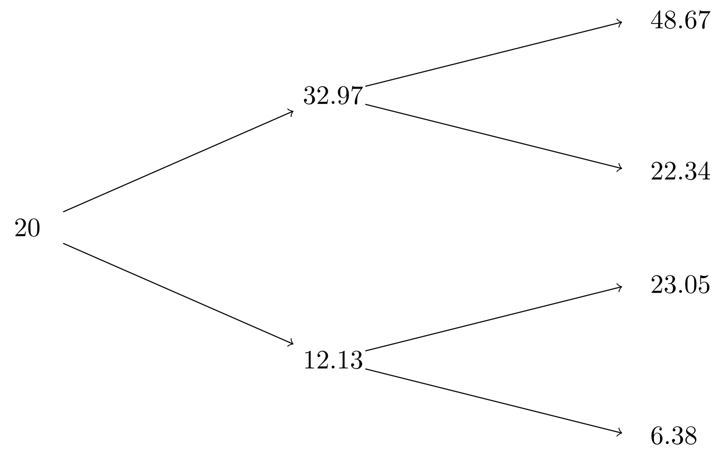
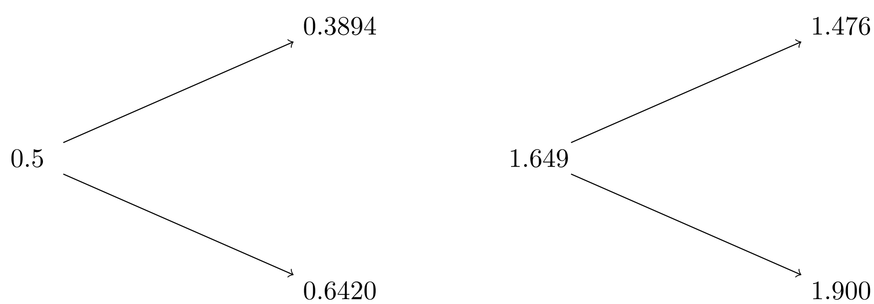


Figure 6: Stock price tree



(a) Volatility Tree (b) u Tree

Figure 7: Volatility and u

there are 4 states at period 2, there are four possible returns and corresponding probabilities. Table 1 represents returns and probabilities at each states. From the table, since returns and probabilities at each state are given, it is possible to calculate expected return and variance. Expected return is calculated as $\mathrm{E}(r) = \sum_{i=1}^{4} r_i p_i = 0.889 \times 0.3025 + 0.111 \times 0.2475 + 0.142 \times 0.2475 - 1.142 \times 0.2025 = 10.03\%$, Variance is calculated as $\mathrm{Var}(r) = \sum_{i=1}^{4} (r_i - \mathrm{E}(r))^2 = 50.13\%$. Therefore, standard deviation is equal to $\sqrt{50.13}\% = 70.81\%$, and annualized variance and volatility is equal to 25.07% and 50.07%, respectively.

b. Since the volatility changes at each node, risk neutral probability q would change at each nodes, and it affects value of the option. Figure 8 represents risk-neutral probabilities at each node. Using risk-neutral probabilities at each node and backward induction formula $c = \frac{1}{1+r_f}(q \times c_u + (1-q) \times c_d)$, it is possible to value the call option. (c = value of option at previous node, c_u, and c_d represents value of option at up-

Stock Price	*Return(r)*	*Probability(p)*
48.67	0.889	0.3025
22.34	0.111	0.2475
23.05	0.142	0.2475
6.38	-1.142	0.2025

Table 1: Returns and probabilities at each state

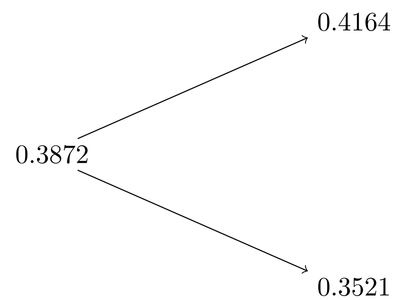


Figure 8: Risk-neutral probabilities

state and down-state. r_f is risk-free rate.) Figure 9 shows valuation process using the backward induction procedure. From the tree, the option value is calculated as 5.696 when volatility is not constant. In order

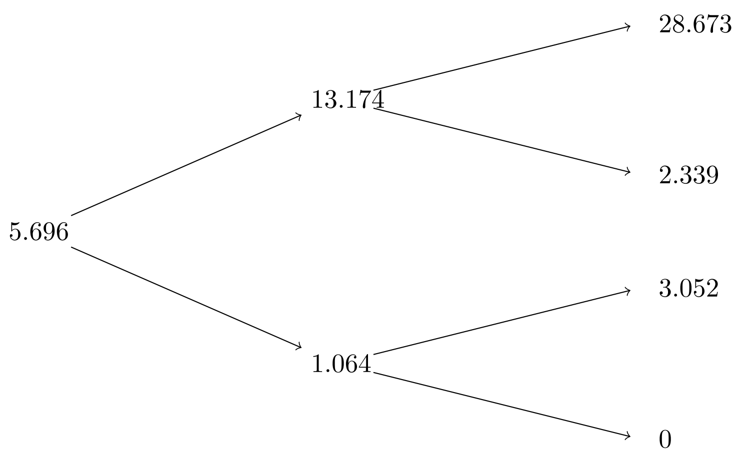


Figure 9: Option price tree

to compare to value of option with constant volatility, stock price tree(Figure 10) is constructed using volatility calculated from (a). Using the same backward induction method, option price was calculated as in Figure 11. The value calculated is 5.055, which is smaller than option price using non-constant volatility.

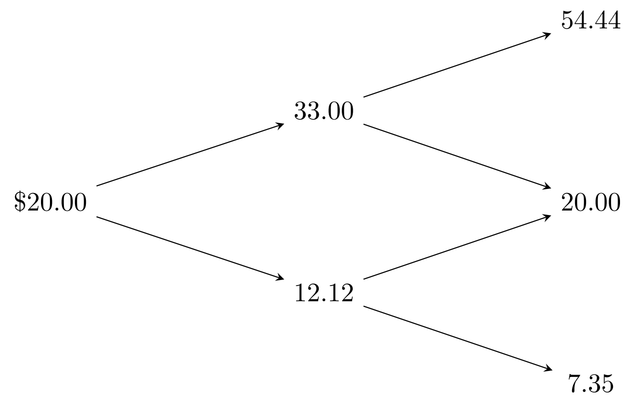


Figure 10: Stock price tree: constant volatility

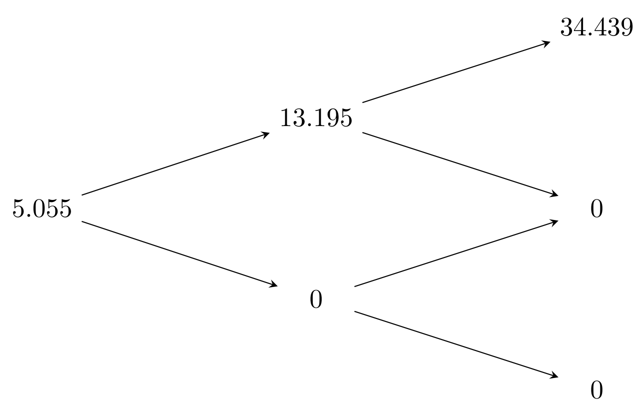


Figure 11: Option price tree: constant volatility