FIN 513: Homework #4

Due on Tuesday, February 20, 2018

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Problem 1

Problem 2

Problem 3

False. It is not necessary to assume the options are not marked to market. The only assumption to derive options prices under binomial model is that underlying asset and riskless bond are tradable(both long and short) at each steps. Regardless of marking-to-market, it is possible to price option only if we can replicate payoff of options by trading underlying asset and riskless bond at each node.

Problem 4

a. Let $\bar{r}^d$ and $\bar{r}^f$ denote gross return of domestic riskless bond and foreign riskless bond, respectively. Then the risk-neutral probability $\bar{q}$ is calculated as $\bar{q} = \frac{\bar{r}^d/\bar{r}^f - d}{u-d}$.

i. *Assume* $\bar{q} > 1$. Since $u - d > 0$, $\bar{q} = \frac{\bar{r}^d/\bar{r}^f - d}{u-d} > 1$ is equivalent to $\frac{\bar{r}^d}{\bar{r}^f} - d > u - d \Rightarrow \frac{\bar{r}^d}{\bar{r}^f} > u > d$. Using this inequality, it is possible to make an arbitrage profit using the following strategies.

(1) Borrow an foreign currency and exchange to domestic currency.

(2) Invest domestic currency from exchange of foreign currency to riskless bond.

After one period, assume that the exchange rate becomes higher to $S_t \times u$. Then the investor have to pay $S_t u\bar{r}^f$ to the lender, and gets $S_t\bar{r}^d$ from the investment of domestic riskless bond. Therefore, the final payoff of the strategy is equal to $S_t\bar{r}^d - S_t u\bar{r}^f = S_t(\bar{r}^d/\bar{r}^f - u)$. Because we assumed that $\bar{r}^d/\bar{r}^f - u > 0$, and there is no initial cost to make the portfolio, the investor can make an arbitrage profit. Case in which exchange rate becomes $S_t \times d$ is analogous.

ii. *Assume* $\bar{q} < 0$. $\bar{q} = \frac{\bar{r}^d/\bar{r}^f - d}{u-d} < 0$ is equivalent to $\frac{\bar{r}^d}{\bar{r}^f} - d < 0 \Rightarrow \frac{\bar{r}^d}{\bar{r}^f} < d < u$. In this case, it is also possible to make an arbitrage profit using the following strategies.

(1) Borrow S_t amount of domestic currency and exchange into a foreign currency.

(2) Invest a foreign currency to riskless bond.

Then after one period, assuming the exchange rate becomes to $S_t \times d$, the investor will get $S_t d\bar{r}^f$ and have an obligation to pay $S_t\bar{r}^d$. Therefore, the final payoff the strategy is equal to $S_t d\bar{r}^f - S_t\bar{r}^d = S_t(d - \bar{r}^d/\bar{r}^f)$, which is positive. Therefore, the investor can make an arbitrage profit since there is no initial amount of investment. Case in which exchange rate becomes $S_t \times u$ is analogous.

b. First, let us assume that exchange rate y becomes $y \times u$ or $y \times d$ after one month. Assuming the expect return of investing in Afghani is zero, in order to match the expected return, the following equation should hold.

$$y = 0.5 \times yu + 0.5 \times yd$$

Cancelling y out on both sides and rearranging the terms, equation $d = 2 - u$ is obtained. Using the result, in order to match standard deviation, we can construct the following equation.

$$0.5 \times u^2 + 0.5 \times d^2 = 0.5 \times u^2 + 0.5 \times (2-u)^2 = \frac{0.1}{12} \approx 0.008$$
$$\Rightarrow u^2 - 2u + 2 - 0.008 = 0$$
$$\Rightarrow u = \frac{2 \pm \sqrt{4 - (1.992)^2}}{2} = 1.089 \text{ or } 0.910$$

Since $u > d$, $u = 1.089$ and $d = 2 - u = 0.910$. By using the given interest rate to calculate risk-neutral probability $\bar{q}$, it is calculated as $\bar{q} = \frac{1.02/1.08 - 0.910}{1.089 - 0.910} = 0.1924$, which is between 0 and 1, satisfies the condition derived in (a). Therefore, if the value of u and d calculated above are used for constructing tree, it is possible to match the moments of exchange rate.

Problem 5

a. Using the formula $u = e^{\sigma \Delta t}$, $d = 1/u$, a tree is constructed(Figure 1). However, since volatility changes at each nodes, u and d changes(as shown on Figure 2), it does no longer recombines, consequently. Since

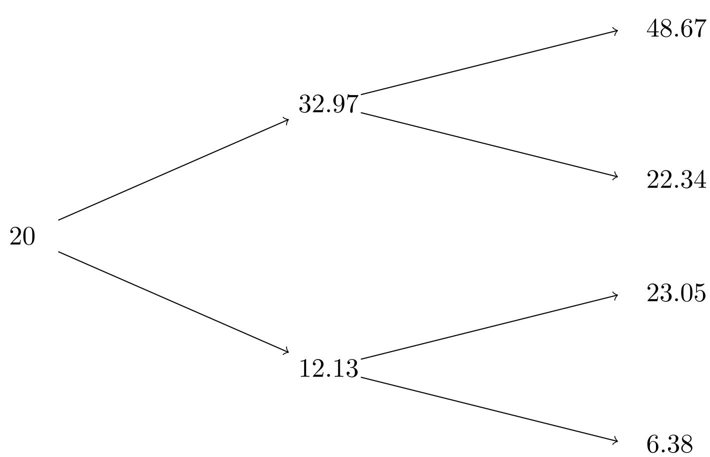
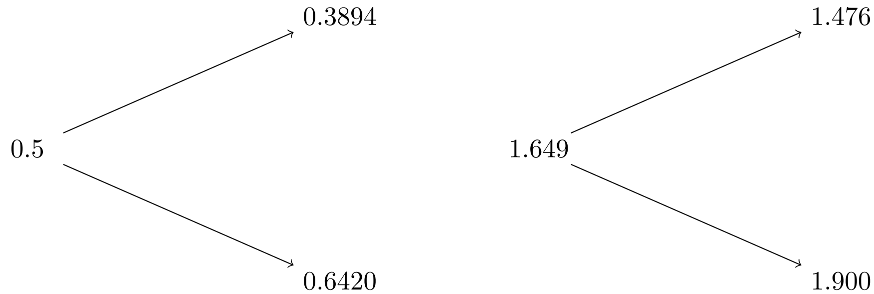


Figure 1: Stock price tree

there are 4 states at period 2, there are four possible returns and corresponding probabilities. Tables 1 represents returns and probabilities at each states. From the table, since returns and probabilities at each state are given, it is possible to calculate expected return and variance. Expected return is calculated as $\mathrm{E}(r) = \sum_{i=1}^{4} r_i p_i = 1.434 \times 0.3025 + 0.117 \times 0.2475 + 0.153 \times 0.2475 - 0.681 \times 0.2025 = 36.25\%$, Variance is calculated as $\mathrm{Var}(r) = \sum_{i=1}^{4} (r_i - \mathrm{E}(r))^2 = 59.33\%$. Therefore, standard deviation is equal to $\sqrt{59.33}\% = 77.03\%$, and annualized variance and volatility is equal to 29.67% and 54.47%, respectively.



(a) Volatility Tree (b) u Tree

Figure 2: Volatility and u

Stock Price	*Return(r)*	*Probability(p)*
48.67	1.434	0.3025
22.34	0.117	0.2475
23.05	0.153	0.2475
6.38	-0.681	0.2025

Table 1: Returns and probabilities at each state

b. Since the volatility changes at each node, risk neutral probability q would change at each nodes, and it affects value of the option. Figure 3 represents risk-neutral probabilities at each node. Using risk-neutral

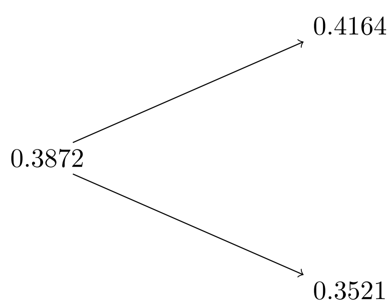


Figure 3: Risk-neutral probabilities

probabilities at each node and backward induction formula $c = \frac{1}{1+r_f}(q \times c_u + (1-q) \times c_d)$, it is possible to value the call option. (c = value of option at previous node, c_u, and c_d represents value of option at up-state and down-state. r_f is risk-free rate.) Figure 4 shows valuation process using the backward induction procedure. From the tree, the option value is calculated as 5.696 when volatility is not constant.

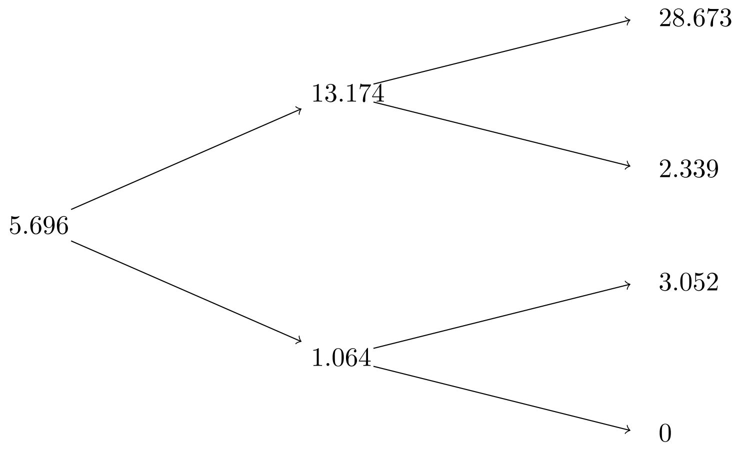


Figure 4: Option price tree

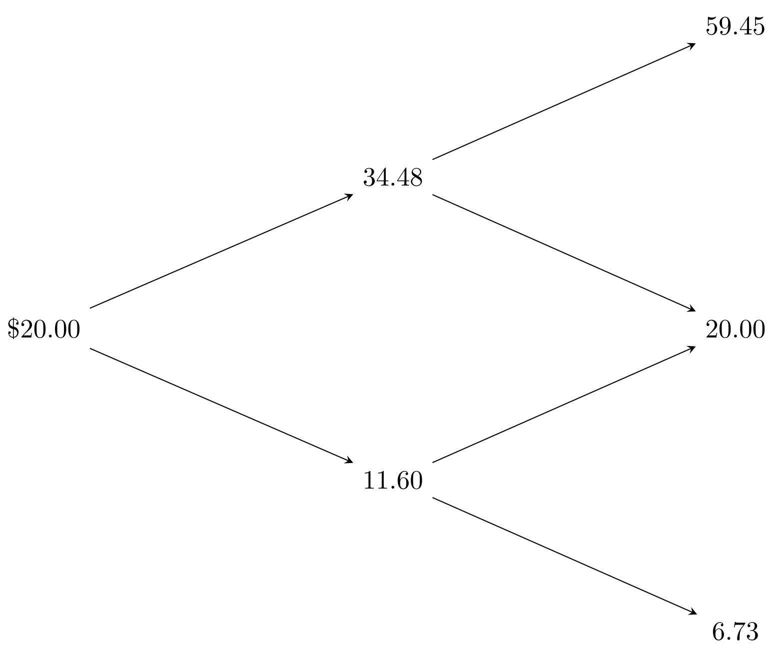


Figure 5: Stock price tree: constant volatility

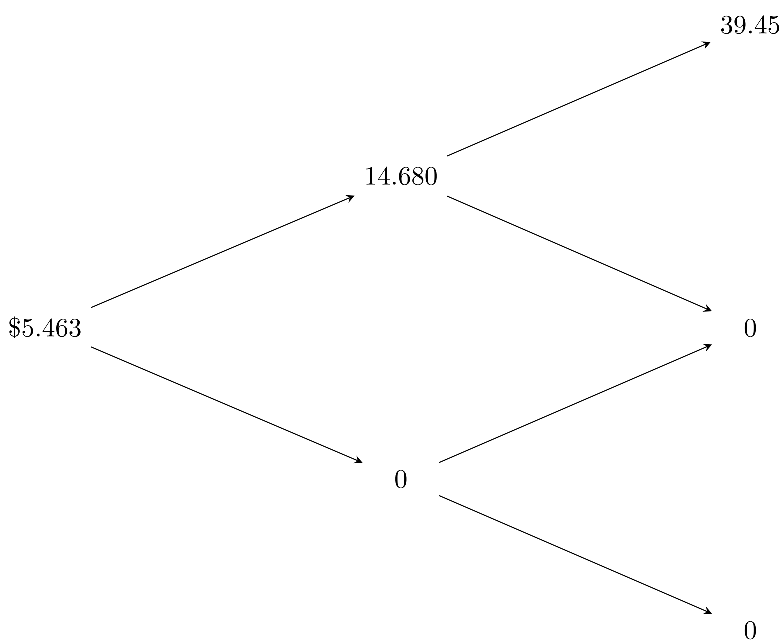


Figure 6: Option price tree: constant volatility