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Lecture Note 7.2: Multiple Sources of Risk

Introduction:

This lecture generalizes the lessons we learned in developing models with stochastic volatilty. The techniques we used there can be used to value derivatives with any number of underlying sources of randomness. This will let us tackle several new types of problems.

Once we have formulated the problems, we also have to learn how to solve them. We will talk about some issues of implementation. Most crucially, we will discuss how the idea of risk-neutral pricing extends to the multidimensional case. This will give us an extremely powerful tool.

In the last part of the lecture, we will see how to apply this tool through some examples, and also discuss some limitations.

Outline:

I. Classes of Multidimensional Problems
II. Implementability Issues
III. Probabalistic Solutions to General Models
IV. Applications
V. Limitations of Monte Carlo
VI. Summary

I. Some classes of multidimensional problems.

- Last week we put a lot of effort into figuring out how to cope with a second source of risk in derivatives pricing. Now let's step back and recognize that how useful that was.

  There are many other situations in which we will want to analyze multiple sources of risk.

- Here are a few.

(A) Other random parameters.

- The motivation for our investigation last time was to address the sensitivity of the B-S formula to the constant volatility assumption – within a consistent framework.

- We could now proceed the same way to examine the other parameter assumptions, like constant interest rates.

- As before, our method will be to specify new stochastic processes for the thing we previously assumed was constant:

$$\begin{aligned} dS &= \mu\, S\, dt \;+\; \sigma\, S\, dW^S \\ dr &= m(\cdot)\, dt \;+\; b(\cdot)\, dW^r. \end{aligned}$$

  (We're keeping σ constant for simplicity.)

- Now, we have our option, C, and it varies with both S and r. So just as we did before, we use the two-dimensional Itô rule to write down how it changes. Then we get rid of the two types of risk, just like we did with stochastic volatility. Then ....

- Hey, the whole argument is the exact same!
- So we already know what the PDE will turn out to be:

$$\frac{1}{2}b^2C_{rr} + \rho b\sigma SC_{rS} + \frac{1}{2}\sigma^2S^2C_{SS} + rSC_S - rC + C_t + [m - \lambda b]C_r = 0.$$

- This is the exact same equation we got last week but with r substituted for σ in the partial derivatives.
  - ▶ And now ρ is the correlation between dW^S and dW^r;
  - ▶ and $\lambda = \lambda^r$ is the **market price of interest-rate risk**.
- The point is, when we derived the stochastic volatility PDE, our argument never really made reference to what σ <u>was</u>.
- So now we know how to handle *any* type of random parameter.

- Recall how the whole thing worked:
  - ▶ The key was there had to be a second security out there that we could use to (continuously) hedge the new risk.
  - ▶ And we have to know the new risk's market price, λ.

(B) Derivatives on Non-traded Underlyings

- Our analysis so far has been concerned with the effect of random parameters in valuing derivatives on some asset S.

- But our methods point us to a further generalization:

  - ▶ We could price derivatives whose payoff also depends on the random parameter.

  - ▶ Our no-arbitrage arguments still works even if there is no traded underlying security – no S – at all.

- For example, interest rate or volatility derivatives.

  - ▶ If r is the ONLY random thing affecting the payoffs of some claim, then our PDE is even simpler:

$$\frac{1}{2}b^2 C_{rr} - rC + C_t + [m - \lambda b]C_r = 0.$$

    Note: all the S terms are gone.

- *Our approach can value derivatives on any quantity.*

- The "underlying can be anything. It doesn't even have to have any connection to finance. It could be:

  - ▶ Weather derivatives.

  - ▶ Catastrophe derivatives.

  - ▶ A firm's accounting results (e.g., sales).

  - ▶ Whatever you can imagine...

(C) Higher Dimensional Models

- Another direction in which we can exploit the work we have done is to add still more types of risk.

- We have shown how it works for $N = 2$ variables, but nothing stops us from building general $N-$dimensional models with $N = 3, 4, \ldots$

- For example, we could handle:

  - Currency options with stochastic volatility and domestic and foreign interest rates ($N = 4$).

  - Multifactor term-structure models, e.g. bond prices that depend on both r and inflation.

  - Options on portfolios with N correlated assets.

- Solving the PDE takes more computer power. But it can be done.

- One goup of researchers has solved a pricing problem in 360 dimensions!

  - The product was a collaterallized mortgage obligation (CMO).

  - This is a security that pools together cash-flows from many underlying loans and re-sells them in packages.

  - They wanted to let the re-payment rates on each of the underlying mortgages be its own random process.

- We can summarize what happens to the no-arbitrage PDE when we add a new factor, X, as follows:

  1. Add a second derivative term $\frac{1}{2}X^2 C_{XX}$ times the squared volatility of X.

  2. For each other factor, Y, add a cross-derivative term XYC_{XY} times the covariance between X and Y.

  3. Add a first derivative term C_X, whose coefficient is

    (i) X times the risk-free rate minus the payout rate **if X is a traded factor**;

    (ii) the drift of X minus the product of the market price of X risk with the diffusion coefficient of X, **if X is a non-traded factor**;

- The derivation of these rules just follows the lines of the argument we went through for the stochastic volatility case.

  ▶ Recall that "X is a traded factor" means that there is a security that we can buy and sell whose price is X. Any other risk factor is "non-traded".

- It is starting to look like we can handle almost unlimited complexity.

II. Implementability Issues

- Before getting carried away, let's just keep track of the practicalities of employing an N-dimensional model on, perhaps, arbitrary quantities.
  1. You have to write down a stochastic model for each random quantity that you believe.
  2. You have to have estimates of the unknown parameters of that model that you trust.
  3. There has to be at least one other security in the market that you can buy and sell to hedge the risk of each new quantity.
  4. The risk of non-hedgeable jumps in the new quantity must be negligible.
  5. You have to have some way of finding the market price of risk for each non-traded quantity.
- Let's think about each of these for a moment.
- Issues 1 and 2 are really about econometrics.
  - ▶ For the models to be <u>improvements</u> over simpler models, we have to be confident that:
    - ∗ Our specification of the model is a better description of the true underlying process that we are trying to describe – as it will evolve in the future; and

- ∗ We can estimate (and forecast) the new parameters required with a high degree of confidence; and
- ∗ Our uncertainty about them has smaller effects on pricing and hedging than a simpler model with fewer parameters.

▶ This will not always be the case. In fact there are dangers from two directions.

i. We can't estimate the new model's parameters with any accuracy at all.

ii. We estimate them "too well" and end up overfitting.

▶ When we have N sources of risk, and N models for them, we could have around $N^2/2$ new parameters to estimate.

- ∗ For each new factor we include, we have to specify its correlation with each old one. Then we have to estimate means, variances, and covariances.

▶ The task of *model validation* is an important part of risk management in any financial engineering exercise.

- ∗ Firms that use proprietary models typically impose formal procedures for assessing both *estimation risk* and *specification risk*.

- Issues 3 and 4 on my list above concern the realism of the continuous trading/hedging assumption.

- Economists call this the *market completeness* condition.

- It is unrealistic if

  * There is not a way to create a pure hedge for each source of risk.

    - For example, an individual homeowner's mortgate repayment decision.

  * The hedge cannot be undertaken from both sides: long and short.

  * Market liquidity is not sufficient to allow frequent re-hedging (without affecting prices).

• This brings us to issue 5. Recall our earlier discussion of λs.

- Once our model involves them, we are no longer in the (nice) position of being able to ignore economics for valuing derivatives.

- We need to know something about people's risk/reward preferences for each type of risk.

- There is no reason why people's preferences can't change over time!

  * Our PDE allows us to incorporate changing market prices of risk, by making them functions of other state variables.

  * Sometimes economic theories can provide guidance.

* **Example:** What does the CAPM assert about the market price of risk for any factor X?

  - CAPM: $\pi_X = \beta_{X,M}\ \pi_M$.

  - Definition: $\beta_{X,M} = \rho_{X,M}\sigma_X/\sigma_M$

  - Hence: $\lambda_X = \rho_{X,M}\ \lambda_M$

▶ I'm not saying this is true! But it is an alternative to computing historical or implied lambdas as we discussed last time.

• To summarize the implementability issues, multidimensional models are exciting, but not always feasible in practice.

• Hence, *more sophisticated models require more sophisticated scrutiny.*

• More complicated isn't necessarily better.

III. Probabalistic Solutions to General Models

- I now want to show you one solution technique for the general multi-dimensional PDE we have developed.
- The idea is to extend to the general case the correspondence between solutions and discounted expectations using suitably adjusted probabilities.
  - ▶ Often, it is possible to find these expectations directly.
  - ▶ Even when it's not, the intuition is important.
- How do we go about "risk-neutralizing" these more complex models?
- To review, when we considered the Black Scholes PDE, we learned that one way to find the solution was *to average all possible terminal payoffs (discounted at the riskless rate), after having adjusted the drift on the underlying process.*
- Specifically, we set the expected percentage change to the risk-free rate (or that rate minus the payout rate).
  - ▶ We interpreted that adjustment as switching to an alternative (false) model in which the stock price was determined by risk-neutrality. Hence we called the pricing approach "taking the discounted risk-neutral expected payoff."
- But recall the subtle underpinnings of the logic that allowed us to reach this conclusion.

- Then, the justification for this result was that the equation we had to solve made no reference to people's preferences for risk, so that risk neutral people (who value all cashflows as their expected present value) must reach the same solution as anybody else. So we could pretend we were risk-neutral.

- Now, the more general equation we have to solve *does* involve people's preferences. That's what that λ was.
  - ▶ So it's not clear whether we should expect to get away with a similar trick.

- Well, luckily for us, there is a theorem in mathematics that tells us precisely what to do.

- I'll describe it in terms of the stochastic volatility model we saw last time. Recall the two processes we specified were

$$\begin{aligned} dS &= \mu S\, dt + \sigma S\, dW^S \\ d\sigma &= \kappa(\sigma_0 - \sigma)\, dt + s_0\sigma\, dW^\sigma. \end{aligned}$$

  and the equation we want to solve is:

$$\frac{1}{2}s_0^2\sigma^2 C_{\sigma\sigma} + \rho s_0\sigma^2 S C_{\sigma S} + \frac{1}{2}\sigma^2 S^2 C_{SS} + rSC_S - rC + C_t + [\kappa(\sigma_0 - \sigma) - \lambda s_0\sigma]C_\sigma = 0.$$

- The result we need is a version of what is called the **Feynman-Kač Theorem.**

- This theorem tells us how to “risk-neutralize” any risk factor to get an adjusted model that we can simulate in order to compute prices.

- The message of FK is:

  The value of any security that satisfies the PDE is the expectation of its payoffs (discounted at the riskless rate), after each of the underlying processes has had its drift changed to **whatever multiplies the first partial derivative term with respect to it in the PDE.**

- In other words, here

(A) Set the drift of the stock to $r \cdot S$, as before (which is the same as setting its expected return to r).

(B) Set the drift of σ to $m() - \lambda b()$.

(C) Don't change anything else. Most notably, this means the volatilities and correlations don't get changed.

- Then all you have to do is figure out the average payoff in present value terms.

- The theorem tells us, once we have shown that our derivative satisfies the PDE, **we can conclude that that adjusted average payoff is the solution.**

- Before looking at examples, I want to call your attention to one subtle point about the recipe. *If the interest rate itself is one of the random variables,* then there is a difference between *the expected payoff, discounted* and *the expected discounted payoff.* The theorem tells us we are supposed to use the latter:
  - ▶ The discount factor goes inside the expectation.

- To put it in terms of equations, let X stand for the vector of all our state variables.

  Then, if the derivative has terminal payoff $C(X_T)$, and $\mathrm{E}^{\star}$ denotes averaging over future outcomes with the adjusted drifts, then the result says its price at t is

  $$\mathrm{E}^{\star}[DF_{t,T} \cdot C(X_T)]$$

  where

  $$DF_{t,T} \equiv e^{-\int_t^T r_u \, du}$$

  is the integrated discount factor.

- Putting it inside the expectation takes into account the possibility that the evolution of X may be correlated with that of r.

- If there are cash-flows $\Gamma_s(X_s)$ at times $t < s < T$, then the formula is

  $$\mathrm{E}^{\star}[\int_t^T DF_{t,s} \cdot \Gamma_s \, ds + DF_{t,T} \cdot C(X_T)]$$

  - ▶ Recall that these cash-flows (such as interest payments) just tack on the additional term Γ to the PDE.

- If we were simulating the process over intervals Δt then the discount factor would be the same as

$$\left(e^{-r_{t_0}\Delta t} \times e^{-r_{t_1}\Delta t} \times e^{-r_{t_2}\Delta t} \times \ldots e^{-r_{t_N}\Delta t}\right)$$

which is approximately equal to

$$\left(\frac{1}{(1+r_{t_0}\Delta t)} \times \frac{1}{(1+r_{t_1}\Delta t)} \times \frac{1}{(1+r_{t_2}\Delta t)} \times \ldots \frac{1}{(1+r_{t_N}\Delta t)}\right)$$

i.e the product of all the little one-period discount factors that will apply between now and T.

- Or, in English, **when interest rates are random, the discount factors between t and T is the product of all the future one-period discount factors multiplied together** – after adjusting the drift of r.

- Of course, if we are using a model with constant r, that is just $e^{-r(T-t)}$.

- Also notice that, if rates are random – but they are independent of the state variables X – then we can observe that

$$\begin{aligned} \mathrm{E}^\star[DF_{t,T} \cdot C(X_T)] &= \mathrm{E}^\star[DF_{t,T}] \cdot \mathrm{E}^\star[C(X_T)] \\ &= B_{t,T} \cdot \mathrm{E}^\star[C(X_T).] \end{aligned}$$

So we don't have to model the bond prices if they don't interact with the variables that determine our cashflows.

- Let's continue with the stochastic volatility example, so that you get the idea of how to use the Feynman-Kač result.

- Our general formula (for a call) requires us to compute

$$B_{t,T} \cdot \mathrm{E}^{\star}[\max(S_T - K, 0)]$$

But it is not clear how to compute that expectation mathematically.

- So just simulate! Here's how it goes.

(A) Adjust the stock and volatility processes according to the rule.

$$\begin{aligned} dS_t &= rS_t\,dt \;+\; \sigma_t S_t\,dW^S \\ d\sigma_t &= [\kappa(\sigma_0 - \sigma_t) - \lambda s_0 \sigma_t]\,dt \;+\; s_0 \sigma_t\,dW^{\sigma}. \end{aligned}$$

(B) Approximate the continuous time model by a discrete time version, i.e. replace dt with Δt and dW^{σ} and dW^S with two normal random variables with mean zero, variance Δt and correlation ρ.

(C) Draw one realization of the random sequences.

(D) Compute the terminal payoff, $\max(S_T - K, 0)$, for this one outcome.

(E) Multiply by the discount factor.

(F) Average the result over a whole lot of draws.

- This is the **Monte Carlo** method of evaluating expectations.

IV. Applications.

- Sometimes we don't need to simulate. We can compute the adjusted expectations directly.

(A) Forwards.

- Go back to the simple, one-variable model. One fact you need to know is that, if we have any process described by the model
$$\frac{dX}{X} = c_0\,dt + c_1\,dW$$
where c_0 and c_1 are constant, then, if we are at time t, for any horizon T, the expected value of X_T is
$$\mathrm{E}[X_T] = X_t e^{c_0(T-t)} \quad \text{or} \quad \frac{1}{(T-t)} \log \frac{\mathrm{E}[X_T]}{X_t} = c_0.$$

- Of course if X is an arithmetic Brownian motion, $dX = d_0\,dt + d_1\,dW$, then $\mathrm{E}[X_T] = X_t + d_0(T-t)$.

▶ **Example:** A stock pays no dividends and has a constant 9% expected return and constant volatility. Today it is at 100. What is its expected price in 10 years?

∗ The model here is $\frac{dS}{S} = 0.09\,dt + \sigma\,dW$

∗ So the formula says $\mathrm{E}[S_T] = 100 \cdot e^{0.09 \cdot 10} = 246.$

- We can use these recipes for expectations to immediately derive some derivatives prices.

- Suppose we want to price an outstanding forward contract with forward price F_0 on a non-dividend paying stock that follows that same type of model.
  - ▶ Assume r is constant.
  - ▶ Then using our recipe, we first change the expected return to r, and then compute the expected discounted payoff

    $$\begin{aligned} V_F &= \mathrm{E}^\star[e^{-r(T-t)}(S_T - F_0)] \\ &= e^{-r(T-t)}(\mathrm{E}^\star[S_T] - F_0) \end{aligned}$$

    where the $\mathrm{E}^\star$ notation is to remind us that this isn't our true expectation.
  - ▶ Now use the result we just learned to evaluate $\mathrm{E}^\star$:

    $$V_F = e^{-r(T-t)}(S_t e^{r(T-t)} - F_0) = B_{t,T}(F_{t,T} - F_0)$$

    which is a formula we saw in week one.
  - ▶ Also, if we solve for the forward rate that makes V_F zero, it's given by $F_0 = S_t e^{r(T-t)}$, which we also knew.
- Old stuff there. But what about a forward on, say, the consumer price index? We couldn't do that in week one. But now we at least have a framework for approaching it.

  Let π_t denote the price level index.
  - ▶ First, we are going to suppose π follows

    $$d\pi = \iota\pi\, dt + \sigma\pi\, dW.$$

That says, essentially that ι is the expected rate of increase, which is the same as the inflation rate. And now we are supposing ι and σ are constant.

- How do we risk-neutralize this thing? What's our recipe?

- Since the index itself isn't traded, our generic formula say just change the drift to $m - \lambda b$, or, here,

$$\pi\iota - \lambda\pi\sigma$$

- So this makes the adjusted price level process

$$\frac{d\pi}{\pi} = (\iota - \lambda\sigma)\, dt + \sigma\, dW$$

- And this is still just a constant-coefficient lognormal process **assuming λ is a constant.**

- Suppose it is. What is the forward worth?

- Do the same steps as for the stock, replacing S by π:

$$\begin{aligned} V(\pi_t, t) &= \mathrm{E}^\star[e^{-r(T-t)}(\pi_T - F_0)] \\ &= e^{-r(T-t)}(\mathrm{E}^\star[\pi_T] - F_0) \\ &= e^{-r(T-t)}(\pi_t e^{(\iota-\lambda\sigma)(T-t)} - F_0) \end{aligned}$$

which is a new result.

- Setting this to zero to solve for the current forward price

$$F = \pi_t e^{(\iota-\lambda\sigma)(T-t)}.$$

- Notice some interesting things about this formula:

  1. **If we know the true process (ι and σ), the forward price would give us λ.** This is an instances of the observation we made last time that we may not need to estimate market prices of risk if we take the prices of other derivatives as given.

  2. **The forward price is the expected spot price if and only if $\lambda = 0$.** This conclusion holds whether or not the process is lognormal. Also, it holds for traded underlyings too. We have already seen that, for currencies, for example:

  $$\lambda = (\mu + r_f - r_d)/\sigma$$

  So $\lambda = 0$ means $\mu = r_d - r_f$ which means

  $$F = S_t \cdot e^{(r_d - r_f)(T-t)} = S_t \cdot e^{\mu \cdot (T-t)} = \mathrm{E}[S_T].$$

(B) Bonds.

- Consider pricing a zero-coupon riskless bond in a one state-variable model.

- The simplest model of interest rates people use is the **Vasicek model**

$$dr = \kappa(\bar{r} - r)\, dt + b\, dW$$

- This is called an Ornstein-Uhlenbeck model. It has several convenient features, including the fact that, given today's rate r_t, the distribution of the rate at every future date is normal:

$$r_T \sim N(r_t\, e^{-\kappa(T-t)} + \bar{r}(1 - e^{-\kappa(T-t)}),\ \frac{b^2}{2\kappa}(1 - e^{-2\kappa(T-t)})\ ).$$

- The risk-neutralized version of the Vasicek process is

$$\begin{aligned} dr &= [\kappa(\bar{r} - r) - \lambda b]\, dt + b\, dW \\ &= \kappa(r^\star - r)\, dt + b\, dW \end{aligned}$$

  where $r^\star \equiv \bar{r} - \lambda b/\kappa$ is the new "long run mean" that results from our risk-neutralization.

- The payoff function for our bond is <u>always</u> one. So, according to our formula, its price must be

$$B_{t,T}(r_t) = \mathrm{E}^\star[DF_{t,T} \cdot 1]$$

  where $DF_{t,T}$ is the riskless discount factor to time T.

- Now that rates are random, this discount factor will be different along each path that r takes.

  ▶ So we need to use

$$DF_{t,T} = e^{-\int_t^T r_u \, du}$$

- So our pricing law just tells us that *zero coupon bond prices are the risk-neutralized expectation of this discount factor.*

- It happens that some mathematician already worked out this expectation for this particular model. The solution is

$$B_{t,T}(r_t) = e^{G_0(t,T) - r_t \, G_1(t,T)}$$

  where

$$\begin{aligned} G_1(t,T) &= \frac{1}{\kappa}(1 - e^{-\kappa(T-t)}) \\ G_0(t,T) &= \frac{1}{\kappa^2}(G_1(t,T) - T + t)(r^\star \kappa^2 - b^2/2) - \frac{(bG_1(t,T))^2}{4\kappa} \end{aligned}$$

  which describes a complete term-structure model in terms of the spot rate and four parameters.

- Do you see how you could now use this to try to extract λ from the government bond curve?

- For more complicated payoffs (like swaptions or interest-rate options) there aren't closed-form formulas.

- But even without formulas, it is still really simple to just program up a routine to simulate future paths of r under the adjusted model, compute the discount factor along each future path, and take the average payoff.
  - ▶ In Vasicek, all we do is change the long-run spot rate before simulating.
  - ▶ **Example:** The volatility of rates is 400 BP/year ($b = 0.04$), the mean reversion, κ, is 30% and the market price of interest rate risk is 15%.
    - $*$ The true long-term rate is 6%.
    - $*$ What is the "risk-adjusted" long term rate?
    - $*$ $r^\star \equiv \bar{r} - \lambda b/\kappa = 0.06 - (0.15\ 0.04/0.30) = 0.04 = 4\%$.
- The point is, the risk adjustment is easy.
- Notice we have totally put the partial differential equation in the background. We're just evaluating some average outcomes for particular random processes.
- So pricing bonds is easy!

(**C**) Energy Swaps.

- In many countries there are now deregulated power markets enabling "spot" trading of electricity. However this does not mean electricity is a "traded risk factor" in the sense that we have been using the term.
  - ▶ You cannot buy *and hold* spot electricity in a portfolio.
  - ▶ There is no feasible way to store it – at least with current technology.
  - ▶ And, of course, if it cannot be stored, it can also not be loaned/borrowed. So short selling is impossible.
- However there are still important markets for electricity derivatives, like swaps and options on swaps.
- Even if these cannot be hedged with spot positions, our theory can still price them as long as there is more than one – and that we can find the appropriate market price of risk.
- Suppose we want to compute the no-arbitrage swap price for electricity to be delivered between T_1 and T_2.
- Two important features that we will need to take into account are (i) seasonality in prices, and (ii) unpredictable long-term price trends.

- So suppose we model the spot price, P, as $f_t + X_t$ where f is the deterministic seasonal mean and X is an OU process with a stochastic long-run mean:

$$dX_t = \kappa(\bar{X}_t - X_t)\; dt \;+\; s\, dW_t$$
$$d\bar{X}_t = \;\bar{s}\, d\bar{W}_t.$$

(Assume these are the specifications under the risk-neutral measure.)

- Then the no-arbitrage fee paid continuously from T_1 to T_2 must equate:

$$\varphi \int_{T_1}^{T_2} e^{-ru}\, du \qquad \text{and} \qquad \int_{T_1}^{T_2} \mathrm{E}_0^{\star}[P_u] e^{-ru}\, du$$

- The expectation in the right-hand expression is $f_u + \mathrm{E}_0^{\star}[X_u]$.

- We know how to compute the expectation of an OU process with constant $\bar{X}$. That is just the expression we saw above for the mean of the Vasicek process.

- We can find the expectation with stochastic $\bar{X}_t$ by observing that $Y = X - \bar{X}$ obeys the simple equation $dY = -\kappa Y\, dt + \sigma_Y\, dW^Y$.

  - ▶ Here I am using the fact that the diffusion terms of dY are $s\, dW_t - \bar{s}\, d\bar{W}_t$ and the sum of two Brownian motions is still a Brownian motion.

  - ▶ In this case, $\sigma_Y^2 = s^2 + \bar{s}^2 + 2\rho s\ \bar{s}$, but we will not need this.

- So now we know $\mathrm{E}_0^\star[X_u] = \mathrm{E}_0^\star[\bar{X}_u + Y_u] = \mathrm{E}_0^\star[\bar{X}_u] + \mathrm{E}_0^\star[Y_u] = \bar{X}_0 + Y_0 e^{-\kappa u} = \bar{X}_0(1 - e^{-\kappa u}) + X_0 e^{-\kappa u}$ – which, perhaps surprisingly, is the same as for nonstochastic $\bar{X}_t$.

- Completing our problem, we find

$$\varphi = \frac{\int_{T_1}^{T_2} [f_u + \bar{X}_0 + (X_0 - \bar{X}_0)e^{-\kappa u}] e^{-ru} \, du}{\int_{T_1}^{T_2} \ e^{-ru} \, du}$$

- This is the sum of (i) the long run level today, (ii) the weighted average seasonal mean over the delivery period, and (iii) a weighted mean-reversion term equal to

$$(X_0 - \bar{X}_0) \frac{\int_{T_1}^{T_2} e^{-(r+\kappa)u} \, du}{\int_{T_1}^{T_2} \ e^{-ru} \, du}$$

- As with forward contracts, swap prices can be used to back out the market price of electricity risk if we know the correct (true) model.

- Another realistic feature of electricity markets is *seasonal volatility*.

  ▶ Starting from the true (not risk neutral) model, what would seasonal (deterministic) volatility do to the computations above?

- Some researchers have suggested that the market price of energy risk might itself be seasonal.

(D) Path-Dependent Options

- Monte Carlo techniques can easily be used to value types of exotic derivatives we haven't even tried to price before.

- A path-dependent option is a claim whose payoff is a function of what happened to the underlying over the time period t to T. The most popular examples are

  Asian Options. These are options which pay the difference between the *average* price of the underlying over the time period and the strike. Alternatively, *average strike options* gives you the right to buy/sell the underlying at the average.

  Lookback Options. Are similar except, instead of the right to buy/sell at the average, they confer the right to sell/buy the the high or low. So a lookback put essentially guarantees you the ability to "sell at the top".

- Now we have to be careful before trying to value these things: *Are we sure their values obey our PDE?* And, if so, what are the state variables and what are their prices of risk?

  ▶ Remember, we are only allowed to apply our expectations recipe if we know the value of our claim is a function of some continuous, Brownian motion type processes, and hence satisifies the PDE.

  ▶ For example, the right to buy at the low has payoff of $S_T - L_T$ where L_T is the low between t and T. It is NOT an Ito diffusion.

- ▶ Luckily, $Z_t \equiv S_t - L_t$ IS a type of diffusion, called a *reflected Brownian motion*, which is a diffusion.
- ▶ And, although Z_t is not the price of a traded asset, it is instantaneously perfectly correlated with S_t. This means that there is really only one source of randomness. (So we do not need to find a λ for Z, for example).

- The tricky thing about these claims is that you have to keep track of the whole history of the underlying's price to know what they are worth at the end.

- But for Monte Carlo methods this is no problem at all! You are already building entire future paths. So it's no extra work (or very little) to figure out what the high was, say, over each path.

- So the payoffs are just as simple to compute as for plain calls and puts. And again all you do is compute a whole bunch of them and average the (discounted) result.

(E) Options on Strategies

- Exotic options like those above still just have one underlying. But we can handle much more than that.

- Consider an option on a portfolio of stocks *whose weights change over time.*

- Suppose you are a fund manager and you want to insure a portfolio whose holdings (shares of each stock) is held fixed. Then your percentage allocations vary with their performances.

- For example, suppose there are just two stocks A and B with prices S_A and S_B, and you hold N_A shares of A and N_B shares of B.

- Then, even if they are independent, the volatility of your portfolio is
$$\sqrt{\frac{N_A^2 S_A^2 \sigma_A^2 + N_B^2 S_B^2 \sigma_B^2}{N_A S_A + N_B S_B}}$$
which is a complicated, time varying function of S_A, and S_B.

- But we know how to price options on it! Just set the expected returns of both stocks to r (no prices of risk here) and start simulating.

- You can extend this idea to price derivatives on *arbitrary* time-varying strategies, as long as you can specify the portfolio weights as functions of the state variables.

(F) Structured Notes

- Structured notes are just ordinary bonds with exotic features determining the "interest" payment and/or the principal repayment.

- Usually the *credit* risk in these is minimal because the issuer will have secured a back-up guarantee from a AAA entity.

- Some examples:

Equity appreciation notes. Popular with retail investors, these notes determine their quarterly coupon based upon the appreciation on some basket of stocks over the quarter. The payment might look like

$$\Gamma_{t_i} = \min(10\%, \max(0, (S_{t_i}/S_{t_{i-1}} - 1)))$$

which also features a cap (ten percent) and a floor (zero). Each coupon thus has the features of an option (or option spread) whose strike price is determined in the future. Options like this are called **cliquets**.

Power notes. These pay a floating coupon linked to *some function of* the rate r (or else LIBOR) such as r^3.

Range notes. The coupon payment on range notes ceases (or else may be capped) if some underlying index goes out of a certain range.

Rainbow notes. These pay off in one of several different currencies, depending on which has appreciated most.

$$F(T) = \max_{n=1:N}(S_T^{(n)}/S_0^{(n)}).$$

One recent issue specified that it would pay off in the *second* most appreciated currency, but only if at least one other currency had depreciated!

- Even with the complex path dependency in these products, simulation methods easily give their payoffs.
- As you simulate each path forward, you just compute each contingent payoff Γ_{t_i} at the coupon dates t_i, multiply them by the discount factor to that date DF_{t_i} and add this amount to the running total for that path.

V. Problems with the Probabalistic Technique

- Before you jump to the conclusion that you now know how to handle any derivatives problems, I have to give you the bad news: Monte Carlo techniques have a number of serious practical drawbacks.

- Roughly, these fall into two categories: computational difficulties, and inherent limitations.

- The computational difficulties arise because the averages you calculate are, by definition, *approximations*. That means there is always approximation error. That means you have to (a) keep track of the error; and (b) make it small.

- In fact there are two sources of approximation error:
  - ▶ The error that comes from only being able to approximate an expectation with an average of a finite number of draws; and

  - ▶ The error that comes from approximating a continuous-time process by a discrete-time one.

- In principle, we can deal with both issues with enough computer power. Just scale up the number of paths, and scale down the time interval Δt.

- And, in fact, the second type of error declines quickly as you shrink the time interval. It is essentially zero when you use $\Delta t =$ one day.

- But, an unfortunate mathematical law tells us that the first type of error goes down at a slow rate. Specifically, the error in the average for a payoff function $g(S_T)$ with M simulations of S is approximately

$$\sqrt{\frac{\mathrm{Var}[g(S_T)]}{M}}.$$

The numerator is the variance of the payoffs experienced over the M draws. It doesn't change (much) with M. The problem is the denominator only gets large very slowly.

- For instance, if you start with 1000 paths and discover that your error is 10%, and your boss tells you to get it down to 1%, you have to draw

$$\frac{\text{old error}}{\text{new error}} = \frac{.10}{0.01} = 10 = \sqrt{\frac{\text{new } M}{\text{old } M}}$$

which means new $M = 100\times$ old M or 100,000 draws!

- Anyhow, there are various numerical tricks for making things better, which we won't go into. The situation can be summarized by saying

  Monte Carlo techniques are still too slow for valuing lots of derivatives at a time, or doing so in real-time.

- On the other hand, they are definitely the method of choice for handling one-shot valuation of specific securities, especially very complicated ones.

- However the nature of derivatives problems also imposes some inherent limitations on the use of Monte Carlo.

- There are two things you need to be aware of:
  1. **It's very hard to value American options with Monte Carlo.** This seems strange, given all the other exotics we can do. But Americans are different, because the early exercise floor imposes a boundary condition that depends on the value of the derivative at the time. If you think about a grid, there we valued by going backwards in time. So at each date we knew what the option's value was. But now we are trying to value by simulating forward in time. And, at intermediate time-steps, we don't have any way of knowing what the derivative will turn out to be worth.
  2. **Monte Carlo only delivers the price of the derivative.** For nearly every derivative application, we need to know more than the price. We need all the hedge ratios, to tell us how to replicate it, or take away our risk. Monte Carlo tells you what the thing is worth starting from the world right now. It can't tell you what it would be worth if things changed a little bit (like a small move in the underlying). To get those, we just have to change the starting conditions a little bit *and run the whole thing again.* If we want to know hedge ratios with respect to 5 quantities (perhaps volatility, interest rates, time, and two underlyings), we have to repeat the process 5 times.
- So if you were wondering why we had to go through all that tedium with partial differential equations, now you know. Sometimes "risk-neutral" pricing doesn't help much.

VI. Summary

- We learned that, in principle, we can handle derivative problems involving any number of sources of risk. Three applications we saw were
  1. Random parameters in the model.
  2. Non-traded underlyings.
  3. Multiple underlyings.

- In practice, these models are tricky to fit and require strong assumptions. **Use the simplest model you can get away with.**

- The method of pricing derivatives by taking "risk-neutral" discounted expectations applies to all problems which can be described in terms of our fundamental PDE
  - ▶ We can't do this with any arbitrary model and any arbitrary derivative.

- The general transformation is to change the <u>expected returns</u> of all traded processes to r (or $r - y$), and to change the <u>drifts</u> of all non-traded processes to $m - \lambda b$. Also:
  - ▶ Volatilities and correlations don't get changed.
  - ▶ The discount factor goes inside the expectation.

- When we can't figure out the exact expectation, we can approximate it by Monte Carlo simulation.

Lecture Note 7.2: Summary of Notation

Symbol	Page	Meaning
dr	p2	*change in instantaneous risk-free rate over* dt
$m(), b()$	p2	*drift and diffusion functions of* dr
ρ, λ	p3	*correlation between* r *and* S*, and market price of* r *risk*
m, b, ρ, λ	p4	*analogous parameters when payout* dy *is random*
X	p8	*an arbitrary risk factor obeying a diffusion process*
C_X, etc	p8	*sensitivity* $(\partial C/\partial X)$ *of security* C *to* X
$DF_{t,T}$	p15	*discount factor to apply to random payouts at time* T
		when future interest rates are unknown
$\mathrm{E}^\star[\ ]$	p15	*expectation using risk-neutralized processes*
$C(), \Gamma()$	p15	*payoff and cash-flow as a function of state-variables* X
c_0, c_1	p19	*expected growth and volatility of generic process* dX
$\mathrm{E}[\ ]$	p19	*expectation using true model for* X
V_F	p20	*value of an old forward contract at forward price* F_0
		when the current spot price is S_t *at* t
π, ι	p20	*consumer price index and inflation rate*
σ, λ	p21	*volatility of price index and market price of* π *risk*
κ, r_0	p23	*mean-reversion rate and long-run level of riskless rate* r
b, λ	p23	*diffusion coefficient of riskless rate; market price of* r *risk*
$r^\star$	p23	*risk-neutralized long-run level of riskless rate*
$\int_t^T r_u\,du$	p23	*sum of future instantaneous rates between* t *and* T
G_0, G_1	p24	*coefficient functions in Vasicek bond solution*
P	p27	*spot price of electricity*
f	p27	*seasonal mean of electricity spot price*
Z_t	p29	*difference between* S_t *and the low,* L_t*, between 0 and* t
S_A, S_B, N_A, N_B	p31	*prices and number of shares of two stocks* A *and* B
Γ_{t_i}	p32	i*th coupon payment of structured note.*
$F(T)$	p33	*Principal repayment on rainbow note*
$\mathrm{Var}[g(S_T)]$	p35	*variance of realized payoffs* $g(S_T)$ *in a sample of* M