University of Illinois, Urbana-Champaign *Professor Tim Johnson*
FIN 513 - Spring 2018 *tcj@illinois.edu*

Lecture Note 4.1: Introduction to Dynamic Arbitrage

We now introduce our first stochastic model of asset prices. This is something we did not do in the first part of the course.

By assuming that the underlying price moves in a particular way, we will be able to find a *dynamic trading strategy* that replicates an option's payoff. Even though the replicating portfolio changes through time, it won't involve us injecting more capital. So its value is just the cost of the initial positions we have to take. Then, by no-arbitrage, that must also be the value of the option it replicates.

The model at first seems simple-minded, but is, in fact, extremely flexible and powerful, and it is used in practice for real financial engineering applications.

Outline:

I. The Nature of the Model

II. Hedging and Arbitrage in a Two-State World

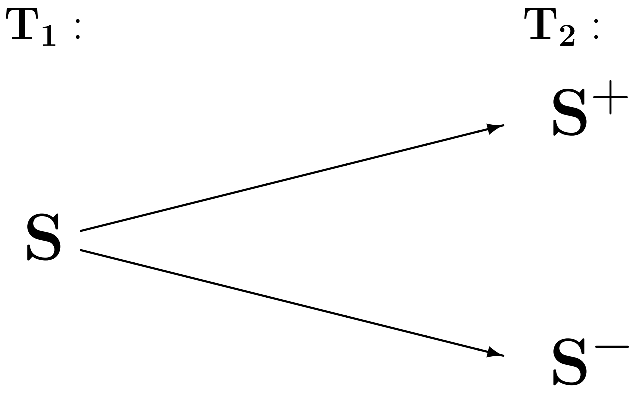
III. General Options Pricing Formulas

IV. The State-Price Density

V. Summary

I. The Nature of the Model

- The binomial model postulates that time is divided up into discrete periods, and, at each instant in time, the price of the underlying asset (call it a stock) can only go up to a specified higher price or down to a specified lower one. It can't stay the same. It can't do anything else.



- This sounds absurd, until you realize that the length of the time step could be interpreted as half a second. And the stock step size could be one "tick". Then, looking one minute ahead, the permitted behavior is quite plausible.

- What determines whether it goes up or down?

- It's random
  - ▶ At each instant in time, Nature flips a coin.
  - ▶ The coin has probability π of landing "up" and $1 - \pi$ of landing "down".
  - ▶ Each flip is independent of the ones that came before.

- The reason the binomial model is really so flexible is that you can make the grid (or **tree**) that the stock traverses have differing step sizes, and you can allow the probability π to be different at different points (or **nodes**) on the grid.

- Our examples will use grids having constant <u>percentage</u> step sizes.

  ▶ We fix numbers $u > 1$ (say 1.10) and $d < 1$ (say 0.90), and then allow the stock to move from S to either
  $u \cdot S$ or $d \cdot S$.

  ▶ So, for example, after four "ups" and two "downs" the stock will have gotten to $u^4 d^2 S$.

  ▶ Notice that you get to the same place regardless of the order that the ups and downs come in. This is called a **recombining tree.**

  ▶ In terms of continuously compounded returns (log differences), our grid has fixed absolute step sizes $\log u$ and $\log d$.

$\mathbf{T_1}$: $\qquad\qquad\qquad\qquad$ $\mathbf{T_2}$:

$$\begin{tikzcd} & \textbf{log S + log u} \\ \textbf{log S} \arrow[ur] \arrow[dr] & \\ & \textbf{log S + log d} \end{tikzcd}$$

- Once we fix the tree, the choice of π determines the mean and variance of the stock price (and/or its returns) at each step.

$$\mathrm{E}[S_{t+1}] = \pi S_t u + (1-\pi) S_t d \quad \text{so}$$
$$\frac{\mathrm{E}[S_{t+1}]}{S_t} \equiv \bar{\mu} = \pi u + (1-\pi) d$$

$$\mathrm{E}\left[\log(S_{t+1}/S_t)\right] = \pi \log u + (1-\pi) \log d$$
$$\mathrm{Var}[\log(S_{t+1}/S_t)] = \pi(1-\pi)[\log(u/d)]^2$$

- The last formula follows from the law:

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - \mathrm{E}\left[X\right]^2$$

and a little algebra.

- Suppose π is fixed for a tree of N time steps, each of length $\Delta t \equiv T/N$. There will be $N+1$ possible final prices S_N.

  - ▶ Pick one of the terminal nodes. What is the probability that S ends up there?

- We can compute the probability of ending at the point $u^j d^{N-j} S_0$ by multiplying the probability of any one path that ends there (which is just $\pi^j(1-\pi)^{N-j}$) times the number of such paths.
- There turn out to be

$$\frac{N!}{j!(N-j)!}$$

  of them, where

$$N! \equiv N \cdot (N-1) \cdot (N-2) \cdots 2 \cdot 1.$$

- So the formula for the distribution (with $j = 0, 1, \ldots N$) is:

$$\text{Prob}(S_N = u^j d^{N-j} S_0) = \frac{N!}{j!(N-j)!} \pi^j (1-\pi)^{N-j}.$$

- This is called the N-step binomial distribution.

- ▶ On a log scale, the individual returns are independent and identically distributed. So it's easy to calculate the mean and variance of $\log S_N/S_0$:

$$\mu_N \equiv \mathrm{E}\left[\log(S_N/S_0)\right] = \mathrm{E}\left[\sum_{k=0}^{N-1} \log(S_{k+1}/S_k)\right]$$

$$= N \cdot (\pi \log u + (1-\pi)\log d).$$

$$\sigma_N^2 \equiv \mathrm{Var}[\sum_{k=0}^{N-1} \log(S_{k+1}/S_k)] = N \cdot (\pi(1-\pi)[\log(u/d)]^2).$$

- ▶ In other words, **the mean and variance of returns are just their one-period values times the number of periods.**

- ▶ Remember that, for sums of independent random variables, you sum the variances, but not the standard deviations.

- **Example.** Suppose the time interval is a day, and the stock can only go up or down by 1% each day, and the probability that it goes up is 51%.

  What are the mean and variance of returns per day and per year?

Parameters: $\pi = 0.51, u = 1.01, d = .99,$

$N = 256$ (= number of trading days per year. And a nice number.)

One-day mean: $0.51 \log(1.01) + 0.49 \log(0.99) = 0.00015.$

One-year mean: $256 \times 0.00015 = 0.0384$

= 3.8% expected return per year = μ.

One-day variance: $.51\,.49\,[\log(1.01/.99)]^2 = .51\,.49\,.02^2 = 0.00010.$

One-year variance: 0.0256.

Usually we are interested in the square root of this (which is 0.16).

- For the rest of the course, **the annualized standard deviation of returns will be called the volatility.** denoted σ.

- Frequently, in practice, we will want to set up a grid to model an asset with a given mean and volatility. So, with N/T fixed, we will want to come up with u, d and π to match μ and σ at horizon T.
  - ▶ Since we have three things to play with, there is an extra degree of freedom. We can use it to simplify calculations.
    - ∗ Set $d = 1/u$.
  - ▶ Now have two equations in two unknowns:

$$N \log(u)(2\pi - 1) = \mu\, T, \quad N[\log(u^2)]^2 \pi(1-\pi) = \sigma^2\, T$$

- Solution (you don't need to remember this):

$$
\begin{aligned}
\ln(u) &= \sigma\sqrt{\frac{T}{N}}\left(1+\frac{\mu^2 T}{\sigma^2 N}\right)^{1/2} \\
\ln(d) &= -\ln(u) = -\sigma\sqrt{\frac{T}{N}}\left(1+\frac{\mu^2 T}{\sigma^2 N}\right)^{1/2} \\
\pi &= \frac{1}{2}+\frac{\mu}{2\sigma}\sqrt{\frac{T}{N}}\left(1+\frac{\mu^2 T}{\sigma^2 N}\right)^{-1/2}
\end{aligned}
$$

- For large N/T this is approximately:

$$
u = e^{\sigma\sqrt{\Delta t}} \approx 1+\sigma\sqrt{\Delta t}
$$
$$
\pi = \frac{1}{2}\left(1+\frac{\mu}{\sigma}\sqrt{\Delta t}\right)
$$

  where $\Delta t \equiv T/N$ is the length of each time step.

- **Example.** Build a grid with $u = 1/d$ for a stock with expected return 20% p.a. and volatility 40% p.a. with weekly time-steps.

  Parameters: $\mu = 0.2, \sigma = 0.40, \Delta t = 1/52 = 0.0192$ years.

  $\mathbf{u} \; = \; 1 + .4 \cdot \sqrt{0.0192} = 1.055$ so $d = 0.947$.

  $\pi \; = \; 0.50\,(1 + (.2/.4)\sqrt{0.0192}) = 0.5346.$

- Now suppose that we divide time up more finely, i.e. increasing N while keeping T fixed.
  - If we keep adjusting u and π as above we keep the mean and variance per unit time fixed.
  - The Central Limit Theorem guarantees that the time-T distribution of the return (i.e. $\log(S_T/S_0)$) converges to a normal distribution:

$$\log(S_T/S_0) = r_T \sim \mathcal{N}\left(\mu T, \sigma^2 T\right).$$

  - This holds for any time horizon, not just the end, since there are an infinite number of steps in any small interval.
  - In fact, the percentage change in the stock from any t to any $s > t$ is $\mathcal{N}\left(\mu(s-t), \sigma^2(s-t)\right)$. And, of course, each instant's change is still independent.
  - In the limit, the process looks the same no matter how small the time interval. It is infinitely wiggly and continuous.
  - We describe this process by a **stochastic differential equation.**

$$\frac{\Delta S}{S} \approx \frac{dS}{S} = \mu\, dt \;+\; \sigma\, dW.$$

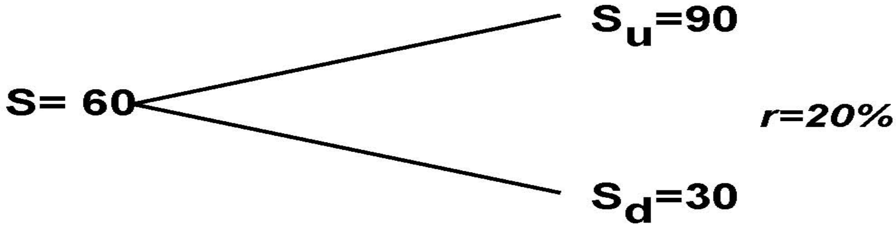
    - $*$ the dt is an infinitely small Δt.
    - $*$ dW is the random part of the change:$\sim \mathcal{N}(0, dt)$.
    - $*$ The cumulatives sum, W, is called a **Wiener process**.
  - This S process is a **geometric Brownian motion**.
  - we will come back to it often.

II. Hedging and Arbitrage in a Binomial World

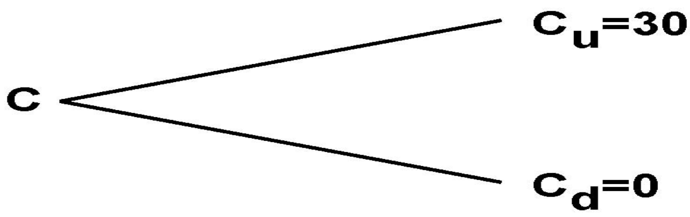
(A) The one-step argument.

Consider the problem of valuing a call in a binomial world when there is only one time step remaining before expiration.

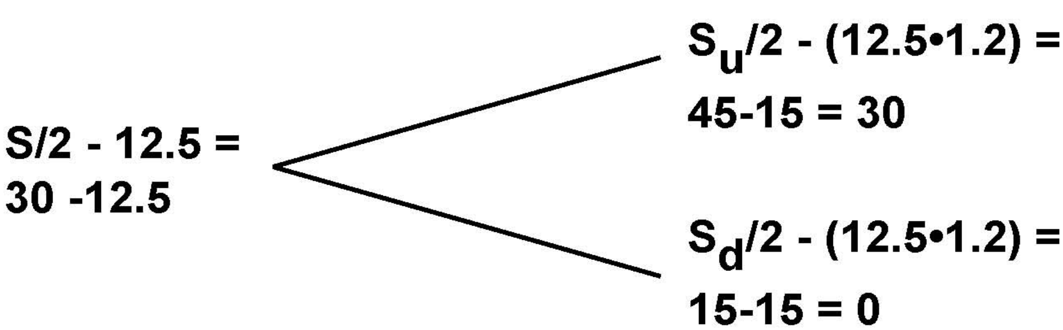
- Say the current per-period interest rate is 20%, the current stock price is 60 and the stock price can either fall to 30 or rise to 90.



- At the end of the period a call with a strike price of 60 will be worth either 0 or 30.



- Suppose we buy one-half share and borrow \$12.50.



- Since this portfolio represents the payoff to the call, the call must have the same value as the portfolio.

$$C = \frac{S}{2} - 12.50 = 30 - 12.50 = 17.50$$

- Therefore, this call is worth **17.50.**

- **What if the call option is trading at \$18.50?**

  - ▶ We then have an arbitrage opportunity, since we can replicate the payoff of the call option by spending only \$17.50. Specifically,

    - ∗ Sell one call option;

    - ∗ Buy the replicating portfolio of stocks and bonds for the call option; That is, buy one half share of the stock and borrow \$12.50.

  - ▶ The cashflow at time 0 is \$1.00.

  - ▶ The payoff from the replicating portfolio, by design, will exactly cover the written-option obligation.

- Before formalizing all this in mathematics, we should consider some deep questions about this simple little set-up.

1. How did we guess the replicating portfolio?

- We need to solve for the replicating portfolio by using the conditions that the portfolio generate the same payoffs as the options.

- Suppose a portfolio of δ shares of stocks plus L in borrowing can replicate the call payoff.
  If the stock goes up to \$90,

$$90\delta - 1.2L = 30; \quad (1)$$

  If the stock goes down,

$$30\delta - 1.2L = 0. \quad (2)$$

- Solving (1) and (2) will give us $\delta = 0.5$ and $L = 12.5$.

- Interpreting δ.

  - ▶ The portfolio should have the same sensitivity to the stock price as that of the call.

  - ▶ This is the hedge ratio or the "delta" of the call price with respect to the stock price:

$$\delta \equiv \frac{\Delta C}{\Delta S} = \frac{30 - 0}{90 - 30} = \frac{1}{2}.$$

  - ▶ (It's just a coincidence that it happened to be 1/2 with the numbers here.)

  - ▶ Note: some people (e.g. Hull) use Δ instead of δ.

2. Would the argument work in a trinomial world?

3. Why doesn't the option price $C(S_t)$ depend on the probability π?

- Mechanical Answer: To figure out a replication, you have to match cash flows in every possible state. But the probabilities of those states don't enter into the construction.

3.1: Shouldn't the call price depend on the stock's expected return?

- It does, in so far as the stock's **current** price is determined by the returns people demand to hold it. But **given the current stock price**, we don't need any more.

3.2: But, holding the current stock price fixed, if the probability π for an up stock move is higher, then the expected payoff of the call is also higher. Why then shouldn't the call price be higher?

- Holding S_t fixed, if π is higher, then it means that the market demands a higher return for the stock, hence the market should also demand a higher return for holding the call option which, after all, is a leveraged position in the stock. The effect of a higher π is canceled by the effect of the higher discount rate.

3.3: Does this mean that a news event that leads to an increase in the assessed probability for an up-move in the stock price does not affect the value of the call option?

(B) Multi-period case

We now know how to value a call at any stock price (node) on our grid with one-step left in its life.

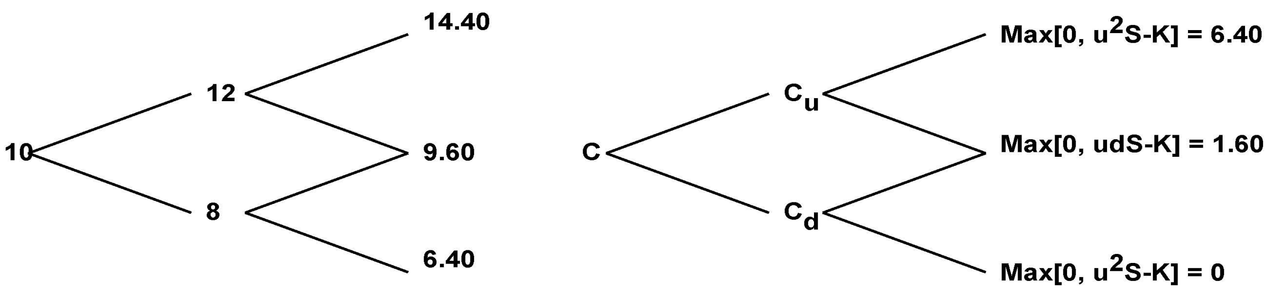
As you will have guessed, there's nothing special about the last time-step. To get the price one period earlier, we just repeat the replication strategy, this time choosing δ and L at each node to match the next period call values, whether the stock moves up or down.

Then, for the next time step......

- In other words, we're done! There really is no more to the pricing argument than what we saw in the one-step example.
- The key is that at each node there are two outcomes to hedge against and we have two securities to hedge with.
- As we work backward in time, we will get *both* the option price and the replicating portfolio (hedge ratio plus amount of borrowing).
- We not only determine what the no-arbitrage price *should be*, but also the exact strategy we will follow if it *isn't*.
- Let's look in detail at how it all works.

1. Two-period example:

- **Parameters:** $u = 1.2, d = 0.8, S = 10.0, K = 8.0, \quad \bar{r} \equiv e^{r\Delta t} = 1.1.$



- **At node u at time 1:** The replicating portfolio for the call option has δ_u shares and L_u in borrowing:

$$\delta_u = \frac{C_{uu} - C_{ud}}{uuS - udS} = \frac{6.4 - 1.6}{14.4 - 9.6} = 1.0$$

$$L_u = \frac{\delta udS - C_{ud}}{\bar{r}} = \frac{1.0 \cdot 9.6 - 1.6}{1.1} = 7.27$$

  - Check that 1.0 share plus 7.27 in borrowing indeed replicates the option's payoff at time 2:

$$\text{At node } uu \; : \; 1.0 \cdot uuS - \bar{r} \cdot 7.27 = 6.40$$
$$\text{At node } ud \; : \; 1.0 \cdot udS - \bar{r} \cdot 7.27 = 1.60$$

  - The cost of the node-u replicating portfolio at u is

$$\delta_u \cdot uS - L_u = 1.0 \cdot 12 - 7.27 = 4.73$$

- **At node d at time 1:** The replicating portfolio for the call option has δ_d shares and L_d in borrowing:

$$\begin{aligned}\delta_d &= (1.6 - 0)/(9.6 - 6.4) = 0.5 \\ L_d &= (0.5 \cdot 6.4 - 0)/1.1 = 2.91\end{aligned}$$

  - ▶ Check that 0.5 shares plus 2.91 in borrowing indeed replicates the option's payoff at time 2:

$$\begin{aligned}\text{At node } ud &: 0.5 \cdot udS - \bar{r} \cdot 2.91 = 1.60 \\ \text{At node } dd &: 0.5 \cdot udS - \bar{r} \cdot 2.91 = 0\end{aligned}$$

  - ▶ The cost of the node-d replicating portfolio at d is

$$\delta_d \cdot dS - L_d = 0.5 \cdot 8 - 2.91 = 1.09$$

- **At time 0:** The replicating portfolio at time 0, with δ shares and L in borrowing, should have value 4.73 if the stock goes up and 1.09 if the stock goes down.

$$\begin{aligned}\delta &= \frac{C_u - C_d}{uS - dS} = \frac{4.73 - 1.09}{12 - 8} = 0.91 \\ L &= \frac{\delta dS - C_d}{\bar{r}} = \frac{0.91 \cdot 8 - 1.09}{1.1} = 5.63\end{aligned}$$

  - ▶ Check that 0.91 shares plus 5.63 in borrowing indeed provides exactly the required value to replicate the option at time 1:

$$\begin{aligned}\text{At node } u &: 0.91 \cdot uS - \bar{r} \cdot 5.63 = 4.73 \\ \text{At node } d &: 0.91 \cdot dS - \bar{r} \cdot 5.63 = 1.09\end{aligned}$$

  - ▶ The cost of the replicating portfolio at time 0 is

$$\delta \cdot S - L = 0.91 \cdot 10 - 5.63 = 3.47$$

2. Key points to notice about the adjustment strategy:

a. It is self-financing. The value of the extra stock we buy/sell is exactly the same as the value of our extra borrowing/lending. Another way of saying this is that we don't put any of our own money in (or take any out) of our hedge position. If we did have to, then we could not conclude that today's value of the replicating position must be the price of the derivative.

b. We have to buy if the stock goes up and sell if it goes down. And, if there were more steps, we'd have to keep doing this. In other words, it's going to costs us money. It would cost less if the stock didn't move as much. But in a binomial world that can't happen. The degree of volatility is fixed by nature.

c. We are always borrowing money. So we have costs here too. A call is like a levered position in the stock. Leverage costs money.

d. If we reversed all the trades, we'd replicate the payoff to a short call position. Then we'd make money on our hedge adjustments. How much? \$3.47! An option is worth what it costs to hedge it.

III. General Binomial Formulas

Our methodology is now complete. To make it useful, however, we need to derive some mathematical formulas or rules for the procedure that we could program.

(A) The One-Period Formula

- As usual, let:

$$\begin{aligned} u &= 1 + \text{rate of return if stock goes up} \\ d &= 1 + \text{rate of return if stock goes down} \\ \bar{r} &= 1 + \text{interest rate for borrowing and lending} \end{aligned}$$

- Note that the rates here are simple, per-period rates. So if the continuously-compounded annualized rate is r, $\bar{r} \approx e^{r\,\Delta t}$.

- We assume we know the two possible values the call could have next period. Call them C_u and C_d.

- We want to replicate the call, so we require that:

$$\begin{aligned} \delta u S - \bar{r} L &= C_u \\ \delta d S - \bar{r} L &= C_d \end{aligned} \qquad (3)$$

- Computing the hedge ratio yields:

$$\begin{aligned} \delta(uS - dS) &= C_u - C_d \\ \delta &= \frac{C_u - C_d}{S(u-d)} \end{aligned} \qquad (4)$$

- Computing the required amount of the debt (using equations (3) and (4)):

$$
\begin{aligned}
& \frac{C_u - C_d}{S(u-d)} \cdot uS - \bar{r}L = C_u \\
\Longrightarrow \quad & (C_u - C_d) \cdot u - \bar{r}L \cdot (u-d) = C_u \cdot (u-d) \\
\Longrightarrow \quad & \bar{r}L \cdot (u-d) = dC_u - uC_d \\
\Longrightarrow \quad & L = (dC_u - uC_d)/(\bar{r}(u-d))
\end{aligned} \tag{5}
$$

- So the value of the call is the value of the hedge:

$$
\begin{aligned}
C &= \delta S - L \\
&= \frac{C_u - C_d}{u-d} - \frac{dC_u - uC_d}{\bar{r}(u-d)} \\
&= \left(\frac{1 - d/\bar{r}}{u-d}\right) C_u + \left(\frac{u/\bar{r} - 1}{u-d}\right) C_d \\
&= \frac{1}{\bar{r}}\left(\frac{\bar{r} - d}{u-d}\right) C_u + \frac{1}{\bar{r}}\left(\frac{u - \bar{r}}{u-d}\right) C_d \\
&= \frac{\mathbf{q}\,\mathbf{C_u} + (\mathbf{1}-\mathbf{q})\,\mathbf{C_d}}{\bar{\mathbf{r}}}
\end{aligned} \tag{6}
$$

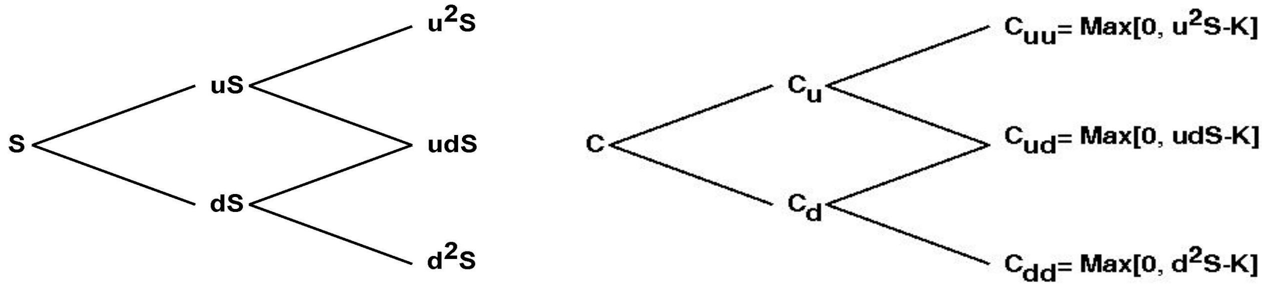
- where we have defined

$$
q = \frac{\bar{r} - d}{u-d} \quad \text{and} \quad (1-q) = \frac{u - \bar{r}}{u-d}.
$$

- **Notice:** q does not depend on where we are in the tree or what any of the payoffs are.

- Equation (6) tells you how to move backwards along the grid. All you need to know is the values one step ahead!

(B) The Two-period Case



- From the one-period case

$$
\begin{aligned}
C_u &= \frac{qC_{uu} + (1-q)C_{ud}}{\bar{r}} \\
C_d &= \frac{qC_{ud} + (1-q)C_{dd}}{\bar{r}}
\end{aligned}
$$

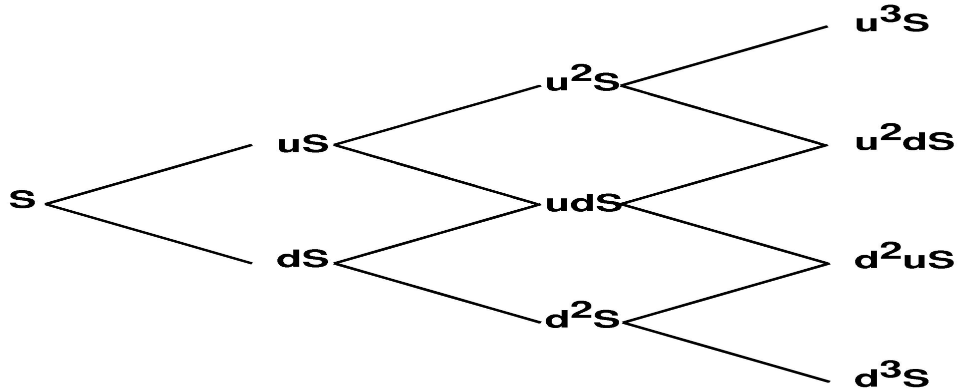
- Once we have computed C_u and C_d, we have a one-period case.

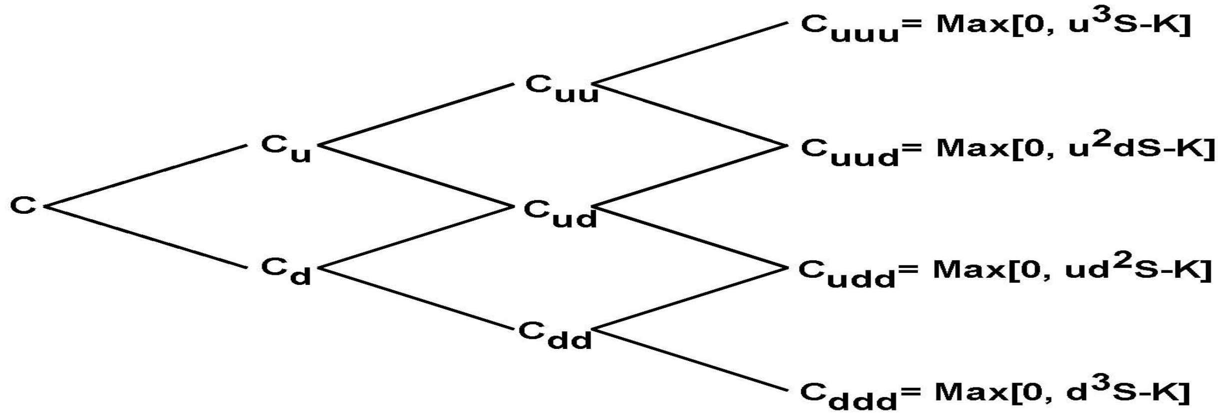
$$
\begin{aligned}
C &= \frac{qC_u + (1-q)C_d}{\bar{r}} \\
&= \frac{q^2C_{uu} + q(1-q)C_{ud} + (1-q)qC_{du} + (1-q)^2C_{dd}}{\bar{r}^2} \\
&= \frac{q^2C_{uu} + 2q(1-q)C_{ud} + (1-q)^2C_{dd}}{\bar{r}^2}
\end{aligned}
$$

using the fact that $C_{ud} = C_{du}$.

- We're using the fact that the one-step formula only involved q, which doesn't change. So we didn't have to solve for δ and L all over again.

(C) The Three Period Case





$$C = \frac{1}{\bar{r}^3}\left[q^3 C_{uuu} + 3q^2(1-q)C_{uud} + 3q(1-q)^2 C_{udd} + (1-q)^3 C_{ddd}\right]$$

- We could keep going but it would be tedious.
- What are we accomplishing?

(D) A General Pricing Formula for European Contingent Claims

- What we just did in the two and three period cases is combine the one-period formulas to eliminate the intermediate values of the derivative.

- This shows something remarkable: *we can write the value of the derivative today solely as a function of its payoffs at expiration.*

  - ▶ All we are doing is multiplying the different possible payoffs by weighting factors, and then adding them up.

  - ▶ This will work for any **European** derivative, because all the cash-flows they generate are at expiration.

- The formulas are getting messy. But we can simplify them for the general case of N periods to go – *if we can see the pattern. in those weighting factors.*

- The key is to recognize the ones we just derived for $N = 2$ and $N = 3$. **They are the binomial probabilities.**

- We met them earlier in the lecture.

- For each node $u^j d^{N-j} S_0$ in the N-step tree, with j up steps and $N-j$ down steps, the weights in our multi-period pricing formula turn out to be

$$\left(\frac{N!}{j!(N-j)!}\right) q^j (1-q)^{N-j}$$

- Compare that to the <u>true</u> probability of getting to that node:

$$\left(\frac{N!}{j!(N-j)!}\right)\pi^j(1-\pi)^{N-j}$$

and we can conclude:

> **The weighting factors applied to each terminal payoff are just the probabilities of those payoffs, when π has been replaced by $\mathbf{q}$.**

- So we can now jump straight to the general formula that must hold for an ARBITRARY derivative that pays off the amount $C_T(S_T(j))$ in N periods if the underlying ends up at $S_T(j) = S_t\ u^j d^{N-j}$.

- It must be:

$$C_t \;=\; \left(\frac{1}{\bar{r}^N}\right)\sum_{j=0}^{N}\left(\frac{N!}{j!(N-j)!}\right)q^j(1-q)^{N-j}C_T(S_T(j))$$

- This formula applies to ANY payoff function $C_T(\ )$.

- It tells us that a European claim's value can be interpreted as **the mathematical expectation of its payoff** discounted at the risk free rate...

  - ...using the wrong probabilities!

IV. The State Price Density.

- If you think back to last week's lecture, you may recall seeing a formula a lot like that before. **It is a special case of our general derivatives pricing equation:**

$$C_t = \sum_{x=0}^{\infty} C_T(x)\ b_T(x) = B_{t,T}\ \sum_{x=0}^{\infty} C_T(x)\ q_T(x).$$

where $C_T(x)$ is the payoff function if the underlying asset ends at $S_T = x$, and $b_T(x)$ are the butterflies – that is – the prices of claims paying off 1 if $S_T = x$ and 0 otherwise.

- Now the formula on the previous page was also a sum over all the final stock prices, which were indexed by j the number of up-steps.

- So for this model, we can immediately see

$$b_T(j) = \left(\frac{1}{\bar{r}^N}\right)\ \left(\frac{N!}{j!(N-j)!}\right) q^j (1-q)^{N-j}.$$

- But $\left(\frac{1}{\bar{r}^N}\right)$ is just $B_{t,T}$, the price of a riskless bond maturing at T. (Recall $\bar{r}$ was one plus the per-period simple rate.)

- And we defined $b_T(x)/B_{t,T}$ to be $q_T(x)$ the state-price (or risk-neutral) probability that $S_T = x$.

- So for this model

$$\begin{aligned}\text{Risk-Neutral Probability that } S_T = S_t\, u^j d^{N-j} &= \\ \text{Risk-Neutral Probability of } j \text{ up moves out of } N &= \left(\tfrac{N!}{j!(N-j)!}\right) q^j (1-q)^{N-j}.\end{aligned}$$

- And, as we saw earlier, this is just another case of the N-step binomial distribution with parameter q.

  This is our first example of an explicit recipe for the state price density (SPD).

- For this model, the SPD has the same form as the "true" stock price distribution, but with π replaced by q.

- How are these two distributions related to each other?
  - Consider the true one-period expected simple return on the stock:
    $$\mathrm{E}\left[\frac{S_{t+1}}{S_t}\right] = \pi u + (1-\pi)d \equiv \bar{\mu}.$$
    $$\text{So } \ \pi = \frac{\bar{\mu} - d}{u - d}.$$
  - Compare that to
    $$q \equiv \frac{\bar{r} - d}{u - d}.$$
  - We see that q *would be* the same as π if the stock's expected return were $\bar{r}$ instead of $\bar{\mu}$.

- We will find that this relationship holds for many models: **To get the RND from the true distribution, just change the true expected return of the stock to the risk-free rate.**

- If we lived in a world where q actually *was* the up-move probability, then it would look like the stock *or any other asset* had expected return $\bar{r}$, i.e. the riskless rate.

- This is why q and $1-q$ are also called risk-neutral probabilities of up and down moves on the tree.

- As we emphasized last lecture, these aren't really the probabilities of anything. For our purposes, they are just terms in a formula: a calculating device.

V. Summary.

- In this lecture we introduced our first stochastic process model for stock price behavior: the binomial model.
- It's not as silly as it first looks, because we can build complicated trees from simple ones.
- In a binomial world, we can hedge perfectly over each time step using two securities, so no-arbitrage pricing is easy.
- By iterating backwards in time, we get the options price and the replicating portfolio at every point on a grid.
- To do the arbitrage (or replicate the payoff), we just follow whatever path nature takes, changing our hedge as we go forward. This is a self-financing, dynamic replication strategy.
- The strategy generates interests costs and hedge-adjustment (trading P&L) cost. These are certain in advance, and **the option is worth what it costs to hedge it.**
  - ▶ Amazing fact. Holding the current stock price fixed, *the no-arbitrage price of a derivative is not affected by the probability that the underlying will go up or down.*
- We also derived the exact binomial price formula for ANY European contingent-claim.
- From it we learned that the state-price density is the same as the stock price density, with the expected return on the stock replaced by the risk-free rate.