Fin 514: Financial Engineering II

Lecture 7: Brownian Motion and a model for stock price movements

Dr. Martin Widdicks

UIUC

Spring, 2018

Outline

- We have shown how to value derivatives in a multi-period, discrete state model. However, we are left with two problems:
  1. Even in this model how do we estimate the future prices (matrix D in Lecture 6) for the securities in our market?
  2. Can we generalize this to a more realistic model?
- In Lecture 7 exactly this by constructing a basic continuous time model for stock prices, Geometric Brownian Motion, which has many useful features as a starting point. This is the starting point of our PDE/continuous time model.
- To get there we need to work through the basics of stochastic processes as well as determining what features we require for our model.

Recap: returns

- We have two possible notations for the stock price return. For a stock with price S_t at time t, receiving cash flows D (dividends or interest) over a given time period, Δt we have:
- **Ordinary returns** ($\bar{r}$),

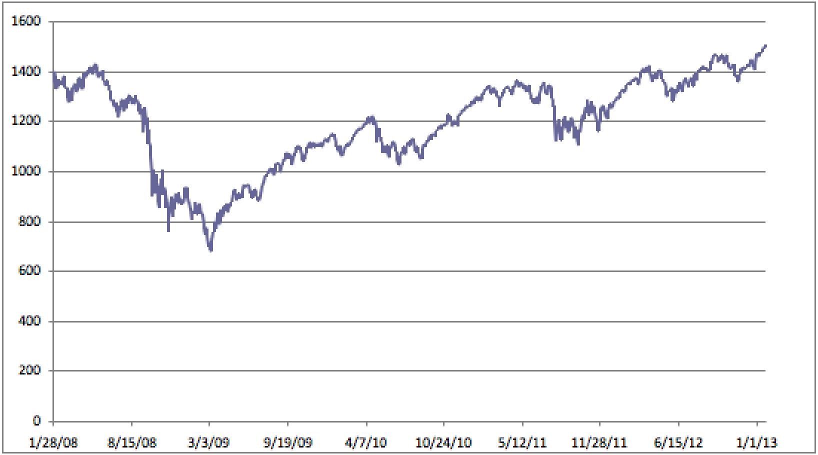
$$\begin{aligned}\bar{r} &= \frac{S_{t+\Delta t} - S_t + D}{S_t} \\ &= \frac{S_{t+\Delta t} + D}{S_t} - 1\end{aligned}$$

- and **Continuously compounded returns** (r)

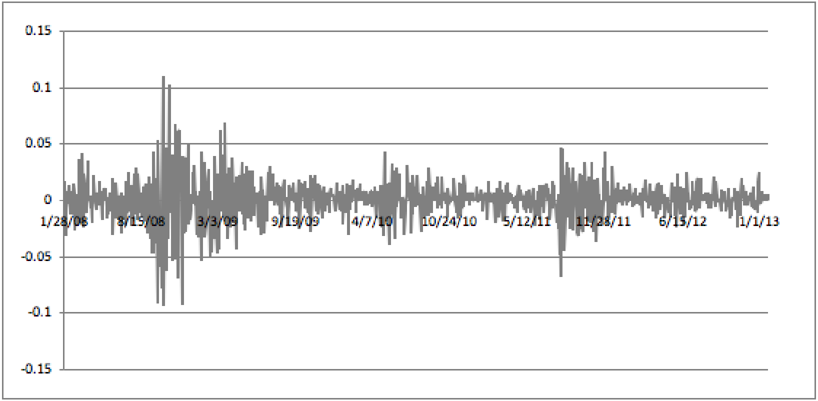
$$r = \ln\left(\frac{S_{t+\Delta t} + D}{S_t}\right)$$

Stock price returns: S&P 500 levels

The next two graphs show how the S&P 500 has changed over a 5 year period and the daily returns data.



S &P 500 Daily returns



Return properties

- Statistics of S&P 500 returns:

	1 day returns	5 day returns
Mean	0.0000830	0.000349
Variance	0.000273	0.001037
Standard deviation	0.01651	0.03221

- Mean increases roughly proportionally to time.
- Variance increases roughly proportionally with time.
- Standard deviation increases proportionally with the square root of time.

QUIZ

For a risky asset what do we expect to see with mean returns over a time period?

- Mean $= 0$
- Mean > 0
- Mean < 0

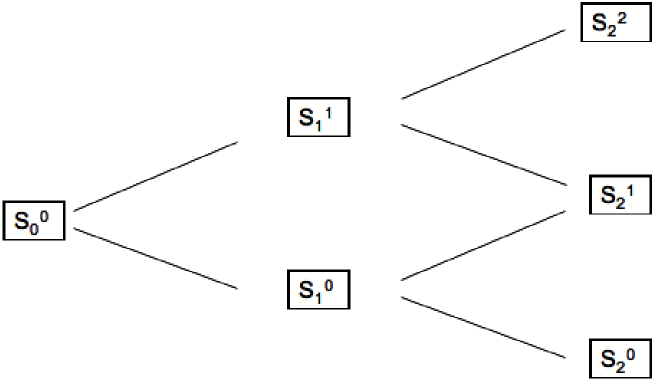
QUIZ

For a risky asset what do we expect to see with mean returns over a time period?

- Mean $= 0$
- Mean > 0 CORRECT
- Mean < 0

Stochastic process

- A stochastic process is a family of random variables, indexed by time.
- An example is our multistep binomial tree where the random variables are S_j^i and the index is time (j).



Continuous time stochastic processes

- Continuous stochastic processes where the time index will apply across an interval of time [0,T] and changes in the random variable values will happen over infinitesimal changes in time.
- A typical continuous stochastic process will look like these:

Link with the binomial model

- Recall that our natural choice of u and d in the binomial world was:

$$\begin{aligned} u &= e^{\overline{\mu}\Delta t+\sigma\sqrt{\Delta t}} \quad \text{with probability } q \\ d &= e^{\overline{\mu}\Delta t-\sigma\sqrt{\Delta t}} \quad \text{with probability } 1-q \end{aligned}$$

  where $\overline{\mu}$ is our choice of first term and so the cont. comp. return over a time step (Δt) is given by:

$$r_{\Delta t} = \ln \frac{S_{t+\Delta t}}{S_t} = \begin{cases} \overline{\mu}\Delta t + \sigma\sqrt{\Delta t} & \text{with probability } q \\ \overline{\mu}\Delta t - \sigma\sqrt{\Delta t} & \text{with probability } 1-q \end{cases}$$

- By way of verification, with this set-up, then

$$\begin{aligned} E[S_{\Delta t}] &= S_0 e^{\overline{\mu}\Delta t} \\ Var[r_{\Delta t}] &= \sigma^2\Delta t \end{aligned}$$

From binomial to continuous time

- Thinking of returns, $\ln(S_{t+\Delta t}/S_t)$ gives us that

$$\ln(S_{t+\Delta t}/S_t) = \overline{\mu}\Delta t \pm \sigma\sqrt{\Delta t}$$

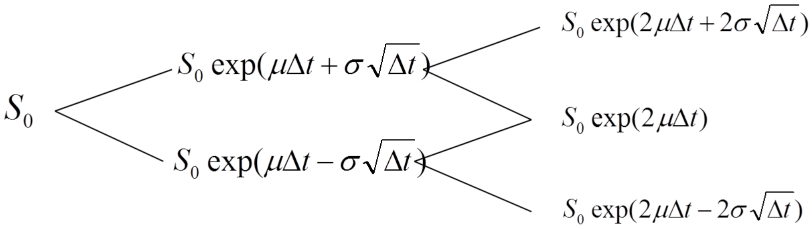
  and so

$$\ln(S_{t+\Delta t}) = \ln(S_t) + \overline{\mu}\Delta t \pm \sigma\sqrt{\Delta t}$$

- This is discrete random walk (with drift) for the log stock price where the next period price, $\ln(S_{t+\Delta t})$ is determined by the current price, $\ln(S_t)$, plus an expected change or drift term, $\overline{\mu}\Delta t$ plus a random component $\pm\sigma\sqrt{\Delta t}$.

More steps

Stepping through the binomial tree amounts to multiplying the stock price by factors of the form $e^{\overline{\mu}\Delta t \pm \sigma\sqrt{\Delta t}}$ giving us the tree for S_t .



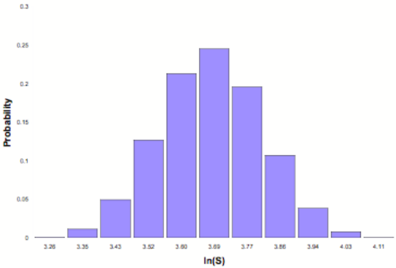
More steps

Or, alternatively, stepping through the binomial tree amounts to adding up random components of form $\overline{\mu}\Delta t \pm \sigma\sqrt{\Delta t}$ giving us the tree for $\ln(S_t)$.

$$
\begin{array}{ccccc}
 & & & & \ln S_0 + 2\mu\Delta t + 2\sigma\sqrt{\Delta t} \\
 & \ln S_0 + \mu\Delta t + \sigma\sqrt{\Delta t} & & & \\
\ln S_0 & & & & \ln S_0 + 2\mu\Delta t \\
 & \ln S_0 + \mu\Delta t - \sigma\sqrt{\Delta t} & & & \\
 & & & & \ln S_0 + 2\mu\Delta t - 2\sigma\sqrt{\Delta t}
\end{array}
$$

Continuous time binomial

If we take a fixed period of length T, divide it into N periods each of length $\Delta t = T/N$. Let N become large (Δt small), the resulting binomial distribution is an approximation of Normal distribution:



Continuous time binomial

- If we consider limit as $N \to \infty$, a version of central limit theorem tells us that limiting distribution is the Normal distribution and so the distribution of returns or $\ln(S)$ is normal.
- But if $\ln(S)$ is Normal, then the distribution of S is lognormal (this is the definition of the lognormal distribution).

Quiz

Do we think that returns are normally distributed?

- Yes
- No
- Maybe

Quiz

Do we think that returns are normally distributed?

- Yes
- No CORRECT
- Maybe

Very large number of time steps

- Now let's force the probabilities of an up and down movement (or the risk-neutral probabilities) to both be 0.5.
- So by combining binomial jumps we have

$$\ln S_T = \ln S_0 + \sum_{j=1}^{N} \overline{\mu}\Delta t + \sum_{j=1}^{N} \sigma\sqrt{\Delta t} Z_j$$

  where Z_j can take on values of -1 and 1 respectively, each with probability $= 0.5$.
- This simplifies to

$$\ln S_T = \ln S_0 + \overline{\mu} T + \sigma\sqrt{T}\frac{1}{\sqrt{N}}\sum_{j=1}^{N} Z_j$$

Very large number of time steps

- From the central limit theorem the sum of N binomially distributed values taking on values of 1 and -1 with probability 0.5 converges to a normal distribution with mean 0 and variance N.
- Thus, $\frac{1}{\sqrt{N}} \sum_{j=1}^{N} Z_j$ will be normally distributed with mean 0 and variance 1.
- And so, we rewrite our S_T value as:

$$\ln S_T = \ln S_0 + \overline{\mu} T + \sigma \sqrt{T} \phi$$

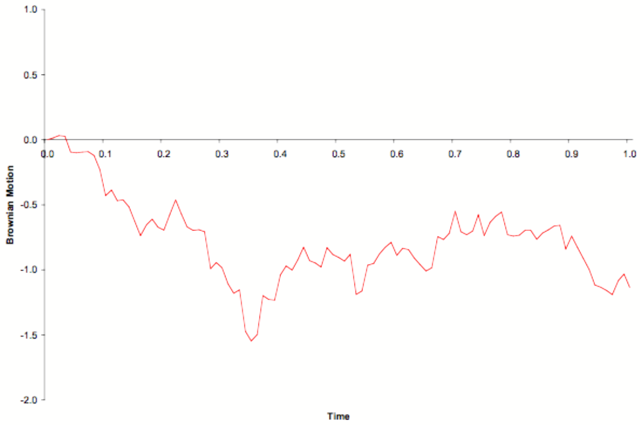
where $\phi \sim N(0, 1)$ has a link to **Brownian motion**.

Brownian Motion (or Wiener Process)

- A standard Brownian motion, X (or sometimes W, or B), is a continuous time process defined by the following properties:
  - $X(0) = 0$.
  - For any times t and $s > t$, $X(s) - X(t)$ is normally distributed with mean 0 and variance $(s - t)$.
  - For any times $0 \leq t_0 < t_1 < \cdots < t_n < \infty$ the random variables $X(t_0), X(t_1) - X(t_0), \ldots, X(t_n) - X(t_{n-1})$ are independently distributed.
  - The sample paths are continuous.
- Note that the general Brownian motion does not need to have mean 0 and variance $s - t$, this standard BM is also called a **Wiener process**, so we can use the terms interchangeably.

Sample path of Brownian Motion

See Also: `http://www.stat.umn.edu/~charlie/Stoch/brown.html`



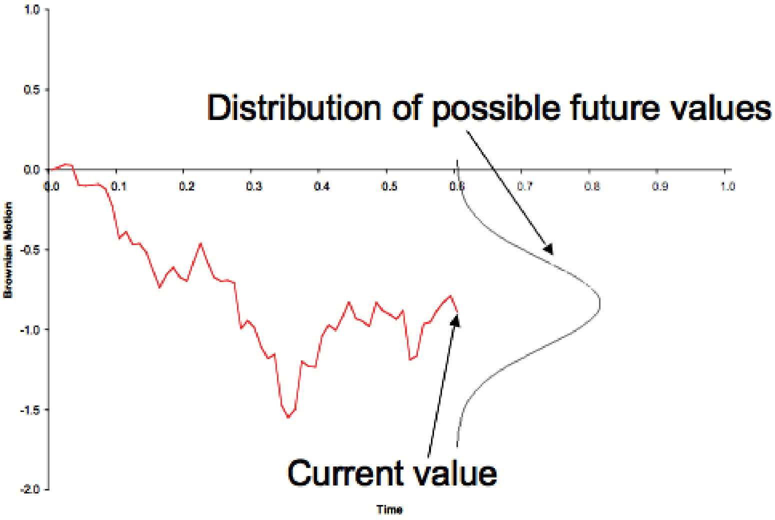
Properties of Brownian motion

- Increments are independent.
- Sample paths are continuous.
- Increments normally distributed.
- Increments have mean of 0.
- Increments have variances of s-t.
- Sample paths are non-differentiable with respect to time (See Aside at end)
- Sample paths have infinite total variation over finite time periods.
- If the path hits a value then it hits it again in an arbitrarily short period afterwards.

QUIZ

Of the properties of Brownian motion, which are desirable for a stock price model and which are not? Explain why.

Development of Brownian motion



A model for stock prices

- If we now consider a new process, called **Arithmetic Brownian motion**, $Y(t) = \overline{\mu}t + \sigma X(t)$
- This process has the following properties:

$$\begin{aligned} E[Y(s) - Y(t)] &= \overline{\mu}(s-t) + \sigma E[X(s) - X(t)] = \overline{\mu}(s-t) \\ Var[Y(s) - Y(t)] &= \sigma^2 Var[X(s) - X(t)] = \sigma^2 (s-t) \\ sd[Y(s) - Y(t) &= \sigma\sqrt{s-t}. \end{aligned}$$

- But this still gives us the possibility of negative stock prices.

Geometric Brownian Motion GBM

- We now introduce Geometric Brownian Motion (GBM), which looks as follows:

$$
\begin{aligned}
S(t) &= S(0)\exp[Y(t)] \\
&= S(0)\exp[\overline{\mu}t + \sigma X(t)]
\end{aligned}
$$

- This has the following properties:
  - $S(t) > 0$
  - Increments to $\ln S(t)$ are independent
  - $\ln S(s) - \ln S(t)$ is normally distributed with the following properties:

$$
\begin{aligned}
E[\ln S(s) - \ln S(t)] &= E[\ln S(0) - \ln S(0) + Y(s) - Y(t)] \\
&= \overline{\mu}(s-t) \\
Var[\ln S(s) - \ln S(t)] &= Var[Y(s) - Y(t)] = \sigma^2(s-t) \\
sd[\ln S(s) - \ln S(t)] &= \sigma\sqrt{s-t}
\end{aligned}
$$

More on GBM

- Consider a discrete time period Δt, and let us see what GBM suggests about stock returns. We have:

$$
\begin{aligned}
\ln\left(\frac{S(t+\Delta t)}{S(t)}\right) &= \ln S(t+\Delta t) - \ln S(t) \\
&= \overline{\mu}\Delta t + \sigma(X(t+\Delta t) - X(t)) \\
&= \overline{\mu}\Delta t + \sigma\phi\sqrt{\Delta t}
\end{aligned}
$$

as Brownian motion increments have variance equal to time increment and where ϕ is normally distributed with mean 0 and variance 1. Thus, by a Taylor expansion:

$$
\begin{aligned}
\frac{S(t+\Delta t)}{S(t)} &= \exp[\overline{\mu}\Delta t + \sigma\phi\sqrt{\Delta t}] \\
&= 1 + (\overline{\mu}\Delta t + \sigma\phi\sqrt{\Delta t}) + (\overline{\mu}\Delta t + \sigma\phi\sqrt{\Delta t})^2/2 +
\end{aligned}
$$

GBM and returns

subtracting 1 from both sides gives:

$$\begin{aligned}
\frac{S(t+\Delta t)}{S(t)} - 1 &= (\overline{\mu}\Delta t + \sigma\phi\sqrt{\Delta t}) + (\overline{\mu}\Delta t + \sigma\phi\sqrt{\Delta t})^2/2 + \dots \\
\frac{S(t+\Delta t) - S(t)}{S(t)} &= \sigma\phi\sqrt{\Delta t} + (\overline{\mu} + \sigma^2\phi^2/2)\Delta t + O(\Delta t^{3/2})
\end{aligned}$$

taking expectations gives:

$$\begin{aligned}
E\left[\frac{S(t+\Delta t) - S(t)}{S(t)}\right] &= 0 + \overline{\mu}\Delta t + \sigma^2/2\Delta t \\
&= (\overline{\mu} + \sigma^2/2)\Delta t \\
&= \mu\Delta t
\end{aligned}$$

where $\boxed{\mu = \overline{\mu} + \sigma^2/2}$ and

$$Var\left[\frac{S(t+\Delta t) - S(t)}{S(t)}\right] = \sigma^2\Delta t \qquad (1)$$

New equation for GBM

- So we think of stock prices following the process below:

$$\frac{S(t+\Delta t)-S(t)}{S(t)} = \mu\Delta t + \sigma\phi\sqrt{\Delta t}$$

or

$$S(t+\Delta t)-S(t) = \mu S(t)\Delta t + \sigma S(t)\phi\sqrt{\Delta t}$$

- We typically write this expression as follows:

$$dS(t) = \mu S(t)dt + \sigma S(t)dX(t)$$

where dX can be modelled as $dX = \phi\sqrt{t}$.

Our equation for GBM

This is called a stochastic differential equation. Where we denote:

- μS: drift term
- μ: expected return/growth
- σ: volatility
- σS: diffusion term

Quiz

If the expected stock return is the risk-free rate r_f then what is $\overline{\mu}$ from $S(t) = S(0) \exp[\overline{\mu}t + \sigma X(t)]$

- r_f
- $r_f - \sigma$
- $r_f - \frac{1}{2}\sigma^2$
- $r_f + \frac{1}{2}\sigma^2$

Quiz

If the expected stock return is the risk-free rate r_f then what is $\overline{\mu}$ from $S(t) = S(0)\exp[\overline{\mu}t + \sigma X(t)]$

- r_f
- $r_f - \sigma$
- $r_f - \frac{1}{2}\sigma^2$ Correct
- $r_f + \frac{1}{2}\sigma^2$

Return to binomial

- We left it with
$$\ln S_T = \ln S_0 + \overline{\mu} T + \sigma\sqrt{T}\phi$$
where $\phi \sim N(0,1)$.
- One thing we have not determined is the appropriate value of $\overline{\mu}$. However, in the risk-neutral world we know that $E[S_T] = S_0 e^{r_f T}$, where r_f is the risk-free rate, and so

$$\begin{aligned} E[S_T] &= E[S_0 e^{\overline{\mu} T + \sigma\sqrt{T}\phi}] \\ &= S_0 e^{\overline{\mu} T} E[e^{\sigma\sqrt{T}\phi}] = S_0 e^{r_f T} \end{aligned}$$

What is $\overline{\mu}$?

- But, from probability results we can show that

$$
\begin{aligned}
E[e^{\sigma\sqrt{T}\phi}] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\sigma\sqrt{T}\phi} e^{-\phi^2/2} d\phi \\
&= \frac{e^{\frac{1}{2}\sigma^2 T}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2} dy \\
&= e^{\frac{1}{2}\sigma^2 T}
\end{aligned}
$$

where $y = \phi - \sigma\sqrt{T}$ and so,

$$S_0 e^{\overline{\mu}T + \frac{1}{2}\sigma^2 T} = S_0 e^{r_f T}$$

thus,

$$\overline{\mu} = r_f - \frac{1}{2}\sigma^2$$

which matches with our derivation above!

Conclusion

- We have made our first attempt to formulate a model for stock price movements.
- The basis for this is Brownian Motion, which is a continuous stochastic process with iid increments, distributed normally. To improve the model, we transform this to Geometric Brownian Motion, where we model log-returns as a Brownian Motion with drift.
- We can also think of Brownian motion as the limit of a infinite step binomial tree and it share many of the properties.
- Brownian motion is very oddly behaved which may cause problems when considering functions of S(t), we will attempt to deal with this by using Ito calculus.
- There are many books/papers disputing the use of GBM, see:
Lo, A., MacKinley, A. C., 2001, A Non-random walk down wall street, ISBN: 0691092567
Mandelbrot, B., The Misbehavior or Markets, 2004, Profile Books
Taleb, N., Fooled by Randomness, 2005 (2nd Ed.), Random House

Aside: Non-differentiable I (sketch)

Consider the following limit for a Brownian motion X

$$\begin{aligned}
\lim_{s\to t} E_t\left[\left(\frac{X(s)-X(t)}{s-t}\right)^2\right] &= \lim_{s\to t}\frac{Var[X(s)-X(t)]}{(s-t)^2} \\
&= \lim_{s\to t}\frac{s-t}{(s-t)^2} \\
&= \lim_{s\to t}\frac{1}{s-t} \\
&= \infty
\end{aligned}$$

but if X(t) is a differentiable function we require that

$$\frac{dX(t)}{dt} = X'(t) = \left|\lim_{s\to t} E_t\frac{X(s)-X(t)}{s-t}\right| < \infty$$

but as this variation is unbounded then the differential is not bounded. The full proof is beyond the scope of the course.

Aside: Non-differentiable II (sketch)

- Alternatively we can look at total variation and total squared variation.
- Consider total variation (TV) and total squared variation (TSV) as follows

$$
\begin{aligned}
TV &= \lim_{n\to\infty} E_0 \sum_{k=1}^{n} |X(k/n) - X((k-1)/n)| \\
TSV &= \lim_{n\to\infty} E_0 \sum_{k=1}^{n} [X(k/n) - X((k-1)/n)]^2
\end{aligned}
$$

for a differentiable function TV is finite and TSV is zero

$$
\begin{aligned}
TV &= \lim_{n\to\infty} E_0 \sum_{k=1}^{n} |X(k/n) - X((k-1)/n)| \\
&= \lim_{n\to\infty} n\sqrt{Var_0[X(k/n) - X((k-1)/n)]} \\
&= \lim_{n\to\infty} n\sqrt{1/n} = \lim_{n\to\infty} \sqrt{n} = \infty
\end{aligned}
$$

Aside: Non-differentiable II (sketch)

$$
\begin{aligned}
TSV &= \lim_{n\to\infty} nVar_0[X(k/n) - X((k-1)/n)] \\
&= \lim_{n\to\infty} n/n = 1
\end{aligned}
$$