FIN 514: Problem Set #4

Due on Wednesday, March 7, 2018

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Problem 1

a. By Ito's lemma,

$$\begin{aligned} df(X) &= \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX)^2 \\ &= 2X(t)dX(t) + \frac{1}{2} \times 2dt \\ \Rightarrow X(t)dX(t) &= \frac{1}{2}df(X) - \frac{1}{2}dt \\ \Rightarrow \int_0^t X(\tau)d\tau &= \frac{1}{2}\int_0^t dX^2(\tau) + \int_0^t \frac{1}{2}d\tau \\ &= \frac{1}{2}X^2(t) + \frac{1}{2}t \end{aligned}$$

b. Let $g(t, X) = tX(t)$. Then by Ito's lemma,

$$\begin{aligned} dg(t, X) &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial X}dX + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}(dX)^2 \\ &= X(t)dt + tdX(t) \\ \Rightarrow tdX(t) &= d(tX(t)) - X(t)dt \\ \Rightarrow \int_0^t \tau dX(\tau) &= tX(t) - \int_0^t X(\tau)d\tau \end{aligned}$$

c. Let $h(X) = X^3(t)$. Then by Ito's lemma,

$$\begin{aligned} dh(X) &= \frac{\partial h}{\partial X}dX + \frac{1}{2}\frac{\partial^2 h}{\partial X^2}(dX)^2 \\ &= 3X^2(t)dX(t) + \frac{1}{2}6X(t)dt \\ &= 3X^2(t)dX(t) + 3X(t)dt \\ \Rightarrow X^2(t)dX(t) &= \frac{1}{3}d(X^3(t)) - X(t)dt \\ \Rightarrow \int_0^t X^2(\tau)dX(\tau) &= \frac{1}{3}X^3(t) - \int_0^t X(\tau)d\tau \end{aligned}$$

Problem 2

By Ito's lemma,

$$\begin{aligned} dF &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2 \\ &= -(r-\delta)Fdt + \frac{F}{S}dS(t) \end{aligned}$$

Since $dS(t) = (\mu - \delta)S(t)dt + \sigma S(t)dX(t)$,

$$\begin{aligned} dF &= -(r-\delta)Fdt + \frac{F}{S}((\mu-\delta)S(t)dt + \sigma S(t)dX(t)) \\ &= -(r-\delta)Fdt + (\mu-\delta)Fdt + \sigma FdX(t) \\ &= (\mu - r)Fdt + \sigma FdX(t) \end{aligned}$$

Problem 3

a. By Ito's product rule,

$$\begin{aligned} dY &= d\left(\frac{S_2}{S_1}\right) \\ &= \frac{1}{S_1}dS_2 + S_2 d\left(\frac{1}{S_1}\right) + dS_2 d\left(\frac{1}{S_1}\right) \end{aligned}$$

By Ito's lemma,

$$\begin{aligned} d\left(\frac{1}{S_1}\right) &= -\frac{1}{S_1^2}dS_1 + \frac{1}{2} \times 2(dS_1)^2 \\ &= -\frac{1}{S_1^2}(\mu_1 S_1 dt + \sigma_1 S_1 dX_1) + \frac{1}{S_1^3}\sigma_1^2 S_1^2 dt \\ &= -\frac{\mu_1}{S_1}dt - \frac{\sigma_1}{S_1}dX_1 + \frac{\sigma_1^2}{S_1}dt \\ &= \frac{-\mu_1 + \sigma_1^2}{S_1}dt - \frac{\sigma_1}{S_1}dX_1 \end{aligned}$$

Therefore,

$$\begin{aligned} dY &= \frac{1}{S_1}(\mu_2 S_2 dt + \sigma_2 S_2 dX_2) + S_2\left(\frac{-\mu_1 + \sigma_1^2}{S_1}dt - \frac{\sigma_1}{S_1}dX_1\right) + (\mu_2 S_2 dt + \sigma_2 S_2 dX_2)\left(\frac{-\mu_1 + \sigma_1^2}{S_1}dt - \frac{\sigma_1}{S_1}dX_1\right) \\ &= \mu_2 Y dt + \sigma_2 Y dX_2 + (-\mu_1 + \sigma_1^2) Y dt - \sigma_1 Y dX_1 - \sigma_1\sigma_2\rho Y dt \quad (\because dX_1 dX_2 = \rho dt) \end{aligned}$$

Let $\sigma_3 X_3 = \sigma_1 X_1 + \sigma_2 X_2$, where $\sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}$. Since X_1 and X_2 have expected value zero, $\mathrm{E}[X_3]$ also equal to zero. Furthermore, $\mathrm{Var}[X_3] = \mathrm{Var}[\sigma_1 X_1 + \sigma_2 X_2]/\sigma_3^2 = (\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)t/\sigma_3^2 = t$. Since X_1 and X_2 are normally distributed and have independent increments, X_3 is also normally distributed and has independent increments, so X_3 is a Brownian motion. Therefore, $\sigma_3 X_3 = \sigma_1 X_1 + \sigma_2 X_2$. Finally dY can be represented as follows.

$$\begin{aligned} dY &= (\mu_2 - \mu_1 + \sigma_1^2 - \sigma_1\sigma_2\rho) Y dt + \sigma_3 Y dX_3 \\ &= \mu_Y Y_t dt + \sigma_Y Y_t dX_{3t} \end{aligned}$$

$$\text{where } \mu_Y = \mu_2 - \mu_1 + \sigma_1^2 - \sigma_1\sigma_2\rho, \sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}$$

b. By Ito's product rule,

$$\begin{aligned} dZ_t &= d(Y_t B_{Et}) \\ &= B_{Et} dY_t + Y_t dB_{Et} + dY_t dB_{Et} \\ &= B_E t(\mu_Y Y_t dt + \sigma_Y Y_t dX_{3t}) + Y_t(r_E B_{Et} dt), \quad dB_{Et} = r_E B_{Et} dt \text{ because the bond is riskless.} \\ &= (\mu_Y + r_E) Z_t dt + \sigma_Y Z_t dX_{3t} \\ &= (\mu_Z) Z_t dt + \sigma_Y Z_t dX_{3t} \end{aligned}$$

where $\mu_Z = \mu_Y + r_E$.

Problem 4

a. By Ito's product rule,

$$
\begin{aligned}
dY &= d(S_1 S_2) \\
&= S_1 dS_2 + S_2 dS_1 + dS_1 dS_2 \\
&= S_1(\mu_2 S_2 dt + \sigma_1 S_2 dX_2) + S_2(\mu_1 S_1 dt + \sigma_1 S_1 dX_1) + (\mu_1 S_1 dt + \sigma_1 S_1 dX_1)(\mu_2 S_2 dt + \sigma_2 S_2 dX_2) \\
&= (\mu_1 + \mu_2 + \sigma_1\sigma_2\rho) Y dt + Y(\sigma_1 dX_1 + \sigma_2 dX_2)
\end{aligned}
$$

Let $X_{3t} = \frac{\sigma_1 X_1 + \sigma_2 X_2}{\sigma_3}$, where $\sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho}$, then since $\mathrm{E}[X_{1t}] = \mathrm{E}[X_{2t}] = 0$, $\mathrm{E}[X_{3t}] = 0$, and $\mathrm{Var}[X_{3t}] = \frac{(\sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2\rho)t}{\sigma_3^2} = t$. Since X_{1t} and X_{2t} are normally distributed and have independent increments, X_3 is also normally distributed and has independent increments. Therefore, X_{3t} is a Brownian motion. Finally, dY can be represented as follows.

$$
\begin{aligned}
dY &= (\mu_1 + \mu_2 + \sigma_1\sigma_2\rho) Y dt + \sigma_3 Y dX_3 \\
&= \mu_Y Y dt + \sigma_Y Y dX_3 \\
\text{where } \ \mu_Y &= \mu_1 + \mu_2 + \sigma_1\sigma_2\rho, \sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho}
\end{aligned}
$$

b. Since $Y = S_1 S_2$, the payoff of call option is represented as $\max(Y^{\frac{1}{2}} - K, 0)$.

c. By Ito's lemma,

$$
\begin{aligned}
dZ &= \frac{\partial Z}{\partial Y} dY + \frac{1}{2}\frac{\partial^2 Z}{\partial Y^2}(dY)^2 \\
&= \frac{1}{2} Y^{-\frac{1}{2}}(\mu_Y Y dt + \sigma_Y Y dX_3) + \frac{1}{2}\left(-\frac{1}{2}\right) Y^{-\frac{3}{2}}(\sigma_Y^2 Y^2 dt) \\
&= \frac{1}{2}[\mu_Y Y^{\frac{1}{2}} dt + \sigma_Y Y^{\frac{1}{2}} dX_3 - \frac{1}{2} Y^{\frac{1}{2}} \sigma_Y^2 dt] \\
&= \frac{1}{2}\left[Z\left(\mu_Y - \frac{1}{2}\sigma_Y^2\right) dt + \sigma_Y Z dX_3\right]
\end{aligned}
$$

d. By Ito's lemma,

$$
\begin{aligned}
d\log Z &= \frac{1}{2}[(\mu_Y - \sigma_Y^2)dt + \sigma_Y dX_3] \\
\Rightarrow \log Z_T &= \log Z_0 + \frac{1}{2}[(\mu_Y - \sigma_Y^2)T + \sigma_Y X_{3T}] \\
\Rightarrow \frac{Z_T}{Z_0} &= \exp\left(\frac{1}{2}(\mu_Y - \sigma_Y^2)T + \sigma_Y X_{3T}\right) \\
\Rightarrow \mathrm{E}\left[\frac{Z_T}{Z_0}\right] &= \frac{1}{2}(\mu_Y - \sigma_Y^2)T + \frac{1}{4}\sigma_Y^2 T \\
&= \frac{1}{2}(\mu_Y - \frac{1}{2}\sigma_Y^2)T
\end{aligned}
$$

Therefore, expected return of the geometric average of two stocks is $\frac{1}{2}(\mu_Y - \frac{1}{2}\sigma_Y^2)$.

Problem 5

a. By Ito's lemma,

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \\ &= \left(\frac{\partial V}{\partial t} + (\mu - \delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S \frac{\partial V}{\partial S}dX \end{aligned}$$

b. If the value μ equals to r, drift term of dV becomes equal to left hand side of the pde.

c. Since $\frac{dV}{V}$ can be interpreted to rate of return of option, $\mathrm{E}[\frac{dV}{V}]$ can be interpreted as expected return of the option. If $\mu = r$, then $\mathrm{E}[\frac{dV}{V}] = (\frac{\partial V}{\partial t} + (r-\delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})/V$, which equals r regarding the pde. Therefore, setting μ equals to r includes an assumption that expected rate of return of option is risk-free rate.

d. If $\mu = r$, by Ito's lemma,

$$\begin{aligned} dF &= -(r-\delta)Fdt + \frac{F}{S}((r-\delta)S(t)dt + \sigma S(t)dX(t)) \\ &= -(r-\delta)Fdt + (r-\delta)Fdt + \sigma F dX(t) \\ &= \sigma F dX(t) \end{aligned}$$

It does not have a drift term.