FIN 514: Problem Set #4

Due on Wednesday, March 7, 2018

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Problem 1

a. By Ito's lemma,

$$\begin{aligned}
df(X) &= \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX)^2 \\
&= 2X(t)dX(t) + \frac{1}{2} \times 2dt \\
\Rightarrow X(t)dX(t) &= \frac{1}{2}df(X) - \frac{1}{2}dt \\
\Rightarrow \int_0^t X(\tau)d\tau &= \frac{1}{2}\int_0^t dX^2(\tau) + \int_0^t \frac{1}{2}d\tau \\
&= \frac{1}{2}X^2(t) + \frac{1}{2}t
\end{aligned}$$

b. Let $g(t, X) = tX(t)$. Then by Ito's lemma,

$$\begin{aligned}
dg(t, X) &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial X}dX + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}(dX)^2 \\
&= X(t)dt + tdX(t) \\
\Rightarrow tdX(t) &= d(tX(t)) - X(t)dt \\
\Rightarrow \int_0^t \tau dX(\tau) &= tX(t) - \int_0^t X(\tau)d\tau
\end{aligned}$$

c. Let $h(X) = X^3(t)$. Then by Ito's lemma,

$$\begin{aligned}
dh(X) &= \frac{\partial h}{\partial X}dX + \frac{1}{2}\frac{\partial^2 h}{\partial X^2}(dX)^2 \\
&= 3X^2(t)dX(t) + \frac{1}{2}6X(t)dt \\
&= 3X^2(t)dX(t) + 3X(t)dt \\
\Rightarrow X^2(t)dX(t) &= \frac{1}{3}d(X^3(t)) - X(t)dt \\
\Rightarrow \int_0^t X^2(\tau)dX(\tau) &= \frac{1}{3}X^3(t) - \int_0^t X(\tau)d\tau
\end{aligned}$$