Tree construction

- We will wish to match our binomial distribution to our perceived true distribution of stock returns. If we believe that returns $r_{\Delta t} = \ln(S_{\Delta t}/S_0)$ are normally distributed (!) then stock prices are *lognormally distributed*.

- In the risk-neutral world the mean (μ) and variance (ν) of a normally distributed $r_{\Delta t}$ are:

$$
\begin{aligned}
\mu &= \left(r - \frac{1}{2}\sigma^2\right)\Delta t \\
\nu &= \sigma^2 \Delta t
\end{aligned}
$$

  where σ is the volatility of the stock returns and r is the risk-free rate.

- We have two equations and three unknowns q, u and d, an additional choice/constraint (Joshi, p 57) gives us one possible choice of u and d:

$$u = e^{r\Delta t + \sigma\sqrt{\Delta t}}, \quad d = e^{r\Delta t - \sigma\sqrt{\Delta t}}$$

- In detail, let's consider a general choice of u and d

$$u = e^{\mu\Delta t + \sigma\sqrt{\Delta t}}, \quad d = e^{\mu\Delta t - \sigma\sqrt{\Delta t}}$$

  by expanding the exponential and only considering small steps (i.e ignoring terms of $(\Delta t)^{3/2}$ and smaller:

$$
\begin{aligned}
e^{r\Delta t} &= 1 + r\Delta t + O(\Delta t^2) \\
u &= 1 + \mu\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + O(\Delta t^{3/2}) \\
d &= 1 + \mu\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + O(\Delta t^{3/2}) \\
q &= \frac{(1 + r\Delta t - (1 + \mu\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t))}{1 + \mu\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t - (1 + \mu\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t)} \\
&= \frac{(r-\mu)\Delta t + \sigma\sqrt{\Delta t} - \frac{1}{2}\sigma^2\Delta t}{2\sigma\sqrt{\Delta t}} \\
&= \left(\frac{r - \mu - \frac{1}{2}\sigma^2}{2\sigma}\right)\sqrt{\Delta t} + \frac{1}{2} \\
(1-q) &= \frac{1}{2} - \left(\frac{r - \mu - \frac{1}{2}\sigma^2}{2\sigma}\right)\sqrt{\Delta t}
\end{aligned}
$$

and so

$$
\begin{aligned}
E[r_{\Delta t}] &= q \ln u + (1-q) \ln d \\
&= \left(\left(\frac{r - \mu - \frac{1}{2}\sigma^2}{2\sigma} \right) \sqrt{\Delta t} + \frac{1}{2} \right) \left(\mu \Delta t + \sigma \sqrt{\Delta t} \right) + \left(\frac{1}{2} - \left(\frac{r - \mu - \frac{1}{2}\sigma^2}{2\sigma} \right) \sqrt{\Delta t} \right) \left(\mu \delta t - \sigma \sqrt{\Delta t} \right) \\
&= \mu \Delta t + \left(\frac{r - \mu - \frac{1}{2}\sigma^2}{2\sigma} \right) \sigma \sqrt{\Delta t} + \left(\frac{r - \mu - \frac{1}{2}\sigma^2}{2\sigma} \right) \sigma \sqrt{\Delta t} + O(\Delta t^{3/2}) \\
&= \mu \Delta t + \left(r - \mu - \frac{1}{2}\sigma^2 \right) \Delta t \\
&= \left(r - \frac{1}{2}\sigma^2 \right) \Delta t
\end{aligned}
$$

as required.

- And for the variance:

$$
\begin{aligned}
Var[r_{\Delta t}] &= q \times (\ln u - E[r_{\Delta t}])^2 + (1-q) \times (\ln d - E[r_{\Delta t}])^2 \\
&= q \times \left(\left(r - \mu - \frac{1}{2}\sigma^2 \right) \Delta t + \sigma \sqrt{\Delta t} \right)^2 + (1-q) \times \left(\left(r - \mu - \frac{1}{2}\sigma^2 \right) \Delta t - \sigma \sqrt{\Delta t} \right)^2 \\
&= q\sigma^2 \Delta t + (1-q)\sigma^2 \Delta t + O(\Delta t^{3/2}) \\
&= \sigma^2 \Delta t
\end{aligned}
$$

- And, trivially,

$$
\begin{aligned}
E[S_{\Delta t}] &= quS_0 + (1-q)dS_0 \\
&= S_0 e^{e\Delta t}
\end{aligned}
$$

as we have seen before.