FIN 514: Problem Set #5

Due on Tuesday, March 13, 2018

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Problem 1

Option price must satisfy the following pde. It is analogous that diffusion term has changed from $\sigma S(t)$ to $\sigma(S(t))^\gamma$.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\gamma}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

It can be derived by following procedures.

By Ito's lemma, $dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^{2\gamma}\frac{\partial^2 V}{\partial S^2}dt$. Set $\Pi = V - \Delta S - \beta B$, then by self-financing, $d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\gamma}\frac{\partial^2 V}{\partial S^2})dt + \frac{\partial V}{\partial S}dS - \Delta dS - r\beta B dt$. Choose Δ such that $(\frac{\partial V}{\partial S} - \Delta)dS = 0$, therefore $\Delta = \frac{\partial V}{\partial S}$. Since $\beta B = V - \Delta S$, and by no arbirage argument, $0 = d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\gamma}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV)$.

Problem 2

(a) By Black-Scholes, all derivatives with underlying asset S must follow the following pde.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Terminal boundary condition of this contract is $V(S,T) = \ln(S(T)/S(0))$.

(b) By Ito's lemma,

$$\begin{aligned}
d\ln S &= \frac{1}{S}dS - \frac{1}{2}\frac{1}{S^2}(dS)^2 \\
&= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dX(t) \\
\Rightarrow S(t) &= S(0)\exp((\mu - \frac{1}{2}\sigma^2)t + \sigma X(t)) \\
\Rightarrow V(S,t) &= \ln(S(t)/S(0)) = (\mu - \frac{1}{2}\sigma^2)t + \sigma X(t)
\end{aligned}$$

Problem 3

(a) By Ito's product rule,

$$\begin{aligned}
dS_D &= d(eS) \\
&= edS + Sde + dSde \\
&= e(\mu S dt + \sigma S dX_2) + S(\mu_e e dt + \sigma_e e dX_1) + (\mu S dt + \sigma S dX_2)(\mu_e e dt + \sigma_e e dX_1) \\
&= (\mu e S + \mu_e e S + \sigma\sigma_e \rho e S)dt + \sigma_e e S dX_1 + \sigma e S dX_2 \\
&= (\mu + \mu_e + \sigma\sigma_e\rho)S_D dt + \sigma_e S_D dX_1 + \sigma S_D dX_2
\end{aligned}$$

Problem 3 continued on next page. . .

(b) By Ito's product rule,

$$\begin{aligned}
dB_{KD} &= d(eB_K) \\
&= edB_K + B_K de + dB_K de \\
&= e(r_K B_K dt) + B_K(\mu_e e dt + \sigma_e e dX_1) + (r_K B_K dt)(\mu_e e dt + \sigma_e e dX_1) \\
&= (r_K e B_K + \mu_e e B_K)dt + \sigma_e e B_K dX_1 \\
&= (r_K + \mu_e)B_{KD}dt + \sigma_e B_{KD} dX_1
\end{aligned}$$

Problem 4

(a) By Ito's product rule, $dY_t = d(\frac{S_{1t}}{S_{2t}}) = S_{2t}d(\frac{1}{S_{1t}}) + \frac{1}{S_{1t}}dS_{2t} + d(\frac{1}{S_{1t}})dS_{2t}$. And by Ito's lemma,

$$\begin{aligned}
d(\frac{1}{S_{1t}}) &= -\frac{1}{S_{1t}^2}dS_{1t} + \frac{1}{2} \times 2 \times \frac{1}{S_{1t}^3}(dS_{1t})^2 \\
&= -\frac{1}{S_{1t}^2}(\mu_1 S_{1t}dt + \sigma_1 S_{1t}dX_{1t}) + \frac{1}{S_{1t}^3}\sigma_1^2 S_{1t}^2 dt \\
&= (-\mu_1 + \sigma_1^2)\frac{1}{S_{1t}}dt - \sigma_1\frac{1}{S_{1t}}dX_{1t}
\end{aligned}$$

Therefore, the following equation must hold.

$$\begin{aligned}
dY_t &= S_{2t}(-\mu_1 + \sigma_1^2)\frac{1}{S_{1t}}dt - \sigma_1\frac{1}{S_{1t}}dX_{1t}) + \frac{1}{S_{1t}}(\mu_2 S_{2t}dt + \sigma_2 S_{2t}dX_{2t}) \\
&+ (-\mu_1 + \sigma_1^2)\frac{1}{S_{1t}}dt - \sigma_1\frac{1}{S_{1t}}dX_{1t})(\frac{1}{S_{1t}}(\mu_2 S_{2t}dt + \sigma_2 S_{2t}dX_{2t})) \\
&= (-\mu_1 + \sigma_1^2)Y_t dt - \sigma_1 Y_t dX_{1t} + \mu_2 Y_t dt + \sigma_2 Y_t dX_{1t} - \sigma_1\sigma_2\rho Y_t dt \quad \text{(Since } dX_{1t}dX_{2t} = \rho dt\text{).} \\
&= (-\mu_1 + \mu_2 + \sigma_1^2 - \sigma_1\sigma_2\rho)Y_t dt - \sigma_1 Y_t dX_{1t} + \sigma_2 Y_t dX_{2t}
\end{aligned}$$

If we choose $X_{3t} = \frac{-\sigma_1 X_{1t} + \sigma_2 X_{2t}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}}$, then the equation above is represented as follows.

$$\begin{aligned}
dY_t &= \mu_Y Y_t dt + \sigma_Y Y_t dX_{3t} \\
\text{where } \ \mu_Y &= -\mu_1 + \mu_2 + \sigma_1^2 - \sigma_1\sigma_2\rho, \sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}
\end{aligned}$$

(b) By Ito's product rule,

$$\begin{aligned}
dZ_t &= d(Y_t B_{Et}) \\
&= B_{Et}dY_t + Y_t dB_{Et} + dY_t dB_{Et} \\
&= B_{Et}(\mu_Y Y_t dt + \sigma_Y Y_t dX_{3t}) + Y_t(r_E B_{Et}dt) + (\mu_Y Y_t dt + \sigma_Y Y_t dX_{3t})(Y_t r_E B_{Et} dt) \\
&= (\mu_Y + r_E)Y_t B_{Et}dt + \sigma_Y Y_t B_{Et} dX_{3t} \\
&= \mu_Z Z_t dt + \sigma_Y Z_t dX_{3t}
\end{aligned}$$

where $\mu_Z = \mu_Y + r_E$.

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(c) $U(Y,T) = \frac{V(S_{1T},S_{2T},T)}{S_{1T}} = \frac{1000\times\max(S_{2T}-S_{1T},0)}{S_{1T}} = 1000\times\max(Y_T-1,0)$. Payoff of exchange option denominated in GBP is same as payoff of an European option whose underlying asset is price of Euros in terms of GBP and strike price is 1.

(d) Let $\Pi_t = \frac{\pi_t}{S_{1t}}$, then $\Pi_t = -U_t + \Delta_t Z_t + \beta_t B_{Pt}$. Therefore, the following equation has to follow.

$$
\begin{aligned}
d\Pi_t &= -dU_t + \Delta_t dZ_t + \beta_t dB_{Pt} \quad \text{by self-financing.}\\
&= -(\frac{\partial U_t}{\partial t} + \mu_Y Y_t\frac{\partial U_t}{\partial Y_t} + \frac{1}{2}\sigma_Y^2 Y_t^2\frac{\partial^2 U_t}{\partial {Y_t}^2})dt + \Delta_t\mu_Z Z_t dt + \beta_t r_P B_{Pt}dt - \sigma_Y Y_t\frac{\partial U_t}{\partial Y_t}dX_{3t} + \Delta_t\sigma_Y Z_t dX_{3t}
\end{aligned}
$$

Choose Δ_t so that $-\sigma_Y Y_t\frac{\partial U_t}{\partial Y_t} + \Delta_t\sigma_Y Z_t = 0$, then $\Delta_t = \frac{Y_t}{Z_t}\frac{\partial U_t}{\partial Y_t} = B_{Et}\frac{\partial U_t}{\partial Y_t}$, and therefore,

$$
\begin{aligned}
-(\frac{\partial U_t}{\partial t} + \mu_Y Y_t\frac{\partial U_t}{\partial Y_t} + \frac{1}{2}\sigma_Y^2 Y_t^2\frac{\partial^2 U_t}{\partial {Y_t}^2})dt + \frac{Y_t}{Z_t}\frac{\partial U_t}{\partial Z_t}\mu_Z Z_t dt + r_P(U_t - \frac{Y_t}{Z_t}\frac{\partial U_t}{\partial Y_t}Z_t)dt &= 0\\
\Rightarrow \frac{\partial U_t}{\partial t} + \mu_Y Y_t\frac{\partial U_t}{\partial Y_t} + \frac{1}{2}\sigma_Y^2 Y_t^2\frac{\partial^2 U_t}{\partial {Y_t}^2} - \mu_Z Y_t\frac{\partial U_t}{\partial Y_t} + r_P Y_t\frac{\partial U_t}{\partial Y_t} - r_P U_t &= 0\\
\Rightarrow \frac{\partial U_t}{\partial t} + (\mu_Y - \mu_Z + r_P)Y_t\frac{\partial U_t}{\partial Y_t} + \frac{1}{2}\sigma_Y^2 Y_t^2\frac{\partial^2 U_t}{\partial {Y_t}^2} - r_P U_t &= 0
\end{aligned}
$$

Since $\mu_Y - \mu_Z = -r_E$, the following pde holds.

$$
\frac{\partial U_t}{\partial t} + (r_P - r_E)Y_t\frac{\partial U_t}{\partial Y_t} + \frac{1}{2}\sigma_Y^2 Y_t^2\frac{\partial^2 U_t}{\partial {Y_t}^2} - r_P U_t = 0
$$

(e) Since $U(Y,T) = 1000\times\max(Y_T-1,0)$, by analogy of European call option, $U(Y,0)$ is evaluated as:

$$
\begin{aligned}
U(Y,0) &= e^{-r_P T}(Y_t e^{(r_P-r_E)T}N(d_1) - N(d_2))\\
\text{where } \ d_1 &= \frac{\ln Y_t + (r_P - r_E + \frac{1}{2}\sigma_Y^2)T}{\sigma_Y\sqrt{T}}\\
d_2 &= d_1 - \sigma_Y\sqrt{T}
\end{aligned}
$$

Therefore, $V(S_1,S_2,0) = S_{10}\times U(Y,0)$ is evaluated as:

$$
\begin{aligned}
V(S_1,S_2,0) &= S_{10}e^{-r_P T}(Y_t e^{(r_P-r_E)T}N(d_1) - N(d_2))\\
&= e^{-r_P T}(S_{20}e^{(r_P-r_E)T}N(d_1) - S_{10}N(d_2))
\end{aligned}
$$

Problem 5

(a) By Ito's product rule,

$$
\begin{aligned}
dY &= d(S_1S_2)\\
&= S_1 dS_2 + S_2 dS_1 + dS_1 dS_2\\
&= S_1(\mu_2 S_2 dt + \sigma_2 S_2 dX_2) + S_2(\mu_1 S_1 dt + \sigma_1 S_1 dX_1) + (\mu_1 S_1 dt + \sigma_1 S_1 dX_1)(\mu_2 S_2 dt + \sigma_2 S_2 dX_2)\\
&= (\mu_1 + \mu_2 + \sigma_1\sigma_2\rho)Y_t dt + \sigma_1 Y dX_1 + \sigma_2 Y dX_2\\
&= \mu_Y Y dt + \sigma_Y Y dX_3
\end{aligned}
$$

where $\mu_Y = \mu_1 + \mu_2 + \sigma_1\sigma_2\rho, \sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho}, X_3 = \frac{\sigma_1 X_1 + \sigma_2 X_2}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho}}$

Problem 5 continued on next page...

(b) Payoff can be represented as $\max((S_1S_2)^{\frac{1}{2}} - K, 0) = \max(Y^{\frac{1}{2}} - K, 0)$.

(c) By Ito's lemma,

$$\begin{aligned} dZ &= \frac{\partial Z}{\partial Y}dY + \frac{1}{2}\frac{\partial^2 Z}{\partial Y^2}(dY)^2 \\ &= \frac{1}{2}Y^{-\frac{1}{2}}(\mu_Y Y dt + \sigma_Y Y dX_3) + \frac{1}{2}(-\frac{1}{4})Y^{-\frac{3}{2}}\sigma_Y^2 Y^2 dt \\ &= \frac{1}{2}(\mu_Y - \frac{1}{4}\sigma_Y^2)Y^{\frac{1}{2}}dt + \frac{1}{2}\sigma_Y Y^{\frac{1}{2}}dX_3 \end{aligned}$$

(d) Since we assume self-financing portfolio,

$$\begin{aligned} d\pi &= -dV + \Delta dZ + \beta dB \\ &= -(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial Z}dZ + \frac{1}{2}\sigma_Y^2 Z^2 dt) + \Delta dZ + \beta rBdt \end{aligned}$$

Choose $\Delta = \frac{\partial V}{\partial Z}$, then

$$\begin{aligned} d\pi = -(\frac{\partial Z}{\partial t} + \frac{1}{2}\sigma_Y^2 Z^2)dt + r(V - \frac{\partial V}{\partial Z})dt = 0 \\ \Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma_Y^2 Z^2 + rZ\frac{\partial V}{\partial Z} - rV = 0 \end{aligned}$$

Therefore, $V(Z,t)$ follows Black-Scholes pde in terms of Z.

(e) From the general Black-Scholes pde: $\frac{\partial V}{\partial t} + \frac{1}{2}a^2X^2\frac{\partial V}{\partial X} + bX\frac{\partial V}{\partial X} - cV = 0$, the risk-neutral expected return of underlying asset is b, the coefficient of first partial derivatives. Therefore, risk-neutral expeced return of geometic average of two stocks is r, the risk-free rate.

(f) By analogy of evaluating European call option from Black-Scholes pde, value of geometric average call option is evaluated as follows.

$$\begin{aligned} V(S_1, S_2, t) &= e^{-r(T-t)}(Ze^{r(T-t)}N(d_1) - KN(d_2)) \\ &= e^{-r(T-t)}((S_1S_2)^{\frac{1}{2}})e^{r(T-t)}N(d_1) - KN(d_2)) \\ \text{where} \quad d_1 &= \frac{\ln((S_1S_2)^{\frac{1}{2}}/K) + (r + \frac{1}{2}\sigma_Y^2)(T-t)}{\sigma_Y\sqrt{T-t}} \\ d_2 &= d_1 - \sigma_Y\sqrt{T-t} \end{aligned}$$

(g) If correlation between two stocks increases, option value also increases. It is because if correlation gets higher, volatility of geometric average also gets higher, and option value increases as volatility increases. If there is negative correlation between stocks, although one of them increases, the other decreases, therefore geometric average does not changes much. Mathematically, since $\sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho}$, if correlation ρ increases, σ_Y increases, hence option value increases.