FIN 514: Problem Set #5

Due on Tuesday, March 13, 2018

Wanbae Park

Problem 1

Option price must satisfy the following pde. It is analogous that diffusion term has changed from $\sigma S(t)$ to $\sigma(S(t))^\gamma$.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\gamma}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

It can be derived by following procedures.

By Ito's lemma, $dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^{2\gamma}\frac{\partial^2 V}{\partial S^2}dt$. Set $\Pi = V - \Delta S - \beta B$, then by self-financing, $d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\gamma}\frac{\partial^2 V}{\partial S^2})dt + \frac{\partial V}{\partial S}dS - \Delta dS - r\beta B dt$. Choose Δ such that $(\frac{\partial V}{\partial S} - \Delta)dS = 0$, therefore $\Delta = \frac{\partial V}{\partial S}$. Since $\beta B = V - \Delta S$, and by no arbirage argument, $0 = d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\gamma}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV)$.

Problem 2

(a) By Black-Scholes, all derivatives with underlying asset S must follow the following pde.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Terminal boundary condition of this contract is $V(S,T) = \ln(S(T)/S(0))$.

(b) By Ito's lemma,

$$\begin{aligned} d\ln S &= \frac{1}{S}dS - \frac{1}{2}\frac{1}{S^2}(dS)^2 \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dX(t) \\ \Rightarrow S(t) &= S(0)\exp((\mu - \frac{1}{2}\sigma^2)t + \sigma X(t)) \\ \Rightarrow V(S,t) &= \ln(S(t)/S(0)) = (\mu - \frac{1}{2}\sigma^2)t + \sigma X(t) \end{aligned}$$

Problem 3

(a) By Ito's product rule,

$$\begin{aligned} dS_D &= d(eS) \\ &= edS + Sde + dSde \\ &= e(\mu S dt + \sigma S dX_2) + S(\mu_e e dt + \sigma_e e dX_1) + (\mu S dt + \sigma S dX_2)(\mu_e e dt + \sigma_e e dX_1) \\ &= (\mu e S + \mu_e e S + \sigma\sigma_e \rho e S)dt + \sigma_e e S dX_1 + \sigma e S dX_2 \\ &= (\mu + \mu_e + \sigma\sigma_e\rho)S_D dt + \sigma_e S_D dX_1 + \sigma S_D dX_2 \end{aligned}$$

Problem 3 continued on next page. . .

(b) By Ito's product rule,

$$\begin{aligned}
dB_{KD} &= d(eB_K) \\
&= e dB_K + B_K de + dB_K de \\
&= e(r_K B_K dt) + B_K(\mu_e e dt + \sigma_e e dX_1) + (r_K B_K dt)(\mu_e e dt + \sigma_e e dX_1) \\
&= (r_K e B_K + \mu_e e B_K) dt + \sigma_e e B_K dX_1 \\
&= (r_K + \mu_e) B_{KD} dt + \sigma_e B_{KD} dX_1
\end{aligned}$$