Fin 514: Financial Engineering II

Lecture 14: The Probabilistic Solution

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Outline

- We will now approach the option pricing problem from a probabilistic angle. You can think of this as an extension of our discrete option pricing problem.
- Some problems are a lot simpler when we consider the probability approach and it will finally link us with Monte Carlo methods.
- In particular we will introduce the following ideas:
  - Martingales (discrete and continuous)
  - Changing of probability measure (Girsanov's theorem)
  - Uniqueness of change of measure
  - Uniqueness of risk-neutral or martingale measure and no arbitrage/complete markets
  - Option pricing under this approach

Recap on discrete pricing models

- In earlier sessions we saw that in complete markets then there were no arbitrage opportunities if and only if there existed a unique pricing measure. This pricing measure could easily be interpreted as risk-neutral probabilities in that:

$$p_i = \sum_{j=1}^{M} D_{ij}\pi_j = \frac{1}{1+R}\sum_{j=1}^{M} D_{ij}\hat{\pi}_j = \frac{1}{1+R}E^Q[D_i]$$

- We would like to extend this theory to attempt to price derivatives (or other assets) within our continuous time, stochastic process set up.
- First, we will formalize the discrete time and then progress to the continuous case.

Discrete time martingales

- Consider a discrete stochastic process X_t (you can think of this as a stock price in a binomial tree). X_t is called **a Martingale** if for all $T > t$
$$E_t[X_T] = X_t$$
where E_t means an expectation conditional upon all the information available until time t (technically called a filtration $\mathcal{F}_t$). If all the information is contained in the price of X_t then:
$$E[X_T|X_t] = X_t$$
- If this is within a binomial tree, you can think of this information as knowing the path which X_t took up until time t.
- If we are working in a zero interest rate world then the price of an asset is a martingale (driftless) under the risk-neutral measure as
$$p_i = E^Q[D_i]$$
or
$$X_0 = E_0^Q[X_1]$$

Aside: technical notes

- The conditional operator E_t^Q depends upon a probability measure Q and a history or **filtration**, $\mathcal{F}_t$. The Q tells us which probabilities to use. The filtration tells us where to take the expectation from (or the path already taken) and is itself a random variable.
- A **pre-visible process** $\phi = \phi_i$ is a process (in the same tree) whose value at any time i is dependent only upon the history one time period earlier $\mathcal{F}_{i-1}$.
- The pre-visible process is useful when setting up a replicating portfolio and in dealing with deterministic processes. In general we can think of a pre-visible process as a portfolio or a deterministic price (such as the risk-free asset, B).

Quiz

Which of the following are Martingales?

- Stock price, S.
- Option price, V.
- Stock price, S, under risk-neutral measure Q when $R = 0$
- Bond price, B, under risk-neutral measure Q.
- The total received, X, in a gambling game where you toss a coin a finite number of times, receiving \$1 for each head and paying \$1 for each tail?

Informally: martingales and finance

- Thus, when $R = 0$. no arbitrage implies the existence of a unique martingale probability *measure* (called the **risk-neutral measure**) and we can price assets using this measure.
- What about when $R > 0$?
- If we now introduce our risk-free bond, with initial value \$1 $(= B_0)$ and value $B_t = 1 + R$ (or e^{rt}) at time t, then we see that for an asset with price S_t at time t then

$$E_0^Q\left[\frac{S_t}{B_t}\right] = \frac{1}{1+R}E_0^Q[S_t] = \frac{S_0}{B_0}$$

  and so S/B is a martingale with respect to this risk-neutral measure.
- Thus, even with interest rates, in complete markets with no arbitrage there exists a unique martingale measure Q.

More formally: Binomial representation theorem

- If we have a probability measure Q such that the binomial price process S is a martingale and if N is another martingale under Q then there is a pre-visible process ϕ (think of a portfolio of stocks) such that

$$N_t = N_0 + \sum_{k=1}^{t} \phi_k \Delta S_k$$

where $\Delta S_k = S_k - S_{k-1}$.

- This is similar to a typical delta hedging argument. If we choose:

$$\phi_k = \frac{N_{up} - N_{down}}{S_{up} - S_{down}}$$

then, ϕ_k can be calculated (i.e is deterministic) at any point in time and

$$\Delta N_k = \phi_k \Delta S_k$$

as N and S are both martingales and thus their changes conditional on the path have zero expectation. Thus, as $N_t = N_0 + \sum_{k=1}^{t} \Delta N_k$ then the theorem holds.

Link to discrete time finance

- The first thing to do is to turn our option payoff (technically called **a claim**) X into a stochastic process by defining:

$$\eta_t = E_t^Q[X]$$

- Next the bond price is deterministic and so it is *previsible* and also positive, and we assume that $B_0 = 1$. We can also define the discount process $1/B_t$, which is also previsible.
- Now, to get to martingales, consider $Z_t = S_t/B_t$ which is the discounted stock process.
- Finally introduce the discounted option payoff (claim) X/B_T

Option is a martingale

- For any option payoff X the process $\eta_t = E_t^P(X)$ is always a P martingale, i.e. a martingale under any probability measure.
- To see that this is true use the fact that

$$E_s^P[E_t^P[X]] = E_s^P[X]$$

  so if we condition on history up until s and then conditioning on the history until t is the same as conditioning originally up to time s. This is called the **tower law**.
- This is more obvious than it initially appears!

Quiz

What result would ensure that η_t was a martingale ($s > t$)?

- $\eta_t = \eta_s$.
- $\eta_t = E_t[\eta_s]$
- $X_t = E_t[X_s]$

Tower law: example

- Let's look at a stock price process:

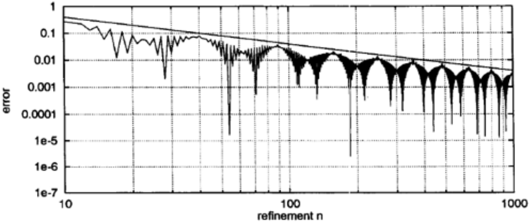


Fig. 4. Graphical representation and examination of the error bound; x-axis and y-axis with log-scale; example with CRR-model and the following selection of parameters: $S = 100$, $K = 110$, $T = 1$, $r = 0.05$, $\sigma = 0.3$, $n = 10, \ldots, 1000$.

and now consider

$$\eta_t = E_t[S_2]$$

for example.

Tower law: example

- And so

$$\eta_0 = \frac{180+80+72+36}{4} = 92$$

- In the up state

$$\eta_1 = \frac{180+80}{2} = 130$$

and in the down state

$$\eta_1 = \frac{72+36}{2} = 54$$

- And so let's analyze η_t,

$$E_0[\eta_1] = \frac{130+54}{2} = 92$$

which is of course, η_0 and so η_t is trivially a martingale.

Replication strategy

- Now choose a probability measure Q such that Z (recall $Z_t = S_t/B_t$) is a Q martingale (this is just our standard risk neutral measure) then there is a Q martingale process $\eta_t = E_t^Q[X/B_T]$.
- There is then a previsible process (portfolio) ϕ such that

$$\eta_t = \eta_0 + \sum_{k=1}^{t} \phi_k \Delta Z_k$$

- So at time t you purchase ϕ_{t+1} units of stock and $\psi_{t+1} = \eta_t - \phi_{t+1} S_t/B_t$ of the cash bond. As we have done in our binomial model
- At time 0, $\Pi_0 = \phi_1 S_0 + \psi_1 B_0 = \eta_0 = E_0^Q[X/B_T]$.
- Then the portfolio rolls forward, consider time $= 1$. Now we have

$$\begin{aligned}
\Pi_1 &= \phi_1 S_1 + \psi_1 B_1 \\
&= B_1(\eta_0 + \phi_1(S_1/B_1 - S_0/B_0)) \\
&= B_1(\eta_0 + \phi_1 \Delta Z_1) \\
&= B_1 \eta_1
\end{aligned}$$

Replication strategy

- This is the cost of the new portfolio as $B_1\eta_1 = \phi_2 S_1 + \psi_2 B_1$, which in turn will go on to be worth $B_2\eta_2$ at the end of the next period and so on.
- Then the portfolio rolls forward again, regardless of how S changes, until at expiry the portfolio is worth $B_T X/B_T = X$ (just as with the binomial model).
- So the current price of the option is the initial value of the replicating portfolio, $\Pi_0 = \eta_0 = E_0^Q[X/B_T]$ or the expected value of the option under the Q (risk-neutral) measure.
- And, in general, $\Pi_t = B_t\eta_t = B_t E_t^Q[X/B_T]$

Quiz

What does this look like?

- An arbitrage strategy
- A self financing portfolio
- An ito process

Self-financing

- What is the definition of self financing here?
- The portfolio is self financing if the closing (i.e time t) value of portfolio Π_{t-1} constructed at t is worth $\Pi_t = \phi_{t-1} S_t + \psi_{t-1} B_t$.
- The financing gap is given by

$$D_t = \Pi_t - \phi_{t-1} S_t - \psi_{t-1} B_t$$

or alternatively:

$$\Delta\Pi_{t-1} = \phi_{t-1}\Delta S_{t-1} + \psi_{t-1}\Delta B_{t-1} + D_t$$

and so the gap D_t is only zero if the changes in the strategy only come from changes in the stock and the bond prices (as we had before).

Recap

- To recap, if we have a binomial tree then (ϕ_t, ψ_t) is a self financing strategy that can construct any payoff if
  - ϕ and ψ are previsible
  - the change in the value of the portfolio, Π, obeys the difference equation:

$$\Delta\Pi_t = \phi_t \Delta S_t + \psi_t \Delta B_t$$

  - $\phi_T S_T + \psi_T B_T = X$
- In this scenario then the time t value, V_t, of the option with payoff X maturing at time T is

$$V_t = B_t E_t^Q \left[\frac{X}{B_T} \right]$$

as under Q the discounted stock price is a martingale then so is the option according to the binomial representation theorem.

- Q exists and is unique under no arbitrage and complete markets.

Quiz

What is true about the process V_t/B_t here?

- It has a positive drift
- It is a martingale
- It mean reverts

Aside: why do we have to make Z the martingale?

- Above, we made $Z = S/B$ the martingale and not S, itself. Why not choose S.
- It is also possible to determine a probability where the stock price is a martingale. Why do we not choose this probability for option valuation?
- Let's see what would happen here: the Binomial representation theorem gives us:

$$\eta_t = \eta_0 + \sum_{k=1}^{t} \phi_k \Delta S_k$$

  as S is now the martingale.
- Now, at $t = 0$, hold ϕ_1 of stock and $\psi_1 = \eta_0 - \phi_1 S_0$ in the risk-free asset, where $B_0 = 1$

Aside: why do we have to make Z the martingale?

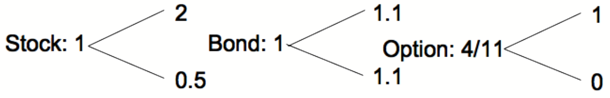
- Then,

$$\begin{aligned} \Pi_1 &= \phi_1 S_1 + \psi_1 B_1 \\ &= \phi_1 S_1 - (\eta_0 - \phi_1 S_0) B_1 \\ &= \phi_1 (S_1 - S_0) + \eta_0 B_1 \\ &= (\eta_1 - \eta_0) + \eta_0 B_1 \end{aligned}$$

- This last expression is not the same as either η_1 or $B_1 \eta_1$.
- So, here, the portfolio would not replicate the final payoff of the derivative, X, at maturity and so is not a self-financing, replicating portfolio.
- Note, it could still be replicating if we moved money in or out of the portfolio but that would not be self-financing.
- Also not that in the special case where $r = 0$ and $B_t = 1$ for all t, then $\Pi_1 = \eta_1$ and the portfolio then replicates the payoff at maturity - see later!

Easy numerical example

- Consider a stock, bond and option in a one period world



- Under the risk-neutral measures, $q = 0.4$ and $1 - q = 0.6$, we have:

$$E_0^Q\left[\frac{S(t)}{B(t)}\right] = \frac{0.4 \times 2 + 0.6 \times 0.5}{1.1} = \frac{1.1}{1.1} = 1 = \frac{S(0)}{B(0)}$$

  thus S/B is a Martingale under the risk-neutral measure Q, where the unique nature of Q means that this also implies no arbitrage opportunities.

- We also have:

$$E_0^Q\left[\frac{V(t)}{B(t)}\right] = \frac{0.4 \times 1 + 0.6 \times 0}{1.1} = \frac{0.4}{1.1} = 4/11 = \frac{V(0)}{B(0)}$$

Easy numerical example

- So under our risk neutral measure V/B is also a martingale, and as we discussed in the fundamental theorem of finance, the risk-neutral measure exists and is unique if and only if we have complete markets and no arbitrage. Also recall that we can then apply the same measure to price all replicable assets (such as derivatives).
- What we intend to do is to extend this to continuous time in which case we can use martingale theory to derive the Black-Scholes PDE and calculate the value of European call and put options.

Quiz

What choice of q and $1-q$ would make the **stock price** a martingale?

- It's not possible
- $q = 2/3, 1-q = 1/3$
- $q = 0.5, 1-q = 0.5$
- $q = 1/3, 1-q = 2/3$

Continuous Martingales

- The martingale representation of the discrete time setting was not completely necessary, as we had derived the pricing methodology without it.
- However, in continuous time martingales will play a very important role. We will summarize the basic results here.
- A stochastic process M_t is a martingale, with respect to a probability measure P if for all $T > t$
  - $E_s[M_t] = M_s$
  - $E[|M_t|] < \infty$

  where E_t denotes an expectation conditional upon all the information available until time t.
- If $W(t)$ (note: change in notation) is a Brownian motion then it is a martingale, as

$$W(T) = W(t) + \int_t^T dW(s)$$

Continuous Martingales

- . . . thus
$$E_t[W(T)] = W(t) + 0 = W(t)$$
- Finally. for any payoff X depending only upon events up to time T the process
$$\eta_t = E_t^P[X]$$
is a martingale.
- This follows from the tower law,
$$E_s^P\left[E_t^P[X]\right] = E_s^P[X]$$
as in the discrete setting.

Quiz

Which of the following are martingales?

- Stock price, S
- Option price, V
- Bond price, B
- $\sigma S dW$
- Stock price, S under risk-neutral measure Q
- Stock price, S under risk-neutral measure Q when $\mathrm{R} = 0$
- Bond price, B under risk-neutral measure Q.

Martingale representation theorem

- We will need a continuous time version of our binomial representation theorem. This is the martingale representation theorem:
- If M_t is a Q martingale process, whose volatility σ_t is always non-zero. Then if N_t is any other Q martingale then there exists a previsible process (portfolio) ϕ and N can be written as:

$$N_t = N_0 + \int_0^t \phi_s dM_s$$

ϕ is also unique.

Driftless processes

- We need a way to determine if processes are martingales from our continuous SDEs.
- In our SDEs we had a drift term. Do martingales always have zero drift, and can martingales always be represented as an SDE with zero drift?
- If X_t is a martingale then it can be written as:

$$X_t = X_0 + \int_0^t \phi_s dW_s$$

where W is a Brownian motion and so $dX_t = \phi_t dW_t$.

- **In general we find that if X is a stochastic process with volatility σ_t, $dX_t = \mu_t dt + \sigma_t dW_t$ then**

$$\boxed{\text{X is a martingale} \Leftrightarrow \text{X is driftless, } \mu_t = 0}$$

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Quiz

Which of the following are martingales?

- S where $dS = \mu S dt + \sigma S dW$.
- B where $dB = rBdt$.
- F where $dF = \sigma F dW$
- V where $dV = rdt + \sigma dW$
- V/B where $d(V/B) = \sigma(V/B)dW$
- r where $dr = k(\theta - r)dt + \sigma dW$

Continuous portfolios

- Now, with analogy to the binomial model, we will define a portfolio. A portfolio is a pair of processes ϕ_t and ψ_t which describe the number of units of stock and bond which we are holding at time t. The stock component, ϕ should be pre-visible, depending upon information available at time t.
- The self financing condition will be the same as in the PDE derivation:

$$\Delta \Pi_t = \phi_t \Delta S_t + \psi_t \Delta B_t$$

- A replicating strategy for a payoff X is a self financing portfolio (ϕ, ψ) such that $\Pi_T = \phi_T S_T + \psi_T B_T = X$.
- To progress further we need a new theorem.

Girsanov's theorem: Part 1

1. **If W is a Brownian motion under the probability measure P, then there exists an equivalent measure Q such that $\tilde{W}_t = W_t + \gamma t$ is a Brownian motion.**

- At this point in time we will not say how to convert between P and Q, only that it is possible to do so.

Example of part 1

- Suppose we have a process:

$$dX_t = \mu_t dt + \sigma_t dW_t$$

  where W is a Brownian motion under P and we wished to change the measure so that the drift was ν_t instead.

- Under Q we have

$$dX_t = \mu_t dt - \gamma\sigma_t dt + \sigma_t d\tilde{W}_t$$

  and so if we choose:

$$\gamma = \frac{\mu_t - \nu_t}{\sigma_t}$$

  then

$$dX_t = \nu_t dt + \sigma d\tilde{W}_t$$

Quiz

If we wished dX_t to have zero drift in the above example, what should γ equal?

- $\gamma = 0$.
- $\gamma = \mu_t$.
- $\gamma = \mu_t / \sigma_t$
- $\gamma = (\mu_t - r_t)/\sigma_t$

Zero Interest rates

- Now consider an abritrary derivative with payoff X at time T, let's say an option. We will want to find a replicating strategy, (ϕ_t, ψ_t). Here, we initially assume that interest rates are zero.
- We carry out a three step process:
  1. Find the probability measure Q under which S_t is a martingale (using Girsanov's theorem).
  2. Form a stochastic process for the current value of X, under this measure, $\eta_t = E_t^Q[X]$.
  3. Find the previsible process (portfolio) ϕ_t such that $d\eta_t = \phi_t dS_t$, to replicate the option payoff. Then conclude that the value of the option at any time must be η_t.

Zero Interest rates

- Changing measure is easy now that we have the Girsanov theorem. Starting from the description of S under P (μ and σ are constants here):

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- According to the Girsanov theorem there is a measure Q such that $\tilde{W} = W_t + \gamma t$ is a Brownian motion, so choosing $\gamma_t = \mu/\sigma$ makes S_t a martingale (as σ is constant):

$$dS_t = \sigma S_t d\tilde{W}_t.$$

- The current expectation of the option value $\eta_t = E_t^Q[X]$ is also a martingale under Q (as it is under any measure) and so we can then use the martingale representation theorem to say that:

$$\eta_t = E_t^Q[X] = E_0^Q[X] + \int_0^t \phi_s dS_s$$

Zero Interest rates

- To replicate the option value we form a portfolio consisting of ϕ_t of stock and $\psi_t = \eta_t - \phi_t S_t$ of the bond at time t. Recall that $B_t = 1$ for all t here.
- At a given time t the value of the portfolio is:

$$\Pi_t = \phi_t S_t + \psi_t = \eta_t$$

- So $d\Pi_t = d\eta_t$ so this portfolio is trivially self financing as $dB_t = 0$ and so

$$\begin{aligned} d\Pi_t &= \phi_t dS_t + \psi_t dB_t \\ &= \phi_t dS_t \end{aligned}$$

- Thus, as $\eta_T = X$ we have a replicating strategy for X and so we have a no arbitrage price for X at all times and at time zero we have, very simply that:

$$\text{Time 0 value of option} = \Pi_0 = \eta_0 = E_0^Q[X]$$

or the time zero expectation under the Q measure.

Quick comments

- First, note that this replication strategy worked for any arbitrary claim - this was not specifically a call option or a put option.
- Second, note how easy the pricing formula is, it's just the current expected value of the claim (under the appropriate probability measure)
- Finally, note that the process followed by S_t under this measure is also remarkably simple, enabling a reasonably straightforward calculation of the expectation of the option payoff, X, if it is a function of S_T, see later.

Quiz

What is S_T under Q, here?

- $S_T = S_0$
- $S_T = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)T + \sigma W(T)\right)$
- $S_T = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma W(T)\right)$
- $S_T = S_0 \exp\left(-\frac{1}{2}\sigma^2 T + \sigma W(T)\right)$

Full derivation ($r \neq 0$)

- If r is non-zero we can't ignore it and it gets in the way of making the replicating portfolio self-financing.
- Instead we change our methodology slightly by considering the discounted value of the stock and option. Call $1/B_t$ the discount process and introduce a discounted stock $Z_t = S_t/B_t$ and a discounted payoff X/B_T.
- First, let's try to make Z_t a martingale, under P:

$$\begin{aligned}
d\left(\frac{S}{B}\right) &= \frac{dS}{B} + Sd\left(\frac{1}{B}\right) \\
d\left(\frac{1}{B}\right) &= -\frac{r}{B}dt \\
d\left(\frac{S}{B}\right) &= (\mu - r)\frac{S}{B}dt + \sigma\frac{S}{B}dW \\
dZ_t &= (\mu - r)Z_t dt + \sigma Z_t dW
\end{aligned}$$

Full derivation ($r \neq 0$)

- Using Girsanov's theorem, choosing $\gamma_t = (\mu - r)/\sigma$ makes Z_t into a martingale:
$$dZ_t = \sigma Z_t d\tilde{W}_t$$

- Now consider the discounted claim process $\eta_t = E_t^Q \left[\frac{X}{B_T} \right]$ which is also a Q martingale.
- Now we need to try to form a replicating portfolio. Our martingale represenation theorem tells us that $d\eta_t = \phi_t dZ_t$ so let's hold ϕ_t of the stock at time t and $\psi_t = \eta_t - \phi_t Z_t$ of the bond.
- This gives that
$$\Pi_t = \phi_t S_t + \psi_t B_t = B_t \eta_t$$

Quiz

What process does the stock price follow under the new measure, Q?

- $dS_t = \sigma S_t d\tilde{W}_t$
- $dS_t = (r - \frac{1}{2}\sigma^2)S_t dt + \sigma S_t d\tilde{W}_t$
- $dS_t = rS_t dt + \sigma S_t d\tilde{W}_t$
- $dS_t = (\mu - \frac{1}{2}\sigma^2)S_t dt + \sigma S_t d\tilde{W}_t$

Quiz

What is does γ_t resemble?

- Stock beta.
- Sharpe ratio.
- Portfolio weight.
- Risk premium.

Full derivation ($r \neq 0$)

- But we know that at all points $\Pi_t = \eta_t B_t$ then:

$$\begin{aligned} d\Pi_t &= B_t d\eta_t + \eta_t dB_t \\ &= \phi_t B_t dZ_t + \eta_t dB_t \\ &= \phi_t B_t dZ_t + (\phi_t Z_t + \psi_t) dB_t \\ &= \phi_t (B_t dZ_t + Z_t dB_t) + \psi_t dB_t \end{aligned}$$

- But $(B_t dZ_t + Z_t dB_t) = d(B_t Z_t) = dS_t$ and so,

$$d\Pi_t = \phi_t dS_t + \psi_t dB_t$$

and so the portfolio is self-financing.

Full derivation ($r \neq 0$)

- Now, $\Pi_T = X$ and so the time t value of the option (or general claim), V_t is given by:

$$V_t = \Pi_t = B_t E_t^Q \left[\frac{X}{B_T} \right]$$

where Q is the measure under which the discounted stock price is a martingale.

- Again, this is a remarkably simple formula and one we can use to value any claim.

- **Note, that if $B_t = B_0 e^{rt}$ then**

$$\boxed{V_t = e^{-r(T-t)} E_t^Q \left[V_T \right]}$$

where $V_T = X$ is the payoff at maturity.

Valuing a call option

- Going back to a call option where the terminal value of the claim $X = V_T = \max(S_T - K, 0)$ then

$$\frac{V_0}{B_0} = E_0^Q \left[\frac{V_T}{B_T} \right]$$

- Under Q we have:

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t$$

but we can solve this SDE directly (and use $B_T = B_0 e^{rT}$) to get

$$V_0 = e^{-rT} E_0^Q \left[\max \left\{ S_0 \exp \left(rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\phi \right) - K, 0 \right\} \right]$$

where ϕ is drawn from a standard Normal distribution.

Valuing a call option

- As we know the pdf of the Normal distribution we can write

$$V_0 = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \max\left[\left\{ S_0 \exp\left(rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x \right) - K, 0 \right\}\right] dx$$

- The integrand is zero unless

$$x \geq \frac{\ln(K/S_0) + \frac{1}{2}\sigma^2 T - rT}{\sigma\sqrt{T}} (= L)$$

thus,

$$\begin{aligned} V_0 &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{L}^{\infty} e^{-x^2/2} S_0 \exp\left(rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x \right) dx \\ &\quad - \frac{e^{-rT}}{\sqrt{2\pi}} \int_{L}^{\infty} e^{-x^2/2} K dx \\ &= A(S_0, t) - B(S_0, t) \end{aligned}$$

Call option valuation: $B(S_0, t)$

- Taking $B(S_0, t)$ first

$$B(S_0, t) = e^{-rT} K \frac{1}{\sqrt{2\pi}} \int_L^{\infty} e^{-\frac{x^2}{2}} dx$$

- This is just a constant term multiplied by the upper tail of a Normal distribution. Thus,

$$\begin{aligned} -B(S_0, t) &= -e^{-rT} K(1 - N(L)) \\ &= -e^{-rT} K N(-L) \\ &= -e^{-rT} K N(d_2) \end{aligned}$$

where

$$d_2 = -L = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Call option valuation: $A(S_0, t)$

- Next,

$$
\begin{aligned}
A(S_0, t) &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_L^\infty S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x} e^{-\frac{x^2}{2}} dx \\
&= \frac{S_0}{\sqrt{2\pi}} \int_L^\infty e^{-\frac{1}{2}\sigma^2 T+\sigma\sqrt{T}x-\frac{x^2}{2}} dx \\
&= \frac{S_0}{\sqrt{2\pi}} \int_L^\infty e^{-(x-\sigma\sqrt{T})^2/2} dx
\end{aligned}
$$

- Introduce a new variable $y = x - \sigma\sqrt{T}$ then we have

$$A(S_0, t) = \frac{S_0}{\sqrt{2\pi}} \int_{L-\sigma\sqrt{T}} {}^{\infty} e^{-y^2/2} dy.$$

Call option valuation: $A(S_0, t)$

- This is just a constant term multiplied by the upper tail of a Normal distribution. Thus,

$$\begin{aligned} A(S_0, t) &= S_0(1 - N(L - \sigma\sqrt{T})) \\ &= S_0 N(-L + \sigma\sqrt{T}) \\ &= S_0 N(d_1) \end{aligned}$$

where

$$d_1 = -L + \sigma\sqrt{T} = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{(T - t)}}$$

Call option valuation

- This then gives us the Black-Scholes-Merton formula at a general time t

$$
\begin{aligned}
V_t &= A(S,t) - B(S,t) \\
&= S_t N(d_1) - e^{-r(T-t)} K N(d_2)
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} \\
d_2 &= \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} = d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

- I personally think that this derivation is far simpler than that from the PDE solution.

The Black-Scholes-Merton PDE

- From this Martingale representation we can also easily derive the Black-Scholes-Merton partial differential equation, consider an option price process V_t and a risk-free bond B_t.
- From Ito's lemma

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2$$

now if we are under the risk-neutral measure, then for no arbitrage:

$$dS = rSdt + \sigma SdW$$

thus

$$dV = \left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW$$

The Black-Scholes PDE

- And so,

$$d\left(\frac{V}{B}\right) \quad = \quad \frac{1}{B}\left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV\right) dt + \sigma \frac{S}{B}\frac{\partial V}{\partial S} dW$$

- But, under the risk neutral world V/B must be a martingale, in which case it must have zero drift and,

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

which is the Black-Scholes PDE.

Very useful: Different numeraires

- **In the examples we have given the risk-free bond plays the role of the Numeraire asset. It is useful because it has no volatility but could we use any asset as the numeraire asset?**
- **In fact we can, we just have a revised pricing formula for any numeraire asset N_t so that now:**

$$\frac{V_t}{N_t} = E_t^Q\left[\frac{V_T}{N_T}\right]$$

- **This is very useful for foreign exchange problems and crucial for term structure work, as we will see later in the course.**
- **We will work through some examples of different numeraires in Problem Set 6.**

Recap

- There are no arbitrage opportunities if and only if assets have a unique pricing measure.
- In a complete market, there are no arbitrage opportunities if and only if the ratio of any tradeable asset to the risk-free (or numeraire) asset is a martingale.
- Girsanov's theorem shows that there is a unique change of measure which made this ratio a martingale.
- All other replicable assets can be priced using the same unique measure as by the ratio of their price to the risk-free bond is also a martingale under the same measure.

Overview

- We have looked at derivative pricing from an entirely probability focussed approach. We have been able to derive the same results as before but it could be argued that the formulas are simpler.
- The crucial insight is that if there is no arbitrage then the current discounted value of an asset is its discounted expectation under the appropriate martingale measure.
- This gives us a simple formula for valuing options and a link back to our fundamental theorems. Next, we shall see some applications of this.

Advanced: Fundamental theorems of finance

- We have mixed together our fundamental theorem of finance and the martingale representation, we should be a little more careful.
- Our fundamental theorems of finance say that
  - Arbitrage $\Leftrightarrow$ Risk-neutral measure.
  - Completeness $\Leftrightarrow$ Unique measure.
- Our measure here is often called the equivalent martingale measure in continuous time finance. These are proved in the famous paper of Harrison and Pliska (1981). We omit the proofs but give a feel for the arguments next.

Advanced: Martingales and arbitrage

- Martingales are essentially the absence of arbitrage.
- To show this, assume that within our self financing replication strategy there is the possibility of an arbitrage.
- The portfolio value at time t is given by:

$$\Pi_t = \phi_t S_t + \psi_t B_t$$

where

$$d\Pi_t = \phi_t dS_t + \psi_t dB_t$$

and the discounted value of the portfolio is $\eta_t = \Pi_t / B_t$ and then

$$d\eta_t = \phi_t dZ_t$$

where $Z_t = S_t / B_t$.

Advanced: Martingales and arbitrage

- Suppose that the strategy starts with zero value, $\Pi_0 = 0$, and finished with a non-negative payoff, $\Pi_T \geq 0$. Is there really an arbitrage opportunity?
- As Z_t is a martingale then η_t is a martingale too and so:

$$E^Q[\eta_T] = E_0^Q[\eta_T] = \eta_0 = \Pi_0 = 0$$

  and so η_T must take value 0, and thus $\Pi_T = 0$ and so if the Martingale measure Q exists then there are no arbitrage opportunities,
- We omit the other direction!

Advanced: Complete markets and uniqueness

- If we can hedge an option then there can only be one measure, Q.
- First, assume that there are two measures Q and Q' and there are digital call options that pay off B_T if event A occurs between now and time T, $X = B_T I_A$. These options must be hedgeable.
- We have a self financing portfolio:

$$\Pi_t = \phi_t S_t + \psi_t B_t$$

and $d\eta_t = \phi_t dZ_t$. Z_t is a martingale under Q and Q' and so η_t must be also and so:

$$\eta_0 = E^Q[\eta_T] = E^{Q'}[\eta_T]$$

- But η_T is just the indicator function of A and so $Q(A) = Q'(A)$. However, A was any event and so the two measures agree completely and so being able to replicate the option means that there is only one martingale measure.

Advanced: Real probs $\rightarrow$ R-N probs

- Consider a very simple two period random walk with, which does not necessarily recombine. To get from time 0 to time 2 there are four possible paths in the table below

Path	Probabilities
$\{0, 1, 2\}$	$p_1 p_2 = \pi_1$
$\{0, 1, 0\}$	$p_1(1 - p_2) = \pi_2$
$\{0, -1, 0\}$	$(1 - p_1)p_3 = \pi_3$
$\{0, -1, -2\}$	$(1 - p_1)(1 - p_3) = \pi_4$

- The π values could denote the probability measure P.
- We could also denote the same paths using a different probability measure Q with path probabilities $\pi'_1, \pi'_2, \pi'_3, \pi'_4$

Advanced: Radon-Nikodym derivative

- Now form the ratios of the probabilities π_i'/π_i which gives us an idea as to how to distort the P measure to get the Q measure.
- We write this mapping as $\frac{dQ}{dP}$ and it is known as the **Radon-Nikodym derivative** of Q with respect to P up to time 2.
- Of course, knowing P (that is $\pi_i, \dots$) and $\frac{dQ}{dP}$ enables us to be able to calculate Q.
- The only problem is if any of the probabilities are 0 as it becomes impossible to use the Radon-Nikodym derivative to switch between probabilities.
- So we need a new definition: Two measures P and Q are said to be **equivalent** if they agree on what is possible in the same sample space:

$$P(A) > 0 \Leftrightarrow Q(A) > 0$$

and $\frac{dQ}{dP}$ only exists if the two measures are equivalent.

Advanced: Radon-Nikodym derivative

- The Radon-Nikodym derivative has some useful properties which are easiest to see in the discrete setting. Consider a claim (option) with payoffs x_i on path i then under P its expected value is:

$$E^P[X] = \sum_i \pi_i x_i$$

where i ranges over the four paths. The expectation with respect to Q is

$$E^Q[X] = \sum_i \pi_i' x_i = \sum_i \pi_i \left(\frac{\pi_i'}{\pi_i} x_i \right) = E^P \left[\frac{dQ}{dP} X \right]$$

- This simple result can be extended to give a useful general result:

$$E_0^Q[X_T] = E_0^P \left[\frac{dQ}{dP} X_T \right]$$

- At intermediate times $0 < t < T$ we need to know more about the Radon-Nikodym derivative, which we can do by converting it into a stochastic process, by letting the time horizon vary.

Advanced: Radon-Nikodym derivative

- Let ζ_t be the Radon-Nikodym derivative taken up to the horizon t. Thus in our simple example ζ_1 can take values π_1'/π_1 and $(1-\pi_1')/(1-\pi_1)$.
- In general we can write:

$$\zeta_t = E_t^P\left[\frac{dQ}{dP}\right]$$

- ζ_t is very useful as it represents the amount of change of measure so far up to time t. So if we wanted to know $E^Q[X_t]$ it would be $E^P[\zeta_t X_t]$ and, more importantly for us, we have:

$$E_s^Q[X_t] = \frac{E_s^P[\zeta_t X_t]}{\zeta_s}$$

Advanced: Changing measure in continuous time

- We extend our nice discrete results to continuous time.
- We need to think a little more deeply about probability measures where we have continuous times and states.
- To do this we need all of the marginal distributions at each time t conditional on every history $\mathcal{F}_s$ for all times $s < t$. We simplify it slightly by considering a large (but finite) set of times $0 = t_0, t_1, \ldots, t_n - 1, t_n = T$ and points $x_1, \ldots, x_n$.
- With one point , $x$and one time, t_1 we could use the normal density function:

$$f_1(x) = \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{x^2}{2t_1}\right)$$

- Extending it to different points at different times we get

$$f_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \Delta t_i}} \exp\left(-\frac{\Delta x_i^2}{2\Delta t_i}\right)$$

where $\Delta x_i = x_i - x_{i-1}$ and $\Delta t_i = t_i - t_{i-1}$.

Advanced: Changing measure in continuous time

- So in a non-elegant way we can define our Radon-Nikodym derivative as a limit:

$$\frac{dQ}{dP} = \lim_{n\to\infty} \frac{f_n^P(x_1, \dots, x_n)}{f_n^Q(x_1, \dots, x_n)}$$

- This still satisfies our earlier conditions:

$$E_0^Q[X_T] = E_0^P\left[\frac{dQ}{dP} X_T\right]$$

$$E_s^Q[X_t] = \frac{E_s^P[\zeta_t X_t]}{\zeta_s}$$

Advanced: Changing measure in continuous time: In practice

- Consider a Brownian motion, W_t, under the probability measure P. Now, as if by magic, define an equivalent probability measure, Q such that:

$$\frac{dQ}{dP} = \exp(-\gamma W_T - \frac{1}{2}\gamma^2 T)$$

- Before progressing, let's consider what a normally distributed random variable looks like. A random variable X is normally distributed (with mean μ and variance σ, if and only if

$$E[\exp(\theta X)] = \exp(\theta\mu + \frac{1}{2}\theta^2\sigma^2)$$

- Given that, let's try to calculate the expectation of θW_T under the measure Q:

$$\begin{aligned} E^Q[\exp(\theta W_T)] &= E^P\left[\exp\left(-\gamma W_T - \frac{1}{2}\gamma^2 T + \theta W_T\right)\right] \\ &= \exp\left(-\frac{1}{2}\gamma^2 T + \frac{1}{2}(\theta - \gamma)^2 T\right) \end{aligned}$$

Advanced: Changing measure in continuous time: In practice

or,

$$E^Q[\theta W_T] = \exp\left(-\theta\gamma T - \frac{1}{2}\theta^2 T\right)$$

Advanced: Changing measure in continuous time: In practice

- This is the same as a normally distributed variable with mean $-\gamma T$ and variance T. Thus the distribution of W_T under Q is also normal, only now with mean $-\gamma T$
- This also holds for W_t which is Brownian motion with respect to P and a Brownian motion with constant drift $-\gamma$ under Q. So, W_t is a Brownian motion (with the same volatility) under both measures!
- This seems remarkable, but it should not be surprising. The possible points on the path are the same (as the measures are equivalent) what the change of measure does is adjust the probabilities of a particular path being chosen. So, under Q the drift is negative and so paths that end up with negative values are more likely than they were under P.
- These results are formalized in Girsanov's theorem, now the full version

Advanced: Girsanov's theorem (full version)

1. If W is a Brownian motion under the probability measure P, then there exists an equivalent measure Q such that $\tilde{W}_t = W_t + \gamma t$ is a Brownian motion.
2. Let W be a Brownian motion under a probability measure P, then under an equivalent measure:

$$Q(A) = E\left[1_A \frac{dQ}{dP}\right]$$

$\tilde{W}$ is a Brownian motion with drift γ. Or $W = \tilde{W} + \gamma T$ is a Brownian motion. (A is an arbitrary set of paths (possible values of W_T) and 1_A is an indicator which takes the value 1 only when path A occurs.)

- In this case,

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right)$$

Advanced: Example of part 2

- Consider switching between measures:

$$P(W_T < x) = P(N(0,1) < x/\sqrt{T}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{T}} e^{-s^2/2} ds$$

transforming,

$$P(W_T < x) = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{x} e^{-s^2/2T} ds$$

and so,

$$\begin{aligned} Q(W_T < x) = E[1_{W_T < x} M(T)] &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{x} e^{-\frac{s^2}{2T}} e^{-\frac{1}{2}\gamma^2 T - \gamma s} ds \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{x} e^{-\frac{(s+\gamma T)^2}{2T}} ds \end{aligned}$$

- Simply changing $r = s + \gamma T$ shows that this is the same as

$$P(W_T < x + \gamma T).$$

Advanced: Example of part 2

- Thus the probability that $W_T < x$ in the new measure is equal to the probability that $W_T - \gamma T$ is less than x in the old measure. That is W_T is a Brownian motion with drift $-\gamma$ in the new measure, as required.