FIN 514: Problem Set #4

Due on Wednesday, March 7, 2018

Wanbae Park

Problem 1

a. By Ito's lemma,

$$\begin{aligned}
df(X) &= \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX)^2 \\
&= 2X(t)dX(t) + \frac{1}{2} \times 2dt \\
\Rightarrow X(t)dX(t) &= \frac{1}{2}df(X) - \frac{1}{2}dt \\
\Rightarrow \int_0^t X(\tau)d\tau &= \frac{1}{2}\int_0^t dX^2(\tau) + \int_0^t \frac{1}{2}d\tau \\
&= \frac{1}{2}X^2(t) + \frac{1}{2}t
\end{aligned}$$

b. Let $g(t,X) = tX(t)$. Then by Ito's lemma,

$$\begin{aligned}
dg(t,X) &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial X}dX + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}(dX)^2 \\
&= X(t)dt + tdX(t) \\
\Rightarrow tdX(t) &= d(tX(t)) - X(t)dt \\
\Rightarrow \int_0^t \tau dX(\tau) &= tX(t) - \int_0^t X(\tau)d\tau
\end{aligned}$$

c. Let $h(X) = X^3(t)$. Then by Ito's lemma,

$$\begin{aligned}
dh(X) &= \frac{\partial h}{\partial X}dX + \frac{1}{2}\frac{\partial^2 h}{\partial X^2}(dX)^2 \\
&= 3X^2(t)dX(t) + \frac{1}{2}6X(t)dt \\
&= 3X^2(t)dX(t) + 3X(t)dt \\
\Rightarrow X^2(t)dX(t) &= \frac{1}{3}d(X^3(t)) - X(t)dt \\
\Rightarrow \int_0^t X^2(\tau)dX(\tau) &= \frac{1}{3}X^3(t) - \int_0^t X(\tau)d\tau
\end{aligned}$$

Problem 2

By Ito's lemma,

$$\begin{aligned}
dF &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2 \\
&= -(r-\delta)Fdt + \frac{F}{S}dS(t)
\end{aligned}$$

Since $dS(t) = (\mu - \delta)S(t)dt + \sigma S(t)dX(t)$,

$$\begin{aligned}
dF &= -(r-\delta)Fdt + \frac{F}{S}((\mu-\delta)S(t)dt + \sigma S(t)dX(t)) \\
&= -(r-\delta)Fdt + (\mu-\delta)Fdt + \sigma F dX(t) \\
&= (\mu - r)Fdt + \sigma F dX(t)
\end{aligned}$$

Problem 3

a. By Ito's product rule,

$$
\begin{aligned}
dY &= d\left(\frac{S_2}{S_1}\right) \\
&= \frac{1}{S_1}dS_2 + S_2 d\left(\frac{1}{S_1}\right) + dS_2 d\left(\frac{1}{S_1}\right)
\end{aligned}
$$

By Ito's lemma,

$$
\begin{aligned}
d\left(\frac{1}{S_1}\right) &= -\frac{1}{S_1^2}dS_1 + \frac{1}{2}\times 2(dS_1)^2 \\
&= -\frac{1}{S_1^2}(\mu_1 S_1 dt + \sigma_1 S_1 dX_1) + \frac{1}{S_1^3}\sigma_1^2 S_1^2 dt \\
&= -\frac{\mu_1}{S_1}dt - \frac{\sigma_1}{S_1}dX_1 + \frac{\sigma_1^2}{S_1}dt \\
&= \frac{-\mu_1+\sigma_1^2}{S_1}dt - \frac{\sigma_1}{S_1}dX_1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
dY &= \frac{1}{S_1}(\mu_2 S_2 dt + \sigma_2 S_2 dX_2) + S_2\left(\frac{-\mu_1+\sigma_1^2}{S_1}dt - \frac{\sigma_1}{S_1}dX_1\right) + (\mu_2 S_2 dt + \sigma_2 S_2 dX_2)\left(\frac{-\mu_1+\sigma_1^2}{S_1}dt - \frac{\sigma_1}{S_1}dX_1\right) \\
&= \mu_2 Y dt + \sigma_2 Y dX_2 + (-\mu_1+\sigma_1^2)Y dt - \sigma_1 Y dX_1 - \sigma_1\sigma_2\rho Y dt \quad (\because dX_1 dX_2 = \rho dt)
\end{aligned}
$$

Let $\sigma_3 X_3 = \sigma_1 X_1 + \sigma_2 X_2$, where $\sigma_3 = \sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}$. Since X_1 and X_2 have expected value zero, $\mathrm{E}[X_3]$ also equal to zero. Furthermore, $\mathrm{Var}[X_3] = \mathrm{Var}[\sigma_1 X_1 + \sigma_2 X_2]/\sigma_3^2 = (\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)t/\sigma_3^2 = t$. Therefore, $\sigma_3 X_3 = \sigma_1 X_1 + \sigma_2 X_2$. Finally dY can be represented as follows.

$$
\begin{aligned}
dY &= (\mu_2 - \mu_1 + \sigma_1^2 - \sigma_1\sigma_2\rho)Y dt + \sigma_3 Y dX_3 \\
&= \mu_Y Y_t dt + \sigma_Y Y_t dX_{3t}
\end{aligned}
$$

$$
\text{where } \mu_Y = \mu_2 - \mu_1 + \sigma_1^2 - \sigma_1\sigma_2\rho, \sigma_Y = \sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}
$$

b. By Ito's product rule,

$$
\begin{aligned}
dZ_t &= d(Y_t B_{Et}) \\
&= B_{Et}dY_t + Y_t dB_{Et} + dY_t dB_{Et} \\
&= B_E t(\mu_Y Y_t dt + \sigma_Y Y_t dX_{3t}) + Y_t(r_E B_{Et} dt), \quad dB_{Et} = r_E B_{Et} dt \ \text{ because the bond is riskless.} \\
&= (\mu_Y + r_E) Z_t dt + \sigma_Y Z_t dX_{3t} \\
&= (\mu_Z) Z_t dt + \sigma_Y Z_t dX_{3t}
\end{aligned}
$$

where $\mu_Z = \mu_Y + r_E$.