

FIN 514: Problem Set #5

Due on Tuesday, March 13, 2018

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Problem 1

Option price must satisfy the following pde. It is analogous that diffusion term has changed from $\sigma S(t)$ to $\sigma(S(t))^\gamma$.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\gamma} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

It can be derived by following procedures.

By Ito's lemma, $dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^{2\gamma} \frac{\partial^2 V}{\partial S^2} dt$. Set $\Pi = V - \Delta S - \beta B$, then by self-financing, $d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\gamma} \frac{\partial^2 V}{\partial S^2})dt + \frac{\partial V}{\partial S} dS - \Delta dS - r\beta Bdt$. Choose Δ such that $(\frac{\partial V}{\partial S} - \Delta)dS = 0$, therefore $\Delta = \frac{\partial V}{\partial S}$. Since $\beta B = V - \Delta S$, and by no arbitrage argument, $0 = d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\gamma} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV)$.

Problem 2

(a) By Black-Scholes, all derivatives with underlying asset S must follow the following pde.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

Terminal boundary condition of this contract is $V(S, T) = \ln(S(T)/S(0))$.

(b) By Ito's lemma,

$$\begin{aligned} d \ln S &= \frac{1}{S} dS - \frac{1}{2} \frac{1}{S^2} (dS)^2 \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dX(t) \\ \Rightarrow S(t) &= S(0) \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma X(t)) \\ \Rightarrow V(S, t) &= \ln(S(t)/S(0)) = (\mu - \frac{1}{2}\sigma^2)t + \sigma X(t) \end{aligned}$$

Problem 3

(a) By Ito's product rule,

$$\begin{aligned} dS_D &= d(eS) \\ &= edS + Sde + dSde \\ &= e(\mu Sdt + \sigma SdX_2) + S(\mu_e edt + \sigma_e edX_1) + (\mu Sdt + \sigma SdX_2)(\mu_e edt + \sigma_e edX_1) \\ &= (\mu_e S + \mu_e eS + \sigma\sigma_e \rho eS)dt + \sigma_e eSdX_1 + \sigma eSdX_2 \\ &= (\mu + \mu_e + \sigma\sigma_e \rho)S_D dt + \sigma_e S_D dX_1 + \sigma S_D dX_2 \end{aligned}$$

(b) By Ito's product rule,

$$\begin{aligned}
 dB_{KD} &= d(eB_K) \\
 &= edB_K + B_K de + dB_K de \\
 &= e(r_K B_K dt) + B_K(\mu_e edt + \sigma_e edX_1) + (r_K B_K dt)(\mu_e edt + \sigma_e edX_1) \\
 &= (r_K eB_K + \mu_e eB_K)dt + \sigma_e eB_K dX_1 \\
 &= (r_K + \mu_e)B_{KD}dt + \sigma_e B_{KD}dX_1
 \end{aligned}$$

Problem 4

(a) By Ito's product rule, $dY_t = d(\frac{S_{1t}}{S_{2t}}) = S_{2t}d(\frac{1}{S_{1t}}) + \frac{1}{S_{1t}}dS_{2t} + d(\frac{1}{S_{1t}})dS_{2t}$. And by Ito's lemma,

$$\begin{aligned}
 d(\frac{1}{S_{1t}}) &= -\frac{1}{S_{1t}^2}dS_{1t} + \frac{1}{2} \times 2 \times \frac{1}{S_{1t}^3}(dS_{1t})^2 \\
 &= -\frac{1}{S_{1t}^2}(\mu_1 S_{1t}dt + \sigma_1 S_{1t}dX_{1t}) + \frac{1}{S_{1t}^3}\sigma_1^2 S_{1t}^2 dt \\
 &= (-\mu_1 + \sigma_1^2)\frac{1}{S_{1t}}dt - \sigma_1 \frac{1}{S_{1t}}dX_{1t}
 \end{aligned}$$

Therefore, the following equation must hold.

$$\begin{aligned}
 dY_t &= S_{2t}(-\mu_1 + \sigma_1^2)\frac{1}{S_{1t}}dt - \sigma_1 \frac{1}{S_{1t}}dX_{1t} + \frac{1}{S_{1t}}(\mu_2 S_{2t}dt + \sigma_2 S_{2t}dX_{2t}) \\
 &\quad + (-\mu_1 + \sigma_1^2)\frac{1}{S_{1t}}dt - \sigma_1 \frac{1}{S_{1t}}dX_{1t} \left(\frac{1}{S_{1t}}(\mu_2 S_{2t}dt + \sigma_2 S_{2t}dX_{2t}) \right) \\
 &= (-\mu_1 + \sigma_1^2)Y_t dt - \sigma_1 Y_t dX_{1t} + \mu_2 Y_t dt + \sigma_2 Y_t dX_{2t} - \sigma_1 \sigma_2 \rho Y_t dt \quad (\text{Since } dX_{1t}dX_{2t} = \rho dt). \\
 &= (-\mu_1 + \mu_2 + \sigma_1^2 - \sigma_1 \sigma_2 \rho)Y_t dt - \sigma_1 Y_t dX_{1t} + \sigma_2 Y_t dX_{2t}
 \end{aligned}$$

If we choose $X_{3t} = \frac{-\sigma_1 X_{1t} + \sigma_2 X_{2t}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho}}$, then the equation above is represented as follows.

$$\begin{aligned}
 dY_t &= \mu_Y Y_t dt + \sigma_Y Y_t dX_{3t} \\
 \text{where } \mu_Y &= -\mu_1 + \mu_2 + \sigma_1^2 - \sigma_1 \sigma_2 \rho, \sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho}
 \end{aligned}$$

(b) By Ito's product rule,

$$\begin{aligned}
 dZ_t &= d(Y_t B_{Et}) \\
 &= B_{Et}dY_t + Y_t dB_{Et} + dY_t dB_{Et} \\
 &= B_{Et}(\mu_Y Y_t dt + \sigma_Y Y_t dX_{3t}) + Y_t(r_E B_{Et}dt) + (\mu_Y Y_t dt + \sigma_Y Y_t dX_{3t})(Y_t r_E B_{Et}dt) \\
 &= (\mu_Y + r_E)Y_t B_{Et}dt + \sigma_Y Y_t B_{Et}dX_{3t} \\
 &= \mu_Z Z_t dt + \sigma_Y Z_t dX_{3t}
 \end{aligned}$$

where $\mu_Z = \mu_Y + r_E$.

- (c) $U(Y, T) = \frac{V(S_{1T}, S_{2T}, T)}{S_{1T}} = \frac{1000 \times \max(S_{2T} - S_{1T}, 0)}{S_{1T}} = 1000 \times \max(Y_T - 1, 0)$. Payoff of exchange option denominated in GBP is same as payoff of an European option whose underlying asset is price of Euros in terms of GBP and strike price is 1.

- (d) Let $\Pi_t = \frac{\pi_t}{S_{1t}}$, then $\Pi_t = -U_t + \Delta_t Z_t + \beta_t B_{Pt}$. Therefore, the following equation has to follow.

$d\Pi_t = -dU_t + \Delta_t dZ_t + \beta_t dB_{Pt}$ by self-financing.

$$= -\left(\frac{\partial U_t}{\partial t} + \mu_Y Y_t \frac{\partial U_t}{\partial Y_t} + \frac{1}{2} \sigma_Y^2 Y_t^2 \frac{\partial^2 U_t}{\partial Y_t^2}\right) dt + \Delta_t \mu_Z Z_t dt + \beta_t r_P B_{Pt} dt - \sigma_Y Y_t \frac{\partial U_t}{\partial Y_t} dX_{3t} + \Delta_t \sigma_Y Z_t dX_{3t}$$

Choose Δ_t so that $-\sigma_Y Y_t \frac{\partial U_t}{\partial Y_t} + \Delta_t \sigma_Y Z_t = 0$, then $\Delta_t = \frac{Y_t}{Z_t} \frac{\partial U_t}{\partial Y_t} = B_{Et} \frac{\partial U_t}{\partial Y_t}$, and therefore,

$$\begin{aligned} & -\left(\frac{\partial U_t}{\partial t} + \mu_Y Y_t \frac{\partial U_t}{\partial Y_t} + \frac{1}{2} \sigma_Y^2 Y_t^2 \frac{\partial^2 U_t}{\partial Y_t^2}\right) dt + \frac{Y_t}{Z_t} \frac{\partial U_t}{\partial Y_t} \mu_Z Z_t dt + r_P \left(U_t - \frac{Y_t}{Z_t} \frac{\partial U_t}{\partial Y_t} Z_t\right) dt = 0 \\ & \Rightarrow \frac{\partial U_t}{\partial t} + \mu_Y Y_t \frac{\partial U_t}{\partial Y_t} + \frac{1}{2} \sigma_Y^2 Y_t^2 \frac{\partial^2 U_t}{\partial Y_t^2} - \mu_Z Y_t \frac{\partial U_t}{\partial Y_t} + r_P Y_t \frac{\partial U_t}{\partial Y_t} - r_P U_t = 0 \\ & \Rightarrow \frac{\partial U_t}{\partial t} + (\mu_Y - \mu_Z + r_P) Y_t \frac{\partial U_t}{\partial Y_t} + \frac{1}{2} \sigma_Y^2 Y_t^2 \frac{\partial^2 U_t}{\partial Y_t^2} - r_P U_t = 0 \end{aligned}$$

Since $\mu_Y - \mu_Z = -r_E$, the following pde holds.

$$\frac{\partial U_t}{\partial t} + (r_P - r_E) Y_t \frac{\partial U_t}{\partial Y_t} + \frac{1}{2} \sigma_Y^2 Y_t^2 \frac{\partial^2 U_t}{\partial Y_t^2} - r_P U_t = 0$$

- (e) Since $U(Y, T) = 1000 \times \max(Y_T - 1, 0)$, by analogy of European call option, $U(Y, 0)$ is evaluated as:

$$\begin{aligned} U(Y, 0) &= e^{-r_P T} (Y_t e^{(r_P - r_E)T} N(d_1) - N(d_2)) \\ \text{where } d_1 &= \frac{\ln Y_t + (r_P - r_E + \frac{1}{2} \sigma_Y^2)T}{\sigma_Y \sqrt{T}} \\ d_2 &= d_1 - \sigma_Y \sqrt{T} \end{aligned}$$

Therefore, $V(S_1, S_2, 0) = S_{10} \times U(Y, 0)$ is evaluated as:

$$\begin{aligned} V(S_1, S_2, 0) &= S_{10} e^{-r_P T} (Y_t e^{(r_P - r_E)T} N(d_1) - N(d_2)) \\ &= e^{-r_P T} (S_{20} e^{(r_P - r_E)T} N(d_1) - S_{10} N(d_1)) \end{aligned}$$