FIN 514: Problem Set #6

Due on Wednesday, April 25, 2018

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Problem 1

(a) By Ito's product rule, $dY(t)$ satisfies the following equation.

$$\begin{aligned} dY(t) &= B_P(t)dS(t) + S(t)dB_P(t) + dB_P(t)dS(t) \\ &= B_P(t)[\mu S(t)dt + \sigma S(t)dX(t)] + r_P S(t)B_P(t)dt \\ &= (\mu + r_P)Y(t)dt + \sigma Y(t)dX(t) \end{aligned}$$

In order to find martingale measure with respect to $B(t)$ as a numeraire, dynamics of $Y(t)/B(t)$ is derived as follows.

$$\begin{aligned} d\left(\frac{Y(t)}{B(t)}\right) &= Y(t)d\left(\frac{1}{B(t)}\right) + \frac{1}{B(t)}dY(t) + dY(t)d\left(\frac{1}{B(t)}\right) \\ d\left(\frac{1}{B(t)}\right) &= -\frac{1}{B^2(t)}dB(t) \\ &= -\frac{1}{B^2(t)}rB(t)dt = -r\frac{1}{B(t)}dt \\ \Rightarrow d\left(\frac{Y(t)}{B(t)}\right) &= Y(t)\left(-r\frac{1}{B(t)}dt\right) + \frac{1}{B(t)}[(\mu + r_P)Y(t)dt + \sigma Y(t)dX(t)] \\ &= (\mu + r_P - r)\frac{Y(t)}{B(t)}dt + \sigma\frac{Y(t)}{B(t)}dX(t) \end{aligned}$$

By Girsanov's theorem, there exists a probability measure such that $\widetilde{X}(t) = X(t) + \int_0^t \frac{\mu + r_P - r}{\sigma}ds$ is a brownian motion under the measure. Therefore, by plugging $dX(t) = d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt$ into the equation above, then $d\left(\frac{Y(t)}{B(t)}\right)$ becomes $\sigma\frac{Y(t)}{B(t)}d\widetilde{X}(t)$, hence becomes martingale because there is no drift. Therefore, from the perspective of U.S dollar investor, under risk-neutral measure, $dX(t) = d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt$. By plugging it into dynamics of $Y(t)$, we can find dynamics of the U.S price of a GBP bond under risk-neutral measure as follows.

$$\begin{aligned} dY(t) &= (\mu + r_P)Y(t)dt + \sigma Y(t)dX(t) \\ &= (\mu + r_P)Y(t)dt + \sigma Y(t)\left[d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt\right] \\ &= rY(t)dt + \sigma Y(t)d\widetilde{X}(t) \end{aligned}$$

And it is consistent with the fact that expected return of every tradable asset is risk-free rate under risk-neutral measure.

(b) By plugging $dX(t) = d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt$ into dynamics of $S(t)$, we can find dynamics of U.S. dollar price of a British pound under risk-neutral probability as follows.

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dX(t) \\ &= \mu S(t)dt + \sigma S(t)\left[d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt\right] \\ &= (r - r_P)S(t)dt + \sigma S(t)d\widetilde{X}(t) \end{aligned}$$

(c) Unlike the assumption of ordinary Black-Scholes-Merton formula, since expected return of underlying asset has changed from r to $r - r_P$, formula for call option should be changed to following equation.

$$e^{-rT}[S_0 e^{(r-r_P)T} N(d_1) - KN(d_2)]$$
$$d_1 = \frac{\log(S_0/K) + (r - r_P + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Problem 2

(a) By Ito's product rule, dynamics of $B(t)/S(t)$ is as follows.

$$\begin{aligned}
d\left(\frac{B(t)}{S(t)}\right) &= d\left(\frac{1}{S(t)}\right)B(t) + \frac{1}{S(t)}dB(t) + d\left(\frac{1}{S(t)}\right)dB(t) \\
d\left(\frac{1}{S(t)}\right) &= -\frac{1}{S^2(t)}dS(t) + \frac{1}{2}\times 2 \times \frac{1}{S^3(t)}(dS(t))^2 \\
&= -\frac{1}{S^2(t)}[(\mu-d)S(t)dt + \sigma S(t)dX(t)] + \frac{1}{S^3(t)}\sigma^2 S^2(t)dt \\
&= [-(\mu-d)+\sigma^2]\frac{1}{S^2(t)}dt - \sigma\frac{1}{S(t)}dX(t) \\
\Rightarrow d\left(\frac{B(t)}{S(t)}\right) &= \frac{B(t)}{S(t)}[-(\mu-d)+\sigma^2]dt - \sigma\frac{B(t)}{S(t)}dX(t) + r\frac{B(t)}{S(t)}dt \\
&= [r-(\mu-d)+\sigma^2]\frac{B(t)}{S(t)}dt - \sigma\frac{B(t)}{S(t)}dX(t)
\end{aligned}$$

(b) By Girsanov's theorem, there exists a probability measure such that $\widetilde{X}(t) = X(t) - \int_0^t \frac{r-(\mu-d)+\sigma^2}{\sigma}ds$ is a brownian motion under the measure. Under the measure, process $d\left(\frac{B(t)}{S(t)}\right)$ changes as follows.

$$\begin{aligned}
d\left(\frac{B(t)}{S(t)}\right) &= [r-(\mu-d)+\sigma^2]\frac{B(t)}{S(t)}dt - \sigma\frac{B(t)}{S(t)}dX(t) \\
&= [r-(\mu-d)+\sigma^2]\frac{B(t)}{S(t)}dt - \sigma\frac{B(t)}{S(t)}\left[d\widetilde{X}(t) + \frac{r-(\mu-d)+\sigma^2}{\sigma}dt\right] \\
&= -\sigma\frac{B(t)}{S(t)}d\widetilde{X}(t)
\end{aligned}$$

Therefore, under the measure, $\frac{B(t)}{S(t)}$ is a martingale since there is no drift term in dynamics. Plugging $dX(t) = d\widetilde{X}(t) + \frac{r-(\mu-d)+\sigma^2}{\sigma}dt$ into the process of $S(t)$, dynamics of $S(t)$ under martingale measure with respect to $S(t)$ as a numeraire is as follows.

$$\begin{aligned}
dS(t) &= (\mu-d)S(t)dt + \sigma S(t)dX(t) \\
&= (\mu-d)S(t)dt + \sigma S(t)\left[d\widetilde{X}(t) + \frac{r-(\mu-d)+\sigma^2}{\sigma}dt\right] \\
&= (r+\sigma^2)S(t)dt + \sigma S(t)d\widetilde{X}(t)
\end{aligned}$$

Problem 2 continued on next page...

(c) Under the martingale measure with respect to $S(t)$ as a numeraire, $V(t)/S(t)$ is also a martingale. Therefore, by definition of martingale, European call option value $V(t)$ is derived as follows.

$$\begin{aligned}
\frac{V(t)}{S(t)} &= \mathrm{E}_t^Q\left[\frac{V(T)}{S(T)}\right] \\
\Rightarrow V(t) &= S(t)\mathrm{E}_t^Q\left[\frac{V(T)}{S(T)}\right] \\
&= S(t)\mathrm{E}_t^Q\left[\frac{\max(S(T)-K,0)}{S(T)}\right] \\
&= S(t)\mathrm{E}_t^Q\left[\max\left(1-\frac{K}{S(T)},0\right)\right]
\end{aligned}$$

Where K is strike price of the option, and Q is a probability measure in which $V(t)/S(t)$ is a martingale.

(d) In order to evaluate call option value, we need to derive the solution of SDE $dS(t) = (r+\sigma^2)S(t)dt + \sigma S(t)d\widetilde{X}(t)$ first. The solution is derived as follows.

$$\begin{aligned}
d\log S(t) &= \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S^2(t)}(dS(t))^2 \\
&= \left(r+\frac{1}{2}\sigma^2\right)dt + \sigma d\widetilde{X}(t) \\
\Rightarrow S(T) &= S(t)\exp\left[\left(r+\frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\phi\right] \\
\phi &\sim N(0,1)
\end{aligned}$$

Plugging the result of equation above, we can evaluate call option value at time t as follows.

$$\begin{aligned}
V(t) &= S(t)\mathrm{E}_t^Q\left[\max\left(1-\frac{K}{S(T)},0\right)\right] \\
&= S(t)\mathrm{E}_t^Q\left[\max\left(1-\frac{K}{S(t)\exp[(r+\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\phi]},0\right)\right]
\end{aligned}$$

Since $1-\frac{K}{S(t)\exp[(r+\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\phi]} \geq 0$ is equivalent to $\phi \geq \frac{\log(K/S(t)-(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \equiv L$, and ϕ follows standard normal distribution, call option value can be calculated as follows.

$$\begin{aligned}
V(t) &= S(t)\int_L^\infty\left(1-\frac{K}{S(t)\exp[(r+\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}x]}\right)\frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}x^2}dx \\
&= S(t)\int_L^\infty\frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}x^2}dx - K\int_L^\infty\frac{1}{\sqrt{2\pi}}e^{-(r+\frac{1}{2}\sigma^2)(T-t)-\sigma\sqrt{T-t}x-\frac{1}{2}x^2}dx \\
&= S(t)(1-N(L)) - Ke^{-r(T-t)}\int_L^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x+\sigma\sqrt{T-t})^2}dx \\
&= S(t)(1-N(L)) - Ke^{-r(T-t)}\int_{L+\sigma\sqrt{T-t}}^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy \\
&= S(t)(1-N(L)) - Ke^{-r(T-t)}(1-N(L+\sigma\sqrt{T-t})) \\
&= S(t)N(-L) - Ke^{-r(T-t)}N(-L-\sigma\sqrt{T-t}) \\
&= S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)
\end{aligned}$$

$$
\begin{aligned}
y &= x + \sigma\sqrt{T-t} \\
d_1 &= \frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

Which is the desired solution. It is not easier than when $B(t)$ was numeraire because it also has some integration and transformation in evaluating procedure.

Problem 3

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

(i)