# Fin 514: Financial Engineering II

## Lecture 17: Some Applications of the Probabilistic Solution

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# Outline

- We have seen two separate approaches to option valuation: the PDE approach and the probabilistic (risk-neutral pricing) approach.
- Here we look at some of the applications of the probabilistic approach and the relationship between the PDE approach and the probabilistic approach.
- In particular we focus on:
  - Transition density functions
  - Feynman-Kac equation
  - Underlying asset paying a dividend
  - Forward contracts and options on forwards
  - Convertible bonds
  - Barrier options

# Transition probability density functions

- If we have a general process:

$$dS = \mu(S,t)dt + \sigma(S,t)dX$$

  this will make it difficult, in general to analytically calculate the expectation of a function of $S$, although we clearly know it for some simple cases ($\mu$, $\sigma$ constants).

- The general, probabilistic solution (for constant r) will be:

$$V(S,t) = e^{-r(T-t)} \int_0^\infty V(S',T)p(S,t;S',T)dS'$$

  where $S'$ are the future possible share prices at $t = T$, $V(S',T)$ is the payoff from the option and, at a minimum,we need to know is $p(S,t,S',T)$ which is the density of the risk-neutral distribution in this scenario.

- In many scenarios this will also **not be** a known formula.

# Transition probability density functions

- The movement of a general stochastic variable is given by

$$dS = A(S,t)dt + B(S,t)dX$$

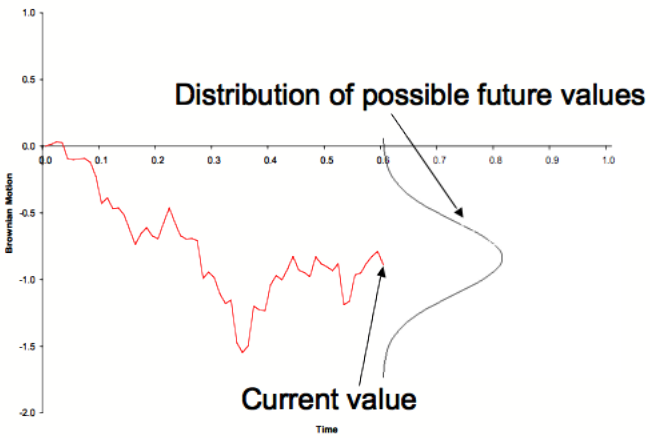
  The transition probability density function is defined by

$$p(a < S' < b \text{ at time } T) = \int_a^b p(S,t;S',T)dy'$$

  $p$ gives probability that $S'$ lies between $a$ and $b$ at time $T$, given that it started out at $S$ at time $t$.

- In the case of a standard Brownian motion $(A = 0, B = 1)$ then we have seen that this is just a normal distribution about which we know a great deal (in fact we used this result to get the value of a european call option in Lecture 14)

# Transition probability density functions



# Transition probability density functions

- For a driftless Geometric Brownian motion, $A = 0$ and $B = \sigma$, then

$$p(S, t; S', T) = \frac{1}{\sigma\sqrt{2\pi(T-t)}} \exp\left[-\frac{(S'-S)^2}{2\sigma^2(T-t)}\right]$$

- When we have regular GBM, $A = \mu S$ and $B = \sigma S$ then this adapts slightly to the lognormal distribution where

$$p(S, t; S', T) = \frac{1}{\sigma S'\sqrt{2\pi(T-t)}} \exp\left[-\frac{(\ln(S'/S) - (\mu - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right]$$

  and in the special case where $\mu = r - \delta$ we get the risk neutral density when there is a continuous dividend yield, $\delta$.

- The density is a function of the future values $S', T$ and current values $S, t$.

# Aside: lognormal

- A lognormal distributed variable, $S'$ is based on $ln(S')$ which has mean $\hat{\mu}$ and a standard deviation $\hat{\sigma}$. The probability density function of $S'$ is:

  $$p(S') = \frac{1}{\hat{\sigma} S' \sqrt{2\pi}} \exp\left[-\frac{(\ln(S') - \hat{\mu})^2}{2\hat{\sigma}^2}\right]$$

  in the case of GBM where

  $$dS = \mu S dt + \sigma S dX$$

  then the density for $S'$ will be based upon $\hat{\mu}$, the expected value of $ln(S')$ at $T$ and $\hat{\sigma}$ is the standard deviation of $ln(S')$ at $T$ which are given by

  $$\hat{\mu} = \ln S + \left((\mu - \frac{1}{2}\sigma^2)(T - t)\right)$$

  and

  $$\hat{\sigma} = \sigma\sqrt{T - t}$$

  where $S$ is the stock price at current time $t$

# Stock price transition density with GBM

- If we now consider the risk-neutral stock price movements where

$$dS = rSdt + \sigma SdX.$$

If at $t = 0, S = 100$ then the probability of the stock price being $S'$ at $T$ is given by:

$$p(100, 0; S', T) = \frac{1}{\sigma S' \sqrt{2\pi T}} \exp\left[ -\frac{(\ln(S'/100) - (r - \frac{1}{2}\sigma^2)T)^2}{2\sigma^2 T} \right]$$

- Note the similarity with the fundamental (or Green's function) solution from Lecture 9. In fact if you include the $e^{-rT}$ term then it is identical.

# Link with A-D prices

- Before we considered a particular AD security (say $S' = 100$, $T = 1$), then the fundamental solution then this gives the value of the AD security at time $t$ for various stock prices $S$.
- So, to back out the probability from the value of an A-D security corresponding to $S' = 100$, $T = 1$ we have:

$$\begin{aligned} p(S, t; 100, 1) &= e^{r(1-t)} \times \text{Fundamental solution} \\ &= \frac{e^{r(1-t)} \times e^{-r(1-t)}}{100\sigma\sqrt{2\pi(1-t)}} \exp\left[-\frac{(\ln(100/S) - (r - \frac{1}{2}\sigma^2)(1-t))^2}{2\sigma^2(1-t)}\right] \end{aligned}$$

  as above.
- Finally, to get to the Black-Scholes formula we can simply integrate over all possible $S'$ at $T = 1$, given a current $S$ and $t$.

# What probabilities would we like to calculate

- Being able to calculate probabilities may be useful for many reasons.
- First, we know from lecture 14 that the option or derivative value is a discounted expectation, so knowing the probability of each payoff occurring can give us the expectation and thus the option value.
- We may wish to know the probability that a certain barrier is hit based on the current stock price. You may also want to know the probability that an option ends up in the money or so on.
- However, each of these probabilities will rely on the underlying stochastic process, so are there ways of generalizing this approach?
- There are three key PDEs that you might wish to use: the Kolmogorov forward and backward equations and the Feynman-Kac equation.

# Transition probability density functions

- It turns out that for a general underlying stochastic process for $S$, the probability density function $p(S, t, S', T)$ satisfies a pair of differential equations:
  - Kolgomorov backward equation.
  - Fokker-Planck (or Kolmogorov) forward equation.
- We can then solve the PDE to find the transition density function and then compute the option price as:

$$V(S, t) = e^{-r(T-t)} \int_0^\infty V(S', T) p(S, t; S', T) dS'$$

although this may have to also be caclulated numerically if the density function is not known analytically.

# Kolmogorov backward equation

- For a *very* general stochastic process

$$dS = A(S,t)dt + B(S,t)dX$$

**the Kolmogorov backward equation** defines the transition probability density function $p(S,t;S',T)$ - or perhaps more easily $P(S,t)$ as $S'$ and $T$ and it is the current $S$ which is uncertain. The density function is the solution to the following pde:

$$\frac{\partial p}{\partial t} + \frac{1}{2}B(S,t)^2\frac{\partial^2 p}{\partial S^2} + A(S,t)\frac{\partial p}{\partial S} = 0$$

with terminal boundary condition for a specific value $S'$ at $T$

$$p(S,T) = \delta(S - S')$$

where the solution $p(S,t)$ tells us the probability at a current $(S,t)$ of reaching a specific value of $S$ at a future time $T$, $(S',T)$

# Quiz

Why might the backward equation be useful for Finance applications

- We have lots of historical stock prices.
- We know the payoff function at maturity but calculate backward in time.
- Stochastic processes move backward in time.

# Kolmogorov Forward equation

- The **Kolmogorov forward equation** is for $P(S', t)$ as the intial values (S,0) are known and it is the future value of $S'$ which is uncertain.

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(pB(S',t)^2\right) - \frac{\partial}{\partial S'}\left(pA(S',t)\right)$$

with initial condition for a specific value $S$ at 0

$$p(S', 0) = \delta(S' - S)$$

where the solution $p(S', t)$ tells us the probability of reaching (or the density function value corresponding to) the current point $(S', t)$ starting from a particular $S$ value at $t = 0$ $(S, 0)$

# Quiz

Why are the initial/final conditions the delta function?

- It is the normal distribution pdf
- Probability of $S' = S$ at $t = 0$ must be one but is not properly defined for a continuous pdf.
- Probability of $S' = S$ is symmetric around $S'$ at $t'$ (or $t$) in the forward case.

# Using these equations

- Consider bank:
- Portfolio includes 5,000 options of USD/JPY exchange rate
- You need to compute market value of portfolio ('fair value')
- Two approaches:
  - Solve BSM PDE 5,000 times (recall that each option will satisfy a version of BSM pde)
  - Solve one the kolmogorov forward equation based on current $S$ and 0 to get $p(S, 0; S', T)$ then you can then numerically (or analytically) solve just the one integral:

$$V(S,0) = e^{-rT} \int_0^\infty V(S',T)p(S,0;S',T)dS'$$

where $V(S',T)$ is the value of the total portfolio given the value of $S'$ at time $T$.

# Feynman-Kac Theorem

- It is actually quite easy to extend the Kolmogorov backward equation to deal with expectations under $S$. This is useful because, of course, option values are discounted expectations.
- The Feynman-Kac theorem says that if:

$$dS = A(S,t)dt + B(S,t)dX$$

  and

$$f(S,t) = E_t[\phi(S_T)]$$

  or the option value is the expected value of some function of future values of $S$. For example, think of $\phi()$ as a payoff function, $S_T$ as stock prices at maturity and $f(S,t)$ as the current option value.
- Then, for $t < T$, then $f(S,t)$ will satisfy the following PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2}B(S,t)^2\frac{\partial^2 f}{\partial S^2} + A(S,t)\frac{\partial f}{\partial S} = 0$$

- We can apply this PDE directly to our option value formula . . .

# Feynman-Kac Theorem

- Consider a share following GBM under the equivalent martingale (or risk-neutral) measure and an option on this share, $V(S,t)$. Define the following function

$$\hat{V}(S,t) = e^{r(T-t)}V(S,t)$$

then we know that this is a Martingale and so

$$\hat{V}(S,t) = E_t^Q[\hat{V}(S,T)]$$

- Thus, by the Feynman-Kac Theorem

$$\frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} + rS\frac{\partial \hat{V}}{\partial S} = 0$$

thus, substituting in the definition of $\hat{V}$ we arrive at the Black-Scholes PDE for $V$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

# Feynman-Kac Theorem

- This Feyman-Kac PDE is practically very useful for estimating key features of derivative contracts.
- For example, applications of the Feynman-Kac formula can be used to determine the expected time to exercise an American put option, the expected time to hit a barrier etc.

# Three approaches

- This treatment illustrates our 3 interpretations of option price:
  - Option price is value of replicating portfolio (PDE solution)
  - Option price can be written in terms of values of Arrow-Debreu securities
  - Option price can be characterized as discounted expected value (probabilistic solution)
- Probabilistic solution is more general than PDE approach
  - Risk-neutral probability exists even when security prices can not be characterized as solutions of PDE's (the interest rate model called the LMM, for example).
- In cases where PDE characterization exists, Feynman-Kac and Kolmogorov solutions allow us to switch back and forth between PDE and probabilistic characterizations at will.

# Feynman-Kac in practice

- The Feynman-Kac solution allowing us to switch back and forth between PDE and probabilistic solution at will is at the heart of what quant groups do.
- If PDE is easier to solve (early exercise for example), rely on PDE characterization
- If probabilistic solution is more convenient, use that (path dependency, large numbers of assets).
- For example, freely switch back and forth between finite difference methods (for numerical solution of PDE); and Monte Carlo methods (relying on probabilistic solution)

# Allowing for dividends

- We can easily extend our probabilistic method to include dividends, if we have

$$\begin{aligned} dS &= (\mu - \delta)Sdt + \sigma SdX \\ dB &= rBdt \end{aligned}$$

- If we let $D$ be the current price of a contract which delivers $S$ at time $T$, then the value of $D$ at time $t$ is $Se^{-\delta(T-t)}$.
- The fundamental theorem of finance applies to $D$ as it is a non-dividend paying asset and so $D/B$ is a martingale, and in the risk-neutral measure:

$$dD = rDdt + \sigma DdX$$

We can get back to $S$ as follows:

$$\begin{aligned} dS &= d(De^{\delta(T-t)}) \\ &= e^{\delta(T-t)}dD - D\delta e^{\delta(T-t)}dt \\ &= (r - \delta)Sdt + \sigma SdX \end{aligned}$$

# Forwards and options on forwards

- We can apply our probabilistic theory in a very elegant way when looking at forward contracts.
- Consider a share under the risk-neutral measure, $r$ constant

$$dS = rSdt + \sigma SdX$$

  the futures price for a contract expiring at $T$ is defined as $F_T(t)$ and has the property that at expiry $F_T(T) = S(T)$.
- There are many ways to determine the value of $F_T(t)$ but consider a forward contract, with value $f_T(t)$, issued at $t$, this has payoff $S(T) - F_T(t)$ and has value 0 today.

# Forwards and options on forwards

- So by our probabilistic solution:

$$
\begin{aligned}
\frac{f_T(t)}{B(t)} &= E_t^Q\left[\frac{S(T)-F_T(t)}{B(T)}\right] \\
f_T(t) &= B(t)\left[\frac{S(t)}{B(t)} - \frac{F_T(t)}{B(T)}\right] \\
0 &= S(t) - \frac{B(t)F_T(t)}{B(T)} \\
F_T(t) &= S(t)e^{r(T-t)}
\end{aligned}
$$

- Thus for a forward issued at time 0, with forward price $K = F_T(0)$ the value of this contract at time $0 < t < T$, is

$$
\begin{aligned}
\frac{f_T(t)}{B(t)} &= E_t^Q\left[\frac{S(T)-K}{B(T)}\right] \\
f_T(t) &= B(t)\left[\frac{S(t)}{B(t)} - \frac{K}{B(T)}\right]
\end{aligned}
$$

# Forwards and options on forwards

- And so

$$
\begin{aligned}
f_T(t) &= S(t) - \frac{B(t)K}{B(T)} \\
f_T(t) &= S(t) - Ke^{-r(T-t)}
\end{aligned}
$$

- Interestingly, if consider the forward price $F_T(t)$ we see that

$$F_T(t) = e^{r(T-t)} S(t)$$

and, under risk-neutral probabilities:

$$dF_T(t) = \sigma F_T(t) dX$$

as you saw in Problem Set 4.

# Quiz

Which of these are martingales under the risk-neutral measure?

- Value of a forward contract.
- Value of a stock.
- Value of a dividend paying stock.
- The forward price.
- Bond price.

# Options on forwards

- Thus $F$ is a martingale under the risk-neutral measure, and so if we consider a call option, $V(F_T, t)$ on the forward price with payoff $V(F_T, T) = \max(F_T(T) - K, 0)$then

$$\begin{aligned} V(F_T, t) &= e^{-r(T-t)} E^Q[\max(F_T(t)\exp(-\sigma^2(T-t) + \sigma\sqrt{T-t}\phi) - K, 0)] \\ &= e^{-r(T-t)}(F_T(t)N(h_1) - KN(h_2)) \end{aligned}$$

where

$$h_1 = \frac{\ln(F_T(t)/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$
$$h_2 = \frac{\ln(F_T(t)/K) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

- This is **Black's formula**.
- Note that if we put $F_T(t) = S(t)e^{r(T-t)}$ we end up back with the Black-Scholes formula.

# Quiz

Is Black's formula any different than Black-Scholes?

- Yes, it is only for options on forwards.
- No, as $F_T = S_T$ the payoffs are the same.
- Yes, but only when interest rates are functions of S, or stochastic.

# Uses of Black's formula

- The discounting occurs purely in the global multiplier and all of the other interest rate (and dividend) effects are hidden in the forward price.
- Thus, Black's formula is very useful in a world with stochastic interest rates or any more complex interest rate model, as the forward price can be viewed as a fundamental price (rather than the spot price).
- This is a very useful construction.
- We have to be more careful with the numeraire as the forward price is not a traded asset but the forward contract is. It is, however, possible to overcome this - see Joshi book.

# Be careful with interest rates

- On the topic of stochastic interest rates, we should briefly revisit option pricing when r is stochastic (or a function of time).
- Under the risk neutral measure $Q$

$$\hat{V}(S,t) = \frac{V(S,t)}{B(t)} = e^{-\int_0^t r(s)ds} V(S,t)$$

is a Martingale.

- This means,

$$\begin{aligned} \hat{V}(S,t) &= E_t^Q[\hat{V}(S,T)] \\ &= E_t^Q\left[e^{-\int_0^T r(s)ds} \max(S(T)-K,0)\right] \end{aligned}$$

thus

$$\begin{aligned} e^{\int_0^t r(s)ds} \hat{V}(S,t) &= E_t^Q\left[e^{-\int_0^T r(s)ds} \max(S(T)-K,0)\right] \\ V(S,t) &= E_t^Q\left[e^{-\int_t^T r(s)ds} \max(S(T)-K,0)\right] \end{aligned}$$

# Be careful with interest rates

- So, when interest rates are deterministic it doesn't matter when you discount.
- However, when the interest rates are stochastic then you must discount first and then take an expectation.

# Intermediate cash flows

- Let's consider the possibility that our **derivative product** might pay dividends or coupons
  - Use $h(S,t)$ to denote the cash flow of the derivative as a function of the stock price and time.
  - For example, if the instrument is a bond that pays a continuous coupon at the rate of \$2 per year, then $h(S,t) = 2$.
  - **It is important to distinguish between dividends/interest paid by the derivative, and dividends/interest paid by the underlying asset.**
  - As usual use $V(S,T)$ to denote the final payoff of the derivative.

# Intermediate cash flows: PDE

- Also let's consider a reasonably general case, under $Q$ (where $S$ and $t$ dependence is suppressed)

$$\begin{aligned} dS &= (r-\delta)Sdt + \sigma S dX \\ dB &= rBdt \end{aligned}$$

- Let $V(S,t)$ denote the price of the derivative, under $Q$

$$dV = \left(\frac{\partial V}{\partial t} + (r-\delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S \frac{\partial V}{\partial S} dX$$

the drift of the option plus the cash flow h should be equal to $rV$ so

$$\frac{\partial V}{\partial t} + (r-\delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + h = rV$$

as the return on the option (including coupons) is equal to the risk-free rate.

# Intermediate cash flows: Prob

- Probabilistic solution is:

2. discounted

3. exp. value

1. final cash flow

$$V(S(t),t) = E_t^Q\left[ e^{-\int_t^T r(s)ds} V(S(T),T) \right]$$

$$+ E_t^Q\left[ \int_t^T \left( e^{-\int_t^\tau r(s)ds} h(S(\tau),\tau) \right) d\tau \right]$$

7. exp. value

6. Integrate over all $\tau$

5. Discount from $\tau$ to $t$

4. intermediate cash flow at time $\tau$

- Think about it by working from the inside (of the integrals/expected values), to the outside.

# Simple Convertible bond

- Usual set-up

$$\begin{aligned} dS &= (\mu - \delta)Sdt + \sigma SdX \\ dB &= rBdt \end{aligned}$$

  where $\mu, d$ and $r$ are constants

- Consider a simple convertible bond (no call or put features, conversion only possible at expiry and constant interest rate) then

$$V(S,T) = \max(aS, K)$$

  also pay coupons or interest of dollar size h.

- PDE:

$$\frac{\partial V}{\partial t} + (r-\delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + h = rV$$

- Prob sol.:

$$V(S,t) = E_t^Q\left[e^{-r(T-t)}\max(aS(T),K)\right] + E_t^Q\left[\int_t^T \left(e^{-r(\tau-t)}h\right) d\tau\right]$$

# Quiz

- How do you think the problem would change if you could convert at any time and receive $aS$?

# Barrier options

- As usual

$$
\begin{aligned}
dS &= (\mu - \delta)Sdt + \sigma SdX \\
dB &= rBdt
\end{aligned}
$$

  Consider a down-and-out call option such that $H < S(0)$ is a constant, $\tau$ is time of first passage to $H$.

- Payoff: $\max(S(T) - K, 0)$ if $\tau > T$, $R$ (rebate) if $\tau \leq T$

# Barrier options

- The PDE is

$$\frac{\partial V}{\partial t} + (r-\delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

boundary conditions:

  - $V(S,T) = \max(S(T)-K,0)$
  - $V(H,t) = R$
  - $V(S,t) = Se^{-\delta(T-t)} - Ke^{-r(T-t)}, S \to \infty$

- Probabilistic solution is:

$$\begin{aligned} V(S,t) &= E_t^Q\left[e^{-r(T-t)}\max(S(T)-K,0)1_{\{\tau>T\}}\right] \\ &\quad + E_t^Q\left[e^{-r(\tau-t)}R1_{\{\tau\le T\}}\right] \end{aligned}$$

where $1_A$ is an indicator function which takes the value 1 if $A$ occurs and 0 if it does not.

# Overview

- We have looked at more applications of the probabilistic solution, with particular emphasis on the relationship between the PDE approach and the probabilistic solution.
- In particular, we looked at the Feynman-Kac formula which will enable us to switch between the two approaches, depending upon which is easier.
- We also looked at applying the probabilistic approach to options on forwards as well as suggesting how it could be extended to such products as convertible bonds and barrier options.