Fin 514: Financial Engineering II

Lecture 11: Generalizations and applications of the BSM PDE.

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Outline

- We have now derived the Black-Scholes formulae which describe the values of European call and put options given a current underlying asset price, risk-free rate, volatility, time to expiry and exercise price.
- Here we consider three extensions to this basic problem:
  - Recap on PDE and European option price
  - Time varying r and σ.
  - Options on foreign exchange.
  - Options on more than one underlying asset, with particular emphasis upon exchange options.
- This is to indicate how to extend the basic theory to different types of options.

BSM equation with dividends

- We have derived the Black-Scholes-Merton PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0$$

- Note that we can link this back to both the Black-Scholes equation for the value of calls and puts. In general, our PDE will be of the form

$$\frac{\partial V}{\partial t} + \frac{1}{2} a^2 X^2 \frac{\partial^2 V}{\partial X^2} + bX \frac{\partial V}{\partial X} - cV = 0$$

  X is the underlying asset of the option, we can interpret a as the volatility of the underlying asset, b is the risk-neutral expected return, so $E_t[X_T] = X_t e^{b(T-t)}$, and c is the discount rate or expected return of the option, $V_t = e^{-c(T-t)} E_t[V_T]$.

European style formula

- In general, the value of a call option on X with a strike price of K has payoff $V(X,T) = \max(X - K, 0)$ and is of the form

$$V(X_0, 0) = e^{-cT}\left(S_0 e^{bT} N(d_1) - K N(d_2)\right)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(X_0/K) + \left(b + \frac{1}{2}a^2\right)T}{a\sqrt{T}} \\ d_2 &= d_1 - a\sqrt{T} \end{aligned}$$

- For example, if

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0$$

European style formula

- Then the value of a call option is

$$V(S_0, 0) = e^{-rT}\left(S_0 e^{(r-\delta)T} N(d_1) - KN(d_2)\right)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_0/K) + \left(r - \delta + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

Binomial set up

- Additionally the general PDE also allows us to quickly determine the appropriate underlying asset and parameters in the binomial model.
- The underlying asset in the binomial will be X, the payoff at maturity, $V_{N,i} = V(X,T)$ will be determined as a function of X.
- Then,

$$
\begin{aligned}
u &= \exp(a\sqrt{\Delta t}) \\
d &= \exp(-a\sqrt{\Delta t}) \\
q &= \frac{exp(b\Delta t) - d}{u - d} \\
V_{j,i} &= exp(-c\Delta t)\left(qV_{j+1,i+1} + (1-q)V_{j+1,i}\right)
\end{aligned}
$$

Time varying parameters

- Our general set up for the BS PDE was:

$$\begin{aligned} dB &= rBdt \\ dS &= (\mu - \delta)Sdt + \sigma SdX \end{aligned}$$

  with μ, r, δ and σ as constants.

- However, the derivation of the PDE did not require that these parameters be constants. There is nothing to prevent us using:

$$\begin{aligned} r &= r(S, t) \\ \sigma &= \sigma(S, t) \\ \delta &= \delta(S, t) \\ \mu &= \ldots \end{aligned}$$

  μ here could depend on anything as it does not appear in the final equation.

Time varying parameters

- The PDE becomes:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma(S,t)^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r(S,t) - \delta(S,t))S\frac{\partial V}{\partial S} - r(S,t)V = 0$$

- In the case where we only have time varying parameters $r(t), \delta(t)$ and $\sigma(t)$ then the new formulae for call and put options are straightforward. Recall, that the original formula is:

$$V(S,t) = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\ln(S_0/K) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \delta - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Time varying parameters

- Note that r, δ and σ only appear as the integrals

$$\int_t^T r ds = r(T-t), \int_t^T \delta ds = \delta(T-t), \int_t^T \sigma^2 ds = \sigma^2(T-t)$$

  and so we can replace these expressions with the following

$$r(T-t) \to \int_t^T r(s) ds, \delta(T-t) \to \int_t^T \delta(s) ds, \sigma^2(T-t) \to \int_t^T \sigma(s)^2 ds$$

- This simplification is not quite so straightforward for $r(S,t), \delta(S,t)$ and $\sigma(S,t)$ as these may be stochastic and so the integral approach only works for $r(t), \delta(t)$ and $\sigma(t)$.

Time varying parameters

- So in the general case

$$V(S,t) = \exp\left(-\int_t^T r(s)ds\right)\left(S\exp\left(\int_t^T r(s)ds - \int_t^T \delta(s)ds\right)N(c$$

where

$$d_1 = \frac{\ln(S/K) + \left(\int_t^T r(s)ds - \int_t^T \delta(s)ds + \frac{1}{2}\int_t^T \sigma(s)^2 ds\right)}{\sqrt{\int_t^T \sigma(s)^2 ds}}$$

$$d_2 = d_1 - \sqrt{\int_t^T \sigma(s)^2 ds}$$

FX options

- This derivation shows how to construct the hedging argument when the underlying asset does not exactly follow GBM, it is a useful exercise to see how to derive the PDE in general.
- If we define the \$ value of a foreign currency (the foreign exchange rate) as S where the \$ risk-free rate is r and the foreign risk-free rate is r_f then we have:

$$\begin{aligned} dB &= rBdt \\ dB_f &= r_f B_f dt \\ dS &= \hat{\mu} S dt + \sigma S dX \end{aligned}$$

where we will often think of $\hat{\mu} = \mu - r_f$ but will be unimportant for the derivation of the PDE as, of course, the drift term disappears from the calculation.

Aside: Drift term

- Foreign risk-free rate r_f plays role of continuous dividend yield
- To understand this, think about the following two questions:

1. What do you miss if you receive the stock late, e.g. at the expiration date of an option?

- Answer: you miss the dividends paid during the life of the option

2. What do you miss if you receive the foreign currency late, e.g. at the expiration date of an option?

- Answer: you miss the opportunity to deposit the foreign currency at the foreign currency risk-free rate r_f ; that is you miss interest paid at the rate r_f.

Derivation of PDE

- The trick is to consider the value of the foreign bond B_f (or foreign Money Market account) in terms of \$ rather than the foreign currency. Denote S_f to be this value, thus:

$$\begin{aligned} S_f &= SB_f \\ dS_f &= B_f dS + SdB_f \\ dS_f &= B_f dS + r_f B_f S dt \end{aligned}$$

- Now construct a portfolio (denominated in \$), consisting of short one option, long Δ of foreign bonds and long β of \$ bonds, thus:

$$\Pi = -V + \Delta S_f + \beta B$$

the self financing condition gives us that

$$d\Pi = -dV + \Delta dS_f + \beta dB$$

Derivation of PDE

- By applying Ito's lemma we have

$$
\begin{aligned}
d\Pi &= -\left(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt\right) \\
&\quad + \Delta B_f(dS + r_f S dt) + \beta dB
\end{aligned}
$$

- Now choose

$$
\Delta = \frac{1}{B_f}\frac{\partial V}{\partial S}
$$

to give

$$
d\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + r_f S \frac{\partial V}{\partial S}dt + \beta r B dt
$$

Derivation of PDE

- Now choose β so that the portfolio has zero value and so

$$\beta B = V - \Delta S_f$$

to give

$$\begin{aligned} 0 &= -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r_f S \frac{\partial V}{\partial S}\right) dt + \beta r B dt \\ &= \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_f S \frac{\partial V}{\partial S} + rV - rS \frac{\partial V}{\partial S}\right) dt \end{aligned}$$

- Rearranging gives:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - r_f) S \frac{\partial V}{\partial S} - rV = 0$$

- Interpretation: value of a hedged portfolio of option and foreign currency grows at riskless rate r, minus foreign riskless rate r_f.

QUIZ

How can we interpret the $(r - r_f)$ term?

- The true expected return on the foreign currency is $(r - r_f)$
- The foreign interest rate is similar to a dividend yield
- The true expected \$ interest rate is $(r - r_f)$

QUIZ

Write down the value of a European call option on the exchange rate S, with an exercise price of K?

- ?

QUIZ

What would the binomial tree parameters be if you wanted to price an American exchange rate option?

- ?

2 Underlying assets

- If we have two stocks which follow correlated Geometric Brownian Motions (X_1 and X_2) such that:

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dX_1 \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dX_2 \end{aligned}$$

- Where X_2 can constructed to be as follows

$$X_2 = \rho X_1 + \sqrt{1-\rho^2} X_3$$

where ρ is the correlation between the two assets and X_3 is a new, independent Brownian motion.

- To verify that X_2 is still a Brownian motion, consider

$$X_2(t) - X_2(s) = \rho(X_1(t) - X_1(s)) + \sqrt{1-\rho^2}(X_3(t) - X_3(s))$$

2 underlying assets

- Thus, as X_1 and X_3 are independent we have:

$$\begin{aligned} E[X_2(t) - X_2(s)] &= \rho E[X_1(t) - X_1(s)] + \sqrt{1-\rho^2} E[X_3(t) - X_3(s)] \\ &= 0 \end{aligned}$$

  and

$$\begin{aligned} Var[X_2(t) - X_2(s)] &= \rho^2 Var[X_1(t) - X_1(s)] + (1-\rho^2) Var[X_3(t) - X_3(s)] \\ &= \rho^2 (t-s) + (1-\rho^2)(t-s) \\ &= t - s \end{aligned}$$

- To link in with earlier work we have (in simpler notation where $dX_i = (X_i(t+dt) - X_i(dt)))$

$$\begin{aligned} E[dX_1 dX_2] &= \rho E[dX_1^2] + \sqrt{1-\rho^2} E[dX_1 dX_3] \\ &= \rho dt + \sqrt{1-\rho^2} E[dX_1] E[dX_3] \\ &= \rho dt \end{aligned}$$

Self-financing

- In this scenario we have a slightly different self-financing portfolio as follows:

$$\int_0^t d\Pi_\tau = \int_0^t \alpha_\tau dV_\tau + \int_0^t \Delta_{1\tau} dS_{1\tau} + \int_0^t \Delta_{2\tau} dS_{2\tau} + \int_0^t \beta_\tau dB_\tau$$

or in differential notation:

$$d\Pi = \alpha dV + \Delta_1 dS_1 + \Delta_2 dS_2 + \beta dB$$

Derivatives on 2 underlyings

- So if we consider a general derivative $V(S_1, S_2, t)$ on S_1 and S_2 where

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dX_1 \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dX_2 \end{aligned}$$

  and

$$X_2 = \rho X_1 + \sqrt{1-\rho^2} X_3$$

- Consider a portfolio

$$\Pi = -V + \Delta_1 S_1 + \Delta_2 S_2 + \beta B$$

  then by the self-financing condition

$$d\Pi = -dV + \Delta_1 dS_1 + \Delta_2 dS_2 + \beta dB$$

Derivatives on 2 underlyings

- By applying Ito's lemma and choosing

$$\Delta_1 = \frac{\partial V}{\partial S_1}, \quad \Delta_2 = \frac{\partial V}{\partial S_2}$$

we obtain

$$\begin{aligned} d\Pi &= -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}\right. \\ &\qquad \left. + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} - \beta r B\right) dt \end{aligned}$$

- Then we choose β to ensure that the option value is replicated:

$$\beta B = V - \Delta S_1 - \Delta S_2$$

Derivatives on 2 underlyings

- Thus,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2}$$
$$+ rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV = 0$$

Simple adjustments

- If the two underlyings pay continuous dividends of δ_1 and δ_2 respectively then

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2}$$
$$+(r-\delta_1)S_1\frac{\partial V}{\partial S_1} + (r-\delta_2)S_2\frac{\partial V}{\partial S_2} - rV = 0$$

and if there are two foreign currencies with interest rates r_{f1} and r_{f2} then we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2}$$
$$+(r-r_{f1})S_1\frac{\partial V}{\partial S_1} + (r-r_{f2})S_2\frac{\partial V}{\partial S_2} - rV = 0$$

Exchange option

- The exchange option is an illustration of how, in certain instances, you can reduce the dimensionality of the problem (like we saw with lookback options). In general PDE methods this is typically referred to as a similarity solution which exploits certain proportionalities in the boundary conditions.
- However, in probability theory or finance it is referred to as changing the numeraire.
- Consider an option on two underlying assets S_1 and S_2 such that

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dX_1 \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dX_2 \\ X_2 &= \rho X_1 + \sqrt{1-\rho^2} X_3 \end{aligned}$$

and the payoff for the option is

$$V(S_1, S_2, T) = \max(S_2 - S_1, 0)$$

Exchange option

- From above, the PDE for the option value is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2}$$
$$+ rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV = 0$$

- However, we will use a change of variable which will simplify the problem greatly.

$$\begin{aligned} Y &= S_2/S_1 \\ S_1 W(Y,t) &= V(S_1, S_2, t) \end{aligned}$$

- This can be interpreted as the ratio of the two prices. What is interesting is that the final condition can be written entirely in Y, thus:

$$\begin{aligned} S_1 W(Y,T) &= V(S_1, S_2, T) \\ &= \max(S_2 - S_1, 0) \\ W(Y,T) &= \max(Y - 1, 0) \end{aligned}$$

Why is changing from S_1 and S_2 to Y useful?

Why is changing from S_1 and S_2 to Y useful?

- We have more intuition for what is happening financially.
- It is far easier to solve the problem with one underlying than with two.
- The boundary conditions are simpler with Y.

Change of Numeraire

- We may freely change numeraire whenever it is convenient to do.
- If there is a arbitrage opportunity in original numeraire then there is arbitrage opportunity in every other numeraire.
- No arbitrage opportunity in original numeraire implies that there is no arbitrage in every other numeraire (some books, e.g. Duffie, state this a a theorem, but we will wait until the probabilistic model to explain it).
- "Good" choices of numeraire reduce the dimensionality of problem.
- in a moment, we will transform the PDE but first let's think more about Y.

What is the process for Y?

- Using Ito's lemma for Y gives us:

$$
\begin{aligned}
dY &= \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial S_1}dS_1 + \frac{\partial Y}{\partial S_2}dS_2 \\
&+ \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 Y}{\partial S_1^2}dt + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 Y}{\partial S_2^2}dt + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 Y}{\partial S_1 \partial S_2}dt
\end{aligned}
$$

- But

$$
\begin{aligned}
\frac{\partial Y}{\partial t} = 0 \qquad & \frac{\partial^2 Y}{\partial S_1^2} = \frac{2S_2}{S_1^3} \\
\frac{\partial Y}{\partial S_1} = -\frac{S_2}{S_1^2} \qquad & \frac{\partial^2 Y}{\partial S_2^2} = 0 \\
\frac{\partial Y}{\partial S_2} = \frac{1}{S_1} \qquad & \frac{\partial^2 Y}{\partial S_1 \partial S_2} = -\frac{1}{S_1^2}
\end{aligned}
$$

- Thus

$$
dY = (\mu_2 - \mu_1 + \sigma_1^2 - \rho\sigma_1\sigma_2)Ydt - \sigma_1 YdX_1 + \sigma_2 YdX_2
$$

What is the process for Y?

- Now consider a new geometric Brownian motion with drift $= \mu_4$, volatility $= \sigma_4$ and a Brownian motion X_4.
- If we define X_4 as follows:

$$X_4 = \frac{-\sigma_1 X_1 + \sigma_2 X_2}{\sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}} = \frac{-\sigma_1 X_1 + \sigma_2 X_2}{\sigma_4}$$

then

$$\begin{aligned} E[X_4] &= 0 \\ Var[X_4] &= \frac{(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)t}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} = t \end{aligned}$$

and we can write

$$dY = \mu_4 Y dt + \sigma_4 Y dX_4$$

Back to Exchange option

- Consider $V = S_1 W(Y,t)$, $Y = S_2/S_1$, then via implementing product and chain rules we get:

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= S_1 \frac{\partial W}{\partial t} \\
\frac{\partial V}{\partial S_1} &= W - Y\frac{\partial W}{\partial Y} \\
\frac{\partial V}{\partial S_2} &= \frac{\partial W}{\partial Y} \\
\frac{\partial^2 V}{\partial S_1^2} &= \frac{Y^2}{S_1}\frac{\partial^2 W}{\partial Y^2} \\
\frac{\partial^2 V}{\partial S_2^2} &= \frac{1}{S_1}\frac{\partial^2 W}{\partial Y^2} \\
\frac{\partial^2 V}{\partial S_1 \partial S_2} &= -\frac{Y}{S_1}\frac{\partial^2 W}{\partial Y^2}
\end{aligned}
$$

Back to Exchange option

- By substituting these variables into the BS equation, the equation simplifies (see notes on board) to

$$S_1 \left(\frac{\partial W}{\partial t} + (rY - rY) \frac{\partial W}{\partial Y} + \left(\frac{1}{2}\sigma_1^2 Y^2 + \frac{1}{2}\sigma_2^2 Y^2 - \rho\sigma_1\sigma_2 Y^2 \right) \frac{\partial^2 W}{\partial Y^2} + (r - r)W \right) = 0$$

- Dividing through by S_1 and simplifying gives:

$$\frac{1}{2}\sigma_4^2 Y^2 \frac{\partial^2 W}{\partial Y^2} + \frac{\partial W}{\partial t} = 0$$

where

$$\sigma_4 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

- This equation is even simpler to solve than the one underlying BSM equation as we have lost the terms involving r (which is interesting but not obvious)

Back to Exchange option

- The solution, by analogy with Black-Scholes, is then:

$$W(Y,t) = YN(z) - N(z - \sigma_4\sqrt{T-t})$$

where

$$z = \frac{\ln Y + \frac{1}{2}\sigma_4^2(T-t)}{\sigma_4\sqrt{T-t}}$$

as from slides 3-4 above:

$$\begin{aligned} X &= Y \\ a &= \sigma_4 \\ b &= 0 \\ c &= 0 \\ K &= 1 \end{aligned}$$

Back to Exchange option

- Thus converting back to $V(S_1, S_2, t)$ gives:

$$V(S_1, S_2, t) = S_1 W(Y, t) = S_2 N(z) - S_1 N(z - \sigma_4 \sqrt{T - t})$$

- Thus the change of numeraire has greatly simplified the problem, and allowed us to get an analytic solution.
- This technique can only be used for certain options such as lookback options with specific payoffs.

QUIZ

What would the binomial tree parameters be if you wanted to price an American exchange option?

- ?

Overview

- We have looked at some generalizations of the Black-Scholes equations. In particular we have shown how to value FX options, which, as usual, enable you to treat the foreign interest rate as a dividend.
- We also derived the equation for valuing options on more than one underlying. In general this involves solving a PDE in many dimensions. The number of dimensions can be reduced in special circumstances when the payoff is homogeneous in the underlying assets.