# FIN 514: Problem Set #6

Due on Wednesday, April 25, 2018

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# Problem 1

(a) By Ito's product rule, $dY(t)$ satisfies the following equation.

$$\begin{aligned} dY(t) &= B_P(t)dS(t) + S(t)dB_P(t) + dB_P(t)dS(t) \\ &= B_P(t)[\mu S(t)dt + \sigma S(t)dX(t)] + r_P S(t)B_P(t)dt \\ &= (\mu + r_P)Y(t)dt + \sigma Y(t)dX(t) \end{aligned}$$

In order to find martingale measure with respect to $B(t)$ as a numeraire, dynamics of $Y(t)/B(t)$ is derived as follows.

$$\begin{aligned} d\left(\frac{Y(t)}{B(t)}\right) &= Y(t)d\left(\frac{1}{B(t)}\right) + \frac{1}{B(t)}dY(t) + dY(t)d\left(\frac{1}{B(t)}\right) \\ d\left(\frac{1}{B(t)}\right) &= -\frac{1}{B^2(t)}dB(t) \\ &= -\frac{1}{B^2(t)}rB(t)dt = -r\frac{1}{B(t)}dt \\ \Rightarrow d\left(\frac{Y(t)}{B(t)}\right) &= Y(t)\left(-r\frac{1}{B(t)}dt\right) + \frac{1}{B(t)}[(\mu + r_P)Y(t)dt + \sigma Y(t)dX(t)] \\ &= (\mu + r_P - r)\frac{Y(t)}{B(t)}dt + \sigma\frac{Y(t)}{B(t)}dX(t) \end{aligned}$$

By Girsanov's theorem, there exists a probability measure such that $\widetilde{X}(t) = X(t) + \int_0^t \frac{\mu + r_P - r}{\sigma}ds$ is a brownian motion under the measure. Therefore, by plugging $dX(t) = d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt$ into the equation above, then $d\left(\frac{Y(t)}{B(t)}\right)$ becomes $\sigma\frac{Y(t)}{B(t)}d\widetilde{X}(t)$, hence becomes martingale because there is no drift. Therefore, from the perspective of U.S dollar investor, under risk-neutral measure, $dX(t) = d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt$. By plugging it into dynamics of $Y(t)$, we can find dynamics of the U.S price of a GBP bond under risk-neutral measure as follows.

$$\begin{aligned} dY(t) &= (\mu + r_P)Y(t)dt + \sigma Y(t)dX(t) \\ &= (\mu + r_P)Y(t)dt + \sigma Y(t)\left[d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt\right] \\ &= rY(t)dt + \sigma Y(t)d\widetilde{X}(t) \end{aligned}$$

And it is consistent with the fact that expected return of every tradable asset is risk-free rate under risk-neutral measure.

(b) By plugging $dX(t) = d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt$ into dynamics of $S(t)$, we can find dynamics of U.S. dollar price of a British pound under risk-neutral probability as follows.

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dX(t) \\ &= \mu S(t)dt + \sigma S(t)\left[d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt\right] \\ &= (r - r_P)S(t)dt + \sigma S(t)d\widetilde{X}(t) \end{aligned}$$

Problem 1 continued on next page. . . 

(c) Unlike the assumption of ordinary Black-Scholes-Merton formula, since expected return of underlying asset has changed from $r$ to $r - r_P$, formula for call option should be changed to following equation.

$$e^{-rT}[S_0 e^{(r-r_P)T} N(d_1) - KN(d_2)]$$
$$d_1 = \frac{\log(S_0/K) + (r - r_P + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

## Problem 2

(a)

(b)

(c)

(d)

## Problem 3

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

(i)