Fin 514: Financial Engineering II

Lecture 9: The Black-Scholes Merton Equation

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Outline

- We are now equipped with the required tools to derive the Black-Scholes equation.
- This derivation will use all of the theory/techniques which we have considered so far, including:
  - No arbitrage
  - Dynamically complete markets
  - Self financing portfolios
  - Existence of a replicating portfolio
  - Ito integral and Ito's lemma
- A version of the correct derivation in this lecture is given in 'How to Derive Black-Scholes Equation Correctly' by Peter Carr and Akash Bandyopadhyay available on COMPASS.

Black-Scholes ('73) derivation

- Black and Scholes (1973) derived their famous partial differential equation for the value of a contingent claim as follows:
- There exists a stock and a bond (technically a Money Market Account) whose prices, S and B, are described by the following processes (I have dropped the time dependence here as all values are evaluated at t):

$$
\begin{aligned}
dB &= rBdt \\
dS &= \mu Sdt + \sigma SdX
\end{aligned}
$$

  where r is the risk-free rate, μ is the expected return on the stock and σ is the stock volatility.
- A replicating portfolio argument follows where there are no arbitrage opportunities and the number of shares held long is constant across short time periods.

Black-Scholes ('73) derivation

- Now construct a portfolio consisting of a written option, $V(S,t)$, and Δ shares thus

$$\Pi = -V + \Delta S$$

and then consider the change in the value of this portfolio over a change in time:

$$d\Pi = -dV + \Delta dS$$

where Δ is treated as a constant.

- Apply Ito's lemma to V(S,t) to get

$$\begin{aligned} d\Pi &= -dV + \Delta dS \\ &= -\left(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt\right) + \Delta dS \end{aligned}$$

Black-Scholes ('73) derivation

- If we now choose Δ such that

$$\Delta = \frac{dV}{dS}$$

then the change in V becomes deterministic:

$$d\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

as this change is deterministic then it is risk-free and must earn the same return as our money market account (i.e the risk-free rate), thus

$$r\Pi dt = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

Black-Scholes ('73) derivation

- Thus we arrive at the Black-Scholes partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  or with time subscripts

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + rS_t\frac{\partial V_t}{\partial S_t} - rV_t = 0$$

- The problem here is that V is typically a non-linear function of S and t and so Δ is not a constant, which appears to contradict the initial set up of the problem.

Black-Scholes problems

- Mathematical Problem: The total derivative of the portfolio was calculated incorrectly. As Δ is not a constant we need to use the correct product rule to differentiate the portfolio value:

$$d(\Delta S) = S d\Delta + \Delta dS + d\Delta dS$$

- Financial Problem: A riskless price process has been applied to a portfolio whose differential change in value is not deterministic. As Δ randomly changes with time, the Black-Scholes hedge portfolio is a risky position.

Self-Financing

- Consider a general portfolio consisting of at (at non-zero) options, Δ_t shares and β_t invested in the money market account for non-negative t (note that we have reintroduced time dependence notation)
- The portfolio is (Merton-)self-financing if:

$$\int_0^t d\Pi_\tau = \int_0^t \alpha_\tau dV_\tau + \int_0^t \Delta_\tau dS_\tau + \int_0^t \beta_\tau dB_\tau$$

or in more convenient differential notation:

$$\begin{aligned} \Pi_{t+dt} - \Pi_t &= \alpha_t(V_{t+dt} - V_t) + \Delta_t(S_{t+dt} - S_t) + \beta_t(B_{t+dt} - B_t) \\ d\Pi &= \alpha_t dV + \Delta_t dS + \beta_t dB \end{aligned}$$

Self-financing: interpretation

- How can we interpret this self-financing condition?
- Financial meaning: The change in the portfolio value in any time interval is fully financed by the change in option value, gain from the stock position, and adjustment of the bond position. We saw the same idea in the context of the multi-period binomial model. The above equation is the same idea in this continuous model.
- The trading gain from one asset in a self-financing portfolio goes to finance the other assets in the portfolio. In other words, the loss from one position is compensated by the profit from other position. It is a closed portfolio in the sense that no money is ever taken out of it, or went into it from the outside after its creation.

Self-financing: mathematically

- Now when we determine the change in the portfolio value across time we can take the total derivative, as in general:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

thus,

$$\begin{aligned} d\Pi_t &= (\alpha_t dV_t + \Delta_t dS_t + \beta_t dB_t) \\ &\quad + ((V_t + dV_t)d\alpha_t + (S_t + dS_t)d\Delta_t + (B_t + dB_t)d\beta_t) \end{aligned}$$

- However, by the self financing condition we have:

$$(V_t + dV_t)d\alpha_t + (S_t + dS_t)d\Delta_t + (B_t + dB_t)d\beta_t = 0$$

which is not unrealistic as the holdings in each of the assets should be fixed at time t.

Self-financing: further thoughts

- Let's think about the interpretation of the above equation directly.
- It tells us that the number of shares of the stock, number of bonds, and the number of options is adjusted at each instant of time t to maintain the replicating portfolio Π self-financing. This condition is exactly like the adjustment of stock and bond positions at each period in the discrete time binomial model.
- The self-financing condition is a conservation law. No money is taken out or went into the portfolio from outside after it has been created at time $t = 0$. Gain from the trading on some asset in the portfolio has been used to finance growth of the other assets. The above equation is a mathematical statement of this conservation law.

QUIZ

In the binomial model, do Δ and β change between time steps?

- Yes.
- No.
- When there are a large number of steps.

Correct derivation

- As before we have a stock and a bond whose movements are described by (we have dropped t again)

$$\begin{aligned} dB &= rBdt \\ dS &= \mu Sdt + \sigma SdX \end{aligned}$$

  there are no arbitrage opportunities and any risk-free asset must provide the riskless return r.

- There exists a self-financing portfolio consisting of one-written option, Δ of the underlying asset and β in the money market account, thus its value is given by

$$\Pi = -V + \Delta S + \beta B$$

Correct derivation

- The self-financing condition tells us that

$$d\Pi = -dV + \Delta dS + \beta dB$$

- Applying Ito's lemma we have:

$$\begin{aligned} d\Pi &= -dV + \Delta dS + \beta dB \\ &= -\left(\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt\right) + \Delta dS + \beta dB \end{aligned}$$

- Again, choose

$$\Delta = \frac{\partial V}{\partial S}$$

- Thus,

$$d\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \beta r B\right) dt$$

Correct derivation

- Now we have two choices as to how to proceed. Now assume that our holding in the stock and money market account exactly replicate changes in the option, so $\Pi = 0$ then

$$\beta B = V - \Delta S$$

- Additionally, the portfolio always has zero value so regardless of its expected return $d\Pi = 0$ and so

$$0 = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r\left(V - \frac{dV}{dS} S\right)\right) dt$$

- This gives the Black-Scholes-Merton partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

QUIZ

Setting $\Pi = 0$ implies what about the relationship between V and $\Delta S + \beta B$?

- $V = \Delta S + \beta B$
- $V < \Delta S + \beta B$
- $V > \Delta S + \beta B$

Correct derivation

- Alternatively, we could have gone straight to $d\Pi = r\Pi dt$ and then

$$d\Pi \quad = \quad r\left(-V + \frac{\partial V}{\partial S}S + \beta B\right)dt$$

- Then equating this with the earlier definition of $d\Pi$ gives the same value.

Self financing (just to check)

- We would like to verify that our self financing condition was reasonable. If we let $V(S,t)$ be the value of the option, then our self financing portfolio should be able to replicate the value of the option at all points in time.
- Thus $\Pi = \Delta S + \beta B = V$. We would like to show that $\Pi = V$ at any point in time.
- Consider the difference between the two. At an arbitrary point in time t, β can be chosen so that they are the same. So we need to check the change in the difference:

$$\begin{aligned} d(\Pi - V) &= d\Pi - dV \\ &= \frac{\partial V}{\partial S} dS + \beta dB - \frac{\partial V}{\partial t} dt - \frac{\partial V}{\partial S} dS - \frac{1}{2}\frac{\partial^2 V}{\partial S^2} dS^2 \\ &= \beta r B dt - \frac{\partial V}{\partial t} dt - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt \end{aligned}$$

Self financing

- By substituting in the value of Π and the Black-Scholes PDE we get:

$$\begin{aligned} d(\Pi - V) &= r\left(\Pi - S\frac{\partial V}{\partial S}\right)dt - r\left(V - S\frac{\partial V}{\partial S}\right)dt \\ &= r(\Pi - V)dt \\ &= 0 \end{aligned}$$

- So the solution to the SDE is zero and so $\Pi(S,t) = V(S,t)$ for all t. So our portfolio replicates the option value at all points on the path taken by S through time.

Comments: β

- How to Find $\beta(t)$? If we choose $\Pi(0) = 0$ in the hedged portfolio then due to absence of arbitrage we must have $\Pi(t) = 0$ for all $T \geq t \geq 0$. Then:

$$V(S_t, t) = \Delta_t S_t + \beta_t B_t$$

- We can easily obtain β_t from this,

$$\beta_t = \frac{V(S, t) - \Delta_t S_t}{B_t}$$

- The right-hand side is the 'self-financing dynamic option replicating portfolio'. As we learned in the discrete time model, in a arbitrage-free dynamically complete market such a portfolio must exist.

QUIZ

In the Black-Scholes derivation we used a stock to hedge an option. If you wished to use an option to hedge a stock then:

- What would the portfolio be?
- What would be the value of the portfolio weightings?

Parameters and option details

- The risk-free rate r, the expected rate of return μ and the volatility σ don't necessarily have to be constants, we can model them as functions of time (but not stochastic, yet). Although in the original Black-Scholes formulation they were assumed to be constant.
- Also, note that at this point in time we have defined nothing about the option payoff, this is a generic equation for any contract whose value is a function of the stock price and time (and thus can be replicated via a position in stocks and the money market account).

Risk-neutrality

- The Black-Scholes PDE is as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Note that the expected return μ does not appear anywhere in the equation. Thus, the value of a derivative is independent of the expected return of the underlying asset. As it was with the binomial model
- Thus, one can simply replace μ by r (the risk-free rate) for derivative pricing.
- This links in directly with the theory of risk-neutral pricing. We will see more clearly the mechanisms for this when we analyze the probabilistic solution.

QUIZ

- What do dividends do to the stock price
- What is the effect of dividends on the stock holder?

BSM derivation with dividends

- In our original derivation we did not consider the case where the underlying asset paid out any dividends. We now model the case where the underlying pays out a continuous dividend yield:
  - D_t is the dividend paid out until time t
  - δ is the dividend yield, generally not a constant
  - Thus δS is the instantaneous dividend payment and

$$
\begin{aligned}
D_t &= \int_0^t \delta S_\tau d\tau, \quad \delta \geq 0, t \geq 0 \\
dD &= \delta S dt, \quad \delta \geq 0, t \geq 0
\end{aligned}
$$

The stock *price* movements are, thus, given by

$$
\begin{aligned}
dS &= \mu S dt + \sigma S dX - \delta S dt \\
&= (\mu - \delta) S dt + \sigma S dX
\end{aligned}
$$

BSM derivation with dividends

- The new self-financing rule is

$$
\begin{aligned}
\Pi &= \alpha V + \Delta S + \beta B \\
\int_0^t d\Pi_\tau &= \int_0^t \alpha_\tau dV_\tau + \int_0^t \Delta_\tau dS_\tau + \int_0^t \Delta_\tau dD_\tau + \int_0^t \beta_\tau dB_\tau \\
d\Pi &= \alpha dV + \Delta(dS + \delta S dt) + \beta dB
\end{aligned}
$$

but by taking total derivatives

$$
\begin{aligned}
d\Pi &= (\alpha dV + \Delta dS + \beta dB) \\
&\quad + ((V + dV)d\alpha + (S + dS)d\Delta + (B + dB)d\beta)
\end{aligned}
$$

thus, the self financing condition is equivalent to stating that:

$$(V + dV)d\alpha + (S + dS)d\Delta + (B + dB)d\beta = \Delta\delta S dt = \Delta dD$$

BSM derivation with dividends

- As before we have a stock and a bond whose movements are described by:

$$\begin{aligned} dB &= rBdt \\ dS &= (\mu - \delta)Sdt + \sigma SdX \\ dD &= \delta Sdt \end{aligned}$$

  there are no arbitrage opportunities and any risk-free asset must provide the riskless return r.

- There exists a self-financing portfolio consisting of one-written option, Δ of the underlying asset and β bonds, thus its value is given by

$$\Pi = -V + \Delta S + \beta B$$

BSM derivation with dividends

- Thus, as the portfolio is self financing:

$$d\Pi = -dV + \Delta(dS + \delta S dt) + \beta dB$$

- By applying Ito's lemma and the hedging argument gives

$$d\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \delta S \frac{\partial V}{\partial S} - \beta r B\right) dt$$

- We can now choose β so that the portfolio has zero value and so

$$0 = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \delta S \frac{\partial V}{\partial S} - r\left(V - S\frac{dV}{dS}\right)\right) dt$$

BSM equation with dividends

- This gives us the Black-Scholes-Merton PDE which is a generalization of the Black-Scholes equation although all of the notes on Black-Scholes also apply to this equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0$$

Quiz

In a leading Math Finance textbook the portfolio Π is constructed as

$$\Pi = V - \Delta S$$

when dividends are introduced and derivatives are taken this becomes:

$$\begin{aligned} d\Pi &= dV - \Delta dS - \Delta dD \\ d\Pi &= dV - \Delta dS - \Delta \delta S dt \end{aligned}$$

- What should $d\Pi$ be equal to? Are you happy with the above argument?

Overview

- We have now correctly derived the Black-Scholes(-Merton) partial differential equation which defines the value of any contingent claim on an underlying asset S.
- We will spend the next section of the course solving this particular equation for a variety of options.
- Note that its derivation uses both dynamic, self-financing replicating portfolios as well as the tools of stochastic calculus such as Ito's lemma and the Ito integral representation.

Aside: BSM revisited

- We can make the original Black-Scholes derivation work by replacing the mathematical operation of taking a total derivative by computing a ‘gain’.
- Consider a portfolio H, below

$$H = -V + \Delta S$$

the gain on H can be defined as follows:

$$g(H_t) = \int_0^t -dV_\tau + \int_0^t \Delta_\tau (dS_\tau + \delta S_\tau d\tau)$$

- Choosing the usual Δ and applying Ito's lemma to V gives:

$$g(H_t) = \int_0^t \left[-\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S_\tau^2 \frac{\partial^2 V}{\partial S_\tau^2} + \delta S_\tau \frac{\partial V}{\partial S_\tau} \right] d\tau$$

Aside: BSM revisited

- Since this financial gain is deterministic, absence of arbitrage requires that it be the same as the interest gain on a dynamic position in the riskless asset chosen so as to finance all trades in the derivative and the stock:

$$g(B_t) = \int_0^t r \left(-V_\tau + S_\tau \frac{\partial V}{\partial S_\tau} \right) d\tau$$

- Equating the two financial gains then leads to the Black-Scholes-Merton equation. It is worth noting that the portfolio consisting of the option and stock is not self-financing. Similarly, positions in the riskless asset are not self-financing. Nonetheless, by showing that the trading gains between two non-self-financing strategies are always equal under no arbitrage, the value of the derivative security can be determined.