Fin 514: Financial Engineering II

Lecture 10: PDE Solutions to the Black-Scholes equation

Dr. Martin Widdicks

UIUC

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Outline

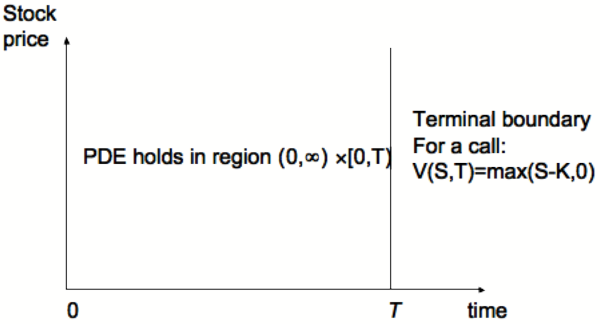
- Having now derived the Black-Scholes equation we will attempt to solve it.
- What follows is not a complete solution but should provide enough information to solve the equation for a range of different derivatives and understand the intuition of the solution for European call and put options.
- In particular we quote, but do not derive, the fundamental solution to the heat conduction equation (the Green's function)
- The solution to the Black-Scholes equation can be viewed directly as an extension of the work on Arrow-Debreu state prices.

The Black-Scholes equation

- When the underlying asset pays no dividends, the Black-Scholes equation is given by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where the solution region is as follows:



Types of PDE

- Depending upon the coefficients, PDEs can be classified as either elliptic, hyperbolic or parabolic.
- The Black-Scholes equation is **Parabolic** as there is a second derivative with respect to one variable (the underlying) and only a first derivative with respect to the other (time).
- Interestingly, it is a **backward** parabolic PDE as the signs of $\partial V/\partial t$ and $\partial^2 V/\partial S^2$are the same.
- This means that unlike forward equations which require initial conditions the Black-Scholes PDE requires a final condition from which you work backward.
- This is entirely consistent with derivative pricing where you know the final conditions at T and need to use these to determine the price of the derivative for $t < T$.

Linearity

- If we have two solutions, V_1, V_2, to the Black-Scholes equation then the linear combination, $V = aV_1 + bV_2$ is also a solution. This can be easily verified by substituting this expression into the PDE.
- Note that this linearity is consistent with finance theory
- First it says that a portfolio value is the sum of its components, i.e. if we hold N_i of instruments with value V_i we have

$$V = \sum_i N_i V_i$$

- Secondly, it is consistent with no arbitrage, as if $V > aV_1 + bV_2$ then sell the portfolio for V and buy a of V_1 and b of V_2, to make a risk-free profit. And vice-versa for $V < aV_1 + bV_2$.
- Note that if there are transaction costs this linear result may not hold, leading to a non-linear PDE.

European option: Boundary conditions

- For a backward parabolic equation we need two conditions in the underlying asset and one terminal condition.
- For a European call with no dividends:

$$\begin{aligned} V(S,T) &= max(S-K,0) \\ V(0,t) &= 0 \\ V(S,t) &= S - Ke^{-r(T-t)} \text{ as } S \to \infty \end{aligned}$$

- For a European put with no dividends:

$$\begin{aligned} V(S,T) &= max(K-S,0) \\ V(0,t) &= Ke^{-r(T-t)} \\ V(S,t) &= 0 \text{ as } S \to \infty \end{aligned}$$

- Generally for calls and puts we don't need the conditions in the underlying assets, as the probability of reaching them is tiny. We relax this when using numerical methods, such as finite-differences.

Trivial solution to BS equation

- Before we focus on solving the Black-Scholes equation for options we should look at a simple example when we have a forward contract.
- Consider a forward contract on a non-dividend paying asset with a delivery price, K. We know the final conditions: $F(S,T) = S - K$ before this we believe that the value is $F(S,t) = S - Ke^{-r(T-t)}$.
- Check this by showing that $F(S,t)$ satisfies the terminal condition (trivial) and the Black-Scholes PDE:

$$\begin{aligned} \partial V/\partial t &= -rKe^{-r(T-t)} \\ \partial V/\partial S &= 1 \\ \partial^2 V/\partial S^2 &= 0 \end{aligned}$$

Upon substitution we get:

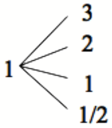
$$-rKe^{-r(T-t)} + 0 + rS - r(S - Ke^{-r(T-t)}) = 0$$

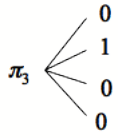
QUIZ

Show that the following expressions satisfy the Black-Scholes equation (A is a constant)?

- $V(S,t) = AS$
- $V(S,t) = Ae^{rt}$
- What do they represent?

Recall A-D prices

- Consider a one-period world, where a stock can take on one of four values: 

- The payoff and current value, π_3 of the third state A-D security are 

- The value of an option or other security can be written as

$$V = \sum_{i=1}^{4} \pi_i D_i$$

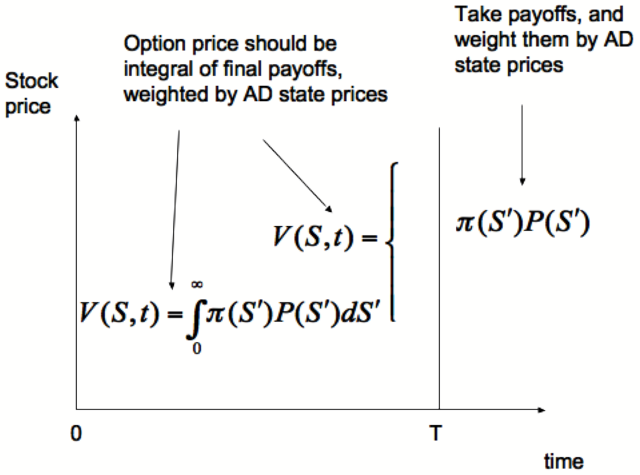
A-D State prices and Black-Scholes

- In the Black-Scholes (continuous time) world there are infinitely many A-D securities, which will cause us some problems.
- To be consistent with our asset pricing models we would like to write

  $$V(S,t) = \int_0^\infty P(S')\pi(S')dS'$$

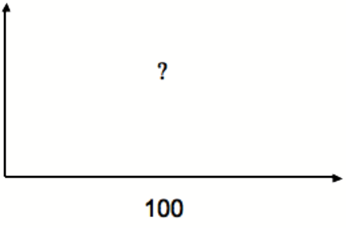
  where S' is the stock price at maturity (to distiguish from S at t) and $P(S')$ gives the payoff of a particular derivative. $\pi(S')$ is the price of an A-D security paying out \$1 when the stock price at maturity is S'.
- The problem we will have is how to define these A-D securities in a continuous setting, the next diagram denotes the problem:
- Let $\eta(S', S^*)$ denote the payoff of an A-D security as a function of the stock price at maturity for a particular state S^*. So if $S^* = 100$, $\eta(S', 100)$ is the payoff of the A-D security that pays \$1 if $S' = 100$ and zero otherwise.

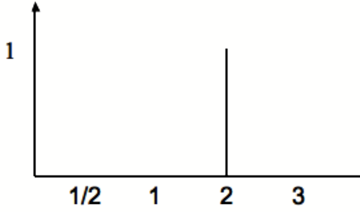
A-D in continuous S space



What does η look like?

- Consider $\eta(S', 100)$, should $\eta(100, 100) = 1$?





- The graph on the right is a discrete analogy
- Considering an interval $[a, b]$, $a < 100 < b$, there will be *infinitely* many A-D securities, what we want to happen is:

$$\int_a^b \eta(S', S')dS' = \infty$$

QUIZ

- If we have a very dense multinomial model then why should the sum of all the payoffs of the A-D securities be ∞?

A-D state prices

- The problem with this integral is that if we integrate all of the payoffs in the region we want to find that the sum of all the A-D payoffs across all possible states/stock prices S' is

$$\int_a^b \eta(S', S') dS' = \infty$$

but this means it is impossible for $\eta(S', S') = 1$ as in this case:

$$\int_a^b \eta(S', S') dS' = b - a < \infty$$

- Additionally, when we consider a single A-D security, e.g $S^* = 100$ we require

$$\int_a^b \eta(S', 100) dS' = 1$$

and if P is a payoff function then we require

$$\int_a^b \eta(S', 100) P(S') dS' = P(100)$$

Dirac Delta function

- These last two properties are key properties of the Dirac delta 'function ', which works as an indicator function.
- This can be defined in the following way, as the limit of a sequence such that:
  - For each ϵ, $\delta_\epsilon(x)$ is piecewise smooth
  - For each ϵ,

$$\int_{-\infty}^{\infty} \delta_\epsilon(x) dx = 1$$

  - For each $x \neq 0$, $\lim_{\epsilon \to 0} \delta_\epsilon(x) = 0$.
  - For a smooth function f(x),

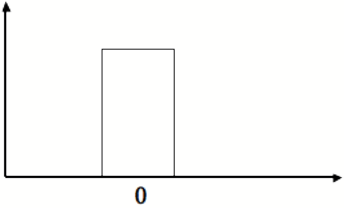
$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \delta_\epsilon(x) f(x) dx = f(0)$$

Dirac Delta function

- A way of visualizing the delta function is as the limit of the following sequence:

$$\delta_\epsilon(x) = \begin{cases} 1/(2\epsilon) \text{ if } -\epsilon \leq x \leq \epsilon \\ 0 \text{ if } |x| > \epsilon \end{cases}$$

- Each function in the sequence integrates to 1 and looks like:



- Another definition is:

$$\delta_\epsilon(x) = \frac{1}{2\sqrt{\pi\epsilon}} e^{-x^2/(4\epsilon)}$$

or a sequence of Normal distribution functions, which integrate to 1.

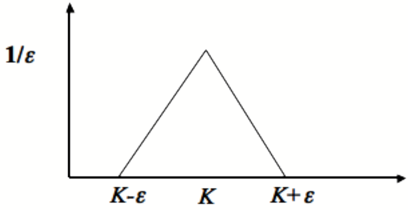
Link back to Finance, see Joshi 6.1

- Consider the following portfolio
  - Buy $1/\epsilon^2$ call options with strike $(K-\epsilon)$.
  - Write $2/\epsilon^2$ call options with strike K.
  - Buy $1/\epsilon^2$ call options with strike $(K+\epsilon)$

$$\begin{aligned}\text{Payoff} &= \frac{1}{\epsilon^2}\max(S-(K-\epsilon),0) - \frac{2}{\epsilon^2}\max(S-K,0) \\ &\quad + \frac{1}{\epsilon^2}\max(S-(K+\epsilon),0)\end{aligned}$$

Which looks as follows and is another sequence which tends to a delta function:

Link back to Finance, see Joshi 6.1



How does it help us?

- Thus we could write the payoff from the A-D security as

$$\eta(S', S^*) = \delta(S' - S^*)$$

and so the $S^* = 100$ A-D claim has payoff

$$\eta(S', 100) = \delta(S' - 100)$$

- Our other conditions are satisfied also as

$$\begin{aligned} \int_a^b \delta(S' - 100) dS' &= 1 \\ \int_a^b \delta(S' - S') dS' &= \infty \\ \int_a^b \delta(S' - 100) P(S') dS' &= P(100) \end{aligned}$$

Fundamental (Green's function) solutions

- Our approach to Black-Scholes formula is to derive a special solution of the pde. The special solution:
  - Has initial (terminal) condition given by a delta function
  - Thus the special solution corresponds to the payoff of a single A-D security
  - In fact, this special solution gives **current value** of a single A-D security
- Then, use fact that the derivative payoff is a sum/integral of weighted payoffs of A-D securities thus, current derivative value must be the sum/integral of current A-D prices weighted by payoffs for different terminal/initial conditions. Now that we have the special solution then the value is the integral of special solutions weighted by payoffs.
- The special solution is called the **fundamental or Green's function solution.**

Link to our problem

- We have a series of A-D securities with payoff $\eta(S', S^*) = \delta(S' - S^*)$ for S^* equal to all possible future stock prices S'. The current value of these A-D securities we will denote as $\pi(S^*)$, which will eventually be a function of the current stock price $S(= S_t)$ and time to maturity $T - t$.
- To calculate the value of each A-D security we will need to solve a partial differential equation with terminal condition $V(S', T) = \delta(S' - S^*)$. We will find this by considering a simpler problem where the PDE is the diffusion equation (see next slide).
- Once we have the value of $\pi(S^*)$ then we can write down the value of the derivative as follows:

$$V(S,t) = \int_0^\infty \pi(S') Payoff(S') dS'$$

or

$$V(S,t) = \int_0^\infty \pi(S') V(S', T) dS'$$

- So the important stage is to figure out $\pi(S^*)$.

Sneak preview

- It will require a few steps to transform the Black-Scholes PDE in the correct form to use the fundamental solution but once we have we will be able to write the solution as follows:

$$
\begin{aligned}
V(S,t) &= \int_0^{\infty} \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \frac{1}{S'} \\
&\times \exp\left(-\frac{(\ln(S'/S) - (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right) \\
&\times \text{Payoff}(S')dS'
\end{aligned}
$$

- The largest term in the integral (the first two rows in blue) is the current value of A-D security paying \$1 when S' occurs - or the fundamental PDE solution. Then this is scaled by the payoff for a given S' and then summed over all of the possible S' values.
- This equation will also have a probabilistic interpretation that requires the existence of a risk-neutral probability measure, see the Probabilistic Solution.

Diffusion or Heat conduction equation

- In order to get to the fundamental solution we need to transform the Black-Scholes equation to the diffusion (or heat conduction) equation, namely

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

- We know a lot about the diffusion equation and it has infinitely many solutions depending upon the initial condition (**we will transform our Black-Scholes pde so that its terminal condition becomes an initial one**).
- We are interested in the solution $u_\delta(x, \tau; x^*)$ where x and τ denote the current (transformed) price and time variables and x^* is associated with the A-D security paying out \$1 at x^* and 0 elsewhere.
- The fundamental solution satisfies the initial condition:

$$u_\delta(x', 0; x^*) = \delta(x' - x^*)$$

Transforming the PDE

- and is

$$u_\delta(x, \tau; x^*) = \frac{1}{2\sqrt{\pi\tau}} \exp\left(-\frac{(x - x^*)^2}{4\tau}\right)$$

where x is the value of x at time τ, which will be an analogue of the current stock price. u_δ is a (scaled) normal density function and thus satisfies the initial condition (see example above).

Aside: Transforming the PDE to the diffusion equation

- So, in order to get to the fundamental solution we need to transform the Black-Scholes equation to the diffusion (or heat conduction) equation, namely

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

- Then we can use our $u_\delta(x', t; x^*)$ from above to write down the fundamental solution to this problem and then reverse the transformations to obtain analytic expressions for the values of European call and put options.

Aside: Transformations I

- Define
$$U(S,t) = e^{r(T-t)}V(S,t)$$
So U is the future value of the derivative price.
- Also (in more compact notation)
$$\begin{aligned} V_S &= e^{-r(T-t)}U_S \\ V_{SS} &= e^{-r(T-t)}U_{SS} \\ V_t &= re^{-r(T-t)}U + e^{-r(T-t)}U_t \end{aligned}$$
- Giving,
$$U_t + \frac{1}{2}\sigma^2 S^2 U_{SS} + rSU_S = 0$$
i.e. the rV term has disappeared.
- Next reverse the direction of time by introducing $t' = T - t$ yielding
$$U_{t'} = \frac{1}{2}\sigma^2 S^2 U_{SS} + rSU_S$$

Aside: Transformations II

- Define $\xi = \ln S$, or $S = e^{\xi}$. Thus,

$$
\begin{aligned}
U_S &= U_\xi \frac{\partial \xi}{\partial S} \\
&= U_\xi \frac{1}{S} \\
USS &= U_{\xi\xi} \left(\frac{\partial \xi}{\partial S}\right)^2 + U_\xi \frac{\partial \xi}{\partial S} \\
&= U_{\xi\xi} \frac{1}{S^2} - U_\xi \frac{1}{S^2}
\end{aligned}
$$

giving,

$$U_{t'} = \frac{1}{2}\sigma^2 U_{\xi\xi} + \left(r - \frac{1}{2}\sigma^2\right) U_\xi$$

- Now introduce $x = \xi + (r - \frac{1}{2}\sigma^2)t'$ and $u(x, t') = U(\xi, t')$

Aside: Transformations III

- Then,

$$\begin{aligned} U_\xi &= u_x \\ U_{\xi\xi} &= u_{xx} \\ U_{t'} &= u_{t'} + u_x \left(r - \frac{1}{2}\sigma^2 \right) \end{aligned}$$

- This gives us:

$$u_{t'} = \frac{1}{2}\sigma^2 u_{xx}$$

- Finally, transform time to $\tau = \frac{1}{2}\sigma^2 t'$ so that

$$\begin{aligned} u_{xx} &= u_{xx} \\ u_{t'} &= \frac{1}{2}\sigma^2 u_\tau \end{aligned}$$

  and so we have

$$u_\tau = u_{xx}, \quad \text{or } \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

  as required.

Aside: Intuition

- The V transformation $U(S,t) = e^{r(T-t)}V(S,t)$ deals with the time value of money and removes the rV term.
- S transformation $(S \to \ln S)$: This changes to constant coefficients should not be surprising, because we know that under the risk-neutral probabilities

$$d \ln S = (r - \frac{1}{2}\sigma^2)dt + \sigma dX$$

  the log of S is the return of the stock, which we know to be normally distributed.
- Note, we could have done the entire analysis using ln S as the underlying variable of the option instead of S. If you repeat the BSM analysis using ln S as the underlying variable, you will get a PDE with constant coefficients.

Aside: Intuition

- Growth term: the rate of growth (mean value) of $\xi = \ln S$ is $\left(r - \frac{1}{2}\sigma^2\right) t$, so the change of variable $x = \ln S + \left(r - \frac{1}{2}\sigma^2\right) t'$ removes the drift of the return and eliminates the first derivative term (rSV_S).
- We switch time $t' = T - t$ to get a problem with an initial condition rather than a terminal condition.
- The scaling of t' by σ^2 then scales the variance to 1, and the $\frac{1}{2}$ is required to move from a normal distribution to the heat conduction equation, via standard convention.

Aside: Fundamental solution

- From above the fundamental solution to this equation is $u_\delta(x, \tau; x^*)$ where x and τ denote the current (transformed) price and x^* is associated with the A-D security paying out \$1 at x^* and 0 elsewhere.
- The fundamental solution is

$$u_\delta(x, \tau; x^*) = \frac{1}{2\sqrt{\pi\tau}} \exp\left(-\frac{(x^* - x)^2}{4\tau}\right)$$

and satisfies the initial condition:

$$u_\delta(x, 0; x^*) = \delta(x^* - x)$$

Aside: Turning this into an option pricing formula

- The fundamental solution on previous page was for the initial condition given by Dirac delta function.
- Now, recall that BS PDE is linear so if we form a new initial condition consisting of a linear combination of some other initial conditions, then the new solution is the linear combination of the solutions corresponding to the other initial conditions.
- Also, note that the initial conditions of options and other securities can be expressed as linear combinations of Dirac delta function thus, option prices can be expressed as linear combinations of the fundamental solutions (or A-D security prices)

Aside: QUIZ

If $u_\delta(x,\tau;x')$ is the current value of an asset paying off 1 when $x = x'$ at maturity ($\tau = 0$) then what is the current value, $V(x,\tau)$ of an option with payoff $V(x')$?

- $V(x,\tau) = u_\delta(x,\tau;x')V(x')$
- $V(x,\tau) = \int_{-\infty}^{\infty} u_\delta(x,\tau;x')V(x')dx'$
- $V(x,\tau) = e^{-r\tau} \int_{-\infty}^{\infty} u_\delta(x,\tau;x')V(x')dx'$

Aside: Turning this into an option pricing formula

- For example, consider a European call option the payoff in terms of the original variables is:

$$V(S', T) = \max(S' - K, 0)$$

- In terms of x at maturity, x', the payoff is

$$V(e^{x'}, 0) = \max(e^{x'} - K, 0)$$

- In terms of the variables (x, τ) the call option price is given by

$$\begin{aligned} u(x,\tau) &= \int_{-\infty}^{\infty} u_\delta(x,\tau;x')u(x',0)dx' \\ &= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\tau}} \exp\left(-\frac{(x'-x)^2}{4\tau}\right) \max(e^{x'} - K, 0)dx' \end{aligned}$$

- Recall from before that Current Price $= \sum_{i=1} N\pi_i D_i$ where here, $\pi_i = \frac{1}{2\sqrt{\pi\tau}} \exp\left(-\frac{(x'-x)^2}{4\tau}\right)$ and $D_i = \max(e^{x'} - K, 0)$.
- Similar formulas also holds for other options, with the only difference being a different payoff function.

Aside: Option price equation

- By now substituting the original expressions for the current t, current S (and potential future values, S') and V we have:

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \frac{1}{S'} \exp\left(-\frac{(\ln(S'/S) - (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right) \mathrm{Payoff}(S')dS'$$

as

$$\begin{aligned} u(x,\tau) &= V(S,t)e^{r(T-t)} \\ x &= \ln S + (r - \frac{1}{2}\sigma^2)(T-t) \\ x' &= \ln S' + (r - \frac{1}{2}\sigma^2)(0) = \ln S' \\ \tau &= \frac{1}{2}\sigma^2(T-t) \\ dx' &= \frac{1}{S'}dS' \end{aligned}$$

Solving for a call option

- By solving this integral (see end for details), the Black-Scholes formula for the price of a European call option is

$$\begin{aligned} V(S,t) &= A(S,t) - B(S,t) \\ &= SN(d_1) - Ke^{-r(T-t)}N(d_2) \end{aligned}$$

- For a European put option, similar computations (or put-call parity) result in

$$V(S,t) = -SN(-d_1) + Ke^{-r(T-t)}N(-d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} \\ d_2 &= d_1 - \sigma\sqrt{(T-t)} \end{aligned}$$

Overview

- By appealing to our work on Arrow-Debreu securities, we know that the current price of an A-D security is the equivalent to the (transformed) fundamental solution to the diffusion equation.
- By using this we can write the value of a derivative by summing (integrating) each of the possible future payoffs multiplied by the A-D price, this gives us expressions for the value of European call and put options.
- This derivation can also be done from a probabilistic approach where we demonstrate the existence of a risk-neutral measure and then value the option as a discounted expectation. This will actually be easier to perform but conceptually more difficult.

Aside: Solving for a call option

- It is possible to simplify this equation for the case of European call and put options. Consider the untransformed equation, namely:

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \frac{1}{S'} \exp\left(-\frac{(\ln(S'/S) - (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right) \mathrm{Payoff}(S')dS'$$

- Now simplify this by reintroducing $x' = lnS'$ and transforming $\max(S-K,0)$ into x' co-ordinates thus:

$$\begin{aligned}
V(S,t) &= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\ln K}^\infty \exp\left(-\frac{(x' - \ln S - (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right)(e^{x'} - K)dx' \\
&= A(S,t) - B(S,t) \\
&= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\ln K}^\infty \exp\left(-\frac{(x' - \ln S - (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right) e^{x'}dx' \\
&\quad - \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\ln K}^\infty \exp\left(-\frac{(x' - \ln S - (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right) Kdx'
\end{aligned}$$

Aside: Solving for a call option

- Taking the second term first gives

$$B(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\ln K}^{\infty}\exp\left(-\frac{(x'-\ln S-(r-\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right)Kdx'$$

now let

$$y = \frac{x'-\ln S-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

$$k_2 = \frac{\ln K-\ln S-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

then this transformation simplifies to give

$$B(S,t) = Ke^{-r(T-t)}\int_{k_2}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-y^2/2)dy$$

Aside: Solving for a call option

- Thus

$$
\begin{aligned}
B(S,t) &= e^{-r(T-t)}K(1-N(k_2)) \\
&= e^{-r(T-t)}KN(-k_2) \\
&= e^{-r(T-t)}KN(d_2)
\end{aligned}
$$

- Where:

$$
d_2 = \frac{\ln(S/K)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}
$$

Aside: Solving for a call option

- It now remains to solve for A(S,t) too,

$$
\begin{aligned}
A(S,t) &= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\ln K}^{\infty}\exp\left(-\frac{(x'-\ln S-(r-\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right)e^{x'}\,dx' \\
&= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\ln K}^{\infty}\exp\left(-\frac{(x'-\ln S-(r-\frac{1}{2}\sigma^2)(T-t))^2-2\sigma^2(T-t)x'}{2\sigma^2(T-t)}\right)dx' \\
&= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\ln K}^{\infty}e^{r(T-t)}S\exp\left(-\frac{(x'-\ln S-(r+\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right)dx'
\end{aligned}
$$

by completing the square.

Aside: Solving for a call option

- Let,

$$
\begin{aligned}
y' &= \frac{x' - \ln S - (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{(T - t)}} \\
k_1 &= \frac{\ln K - \ln S - (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{(T - t)}}
\end{aligned}
$$

- Then

$$
A(S,t) = S \int_{k_1}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-y'^2/2) dy'
$$

and

$$
\begin{aligned}
A(S,t) &= S(1 - N(k_1)) \\
&= SN(-k_1) \\
&= SN(d_1)
\end{aligned}
$$