Fin 512: Financial Derivatives

Lecture 17: Implied volatility

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Outline

- Here we will look at estimating parameters in the Black-Scholes model.
- All of the parameters will be straightforward except for the volatility one.
- We will look at both historic and implied volatilities and in doing so we will see a serious problem with the Black-Scholes model.
- We will look at ways of overcoming this problem and what practitioners choose to do.

BSM formulas

- Last time we derived the Black-Scholes-Merton option pricing formula:
$$C_t = S_t e^{-d(T-t)} N(d_1) - e^{-r(T-t)} K N(d_2)$$
where
$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (r - d + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} \\ d_2 &= \frac{\ln(S_t/K) + (r - d - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} = d_1 - \sigma\sqrt{T-t} \end{aligned}$$
- For a put option we have
$$P_t = -S_t e^{-d(T-t)} N(-d_1) + e^{-r(T-t)} K N(-d_2)$$
where d_1 and d_2 are defined as above.

Input parameters

- The Black-Scholes formulae require six input parameters: S_0, K, T, r, d and σ.
- The first three are very easy to find simply being the current value of the underlying asset, the exercise price of the option and the remaining time to expiry (this is usually denoted as $(T - t)$ where the option is issued at $t = 0$ and the current time is t).
- r is the annual continuously compounded risk-free rate of interest, typically the LIBOR rate for the appropriate period, either quoted or from swap data. This is then converted to a continuously compounded rate.
- d is the continuous dividend yield. This can be estimated from historical data, or more usually from futures prices, where we know that $F_{t,T} = S_t e^{(r-d)(T-t)}$. Thus, knowing the futures price gives us an estimate for d.
- σ is the annual standard deviation of the underlying asset's continuously compounded return.

Determining σ

- As we will see one of the most important parameters in determining the option price will be the volatility of the underlying asset.
- Unfortunately, this is also the parameter which is hardest to estimate. For example, given a typical European call we observe $C(\sigma)$ values:
  $C(0.3) = \$4.31$, $C(0.2) = \$3.26$, $C(0.1) = \$2.27$, $C(0.05) = \$1.92$.
- Option value is very sensitive to the value of σ used and this is not an extreme example. In fact out-of-the-money options are even more sensitive to volatility assumptions.
- Getting a good estimate of σ is very important.

Terminology

- What exactly do financial market practioners mean by "volatility?"
- Usually, volatility or "vol" means standard deviation of continuously compounded returns, expressed on an annual basis. Volatility can be either:
    1. Realized volatility, computed from a time-series of returns; or
    2. Implied volatility, calculated by inverting the Black-Scholes formula or the binomial option pricing model
- Often the word "implied" is dropped, so that "volatility" or "vol" means implied volatility. Thus, volatility typically means the standard deviation of continuously compounded returns, expressed in annual terms, obtained by inverting the Black-Scholes formula or binomial model.

Terminology

- So, volatility is the parameter that you use with an option pricing function.
- However, sometimes "volatility" means the volatility parameter in some other (not Black-Scholes or binomial) option pricing model
- Sometimes "volatility" is used loosely as a synonym for dispersion.
- Nonetheless, volatility usually means standard deviation of continuously compounded returns, expressed on an annual basis, and often "volatility" means implied volatility.

Historical σ

- Typically you should use daily closing stock prices over perhaps the last one or two quarters. You will obtain $n+1$ prices which gives n returns.
- Then calculating the volatility is relatively simple. First calculate the daily returns r_t:
$$r_t = \ln(S_t/S_{t-1})$$
- From this we can calculate the mean return μ and we can estimate the daily volatility using

$$s = \frac{1}{n-1}\sum_{t=1}^{t=n}(r_t - \mu)^2$$

Historical σ

- The Black-Scholes equation requires a yearly estimate of volatility, so we need to multiply the variance by the number of trading days or actual days in the year (depending on the model you are using).
- If we have, for example, 252 trading days in the year then to get the yearly standard deviation we multiply our empirical estimate by $\sqrt{252}$.
- If there are dividends payable during this time period, you can drop the ex-dividend dates or, alternatively use

$$r_t = \ln((S_t + D)/S_{t-1})$$

where D is the estimated drop in the share price caused by the dividend payment.

Implied volatility

- Historical estimates of volatility are inherently backward-looking.
- Is it possible to develop a forward-looking volatility estimate?
- Suppose
  - $S = 40$; $K = 40$; $r = 0.04$; $T - t = 0.3$; $C = 4.25$ (C = call price)
  - The next page graphs the point ($S = 40$, $C = 4.25$), along with a function showing the theoretical value of a call option when $\sigma = 0.20$.

Quiz

Why is option volatility forward looking?

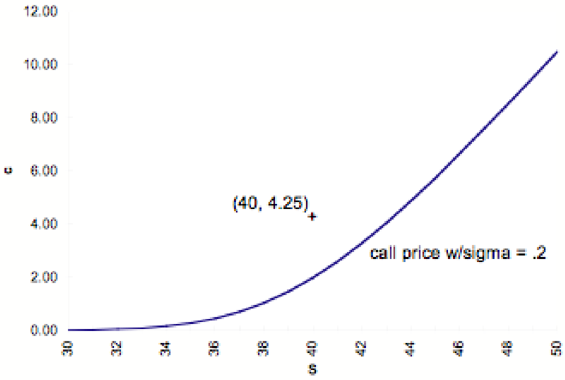
- Because we have to use risk-neutral probabilities to value the option.
- Because an estimate of the volatility between now and maturity is required to value the option.
- Because the option payoff depends upon volatility.
- Because investors are risk averse.

Implied σ

- If the options are market traded then we can use the market price to calculate what the volatility would have to be to arrive at this theoretical value.
- i.e if we think of the option price C as a function of S_0, K, r, d, T and σ, then if we know C then the only unknown is the current predicted volatility σ.

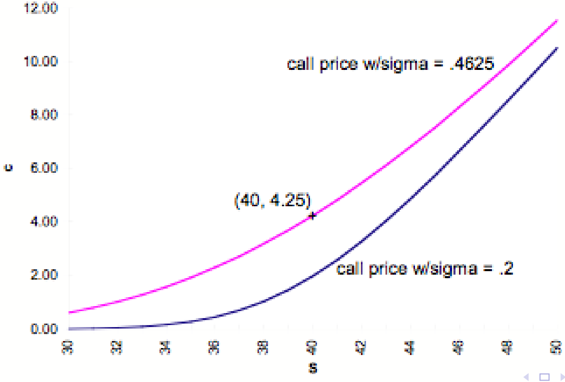
Implied volatility

- Actual option price plus theoretical prices for $\sigma = 0.2$.
- We can see from the graph that 0.20 or 20% is not the market's estimate of volatility over the life of the option.



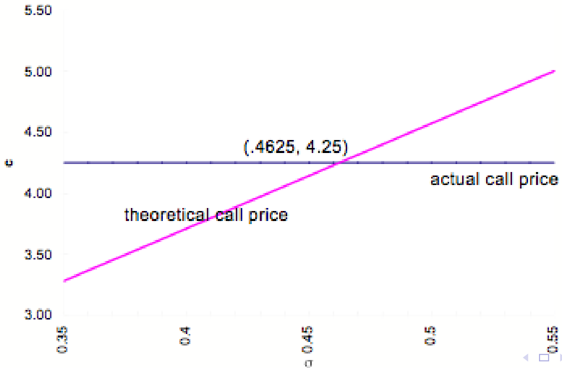
Implied volatility

- Actual option price, along with theoretical prices for $\sigma = 0.2$ and $\sigma = 0.4625$.
- We see from this graph that a volatility of 0.4625 or 46.25% is implicit in the price of the option.



Implied volatility

- Graph shows the actual and theoretical option prices as a function of the volatility input into a pricing model (The σ on the horizontal axis is not the actual volatility, but the input you use with the theoretical pricing model. Thus, the theoretical price depends on σ, but the actual market price does not.)



Implied volatility calculation

- Let σ denote the volatility input.
- $c(\sigma)$ denote the theoretical option price, as a function of the volatility input.
- c_{actual} denote the actual market option price, which does not vary with σ.
- The implied volatility is the σ that satisfies equation:

$$c(\sigma) - c_{actual} = 0$$

Quiz

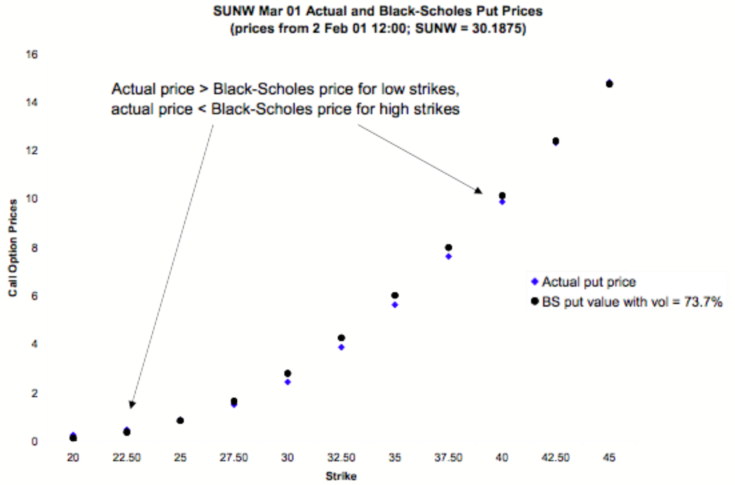
In a BSM world should volatility depend upon the exercise price, K?

- Yes, because out-of-the money options are more risky
- Yes, because the stock price movements depend upon the exercise price.
- No, because it is just a measure of the stock volatility and is independent of other option parameters.

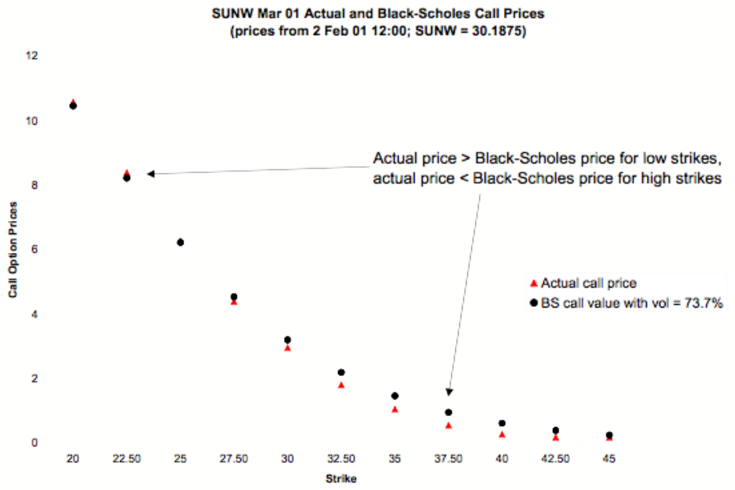
Volatility smile/skew

- As most exchange traded stocks are issued with a range of exercise prices we can use implied volatilities to test the hypothesis that σ is independent of the current strike price.
- Unfortunately, actual option prices are not perfectly consistent with the binomial/Black-Scholes models
- Price of out-of-money puts/in-the-money calls tend to be "too high".
- This means that implied volatilities for these options are "too high".

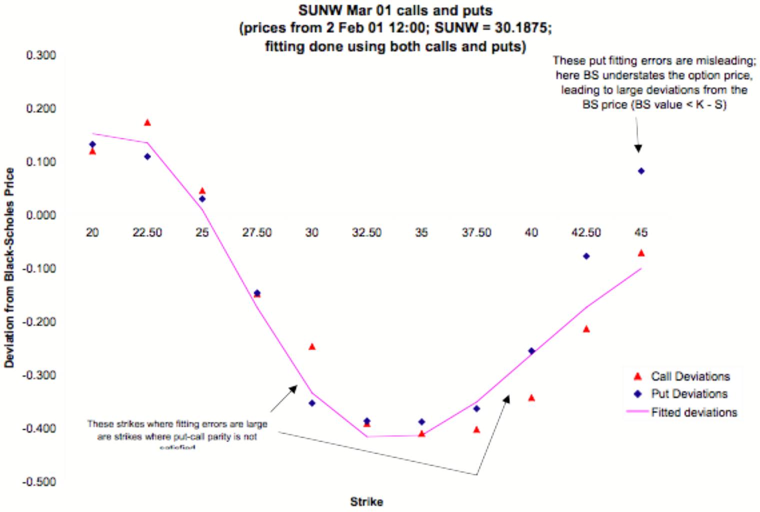
Example: SUNW put option price



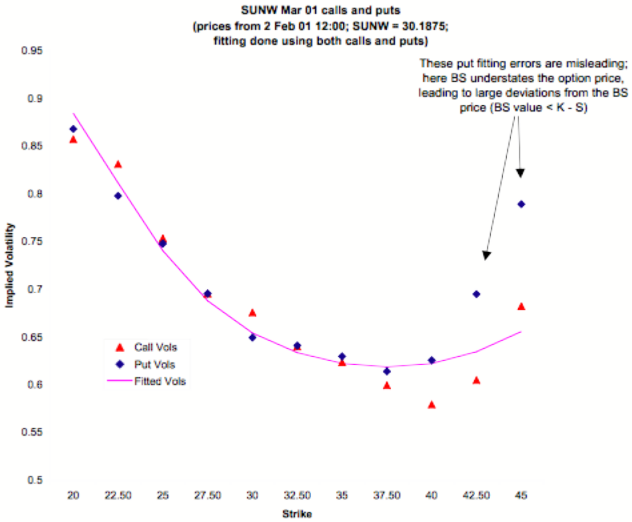
Example: SUNW call option price



Differences between actual and BS prices



Smile/Skew


SUNW Mar 01 calls and puts
(prices from 2 Feb 01 12:00; SUNW = 30.1875;
fitting done using both calls and puts)
These put fitting errors are misleading;
here BS understates the option price,
leading to large deviations from the BS
price (BS value < K - S)
Implied Volatility
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
20
22.50
25
27.50
30
32.50
35
37.50
40
42.50
45
Strike
Call Vols
Put Vols
Fitted Vols

Quiz

If the Black-Scholes model is true and there should be one constant across exercise prices volatility how would you exploit a volatility smile?

- Buy puts with high implied volatility and sell calls with low implied volatility
- Buy calls with high implied volatility and sell puts with low implied volatility?
- Buy options with low implied volatility and sell options with high?
- Buy options with high implied volatility and sell options with low?

Comparison to normal

- Compared with a normal distribution, estimates of the probability density of daily returns have:
  - A higher probability of being within half a standard deviation of the mean
  - A higher probability of being three or more standard deviations away from the mean.
  - Lower probabilities elsewhere.
- In statistical terms this means that daily returns have **excess kurtosis**.
- The next figure illustrates the density for S&P 500 returns from 1950 until October 2010. This is then standardized and compared to a normal distribution with the same mean and standard deviation.
- The table shows how the actual density compares to the normal distribution.

Figure

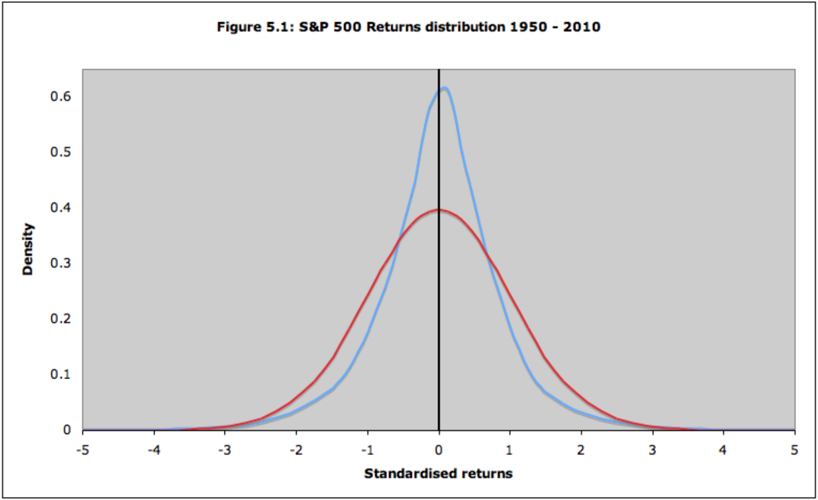
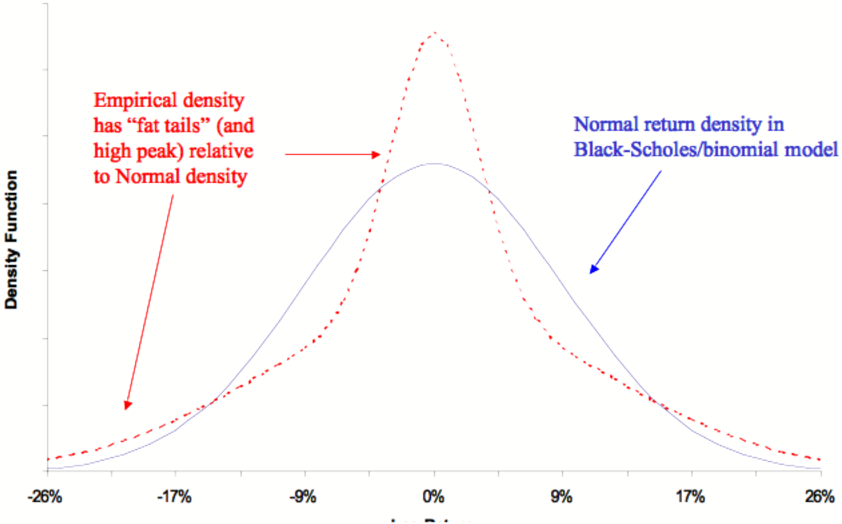


Figure 5.1: S&P 500 Returns distribution 1950 - 2010
Density
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5
Standardised returns

Normal and empirical densities



Table

Range (s.d)	Observed	Expected (Normal)	Observed − Expected
4+	0.26%	0.00%	0.26%
3 to 4	0.38%	0.13%	0.25%
2 to 3	1.55%	2.14%	-0.59%
1.5 to 2	2.39%	4.41%	-2.02%
1 to 1.5	5.58%	9.18%	-3.6%
0.5 to 1	14.09%	14.99%	-0.9%
0.25 to 0.5	11.71%	9.28%	2.43%
0 to 0.25	15.34%	9.87%	5.47%

Here we compared the observed percentages of standardized daily returns

$$\frac{r_t - \overline{r}}{s}$$

and compared to the normal distribution.

Extreme events

Five Largest One Day Percentage Gains S & P 500:

1. 2008-10-13 +11.58%
2. 2008-10-28 +10.79%
3. 1987-10-21 +9.10%
4. 2009-03-23 +7.08%
5. 2008-11-13 +6.92%

Five Largest One Day Percentage Losses S & P 500:

1. 1987-10-19 -20.47% (Black Monday)
2. 2008-10-15 -9.03%
3. 2008-12-01 -8.93%
4. 2008-09-29 -8.81%
5. 1987-10-26 -8.28%

Quiz

Is it plausible to see such extreme movements?

- Yes, these are rare events but they can happen with a normal distribution.
- No, the probability of so many extreme events occurring is too small.

Test for normality

- Observations more than three standard deviations away from the mean are often called outliers or extreme values. For daily returns they occur approximately three times per year.
- The frequency of daily returns more than four standard deviations from the mean is less than one per year; it is one every sixty years for a normal distribution.
- If our null hypothesis is that the distribution is normal, one straightforward z-test is to evaluate

$$z = \frac{k - 3}{\sqrt{24/n}}$$

where $\sqrt{(24/n)}$ is the standard error of a kurtosis estimate for under the null hypothesis of a normal distribution. If we return large values of z (as we nearly always do with daily returns data) then we reject this null hypothesis.

Test for normality

- In general, the returns generating process is not remotely normal and has been shown to hold for almost all series of daily and more frequent returns.

Why are returns not normal?

Volatility clustering explains why the distribution of daily returns is not normal:

- Some returns from a sample come from high volatility periods so the standard deviation of returns is high.
- Other returns come from low volatility periods when the standard deviation of returns is low.
- The full sample is a mixture of these two distributions.
- If the returns do from different normal distributions that have different variances then the mixture of the two will have a larger kurtosis than a normal distribution.

Explanation 1

- If extreme negative returns are more likely than normal/lognormal distribution indicates:
    - Out-of-the-money puts will be more valuable (have higher implied volatilities) then Black-Scholes model suggests
    - Through put-call parity, in-the-money calls will have higher implied volatilities
- If extreme positive returns are more likely than normal/lognormal distribution indicates:
    - Out-of-the-money calls will be more valuable (have higher implied volatilities) then Black-Scholes model suggests.
    - Through put-call parity, in-the-money puts will have higher implied volatilities.

Solution 1

- Use a different model for stock price movements.
- The usual theoretical solution is to make volatility stochastic or to allow the stock price process to have jumps.
- According to trader/academic Rebonato: Stochastic volatility models have two main problems. First, we do not observe volatility directly and so observing its true distribution requires either acts of faith or ‘heroic ’econometric analysis.
- Alternatively we can use simpler adaptations to the Black-Scholes model. The simplest are to use constant elasticity of variance (CEV) and displaced diffusion (DD) models.
- However, the most useful practical model for dealing with volatility is the SABR model, see Hagan et al.(2002)(provided on COMPASS for those who are interested).

Explanation 2: Market participants

- Investors/end-users like certain options, especially out-of-the-money puts (for portfolio insurance).
- As a result, dealer/market-maker community is generally short such options
- In practice, dynamic hedging/replication is difficult and imperfect
- As a result, dealers/market makers are willing to sell these options only if they receive a premium above the theoretical value

Which explanation?

- There is probably some truth to both explanations, But which explanation is most important is unclear:
- Volatility smile/skew could be 90% due to first explanation, and 10% due to second; or Volatility smile/skew could be 10% due to first explanation, and 90% due to second.

Implied volatility: practice

- The usual procedure in using this information is to use a weighted average of the implied sigmas, although you often exclude out-of-the-money options or options with low liquidity. There are also 'model-free'implied volatilities (see VIX).
- Implied volatility gives you a current prediction of volatility over the lifetime of the option and so this should give a better estimate than historic volatility which only factors in past data.
- Often option prices with different exercise prices are quoted relative to their implied volatilities, enabling you to directly compare options with different exercise prices.

Term structure of volatility and volatility surface

- For many options, there is also a 'term structure'of implied volatilities
- If we combine facts:
  - Volatility differs by moneyness S/K
  - Volatility differs by time to expiration $T - t$
- We get a volatility surface, that is

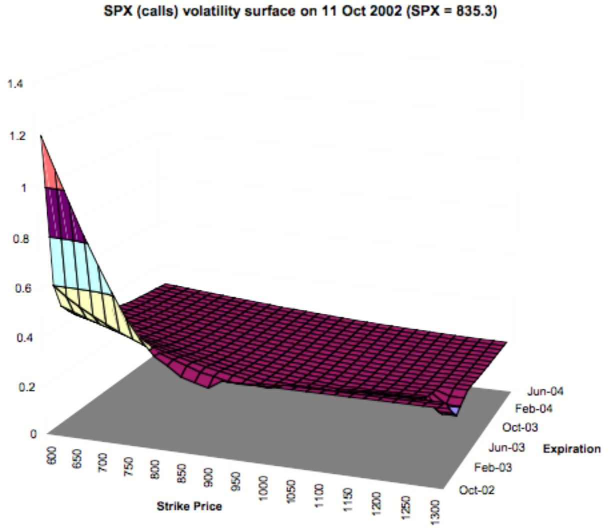
$$\text{implied volatility} = f(S/K, T - t)$$

  or for fixed S and t

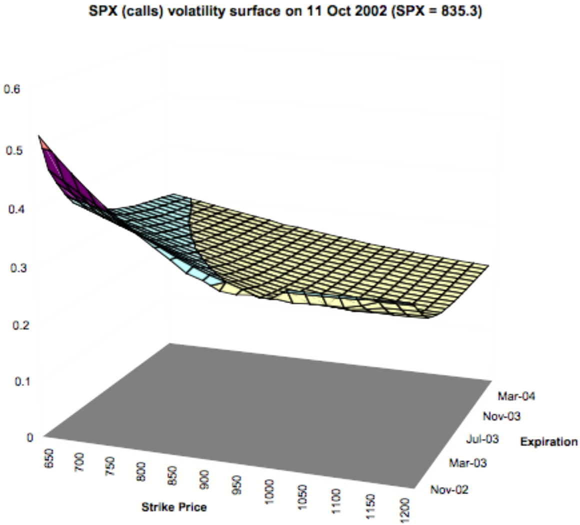
$$\text{implied volatility} = f(K, T)$$

- There is then great skill in deriving a structure for the volatility surface.

Volatility surface in SPX options



A closer look at the center of the graph



Overview

- We have looked at the inputs for the Black-Scholes formula and, in particular, we have looked in detail at volatility.
- We have looked at the definition of implied volatility and how it is calculated.
- Next, we looked at how the implied volatility changes across different strikes and times to maturity.
- Finally,we looked at possible explanations for the volatility smile/skew and ways of possibly modeling it.