Fin 514: Financial Engineering II

Lecture 6: Fundamental Theorem of Finance (discrete)

Dr. Martin Widdicks

UIUC

Spring, 2018

Outline

- Here we return to a simple one period setting but attempt to say something general about the relationship between complete markets, risk-neutral probabilities and no-arbitrage. This relationship is formalized in the two fundamental theorems of asset pricing for multiple discrete states over one period.
- This theorem will be useful for later models and also demonstrates why were are able to use the binomial model for valuation.
- A useful tool for getting to this theorem is to use the Arrow-Debreu securities that we saw in Lecture 2.
- You only need to know the main results, the remainder of the slides are for if you want more information.

States of the world

- Future uncertainty is represented by M possible states of the world, $j = 1, 2, \ldots, M$ as below

		Tomorrow	Probability
		State 1	p_1
	$\nearrow$	State 2	p_2
	$\nearrow$	$\vdots$	$\vdots$
Today	$\rightarrow$	State j	p_j
	$\searrow$	$\vdots$	$\vdots$
		State M	p_M

Assets and payoffs

- There are N financial assets and we assume that they are linearly independent (i.e. you cannot replicate one of the assets by combining the other $N-1$ assets).
- In state of the world j ($j = 1, \ldots, M$), asset i pays $D_{i.j}$ units of whatever the unit of currency (or alternative) is.
- $D_{i,j}$ can take positive and negative values.
- $D_{i,j}$ is called the **payoff of asset i in state of the world j**.
- These payoffs are typically collected together in an $N \times M$ matrix, D, called the **payoff matrix**:

$$D = \begin{bmatrix} D_{1,1} & D_{1,2} & \ldots & D_{1,M} \\ \vdots & & & \vdots \\ D_{N,1} & D_{N,2} & \ldots & D_{N,M} \end{bmatrix}$$

Payoff matrix

- Each row of the payoff matrix

$$D_{i,\bullet} = [D_{i,1}, D_{i,2}, \ldots, D_{i,M}]$$

describes all possible payoffs of the ith asset (in states of the world $j = 1, 2, \ldots, M$.

- Each column of the payoff matrix

$$D_{\bullet,j} = \begin{bmatrix} D_{1,j} \\ D_{2,j} \\ \vdots \\ D_{N,j} \end{bmatrix}$$

describes payoffs of the N assets in the state of nature j

- Note that X^T denoted the *transpose* of the matrix X. The transpose is formed by turning the rows into columns (and columns into rows).

Prices

- Today's prices of the N assets are given by an $N \times 1$ vector, P, given by

$$P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix}$$

where $P_i (i = 1, 2, \ldots, N)$ is today's price of asset i.

First fundamental theorem

- There exists a vector of strictly positive numbers

$$\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_M \end{bmatrix}$$

such that

$$P = D\pi \tag{1}$$

i.e.

$$P_i = \sum_{j=1}^{M} D_{i,j}\pi_j \text{ for all } 1 \leq i \leq N \tag{2}$$

if and only if there are no arbitrage opportunities.

- **In other words: There exists a pricing measure (or state price vector) if and only if there are no arbitrage opportunities.**

QUIZ

How can we interpret π_j ($j = 1, \ldots, M$)?

- Risk-neutral probabilities
- Real world probabilities
- Arrow-Debreu state prices?

QUIZ

What is the price of a bond with a payoff of 1 in this world?

- Unknown
- $\sum_{i=1}^{N} \pi_i$
- $\sum_{i=1}^{N} \pi_i \theta_i$
- $\sum_{j=1}^{M} \pi_j$
- $\sum_{j=1}^{M} \pi_j (1+R)$

Pricing measures and risk neutral probabilities

- It is easy to extend the first fundamental theorem of finance to deal with the existence of risk-neutral probabilities rather than a pricing measure.
- If one of the securities in the market is a riskless bond with payoff 1, then by definition we must have that:

$$P_{\mathrm{bond}} = \sum_{j=1}^{M} \pi_j = \frac{1}{1+R}$$

so by redefining our measure as follows:

$$q_j = (1+R)\pi_j$$

then

$$\sum_{j=1}^{M} q_j = (1+R)\sum_{j=1}^{M} \pi_j = 1$$

Pricing measures and risk neutral probabilities

- Thus we can interpret these q values as risk-neutral probabilities. In particular the price of any particular security, P_i can be written as

$$P_i = \sum_{j=1}^{M} D_{i,j}\pi_j = \frac{1}{1+R}\sum_{j=1}^{M} D_{i,j}q_j = \frac{1}{1+R}E^Q[D_{i,\bullet}]$$

  where this expectation has been taken with respect to these risk-neutral probabilities.

- So, if you prefer then we can write the first fundamental theorem of finance as follows:
  If there is a risk free asset, then there exists a risk-neutral probability measure if and only if there are no arbitrage opportunities.

Portfolio of assets

- A portfolio of assets θ is an $1 \times N$ vector

$$\theta = [\theta_1, \theta_2, \ldots, \theta_N]$$

where each θ represents the amount of asset i held in portfolio.

- The price of portfolio θ is given by

$$\theta P = \sum_{i=1}^{N} \theta_i P_i = \theta_1 P_1 + \cdots + \theta_N P_N.$$

The payoff of portfolio θ in state j is equal to

$$\theta D_{\bullet,j} = \sum_{i=1}^{N} \theta_i P_i = \theta_1 D_{1,j} + \cdots + \theta_N D_{N,j}.$$

The $1 \times M$ matrix θD describes the payoff of portfolio θ in the M states of nature:

$$\theta D = [\theta_1 D_{1,1} + \cdots + \theta_N D_{N,1}, \ldots, \theta_1 D_{1,M} + \cdots + \theta_N D_{N,M}]$$

The second fundamental theorem

- If the no arbitrage market is also complete then there is a unique pricing measure (or state price vector)

$$\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_M \end{bmatrix}$$

(which satisfies equations 1 and 2) where $\pi = D^{-1}P$

- Conversely, if this state pricing vector is unique then markets are complete. Thus,
The pricing measure is unique if and only if the market is complete.
- If we include the risk-free asset among our securities then we have that the risk-neutral measure is unique if and only if markets are complete.

Comments on the second theorem

- Thus using the prices P of traded assets we can unambiguously calculate the pricing measure (state price vector) π. Then using this vector we can compute the fair, unique price of any additional asset introduced into the market, with payoff E as $P_E = E\pi$.
- Additionally we know that using the existing N assets we could build a portfolio θ such that the portfolio payoff is the same as the payoff of this new asset, $E = \theta D$
- Strangely, this suggests that the payoff of this new asset is already offered by the market so why do we need to have this new asset, E?
- Regardless of this, virtually all valuation techniques in finance assume complete markets and no arbitrage opportunities.

Outline

- We have introduced definitions of Arrow-Debreu securities (as connected with a pricing measure) and the full definition of a complete market.
- We have used these within a discrete state, one-period model to show that you can prove the following results:
- A pricing measure exists if and only if there are no arbitrage opportunities.
- The pricing measure is unique if and only if markets are complete.
- If one of the securities is the risk-free asset then we can make the same statements, replacing the pricing measure for a risk-neutral pricing measure.

Arbitrage

- Recall that an arbitrage opportunity is one where it is possible to make an instantaneous risk-free profit by entering into two or more simultaneous trades. It is also referred to as a free lunch.
- We typically believe that in competitive markets that arbitrage opportunities appear only fleetingly and that prices rectify to remove the opportunity.
- Thus, we conclude in equilibrium that there is an absence or arbitrage opportunities and so if different assets produce exactly the same payoffs in the future then they must have the same prices today.

Definition of arbitrage

An arbitrage opportunity is an investment strategy yielding either

1. Strictly positive net gain (or profits) today and no loss in any state of the world in the future.
2. Positive or zero net gain today, no loss in any state of nature in the future and strictly positive gain (profit) in at least one state of nature in the future.

- Both of these definitions are equivalent.

Formal definition

- The value of portfolio A at time t is denoted by V_t^A An arbitrage portfolio is one for which either of the following two conditions hold:

1.

$$\begin{aligned} V_0^A &< 0 \\ V_1^A &\geq 0 \text{ with probability } 1 \end{aligned}$$

2.

$$\begin{aligned} V_0^A &= 0 \\ V_1^A &\geq 0 \text{ with probability } 1 \\ V_1^A &> 0 \text{ for some state of the world} \end{aligned}$$

Arbitrage examples

- Definition 1: No gain or loss in the future, but strictly positive gain today.
- Definition 2: Guaranteed gain in the future, but no cash flows today.
- Definition 1 or 2: Sometimes there will be gains in the future but no matter what happens you will never lose. In addition, your net gain today is positive.

QUIZ

Which of the following four investments are arbitrage opportunities

- Borrowing \$10 today at an interest rate of 10%, then investing it and receiving a guaranteed \$20 in one year.
- Borrowing \$10 today at an interest rate of 10% and investing it in a fund that last year had returns of 100%.
- Borrowing \$10 today at an interest rate of 10% and investing it in a fund that has expected returns of 100%
- Borrowing \$10 today at an interest rate of 10% and investing it in a fund that achieves returns of 100% over the next year.

States of the world

- Future uncertainty is represented by M possible states of the world, $j = 1, 2, \ldots, M$ as below

$$
\begin{array}{lcll}
 & & \textbf{Tomorrow} & \text{Probability} \\
 & & \text{State } 1 & p_1 \\
 & \nearrow & \text{State } 2 & p_2 \\
 & \nearrow & \vdots & \vdots \\
\textbf{Today} & \rightarrow & \text{State } j & p_j \\
 & \searrow & \vdots & \vdots \\
 & & \text{State M} & p_M
\end{array}
$$

Assets and payoffs

- There are N financial assets and we assume that they are linearly independent (i.e. you cannot replicate one of the assets by combining the other $N-1$ assets).
- In state of the world j ($j = 1, \ldots, M$), asset i pays $D_{i,j}$ units of whatever the unit of currency (or alternative) is.
- $D_{i,j}$ can take positive and negative values.
- $D_{i,j}$ is called the **payoff of asset i in state of the world j**.
- These payoffs are typically collected together in an $N \times M$ matrix, D, called the **payoff matrix**:

$$D = \begin{bmatrix} D_{1,1} & D_{1,2} & \ldots & D_{1,M} \\ \vdots & & & \vdots \\ D_{N,1} & D_{N,2} & \ldots & D_{N,M} \end{bmatrix}$$

Payoff matrix

- Each row of the payoff matrix

$$D_{i,\bullet} = [D_{i,1}, D_{i,2}, \dots, D_{i,M}]$$

describes all possible payoffs of the ith asset (in states of the world $j = 1, 2, \dots, M$.

- Each column of the payoff matrix

$$D_{\bullet,j} = \begin{bmatrix} D_{1,j} \\ D_{2,j} \\ \vdots \\ D_{N,j} \end{bmatrix}$$

describes payoffs of the N assets in the state of nature j

- Note that X^T denoted the *transpose* of the matrix X. The transpose is formed by turning the rows into columns (and columns into rows).

Prices

- Today's prices of the N assets are given by an $N \times 1$ vector, P, given by

$$P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix}$$

where $P_i (i = 1, 2, \ldots, N)$ is today's price of asset i.

Example

- If we have an example of a stock of current value 1, a bond also with value 1 and another asset (A-D security) with value 4/11, we could write this *market* as follows.

$$P = \begin{bmatrix} 1 \\ 1 \\ 4/11 \end{bmatrix}$$

$$D = \begin{bmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \\ D_{3,1} & D_{3,2} \end{bmatrix} = \begin{bmatrix} 1.1 & 1.1 \\ 2 & 0.5 \\ 1 & 0 \end{bmatrix}$$

QUIZ

- Construct P and D for the case where we have a Bond paying out 1.10 and a Stock where $R = 0.1, S_0 = 100, u = 1.2, d = 0.9$ and another asset, A, with current price 6.06 and payoffs of 10 in the upstate and 0 in the down state.

QUIZ

Is it necessary to also include asset A in our P and D matrix?

- Yes, as it brings new information
- No, we only have two states we only need two assets

Portfolio of assets

- A portfolio of assets θ is an $1 \times N$ vector

$$\theta = [\theta_1, \theta_2, \ldots, \theta_N]$$

where each θ represents the amount of asset i held in portfolio.

- The price of portfolio θ is given by

$$\theta P = \sum_{i=1}^{N} \theta_i P_i = \theta_1 P_1 + \cdots + \theta_N P_N.$$

The payoff of portfolio θ in state j is equal to

$$\theta D_{\bullet,j} = \sum_{i=1}^{N} \theta_i P_i = \theta_1 D_{1,j} + \cdots + \theta_N D_{N,j}.$$

The $1 \times M$ matrix θD describes the payoff of portfolio θ in the M states of nature:

$$\theta D = [\theta_1 D_{1,1} + \cdots + \theta_N D_{N,1}, \ldots, \theta_1 D_{1,M} + \cdots + \theta_N D_{N,M}]$$

QUIZ

- Depict θP and $\theta D_{\bullet,j}$ $(j = 1, 2)$ from an example of holding 20 of the bond and 0.25 of the share when $B_0 = 1$ and $S_0 = 100$, $R = 0$, $u = 1.2$ and $d = 0.8$.

Arbitrage portfolios

- Thus by using our slightly different notation, we can again describes what it means to have an arbitrage portfolio. We have two definitions again
- Arbitrage portfolio I:

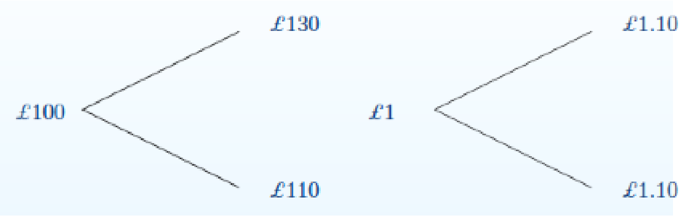
$$
\begin{array}{rcl}
\theta P & < & 0 \\
\theta D_{\bullet,j} \geq 0 & \text{for all} & 1 \leq j \leq M
\end{array}
$$

Arbitrage portfolio II

$$
\begin{array}{rcl}
\theta P & = & 0 \\
\theta D_{\bullet,j} \geq 0 & \text{for all} & 1 \leq j \leq M \\
\theta D_{\bullet,j} > 0 & \text{for some} & 1 \leq j \leq M
\end{array}
$$

Arbitrage example

- Define a stock and a bond so that



- The arbitrage portfolio was to buy the stock and to go short 100 bonds and so

$$\theta = [1, -100]$$

and

$$\theta P = 100 - 100 = 0, \quad \theta D = [20, 0]$$

so we have an arbitrage by definition 2.

QUIZ

- Define the arbitrage portfolio θ for the case above but when $u = 0.9$ and $d = 0.5$, $1 + R = 1.05$ and show that it is an arbitrage

The market

- Our market is well defined by the two matrices P and D. We do not actually need the real world probabilities $p_1, p_2, \ldots, p_M$ in order to define a market.
- This is because finance uses different but equivalent sets of probabilities, depending upon context
  - **Objective** probabilities: historic, empirical averages.
  - **Subjective** probabilities: individual's beliefs.
  - **Risk-neutral** probabilities: pricing financial assets in a no-arbitrage context, etc.
- Prices, P, and payoffs, D, remain the same independent of the choice of probability.

First fundamental theorem

- There exists a vector of strictly positive numbers

$$\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_M \end{bmatrix}$$

  such that

$$P = D\pi \tag{3}$$

  i.e.

$$P_i = \sum_{j=1}^{M} D_{i,j}\pi_j \text{ for all } 1 \leq i \leq N \tag{4}$$

  if and only if there are no arbitrage opportunities.

- **In other words: There exists a pricing measure (or state price vector) if and only if there are no arbitrage opportunities.**

QUIZ

How can we interpret π_j ($j = 1, \ldots, M$)?

- Risk-neutral probabilities
- Real world probabilities
- Arrow-Debreu state prices?

QUIZ

What is the price of a bond with a payoff of 1 in this world?

- Unknown
- $\sum_{i=1}^{N} \pi_i$
- $\sum_{i=1}^{N} \pi_i \theta_i$
- $\sum_{j=1}^{M} \pi_j$
- $\sum_{j=1}^{M} \pi_j (1+R)$

Pricing measures and risk neutral probabilities

- The proof is at the end of these notes.
- It is easy to extend the first fundamental theorem of finance to deal with the existence of risk-neutral probabilities rather than a pricing measure.
- If one of the securities in the market is a riskless bond with payoff 1, then by definition we must have that:

$$P_{\mathrm{bond}} = \sum_{j=1}^{M} \pi_j = \frac{1}{1+R}$$

so by redefining our measure as follows:

$$q_j = (1+R)\pi_j$$

then

$$\sum_{j=1}^{M} q_j = (1+R)\sum_{j=1}^{M} \pi_j = 1$$

Pricing measures and risk neutral probabilities

- Thus we can interpret these q values as risk-neutral probabilities. In particular the price of any particular security, P_i can be written as

$$P_i = \sum_{j=1}^{M} D_{i,j}\pi_j = \frac{1}{1+R}\sum_{j=1}^{M} D_{i,j}q_j = \frac{1}{1+R}E^Q[D_{i,\bullet}]$$

  where this expectation has been taken with respect to these risk-neutral probabilities.

- So, if you prefer then we can write the first fundamental theorem of finance as follows:
  If there is a risk free asset, then there exists a risk-neutral probability measure if and only if there are no arbitrage opportunities.

QUIZ

Does this formula

$$P_i = \frac{1}{1+R} E^Q[D_{i,\bullet}]$$

imply that the fundamental theorem of finance assumes that all investors are risk neutral?

- Yes
- No
- It depends

Proof of theorem 1

- This first theorem is equivalent to saying that **only** one of the following statements is true:
  - There exists a pricing measure (or state price vector).
  - There exists an arbitrage portfolio
- There are two parts to the proof:
  1. No arbitrage $\Rightarrow$ existence of a pricing measure (state price vector).
  2. Existence of a pricing measure (state price vector) $\Rightarrow$ no arbitrage.
- The second proof is straightforward:

Pricing measure $\Rightarrow$ no arbitrage

- If π exists then $P = D\pi$ and the price of each individual asset is $P_i = D_{i,\bullet}\pi$ and so if we have a portfolio of assets θ then the price of the portfolio is

$$\theta P = \sum_{i=1}^{N} P_i \theta_i = \sum_{i=1}^{N} \theta_i \left(\sum_{j=1}^{M} D_{i,j} \pi_j \right)$$

- By changing the order of summation we get

$$\theta P = \sum_{j=1}^{M} \pi_j \sum_{i=1}^{N} D_{i,j} \theta_i$$

or

$$\theta P = \sum_{j=1}^{M} (\theta D_{\bullet,j}) \pi_j$$

Pricing measure $\Rightarrow$ no arbitrage.

- Thus the current price of a portfolio is equal to the sum of its payoffs in the M possible states of the world weighted by the state price vector.
- Thus
  1. If there is no loss in the future in any state of the world:
     - $\Leftrightarrow$ Payoffs of portfolio θ are strictly positive or zero (can be all of them).
     - $\Rightarrow$ Price of portfolio θ is strictly positive or zero.
     - $\Rightarrow$ No arbitrage!
  2. No loss in any state of the world in the future and strictly positive gain in one state of the world:
     - $\Leftrightarrow$ Payoffs of portfolio θ are strictly positive or zero (but not all)
     - $\Rightarrow$ Price of portfolio θ is strictly positive.
     - $\Rightarrow$ No arbitrage!

No arbitrage $\Rightarrow$ Pricing measure

- This side of the proof requires a geometric argument involving separating hyperplanes. Instead of the full proof we will look at a graphical analogy when there are three assets in the market.
- Consider a three dimensional world, $\mathbb{R}^3$, where the position is denoted by the payoff' or D value from each of the securities in the market.
- If we again use our stock, bond and call option form earlier, then the payoff in the upstate can be written as the following co-ordinates $(1.1, 2, 1)$ where the x, y, z co-ordinates denote the payoffs from the bond, stock and call respectively.

No arbitrage $\Rightarrow$ Pricing measure

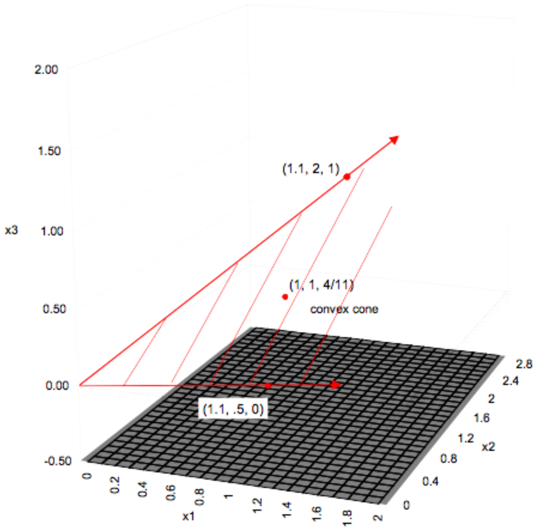
- Note that on the previous slide:
  - $(1.1, 2, 1)$ was payoff in 'up-state'.
  - $(1.1, 0.5, 0)$ was payoff in 'down-state'.
  - $(1, 1, 4/11)$ is current price.
- Our theorem says that:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \pi_1 \begin{pmatrix} 1.1 \\ 2 \\ 1 \end{pmatrix} + \pi_2 \begin{pmatrix} 1.1 \\ 0.5 \\ 0 \end{pmatrix}$$

  which means that the prices of the securities must lie in the interior of the convex cone between the two rays from the origin through (1.1, 2, 1) and (1.1, 0.5, 0).
- Note that the following does not admit arbitrage opportunities

$$\begin{pmatrix} 1 \\ 1 \\ 4/11 \end{pmatrix} = 4/11 \begin{pmatrix} 1.1 \\ 2 \\ 1 \end{pmatrix} + 6/11 \begin{pmatrix} 1.1 \\ 0.5 \\ 0 \end{pmatrix}$$

No arbitrage ⇒ Pricing measure



No arbitrage ⇒ Pricing measure

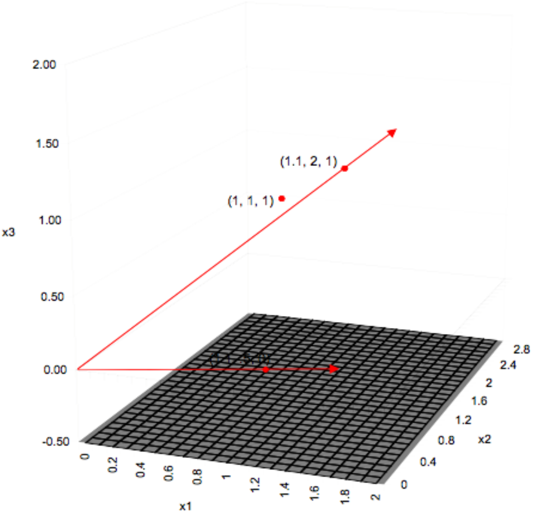
- Our theorem says that if the prices can't be written as:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \pi_1 \begin{pmatrix} 1.1 \\ 2 \\ 1 \end{pmatrix} + \pi_2 \begin{pmatrix} 1.1 \\ 0.5 \\ 0 \end{pmatrix}$$

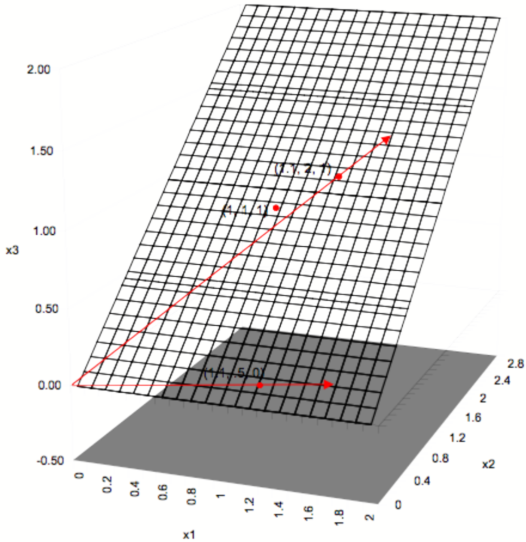
  then there is an arbitrage opportunity.
- Graphically, this means that if the price vector doesn't lie upon the convex cone then there is an arbitrage opportunity.
- To see how this works, consider a point not on the cone, for example

$$p = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

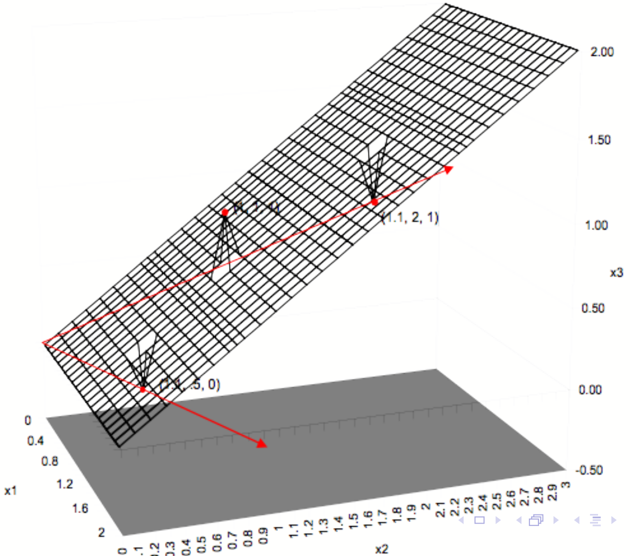
No arbitrage ⇒ Pricing measure



Construct a plane between the cone and (1,1,1)



Here we see that the plane is below (1,1,1) but above the complex cone



No arbitrage $\Rightarrow$ Pricing measure

- How does this geometric argument relate to an arbitrage opportunity (or lack of one)?
- The general equation for a plane is of the form $\theta x = 0$.
- In particular in the previous example, the plane is denoted by $\theta = (0, 2/3, -1)$, thus $(2/3)x_2 - x_3 = 0$.
- The other feature of the plane is that above the plane $\theta x < 0$ and below the plane $\theta x > 0$
- For example if $x = (1, 1, 1), \theta x = -1/3$ and if $x = (1.1, 2, 1), \theta x = 1/3$.
- Note that we can reverse these inequalities by choosing $\theta' = -\theta$ instead.

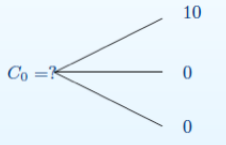
No arbitrage $\Rightarrow$ Pricing measure

- Clearly, the vector θ still relates to a particular portfolio. For the above plane this would be long 2/3 of the second security (a stock in our example) and short the third security (an option in our example).
- For our price vector $p = (1, 1, 1)^T$ we have $(0, 2/3, -1)(1, 1, 1)^T < 0$ as before, which says that the initial value of this portfolio is negative.
- Below the plane we have two possible outcomes:
  - 'up-state': $(0, 2/3, -1)(1.1, 2, 1)^T > 0$.
  - 'down-state': $(0, 2/3, -1)(1.1, 0.5, 0)^T > 0$.

  which both provide a positive cash flow.
- Thus from arbitrage definition I, we have an arbitrage opportunity. Thus we have a contradiction and we conclude the proof.

Three state world

- Consider our usual binomial market with a stock and a bond but now we have three states states. As before $S_0 = 100$ only now it can move to three states uS_0, dS_0 and mS_0 where $u = 1.2, d = 0.8$ as before and, for convenience, $m = 1$. Additionally assume that $R = 0$.
- A third asset is described below:



- Thus our replication relies on us solving:

$$
\begin{aligned}
x + 120y &= 10 \quad (5)\\
x + 100y &= 0 \quad (6)\\
x + 80y &= 0 \quad (7)
\end{aligned}
$$

Pricing the other asset

- There is a problem, here as we have three equations and two unknowns and so the problem may not yield a unique asset price.
- In particular, if we solve equations (4) and (5) we get

$$y = 0, x = 0$$

and thus the portfolio value in the upstate is also 0, which is not the payoff of the asset, and thus the asset price at $t = 0$ is also 0.

- If we use equations (3) and (5) we get

$$y = 0.25, x = -20$$

and thus the portfolio in the middle state at $t = 1$ becomes 5 which is different from the payoff of zero.

- Notice that when we use these x, y values to price the asset we get three possible asset prices: $0, 5$ and 0 (from eqs (3) and (4)).

QUIZ

Which of these do you think is the correct price for the asset

- 5
- 0
- Both
- Neither

Pricing the other asset

- We could try pricing the asset using Arrow-Debreu securities (or a state price vector) which requires that

$$\begin{aligned} 120\pi_1 + 100\pi_2 + 80\pi_3 &= 100 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}$$

  thus $\pi_1 = \pi_3 = \pi$ and $\pi_2 = 1 - 2\pi$ and so $\pi < 0.5$ and so the asset price is 10π and so its price cannot be greater than 5, but we can not be more precise. Thus the no-arbitrage price for this asset is 5 or under.

- This problem occurs because our market is incomplete and we do not have enough basic assets (stocks and bonds) to define the three possible states, thus the new asset price is a range of prices which would be determined by the risk-preference of the investors.

Complete markets

- A market with M states is complete if, for *any* cash flow vector $[D_1, D_2, \ldots, D_M]$, there exists a portfolio of traded securities $[\theta_1, \theta_2, \ldots, \theta_N]$ that has payoff D_j in state j, for all $1 \le j \le M$.
- Market completeness is therefore equivalent to having an $N \times M$ cash payoff matrix D such that the system of linear equations

$$\theta D = E$$

  has a solution θ in $\mathbb{R}^N$ for any E in $\mathbb{R}^M$ e.g. $E = [D_1, D_2, \ldots, D_M]$ for any values of D_j.
- An **incomplete** market is one in which there are assets with payoffs that cannot be replicated by a portfolio of tradeable assets (as in the previous example).

Complete markets: linear algebra

- Thus the market is complete if and only if the rank of the cash flow matrix, $rank(D) = M$, or if the column vectors of D span $\mathbb{R}^M$.
- If we have this property then any new asset can be replicated by trading in existing assets, which is what we were hoping for.
- If we consider this new security with payoffs given by E in $\mathbb{R}^M$. If D is $M \times M$ (if $N > M$ then some rows will be redundant so we can remove them), then to replicate E is straightforward as

$$\theta D = E \text{ and so } \theta = ED^{-1}$$

as D is now invertible (subject to technical constraints).

The second fundamental theorem

- If the no arbitrage market is also complete then there is a unique pricing measure (or state price vector)

$$\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_M \end{bmatrix}$$

(which satisfies equations 1 and 2) where $\pi = D^{-1}P$

- Conversely, if this state pricing vector is unique then markets are complete. Thus,
The pricing measure is unique if and only if the market is complete.
- If we include the risk-free asset among our securities then we have that the risk-neutral measure is unique if and only if markets are complete.

Proof of Theorem 2: Complete $\Rightarrow$ Unique

As the markets are complete and there is no arbitrage then there exists a pricing measure π such that

$$p = D\pi$$

and so if the market is complete, there is a unique solution for p. In particular, if D is M $\times$ M,

$$\pi = D^{-1}p$$

Unique ⇒ Complete

- Suppose there exists a unique state price vector such that equations 1 and 2 are satisfied. We then proceed to argue that the market must be complete by contradiction.
- If the market is not complete, then $rank(D) < M$, which you can consider as there being less than M securities. Now in this case, there is essentially N equations but M unknowns and so there is an infinite number of solutions (see example with three states ($M = 3$) and two assets ($N = 2$), $rank(D) = 2 < M$). In particular,

  $$\text{There exists } \lambda \in R^M \text{ such that } D\lambda = 0, \lambda \neq 0$$

  and so combining these two facts we have

  $$D(\pi + \rho\lambda) = p, \text{ for all } \rho$$

  since π is strictly positive we can choose ρ sufficiently small such that all elements of $\pi + \rho\lambda$ are also strictly positive.

Unique ⇒ Complete

- Thus, there is a new, different pricing measure and so π is no longer unique and so we have a contradiction.
- Thus by contradiction we have shown that a unique pricing measure (or risk-neutral probability measure) implies that markets are complete.

Comments on the second theorem

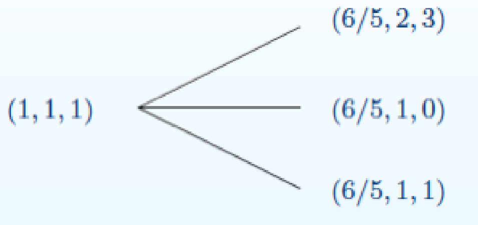
- Thus using the prices P of traded assets we can unambiguously calculate the pricing measure (state price vector) π. Then using this vector we can compute the fair, unique price of any additional asset introduced into the market, with payoff E as $P_E = E\pi$.
- Additionally we know that using the existing N assets we could build a portfolio θ such that the portfolio payoff is the same as the payoff of this new asset, $E = \theta D$
- Strangely, this suggests that the payoff of this new asset is already offered by the market so why do we need to have this new asset, E?
- Regardless of this, virtually all valuation techniques in finance assume complete markets and no arbitrage opportunities.

Incomplete markets

- If markets are incomplete then we will see the introduction of new assets into the market.
- In this case, however, there are N assets but $N < M$ and so at least one of the M Arrow-Debreu securities will not be available in the market and will not be replicable by forming a portfolio of the N assets. This means it will be impossible to determine one of the π_j values.
- Thus, the system of equations: $D\pi = P$ will have infinitely many solutions for π.
- If at least one of these solutions, π, is strictly positive then the incomplete market is arbitrage free. If none are strictly positive then the incomplete market is not arbitrage free.
- If there are assets that can be replicated, their prices will be the same regardless of which of the infinite choices for π we use. Non-replicable assets however will not have unique prices and we will need some more theories...

Example

- Suppose that there are three possible outcomes next period ($M = 3$), and three securities ($N = 3$) with prices as shown on the tree below.

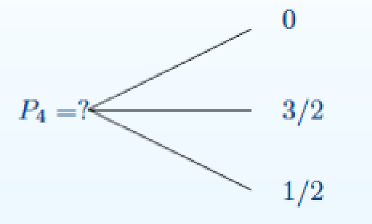


Here $(1, 1, 1)$ gives the current prices of the three securities, and $(6/5, 2, 3)$ gives the prices of the three securities in the "up" state.

a) What is the pricing measure (Arrow-Debreu state prices) consistent with the absence of arbitrage opportunities? What are the corresponding risk-neutral probabilities?

Example

b) Now introduce a fourth security, with current price P_4 and final values as shown on the tree below. What current price P_4 is consistent with the absence of arbitrage opportunities?



c) Suppose that the price of the fourth security is $P_4 = 1$. What is a trading strategy to exploit this arbitrage opportunity? (There is more than one trading strategy.)

Outline

- We have introduced definitions of Arrow-Debreu securities (as connected with a pricing measure) and the full definition of a complete market.
- We have used these within a discrete state, one-period model to show that you can prove the following results:
- A pricing measure exists if and only if there are no arbitrage opportunities.
- The pricing measure is unique if and only if markets are complete.
- If one of the securities is the risk-free asset then we can make the same statements, replacing the pricing measure for a risk-neutral pricing measure.