Fin 514: Financial Engineering II

Lecture 12: Finite Difference Methods I

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Outline

- We now introduce the numerical scheme which is related to the PDE solution.
- Finite difference methods are numerical solutions to parabolic PDEs. They work by approximating the derivatives at each point in time and then rearranging the equations to solve backward in time.
- There are three types of methods - the explicit method, which is analogous to the trinomial tree, the implicit method, and the Crank-Nicholson method, which has the best convergence characteristics.

Finite-difference approximations

- Consider a function of two variables $V(S,t)$, if we consider small changes in S and t we can use a Taylors series to express $V(S+\Delta S,t), V(S-\Delta S,t), V(S,t+\Delta t)$ as follows (all the derivatives are evaluated at (S,t))

$$
\begin{aligned}
V(S+\Delta S,t) &= V(S,t)+\Delta S\frac{\partial V}{\partial S}+\frac{1}{2}(\Delta S)^2\frac{\partial^2 V}{\partial S^2}+O(\Delta S^3) \quad (1)\\
V(S-\Delta S,t) &= V(S,t)-\Delta S\frac{\partial V}{\partial S}+\frac{1}{2}(\Delta S)^2\frac{\partial^2 V}{\partial S^2}+O(\Delta S^3) \quad (2)\\
V(S,t+\Delta t) &= V(S,t)+\Delta t\frac{\partial V}{\partial t}+\frac{1}{2}(\Delta t)^2\frac{\partial^2 V}{\partial t^2}+O(\Delta t^3) \quad (3)
\end{aligned}
$$

- In order to use a finite difference scheme we need to use these expansions to approximate the first and second derivatives with respect to S and t.

Finite difference approximations

- For S, we have two options for the first derivative:
- From the first (or third) equation:

$$\begin{aligned}\frac{\partial V}{\partial S} &= \frac{V(S+\Delta S,t)-V(S,t)}{\Delta S}+\frac{1}{2}\Delta S\frac{\partial^2 V}{\partial S^2}+O((\Delta S)^2)\\ &= \frac{V(S+\Delta S,t)-V(S,t)}{\Delta S}+O(\Delta S)\end{aligned}$$

- From equations 1 and 2:

$$\frac{\partial V}{\partial S} = \frac{V(S+\Delta S,t)-V(S-\Delta S,t)}{2\Delta S}+O((\Delta S)^2)$$

- For the second derivative we use equations 1 and 2 to get

$$\frac{\partial^2 V}{\partial S^2} = \frac{V(S+\Delta S,t)-2V(S,t)+V(S+\Delta S,t)}{(\Delta S)^2}+O((\Delta S)^2)$$

Quiz

Which is the better approximation for the first derivative?

- First one
- Second one
- Both as good as each other?

Finite-difference approximations

- For t we have

$$\begin{aligned}\frac{\partial V}{\partial t} &= \frac{V(S,t+\Delta t)-V(S,t)}{\Delta t}+\frac{1}{2}\Delta t\frac{\partial^2 V}{\partial t^2}+O((\Delta t)^2)\\ &= \frac{V(S,t+\Delta t)-V(S,t)}{\Delta t}+O(\Delta t)\end{aligned}$$

- This particular approximation is called forward differencing whilst the preferred method for S is called central differencing. In general central differencing is the most accurate.

How does this help us?

- Reconsider the Black-Scholes equation and in particular the Black-Scholes equation for a European options where there are continuous dividends:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0$$

- The boundary conditions are, for a call:

$$\begin{aligned} V(S,T) &= \max(S-K,0) \\ V(0,t) &= 0 \\ V(S,t) &= Se^{-\delta(T-t)} - Ke^{-r(T-t)} \text{ as } S \to \infty \end{aligned}$$

For a put:

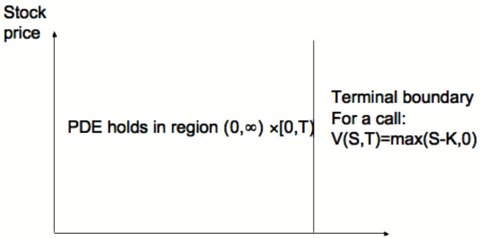
$$\begin{aligned} V(S,T) &= max(K-S,0) \\ V(0,t) &= Ke^{-r(T-t)} \\ V(S,t) &= 0 \text{ as } S \to \infty \end{aligned}$$

Quiz)

- What are the boundary conditions for an American Put option?

How does this help us?

- We will now form a finite difference grid that describes the S-t space in which we need to solve the Black-Scholes equation.



- For a numerical method we need to truncate the range of S to $[S_L, S_U]$ where S_L is typically chosen to be zero and S_U needs to be sufficiently large.

Constructing the grid

- We now need the ranges of S and t to ensure that we have a fine enough grid to allow for most possible movements in S and enough time steps t.
- As for the binomial and Monte-Carlo method we will discuss later what is a suitable size/number for these steps.
- We split up the possible stock prices $[0, S_U]$ into j_{max} intervals each of size $\Delta S = S_U/j_{max}$. Thus possible stock prices (or grid points) are given by $0, \Delta S, 2\Delta S, \ldots, (j_{\text{max}} - 1)\Delta S, j_{\text{max}}\Delta S = S_U$.
- We also divide time to maturity $[0, T]$ into i_{max} time intervals each of length $\Delta t = T/i_{\text{max}}$, thus the time steps (grid points) are $0, \Delta t, \ldots, (i_{\text{max}} - 1)\Delta t, i_{\text{max}}\Delta t = T$.

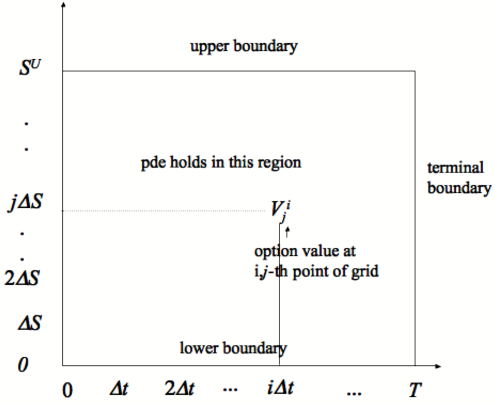
Constructing the grid

- Then to simplify notation **we will denote the option price at each node $V(j\Delta S, i\Delta t)$ as V_j^i.** Where $0 < i < i_{max}$ and $0 < j < j_{max}$.
- Note that the lowest possible stock price does not necessarily have to be 0. If the option values for low stock prices are already known then we can reduce the range to $[S_L, S_U]$ for a suitable S_L.

Quiz)

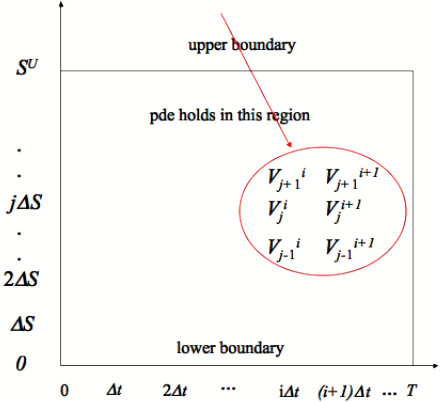
- What would be an appropriate choice of S_L for a down-and-out barrier option?

Finite difference grid



Finite difference grid

Focus attention on i, j-th value V_j^i, and a little piece of the grid around that point



Using equations

- We clearly know the information at $t = T$ as this is the payoff from the option, by limiting our focus to

$$\begin{array}{cc} & V_{j+1}^{i+1} \\ V_j^i & V_j^{i+1} \\ & V_{j-1}^{i+1} \end{array}$$

  we can approximate the derivatives in the Black-Scholes equation by using our difference equations and from this we can write V_j^i in terms of the other three terms.

- Recall the BSM equation again:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0$$

Explicit finite difference method

- The BSM equation approximates to:

$$\begin{aligned} 0 &= \frac{V_j^{i+1} - V_j^i}{\Delta t} + \frac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}}{(\Delta S)^2} \\ &\quad + (r-\delta) j \Delta S \frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2\Delta S} - rV_j^i \end{aligned}$$

the only unknown here is V_j^i, as we have been working backward in time.

- So, we can rearrange in terms of this unknown:

$$V_j^i = \frac{1}{1 + r\Delta t}\left(AV_{j+1}^{i+1} + BV_j^{i+1} + CV_{j-1}^{i+1}\right) \qquad (4)$$

where:

$$\begin{aligned} A &= \left(\frac{1}{2}\sigma^2 j^2 + \frac{(r-\delta)j}{2}\right)\Delta t \\ B &= 1 - \sigma^2 j^2 \Delta t \\ C &= \left(\frac{1}{2}\sigma^2 j^2 - \frac{(r-\delta)j}{2}\right)\Delta t \end{aligned}$$

Aside: alternate equations

- Note that really, the BSM equation approximates to:

$$\begin{aligned} 0 &= \frac{V_j^{i+1} - V_j^i}{\Delta t} + \frac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}}{(\Delta S)^2} \\ &\quad + (r-\delta) j \Delta S \frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2\Delta S} - rV_j^{i+1} \end{aligned}$$

- Rearranging in terms of this unknown gives a slightly less elegant form:

$$V_j^i = \left(AV_{j+1}^{i+1} + BV_j^{i+1} + CV_{j-1}^{i+1} \right) \tag{5}$$

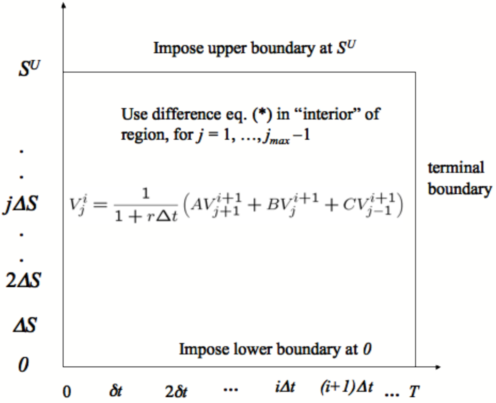
where now

$$\begin{aligned} A &= \left(\frac{1}{2}\sigma^2 j^2 + \frac{(r-\delta)j}{2} \right) \Delta t \\ B &= 1 - r\Delta t - \sigma^2 j^2 \Delta t \\ C &= \left(\frac{1}{2}\sigma^2 j^2 - \frac{(r-\delta)j}{2} \right) \Delta t \end{aligned}$$

Explicit finite difference method

- Thus just as with a binomial tree we have a way of calculating the option value at expiry - the known payoff, and we have an iterative scheme for calculating the option value for all values of S at the previous time step.
- Thus we can use backward iteration and equation 4 (or 5) for V_j^i to calculate the option value all the way back to $t = 0$.
- The differences between the binomial and explicit finite difference method are
  - The binomial uses two nodes to the explicit finite difference's three.
  - You get to choose the specifications of the grid in the finite difference method
  - You also need to specify the behavior on the upper and lower S boundaries.

The grid again:



Upper and Lower boundaries

- If we attempt to use equation **??** to calculate V_0^i then we need to have values of V_{-1}^i which we dont have.
- So for V_0^i and $V_{j_{\max}}^i$ we need to use our boundary conditions. For this call option we have:

$$\begin{aligned} V_0^i &= 0 \\ V_{j_{\max}}^i &= S_U e^{-\delta(T-i\Delta t)} - K e^{-r(T-i\Delta t)} \end{aligned}$$

- These conditions will naturally be different for different options, such as barrier options, put options etc.
- It is possible to have derivative conditions on these boundaries, as you can approximate them using V_1^i etc. For example if at $S=0$:

$$\frac{\partial V}{\partial S} = 0 \rightarrow \frac{V_1^i - V_0^i}{\Delta S} = 0$$

Probabilistic interpretation

- You will see that it is possible to think of the explicit finite difference scheme as a trinomial tree and A, B and C as probabilities.
- First note that $A + B + C = 1$, second consider what the expected value of S is at time $i\Delta t$:

$$
\begin{aligned}
E[S_j^i] &= \frac{1}{1 + r\Delta t}\left(A(S_j^i + \Delta S)) + B(S_j^i) + C(S_j^i - \Delta S)\right) \\
&= \frac{1}{1 + r\Delta t}\left(S_j^i(1 + (r - \delta)\Delta t)\right) \\
&= \frac{1}{1 + r\Delta t} E[S_j^{i+1}]
\end{aligned}
$$

the expected future value of S, following GBM, under the risk-neutral probability discounted at the risk-free rate. So A, B and C can also be interpreted as risk-neutral probabilities (you can also check that the variance works).

Quiz

What is the interpretation if we use the equations on Slide 17 instead?

- The same, $E[S_{\Delta T}] = S_0(1 + (r - \delta)\Delta T)$
- There isn't one as A, B, C cannot be interpreted as probabilities
- $E[S_{\Delta T}] \neq S_0(1 + (r - \delta)\Delta T)$

Stability

- Unfortunately, the explicit finite difference scheme is occasionally unstable, in that for particular choices of Δt and ΔS, the scheme will not give an option value even close to the correct answer as small errors magnify during the iterative procedure.
- There is a mathematical method that can be used to determine what the constraint is, however, we can also appeal to our probabilistic explanation to see what the constraint is for the explicit finite difference method.
- If we consider A, B and C as probabilities, we require than $0 \leq A, B, C \leq 1$.

Stability

- For A and C this requires:

$$j > \left| \frac{r-\delta}{\sigma^2} \right|$$

  which is not too large a constraint unless σ is very small and $|r-\delta|$ is very large, which are rare.

- A far bigger problem is for B where this says that

$$\Delta t < \frac{1}{\sigma^2 j^2}$$

  which means that you need to ensure that the time interval is small enough.

- Specifically the required size of the time interval will shrink as you increase the number of S steps or the volatility increases.

Stability

- The stability often severely restricts choice of $\Delta t, \Delta S, \Delta t$ cant be too small, or else computation will take too long then this puts lower bound on size of ΔS.
- Nonetheless, we have some flexibility. Two common choices:
- choose $\Delta t, \Delta S$ so that $B = 2/3$ (means A, C approx. $1/6$)
- choose $\Delta t, \Delta S$ so that $B = 1/3$ (means A, C approx. $1/3$)

Quiz

If $T = 0.5, j_{\text{max}} = 500$ and $\sigma = 0.2$, what is the minimum required steps in t?

- 0.0001
- 50
- 500
- 5000

Quiz

If we have 500 time steps, what is the maximum number of steps in S?

- $1/1580$
- 15.8
- 158
- 1580

Convergence

- Assuming that the scheme is stable then we would like to analyze the accuracy of the method.
- The errors will arise from only approximating the derivatives, in particular, in the explicit finite difference method:

$$
\begin{aligned}
\frac{\partial V}{\partial S} &= \frac{V(S+\Delta S,t)-V(S-\Delta S,t)}{2\Delta S} + O((\Delta S)^2) \\
\frac{\partial^2 V}{\partial S^2} &= \frac{V(S+\Delta S,t)-2V(S,t)+V(S+\Delta S,t)}{(\Delta S)^2} + O((\Delta S)^2) \\
\frac{\partial V}{\partial t} &= \frac{V(S,t+\Delta t)-V(S,t)}{\Delta t} + O(\Delta t)
\end{aligned}
$$

- And so we would expect the error to decrease linearly with the number of time steps (as with the binomial model) and quadratically with the number of steps in S.

Non-linearity error

- One should be careful when assuming that this is how the scheme converges, if we write out the Taylor expansion in full we see that this convergence depends upon all of the derivatives being well behaved (e.g. not infinite).
- However, we know that in the case of European options, the payoff at expiry is discontinuous leading to an infinite first derivative - and so it seems likely that our approximation may not work as well here.
- Additionally, in the case of an American option, across the early exercise boundary, the second derivative is infinite, this again will lead to difficulty in the approximations - in particular, the assumption as to the error from finite-difference methods.
- There are ways of overcoming this as we shall see.

Quiz

Is it likely that derivatives are well behaved?

- Yes
- No

Non-linearity error

- The nice thing about finite difference methods in general are that you have the freedom to construct the grid as desired and so it is quite simple to construct the grid so that you have a grid point upon any discontinuities.
- For example, if we consider an European call or put option then the only source of non-linearity error is at $S = K$ at expiry. Thus you should always choose ΔS so that $K = j\Delta S$ for an integer value of j, on any other fixed proportion - perhaps you want the strike price between two grid points, this is also possible.
- So if you want a grid point at K in this case where $S_0 = 100$ and $K = 95$, you need a suitably large S^U and a ΔS which is a divisor of 95. Thus if you want 500 S-steps then a reasonable choice for ΔS may be 0.76, which give $S^U = 4x95 = 380$ and $j = 125$ is the position of the exercise price.

Continuous barrier options

- As we have seen when pricing barrier options with the binomial method, there is a large amount of non-linearity error that comes from not having the nodes in the tree aligned with the position of the barrier.
- Thus with barrier options we have two sources of non-linearity error, the error from the barrier and the error from the discontinuous payoff.
- However, with the finite difference grid it is relatively easy to align the grid so that you have Stock price grid points on the barrier *and* on the exercise price at expiry.
- For a down and out barrier option choose S^L (the lower value of S) to be on the barrier and then, as in the previous example, choose ΔS so that the exercise price is also on a grid point.
- For example if $B = 90$ and $K = 95$ if you require 5000 steps, choose $\Delta S = 0.05$, so that $S^U = 340$, and the barrier is at $j = 0$ and the exercise price is at $j = 100$.

Discrete barrier options

- With discretely observed barrier options the problem becomes more difficult as the imposition of the barrier can also affect the time derivative.
- Cheuk and Vorst (1996) and Zvan et al. (2000) (available on COMPASS) find that a good rule of thumb is to have the discrete barrier halfway between two Stock price grid points - just as we saw for the binomial. Even in this case this can create some problems for the Crank-Nicolson method that we see next (see Zvan et al. for more details on these types of options).

Conclusion

- We have introduced the finite difference method which is a way of solving partial differential equations by estimating the first and second derivatives and then substituting the estimates in the PDE.
- This scheme was explained for the Black-Scholes PDE and in particular we derived the explicit finite difference scheme to solve the European call and put option problems.
- The convergence of the method is similar to the binomial tree and, in fact, the method can be considered a trinomial tree. Unfortunately, however, the method can be unstable which puts constraints on our grid size.
- Finally, we saw that with finite-difference methods as you can choose the dimensions of the grid so as to remove the non-linearity error.