

FIN 514: Problem Set #6

Due on Wednesday, April 25, 2018

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Problem 1

(a) By Ito's product rule, $dY(t)$ satisfies the following equation.

$$\begin{aligned} dY(t) &= B_P(t)dS(t) + S(t)dB_P(t) + dB_P(t)dS(t) \\ &= B_P(t)[\mu S(t)dt + \sigma S(t)dX(t)] + r_P S(t)B_P(t)dt \\ &= (\mu + r_P)Y(t)dt + \sigma Y(t)dX(t) \end{aligned}$$

In order to find martingale measure with respect to $B(t)$ as a numeraire, dynamics of $Y(t)/B(t)$ is derived as follows.

$$\begin{aligned} d\left(\frac{Y(t)}{B(t)}\right) &= Y(t)d\left(\frac{1}{B(t)}\right) + \frac{1}{B(t)}dY(t) + dY(t)d\left(\frac{1}{B(t)}\right) \\ d\left(\frac{1}{B(t)}\right) &= -\frac{1}{B^2(t)}dB(t) \\ &= -\frac{1}{B^2(t)}rB(t)dt = -r\frac{1}{B(t)}dt \\ \Rightarrow d\left(\frac{Y(t)}{B(t)}\right) &= Y(t)\left(-r\frac{1}{B(t)}dt\right) + \frac{1}{B(t)}[(\mu + r_P)Y(t)dt + \sigma Y(t)dX(t)] \\ &= (\mu + r_P - r)\frac{Y(t)}{B(t)}dt + \sigma\frac{Y(t)}{B(t)}dX(t) \end{aligned}$$

By Girsanov's theorem, there exists a probability measure such that $\tilde{X}(t) = X(t) + \int_0^t \frac{\mu + r_P - r}{\sigma} ds$ is a brownian motion under the measure. Therefore, by plugging $dX(t) = d\tilde{X}(t) - \frac{\mu + r_P - r}{\sigma} dt$ into the equation above, then $d\left(\frac{Y(t)}{B(t)}\right)$ becomes $\sigma\frac{Y(t)}{B(t)}d\tilde{X}(t)$, hence becomes martingale because there is no drift. Therefore, from the perspective of U.S dollar investor, under risk-neutral measure, $dX(t) = d\tilde{X}(t) - \frac{\mu + r_P - r}{\sigma} dt$. By plugging it into dynamics of $Y(t)$, we can find dynamics of the U.S price of a GBP bond under risk-neutral measure as follows.

$$\begin{aligned} dY(t) &= (\mu + r_P)Y(t)dt + \sigma Y(t)dX(t) \\ &= (\mu + r_P)Y(t)dt + \sigma Y(t)\left[d\tilde{X}(t) - \frac{\mu + r_P - r}{\sigma} dt\right] \\ &= rY(t)dt + \sigma Y(t)d\tilde{X}(t) \end{aligned}$$

And it is consistent with the fact that expected return of every tradable asset is risk-free rate under risk-neutral measure.

(b) By plugging $dX(t) = d\tilde{X}(t) - \frac{\mu + r_P - r}{\sigma} dt$ into dynamics of $S(t)$, we can find dynamics of U.S. dollar price of a British pound under risk-neutral probability as follows.

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dX(t) \\ &= \mu S(t)dt + \sigma S(t)\left[d\tilde{X}(t) - \frac{\mu + r_P - r}{\sigma} dt\right] \\ &= (r - r_P)S(t)dt + \sigma S(t)d\tilde{X}(t) \end{aligned}$$

- (c) Unlike the assumption of ordinary Black-Scholes-Merton formula, since expected return of underlying asset has changed from r to $r - r_P$, formula for call option should be changed to following equation.

$$e^{-rT}[S_0 e^{(r-r_P)T} N(d_1) - K N(d_2)]$$

$$d_1 = \frac{\log(S_0/K) + (r - r_P + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Problem 2

- (a) By Ito's product rule, dynamics of $B(t)/S(t)$ is as follows.

$$\begin{aligned} d\left(\frac{B(t)}{S(t)}\right) &= d\left(\frac{1}{S(t)}\right) B(t) + \frac{1}{S(t)} dB(t) + d\left(\frac{1}{S(t)}\right) dB(t) \\ d\left(\frac{1}{S(t)}\right) &= -\frac{1}{S^2(t)} dS(t) + \frac{1}{2} \times 2 \times \frac{1}{S^3(t)} (dS(t))^2 \\ &= -\frac{1}{S^2(t)} [(\mu - d)S(t)dt + \sigma S(t)dX(t)] + \frac{1}{S^3(t)} \sigma^2 S^2(t)dt \\ &= [-(\mu - d) + \sigma^2] \frac{1}{S^2(t)} dt - \sigma \frac{1}{S(t)} dX(t) \\ \Rightarrow d\left(\frac{B(t)}{S(t)}\right) &= \frac{B(t)}{S(t)} [-(\mu - d) + \sigma^2] dt - \sigma \frac{B(t)}{S(t)} dX(t) + r \frac{B(t)}{S(t)} dt \\ &= [r - (\mu - d) + \sigma^2] \frac{B(t)}{S(t)} dt - \sigma \frac{B(t)}{S(t)} dX(t) \end{aligned}$$

- (b) By Girsanov's theorem, there exists a probability measure such that $\tilde{X}(t) = X(t) - \int_0^t \frac{r - (\mu - d) + \sigma^2}{\sigma} ds$ is a brownian motion under the measure. Under the measure, process $d\left(\frac{B(t)}{S(t)}\right)$ changes as follows.

$$\begin{aligned} d\left(\frac{B(t)}{S(t)}\right) &= [r - (\mu - d) + \sigma^2] \frac{B(t)}{S(t)} dt - \sigma \frac{B(t)}{S(t)} dX(t) \\ &= [r - (\mu - d) + \sigma^2] \frac{B(t)}{S(t)} dt - \sigma \frac{B(t)}{S(t)} \left[d\tilde{X}(t) + \frac{r - (\mu - d) + \sigma^2}{\sigma} dt \right] \\ &= -\sigma \frac{B(t)}{S(t)} d\tilde{X}(t) \end{aligned}$$

Therefore, under the measure, $\frac{B(t)}{S(t)}$ is a martingale since there is no drift term in dynamics. Plugging $dX(t) = d\tilde{X}(t) + \frac{r - (\mu - d) + \sigma^2}{\sigma} dt$ into the process of $S(t)$, dynamics of $S(t)$ under martingale measure with respect to $S(t)$ as a numeraire is as follows.

$$\begin{aligned} dS(t) &= (\mu - d)S(t)dt + \sigma S(t)dX(t) \\ &= (\mu - d)S(t)dt + \sigma S(t) \left[d\tilde{X}(t) + \frac{r - (\mu - d) + \sigma^2}{\sigma} dt \right] \\ &= (r + \sigma^2)S(t)dt + \sigma S(t)d\tilde{X}(t) \end{aligned}$$

(c) Under the martingale measure with respect to $S(t)$ as a numeraire, $V(t)/S(t)$ is also a martingale.

Therefore, by definition of martingale, European call option value $V(t)$ is derived as follows.

$$\begin{aligned}\frac{V(t)}{S(t)} &= E_t^Q \left[\frac{V(T)}{S(T)} \right] \\ \Rightarrow V(t) &= S(t) E_t^Q \left[\frac{V(T)}{S(T)} \right] \\ &= S(t) E_t^Q \left[\frac{\max(S(T) - K, 0)}{S(T)} \right] \\ &= S(t) E_t^Q \left[\max \left(1 - \frac{K}{S(T)}, 0 \right) \right]\end{aligned}$$

Where K is strike price of the option, and Q is a probability measure in which $V(t)/S(t)$ is a martingale.

(d) In order to evaluate call option value, we need to derive the solution of SDE $dS(t) = (r + \sigma^2)S(t)dt + \sigma S(t)d\tilde{X}(t)$ first. The solution is derived as follows.

$$\begin{aligned}d \log S(t) &= \frac{1}{S(t)} dS(t) - \frac{1}{2} \frac{1}{S^2(t)} (dS(t))^2 \\ &= \left(r + \frac{1}{2} \sigma^2 \right) dt + \sigma d\tilde{X}(t) \\ \Rightarrow S(T) &= S(t) \exp \left[\left(r + \frac{1}{2} \sigma^2 \right) (T - t) + \sigma \sqrt{T - t} \phi \right] \\ \phi &\sim N(0, 1)\end{aligned}$$

Plugging the result of equation above, we can evaluate call option value at time t as follows.

$$\begin{aligned}V(t) &= S(t) E_t^Q \left[\max \left(1 - \frac{K}{S(T)}, 0 \right) \right] \\ &= S(t) E_t^Q \left[\max \left(1 - \frac{K}{S(t) \exp[(r + \frac{1}{2} \sigma^2)(T - t) + \sigma \sqrt{T - t} \phi]}, 0 \right) \right]\end{aligned}$$

Since $1 - \frac{K}{S(t) \exp[(r + \frac{1}{2} \sigma^2)(T - t) + \sigma \sqrt{T - t} \phi]} \geq 0$ is equivalent to $\phi \geq \frac{\log(K/S(t) - (r + \frac{1}{2} \sigma^2)(T - t))}{\sigma \sqrt{T - t}} \equiv L$, and ϕ follows standard normal distribution, call option value can be calculated as follows.

$$\begin{aligned}V(t) &= S(t) \int_L^\infty \left(1 - \frac{K}{S(t) \exp[(r + \frac{1}{2} \sigma^2)(T - t) + \sigma \sqrt{T - t} x]} \right) \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2} x^2} dx \\ &= S(t) \int_L^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2} x^2} dx - K \int_L^\infty \frac{1}{\sqrt{2\pi}} e^{-(r + \frac{1}{2} \sigma^2)(T - t) - \sigma \sqrt{T - t} x - \frac{1}{2} x^2} dx \\ &= S(t)(1 - N(L)) - K e^{-r(T - t)} \int_L^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} (x + \sigma \sqrt{T - t})^2} dx \\ &= S(t)(1 - N(L)) - K e^{-r(T - t)} \int_{L + \sigma \sqrt{T - t}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} dy \\ &= S(t)(1 - N(L)) - K e^{-r(T - t)} (1 - N(L + \sigma \sqrt{T - t})) \\ &= S(t)N(-L) - K e^{-r(T - t)} N(-L - \sigma \sqrt{T - t}) \\ &= S(t)N(d_1) - K e^{-r(T - t)} N(d_2)\end{aligned}$$

$$\begin{aligned}
y &= x + \sigma\sqrt{T-t} \\
d_1 &= \frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}$$

Which is the desired solution. It is not easier than when $B(t)$ was numeraire because it also has some integration and transformation in evaluating procedure.

Problem 3

(a) Using Ito's lemma, dynamics of $\log B_e$ is as follows.

$$\begin{aligned}
d\log B_e &= \frac{1}{B_e}dB_e \\
&= \frac{1}{B_e}r_e B_e dt \\
&= r_e dt
\end{aligned}$$

Since $B_e(0) = 0$, $B_e(T)$ is evaluated as $B_e(0)e^{r_e T} = e^{r_e T}$.

(b) By Ito's product rule, dynamics of Z_e and Z_g is derived as follows.

$$\begin{aligned}
dZ_e &= d(eB_e) = B_e de + e dB_e + dB_e de \\
&= B_e(\mu_e edt + \sigma_e edX_2) + er_e B_e dt \\
&= (\mu_e + r_e)Z_e dt + \sigma_e Z_e dX_2 \\
dZ_g &= d(gB_g) = B_g dg + g dB_g + dB_g dg \\
&= B_g(\mu_g gdt + \sigma_g g dX_1) + gr_g B_g dt \\
&= (\mu_g + r_g)Z_g dt + \sigma_g Z_g dX_1
\end{aligned}$$

(c) By Ito's product rule, dynamics of X is derived as follows.

$$\begin{aligned}
dX &= d\left(\frac{g}{e}\right) = \frac{1}{e}dg + gd\left(\frac{1}{e}\right) + dgd\left(\frac{1}{e}\right) \\
d\left(\frac{1}{e}\right) &= -\frac{1}{e^2}de + 2 \times \frac{1}{2} \times \frac{1}{e^3}(de)^2 \\
&= -\frac{1}{e^2}(\mu_e edt + \sigma_e edX_2) + \frac{1}{e^3}\sigma_e^2 e^2 dt \\
&= (-\mu_e + \sigma_e^2)\frac{1}{e}dt - \sigma_e \frac{1}{e}dX_2 \\
\Rightarrow dX &= \frac{1}{e}(\mu_g gdt + \sigma_g g dX_1) + g[(-\mu_e + \sigma_e^2)\frac{1}{e}dt - \sigma_e \frac{1}{e}dX_2] + (\mu_g gdt + \sigma_g g dX_1)[(-\mu_e + \sigma_e^2)\frac{1}{e}dt - \sigma_e \frac{1}{e}dX_2] \\
&= (\mu_g - \mu_e + \sigma_e^2 - \rho\sigma_e\sigma_g)X dt + X(\sigma_g dX_1 - \sigma_e dX_2)
\end{aligned}$$

Let $dX_3 = \frac{\sigma_g dX_1 - \sigma_e dX_2}{\sqrt{\sigma_g^2 - 2\sigma_g\sigma_e + \sigma_e^2}}$. Then by Levy's theorem, X_3 is a brownian motion. Let $\sigma_X^2 = \sigma_g^2 - 2\sigma_g\sigma_e + \sigma_e^2$, then dX can be represented as follows.

$$dX = (\mu_g - \mu_e + \sigma_e^2 - \rho\sigma_e\sigma_g)X dt + \sigma_X X dX_3$$

(d) By Ito's product rule, dynamics of Y is derived as follows.

$$\begin{aligned}
 dY &= d\left(\frac{Z_g}{Z_e}\right) = Z_g d\left(\frac{1}{Z_e}\right) + \frac{1}{Z_e} dZ_g + dZ_g d\left(\frac{1}{Z_e}\right) \\
 d\left(\frac{1}{Z_e}\right) &= -\frac{1}{Z_e^2} dZ_e + \frac{1}{Z_e^3} (dZ_e)^2 \\
 &= -\frac{1}{Z_e} [(\mu_e + r_e)Z_e dt + \sigma_e Z_e dX_2] + \frac{1}{Z_e^3} \sigma_e^2 Z_e^2 dt \\
 &= [-(\mu_e + r_e) + \sigma_e^2] \frac{1}{Z_e} dt - \sigma_e \frac{1}{Z_e} dX_2 \\
 \Rightarrow dY &= \frac{Z_g}{Z_e} [-(\mu_e + r_e) + \sigma_e^2] dt - \sigma_e dX_2 \\
 &\quad + \frac{Z_g}{Z_e} [(\mu_g + r_g) dt + \sigma_g dX_1] \\
 &\quad + \frac{Z_g}{Z_e} [-(\mu_e + r_e) + \sigma_e^2] dt - \sigma_e dX_2 [(\mu_g + r_g) dt + \sigma_g dX_1] \\
 &= [(\mu_g + r_g) - (\mu_e + r_e) + \sigma_e^2 - \rho\sigma_e\sigma_g] Y dt + \sigma_X Y dX_3
 \end{aligned}$$

(e) By fundamental theorem of finance, if market is complete, there exists a martingale measure such that Y is a martingale under the measure. Therefore, under \mathbf{Q} , there is no drift term in dY so that Y is a martingale.

(f) By Girsanov's theorem, if we choose $\tilde{X}_3(t)$ such that $\tilde{X}_3(t) = X_3(t) + \int_0^t \frac{(\mu_g + r_g) - (\mu_e + r_e) + \sigma_e^2 - \rho\sigma_e\sigma_g}{\sigma_X} ds$, then there exists a probability measure \mathbf{Q} so that \tilde{X}_3 is a brownian motion under \mathbf{Q} . Plugging $dX_3 = d\tilde{X}_3 - \frac{(\mu_g + r_g) - (\mu_e + r_e) + \sigma_e^2 - \rho\sigma_e\sigma_g}{\sigma_X} dt$ into dY , it becomes to $dY = \sigma_X Y d\tilde{X}_3$, which is a martingale.

(g) Under \mathbf{Q} , $\tilde{X}_3(t) = X_3(t) + \int_0^t \frac{(\mu_g + r_g) - (\mu_e + r_e) + \sigma_e^2 - \rho\sigma_e\sigma_g}{\sigma_X} ds$ is a brownian motion. Plugging $dX_3 = d\tilde{X}_3 - \frac{(\mu_g + r_g) - (\mu_e + r_e) + \sigma_e^2 - \rho\sigma_e\sigma_g}{\sigma_X} dt$ into dX , dynamics of X is derived as follows.

$$\begin{aligned}
 dX &= (\mu_g - \mu_e + \sigma_e^2 - \rho\sigma_e\sigma_g) X dt + \sigma_X X dX_3 \\
 &= (\mu_g - \mu_e + \sigma_e^2 - \rho\sigma_e\sigma_g) X dt + \sigma_X X (d\tilde{X}_3 - \frac{(\mu_g + r_g) - (\mu_e + r_e) + \sigma_e^2 - \rho\sigma_e\sigma_g}{\sigma_X} dt) \\
 &= (r_e - r_g) X dt + \sigma_X X d\tilde{X}_3
 \end{aligned}$$

(h) Let $V(t)$ denote the option value at time t . Under \mathbf{Q} , since V/Z_e is a martingale, $V(t)$ can be represented as $V(t) = Z_e(t) \mathbb{E}_t^Q \left[\frac{V(T)}{Z_e(T)} \right]$. Since $Z_e = eB_e$, and $V(T) = 1000 \times \max[g_T - e_T, 0]$, $V(t)$ can also be represented as follows.

$$\begin{aligned}
 V(t) &= Z_e(t) \mathbb{E}_t^Q \left[\frac{V(T)}{Z_e(T)} \right] \\
 &= e_t B_e(t) \times 1000 \times \max \left[\frac{g_T - e_T}{e_T B_e(T)}, 0 \right] \\
 &= e_t e^{-r_e(T-t)} \times 1000 \times \max[X(T) - 1, 0]
 \end{aligned}$$

(i) By analogy to the Black-Scholes-Merton formula, $V(g, e, 0)$ is represented as follows.

$$\begin{aligned}
 V(0) &= e_0 e^{-r_e T} \times 1000 \times [X_0 e^{(r_e - r_g)T} N(d_1) - N(d_2)] \\
 &= e_0 e^{-r_e T} \times 1000 \times \left[\left(\frac{g_0}{e_0} \right) e^{(r_e - r_g)T} N(d_1) - N(d_2) \right] \\
 &= 1000 e^{-r_e T} [g_0 e^{(r_e - r_g)T} N(d_1) - e_0 N(d_2)] \\
 d_1 &= \frac{\log X_0 + (r_e - r_g + \frac{1}{2} \sigma_X^2)T}{\sigma_X \sqrt{T}} \\
 &= \frac{\log(g_0/e_0) + (r_e - r_g + \frac{1}{2} \sigma_X^2)T}{\sigma_X \sqrt{T}} \\
 d_2 &= d_1 - \sigma_X \sqrt{T} \\
 \sigma_X^2 &= \sigma_g^2 - 2\sigma_g \sigma_e + \sigma_e^2
 \end{aligned}$$