Fin 514: Financial Engineering II

Lecture 8: Stochastic calculus and Ito's lemma

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Outline

- We will now start formalizing our work on Brownian motion and justify our definition of the previous stochastic differential equation.
- To do this we will attempt to construct a meaningful representation of a Brownian motion over a general period of time.
- This will give rise to the Ito integral which will show us what is meant by a stochastic differential equation as well as having close links with financial problems.
- Then We will introduce the most useful result from stochastic calculus - Ito's lemma. Ito's lemma will enable us to write a general function of a stochastic process, F(S(t),t) as a stochastic differential equation which we will *occasionally* be able to solve.
- This is useful for derivative pricing as the price of a derivative is a function of the share price (stochastic) and time and so Ito's lemma will be invaluable.

Stock price model

- Within our Geometric Brownian Motion model we can write the change in the stock price as follows:

$$S(t+\Delta t) - S(t) \quad = \quad \mu S(t)\Delta t + \sigma S(t)\phi\sqrt{\Delta t}$$

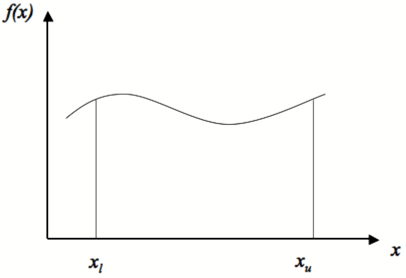
  where ϕ is normally distributed with mean 0 and variance 1.

- Without much rigor we also defined it as follows:

$$dS(t) = \mu S(t)dt + \sigma S(t)dX(t)$$

Riemann Integration

- Integration of deterministic variables is typically carried out by Riemann integration. Consider a general function $f(x)$ of a deterministic variable x, which is continuous in $[x_l, x_u]$.
- The Riemann integral will attempt to calculate the area under the curve, see below diagram:

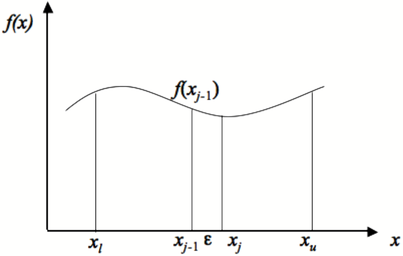


Riemann Integration: process

- To estimate this area, the region $[x_l, x_u]$ is divided into n equally sized intervals of length $\epsilon = (x_u - x_l)/n$.
- If we simplify the notation such that $x_0 = x_l$ then we can write:

$$\begin{aligned} x_j &= x_0 + j\epsilon, j = 1, \ldots, n-1 \\ x_n &= x_u \end{aligned}$$

- Similarly, $f(x_j)$ will be the corresponding values of the function:



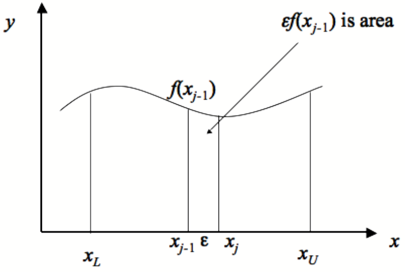
Riemann Integration: process

- The area under the curve will be given by the sum:

$$\sum_{j=1}^{n} f(x_{j-1})\epsilon \quad = \quad \epsilon \sum_{j=1}^{n} f(x_{j-1})$$

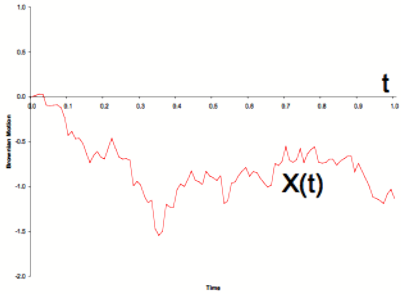
- The integral is defined as the limit as $\epsilon \to 0$

$$\int_{x_l}^{x_u} f(x)dx \quad = \quad \lim_{\epsilon \to 0} \epsilon \sum_{j=1}^{n} f(x_{j-1})$$



Stochastic process as a sum

- With deterministic variables, we had $[x_l, x_u]$ and a function $f(x)$.
- With our stochastic process the set-up is slightly different. The interval is over time $[0, t]$, where we have a Brownian motion $X(t)$



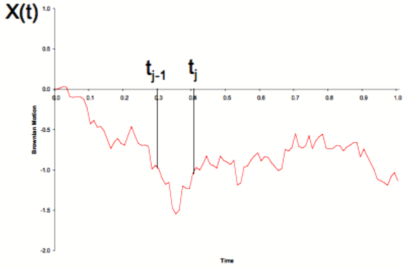
and a function $f(t)$.

Stochastic process as a sum

- As before, we divide $[0, t]$ into n equal intervals.
- Notation of the interval is:

$$
\begin{aligned}
t_0 &= 0 \\
t_j &= t_0 + j\epsilon = j(t/n) \text{ for } j = 1, , n-1 \\
t_n &= t
\end{aligned}
$$

- Consider a Brownian Motion increment $X(t_j) - X(t_{j-1})$



Stochastic process as an integral

- Consider sum:

$$\sum_{j=1}^{n} f(t_{j-1})(X(t_j) - X(t_{j-1}))$$

where $f(t_{j-1})$ can depend upon any information at time t_{j-1} including the value of X at this point and so is a deterministic function of time.

- Due to the potential random nature of $f(t)$ increments then it is crucial that $f(t)$ is evaluated at the left end-point of the interval (i.e. at t_{j-1}) before the $X(t_j) - X(t_{j-1})$ size is known.
- Then the **Ito integral** will be the limit:

$$\int_0^t f(\tau)dX(\tau) = \lim_{n\to\infty} \sum_{j=1}^{n} f(t_{j-1})(X(t_j) - X(t_{j-1}))$$

Ito Integral

- This limit is technically defined in the mean square sense, which is as $n \to \infty$,

$$E\left(\sum_{j=1}^{n} f(t_{j-1})(X(t_j) - X(t_{j-1})) - \int_0^t f(\tau)dX(\tau)\right)^2 \quad \to \quad 0$$

Ito Integral

- The Ito integral is as follows:

$$\int_0^t f(\tau)dX(\tau) = \lim_{n\to\infty}\sum_{j=1}^{n} f(t_{j-1})(X(t_j) - X(t_{j-1}))$$

- Riemann was:

$$\begin{aligned}\int_{x_l}^{x_u} f(x)dx &= \lim_{\epsilon\to 0}\epsilon\sum_{j=1}^{n} f(x_{j-1}) \\ &= \sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1})\end{aligned}$$

- The only difference is that the increments in the Ito integral are random whereas they are deterministic in the Riemann integration.

QUIZ

By considering an integral as the limit of sums, what is the following integral?

$$\int_0^t dX(\tau)$$

- $X(t) - X(0)$
- Impossible to say
- $X(t)$
- $X(t) - X(t_{n-1})$

Ito Integral as a Stochastic process

- We will now see how the Ito integral is the natural way to notate a general stochastic process.
- Consider W(t) which is defined as follows:

$$Y(t) = \int_0^t f(\tau) dX(\tau)$$

  where we let the upper limit of integration, t, increase.
- Then $Y(t)$ defines a stochastic process, as t increases we add random increments of the form

$$f(t_{j-1})(X(t_j) - X(t_{j-1})) \equiv f(t)dX(t)$$

Ito Integral as a Stochastic process

- In general an Stochastic (or Ito) process is obtained by summing an initial value $Y(0)$ with a deterministic integral and an Ito integral, where we let t, the upper limit of integration, increase:

$$Y(t) = Y(0) + \int_0^t g(\tau)d\tau + \int_0^t f(\tau)dX(\tau)$$

- Here, $f(\tau)$ and $g(\tau)$ values are known at time τ even though they may depend upon stochastic variables. These may well be functions of the stock price, interest rate, or volatility which may also be stochastic.

Ito process and our Stock price model

- Reconsider our model of stock price movements which we wrote as:

$$\boxed{dS(t) = \mu S(t)dt + \sigma S(t)dX(t)}$$

- This process can be understood as adding random increments to the initial price, as follows:

$$\begin{aligned} S(t_j) &= S(t_{j-1}) + \mu S(t_{j-1})\Delta t + \sigma S(t_{j-1})\phi\sqrt{\Delta t} \\ &= S(t_{j-1}) + \mu S(t_{j-1})(t_j - t_{j-1}) + \sigma S(t_{j-1})(X(t_j) - X(t_{j-1})) \end{aligned}$$

this gives us step sizes ($n = t/\epsilon$)

$$S(t) = S(0) + \sum_{j=1}^{n} \mu S(t_{j-1})(t_j - t_{j-1}) + \sum_{j=1}^{n} \sigma S(t_{j-1})(X(t_j) - X(t_{j-1}))$$

Ito process and our Stock price model

- If we let the time step become small ($\epsilon \to 0$) and take limits then this becomes a stochastic process

$$\boxed{S(t) = S(0) + \int_0^t \mu S(\tau) d\tau + \int_0^t \sigma S(\tau) dX(\tau)}$$

compare to

$$Y(t) = Y(0) + \int_0^t g(\tau) d\tau + \int_0^t f(\tau) dX(\tau) \tag{1}$$

- The shorthand for the general stochastic process, Y(t) above is **the stochastic differential equation** (or SDE)

$$dY(t) = g(t)dt + f(t)dX(t) \tag{2}$$

- In general we will use the shorthand notation (2) but the full definition is given in the Ito process (1)

Solving an SDE: Simple example

- Our simplest Brownian motion was

$$
\begin{aligned}
Y(t) &= Y(0) + \int_0^t \mu d\tau + \int_0^t \sigma dX(\tau) \\
&= Y(0) + \mu t + \sigma X(t)
\end{aligned}
$$

- Now since the SDE is shorthand for this process then this value for Y(t) is also **the solution** of the following SDE

$$dY(t) = \mu dt + \sigma dX(t)$$

QUIZ

- Solve the following SDE to find Y(t) for the case when

$$dY(t) = tdt + \sigma dX(t)$$

Stochastic processes and portfolios

- Consider the following process

$$Y(t) = Y(0) + \int_0^t \theta(\tau) dS(\tau) \tag{3}$$

where S is itself follows a stochastic (Ito) process, where S can be considered to be a stock price, in SDE notation:

$$dS(t) = \mu S(t) dt + \sigma S(t) dX(t)$$

- This gives:

$$Y(t) = Y(0) + \int_0^t \theta(\tau) \mu S(\tau) d\tau + \int_0^t \theta(\tau) \sigma S(\tau) dX(\tau)$$

Ito integrals and portfolios

- So (3) is also a stochastic (Ito) process. If we consider (3) then you will observe that

$$\int_0^t \theta(\tau) dS(\tau) = \lim_{n \to \infty} \sum_{j=1}^{n} \theta(t_{j-1})(S(t_j) - S(t_{j-1}))$$

and so we can write (3) as the limit of sums of the form:

$$Y(t) = Y(0) + \sum_{j=1}^{n} \theta(t_{j-1})(S(t_j) - S(t_{j-1}))$$

QUIZ

What can we interpret θ as

- A risk-neutral probability
- An Arrow-Debreu state price
- A portfolio weighting
- A future cash flow?

Why is this relevant?

- If we consider (3) as the limit of sums then can relate this to holding a portfolio of shares, namely the term $\theta(t_{j-1})(S(t_j) - S(t_{j-1}))$ can be viewed as the gain/loss from holding $\theta(t)$ shares from t_{j-1} to t_j and (3) gives the final wealth.
- This is the advantage of evaluating $f(t)$ at the start of the period as $\theta(t)$ will only depend upon the current share price (i.e. estimated at time t) rather than at the midpoint or end of the interval.
- This is a natural definition from a financial point of view as you can choose the number of shares to hold based on current information.
- Thus the Ito process is a natural way to model stock prices/portfolio value processes.

A few rules of ito processes

- First moment, as $dX(\tau)$ has expected value 0 and $f(\tau)$ is known,

$$E\left[\int_0^t f(\tau)dX(\tau)\right] = 0$$

- Second moment, also known as **Iso Isometry** (we'll use this more than you would imagine!)

$$E\left[\int_0^t f(\tau)dX(\tau)\int_0^t g(\tau)dX(\tau)\right] = \int_0^t E[f(\tau)g(\tau)]d\tau$$

  or essentially $(dX(\tau))^2 \approx d\tau$

- Addition

$$\int_0^t [f(\tau)+g(\tau)]dS(\tau) = \int_0^t f(\tau)dS(\tau) + \int_0^t g(\tau)dS(\tau)$$

Derivatives prices

- A derivative such as an option will have value defined by $F(S, t)$, so we would like to to form an SDE for F of the form:

$$dF(t) = a(S(t), t)dt + b(S(t), t)dX(t)$$

- Ideally, we would also like to solve this equation so that we can determine the value of F (the option price or other derivative price) at different points in time. Perhaps sometimes we will be able to solve it analytically, other times we may have to simulate or use other numerical methods.

Problem of differentiation

- Consider a **Brownian motion** X and a function $F(X) = \sigma^2 X^2$.
- If we use ordinary calculus we would obtain:

$$\frac{dF}{dX} = 2\sigma^2 X \Rightarrow dF = 2\sigma^2 X dX$$

  Can this be the correct answer?
- By writing out the stochastic process as an integral we get the following results

$$\begin{aligned} F(X(t)) &= \int_0^t 2\sigma^2 X(\tau) dX(\tau) \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} 2\sigma^2 X(t_{j-1})(X(t_j) - X(t_{j-1})) \end{aligned}$$

Problem of differentiation

- By considering the sum $(F(X(t))$ from equation (4) this gives us (from our above equations):

$$E[F(X(t))] = 0$$

- However, looking at the known value of $F(X(t), t)(= \sigma^2 X^2)$ we can write

$$E[F(X(t))] = E[\sigma^2 X^2(t)] = E[(\sigma X(t))^2]$$

  but by considering the variance of $\sigma X(t)$ we have

$$E[(\sigma X(t))^2] = Var[\sigma X(t)] - E[\sigma X(t)]^2 = \sigma^2 Var[X(t)] = \sigma^2 t$$

  thus this doesn't match with our solution.

- As probably expected, the rules of deterministic calculus are not going to work for stochastic variables, we will need a new technique. This is where **Ito's lemma** comes in.

Taylor Series

- Let's consider a Taylor series (See Problem Set 2) expansion of $F(X(t))$

$$dF(X(t)) = F'(X(t))dX(t) + \frac{1}{2}F''(X(t))(dX(t))^2 + O((dX(t))^3)$$

  we assume for now that we can ignore terms of $(dX(t))^3$and higher.

- Let's consider the $(dX(t))^2$ term (in deterministic calculus we can ignore it), we can estimate its integral as

$$\int_0^t (dX(t))^2 = \sum_{i=1}^{n} \left(X\left(\frac{ti}{n}\right) - X\left(\frac{t(i-1)}{n}\right)\right)^2$$

  and introduce $Z_{n,i}$ as

$$Z_{n,i} = \frac{X\left(\frac{ti}{n}\right) - X\left(\frac{t(i-1)}{n}\right)}{\sqrt{t/n}}$$

Taylor Series

- For each n the sequence $Z_{n,1}, Z_{n,2}, \ldots$ is a set of iid normal variables $N(0,1)$ as each increment is an independent normal variable $N(0, t/n)$.
- In which case

$$\int_0^t (dX(t))^2 = t \sum_{i=1}^{n} \frac{Z_{n,i}^2}{n}$$

- By the (weak) law of large numbers the *distribution* of the right hand side summation converges towards the the constant expectation of each $Z_{n,i}^2$, or 1 with zero variance (see notes at the end of these slides).
- Thus,

$$\int_0^t (dX(t))^2 = t$$

or $(dX(t))^2 \approx dt$.

Taylor Series

- Thus we can't ignore $(dX(t))^2$ but what about the other terms? It turns out that the other terms are zero as $(dX(t))^3$ is of order $dt^{3/2}$ and so can be ignored. So, our modified Taylor series is:

$$dF(X(t)) = F'(X(t))dX(t) + \frac{1}{2}F''(X(t))dt + 0$$

- **This gives Ito's Lemma, if have a stochastic process for S (not necessarily a stock price) such that:**

$$dS = a(S,t)dt + b(S,t)dX$$

then a function F(S) which is twice differentiable in S is also a stochastic process and is given by,

$$\begin{aligned} dF &= \frac{\partial F}{\partial S}a(S,t)dt + \frac{\partial F}{\partial S}b(S,t)dX + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}b^2(S,t)dt \\ &= \frac{\partial F}{\partial S}dS + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}b^2(S,t)dt \end{aligned}$$

QUIZ

For a general function $F(\cdot)$ use Ito's lemma to find the process followed by F for the following stochastic processes

- $dY = \mu dt + \sigma dX$
- $dS = \mu S dt + \sigma S dX$
- $dX = dX$

Return to our example, $F(X) = \sigma^2 X^2$

- In our example a(X,t) = 0 and b(X,t) = 1 thus from Ito's lemma we have:

$$\begin{aligned} dF &= \frac{\partial F}{\partial X} dX + \frac{1}{2}\frac{\partial^2 F}{\partial X^2} dt \\ &= 2\sigma^2 X dX + \sigma^2 dt \end{aligned}$$

- Notice that we have a second term added on to our solution from ordinary calculus.
- To check this solution apply the same argument as before:

$$F(X(t)) = \int_0^t 2\sigma^2 X(\tau) dX(\tau) + \int_0^t \sigma^2 d\tau$$

thus

$$E[F(X(t),t)] = E[(\sigma X^2(t))^2] = \sigma^2 t$$

as it should.

Remark

- Note, that in Problem Set 2, question 2 we considered the second order S terms in the derivation.
- We can think of the extra term in Ito's lemma as an adjustment because we have a stochastic term dX in our variable. Recall that $X(t)$ is not differentiable in the usual way and so the addition of the second derivative term is a way of overcoming this.
- Note that for the special case where X is a Brownian motion and we let $F(X,t)$ be twice cont. differentiable in X and continuously differentiable in t then:

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}dt$$

Review

- **For a general function $F(S,t)$, perform a Taylor expansion:**

$$dF \approx \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 + \ldots$$

- **Over any time interval the infinitely many terms $(dX)^2$ integrate (or average out) to their expected value (with reducing variance), thus**

$$\int_t^{t+\delta t} (dX(\tau))^2 = \delta\tau$$

- **This can be seen as a general rule of thumb to replace all $(dX)^2$ terms by dt, giving**

$$dF = \frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}dt$$

Does this make sense?

- Note that if our $(dX)^2$ term

$$\sum_{i=1}^{n}\left(X\left(\frac{ti}{n}\right)-X\left(\frac{t(i-1)}{n}\right)\right)^2$$

actually converged to zero then ordinary calculus would apply, what is interesting here is that as $(dX)^2$ is roughly the size of dt, then dX will be large relative to dt, or our random term is large relative to our deterministic term.

- Note that from the data on the S&P 500 we had

	1 day returns	5 day returns
Mean	0.0000830	0.000349
Variance	0.000273	0.001037
Standard deviation	0.01651	0.03221

i.e the standard deviation was large relative to the mean.

Rules of thumb

- Given a general Brownian motion dX_i then we will employ the following rules of thumb

$$
\begin{aligned}
dX_i dt &= o(dt) \\
(dX_i)^2 &= dt \\
dX_i dX_j &= \rho_{ij} dt \\
(\rho_{ii} &= 1)
\end{aligned}
$$

where ρ is the correlation coefficient between the Brownian motions.

Solving SDEs using Ito's lemma

- Consider the following simple stochastic process

$$dY(t) = \mu dt + \sigma dX(t)$$

can we write down its solution?

- In general solving SDEs is similar to solving ODEs, there is not always a straightforward solution, and sometimes we have to make a lucky guess. In this scenario, however, we already know the answer by explicitly solving the Ito integrals

$$Y(t) = Y(0) + \mu t + \sigma X(t)$$

- Now let's make things a little more interesting and try to solve,

$$dY(t) = \sigma Y(t) dX(t)$$

Solving SDEs

- Let's rewrite the equation as
$$\frac{dY(t)}{Y(t)} = \sigma dX(t)$$
and so this looks like taking logs might be helpful. So let's set $Z = \ln(Y)$ and consider dZ, however, now we must use Ito's lemma.
$$\begin{aligned} dZ &= \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial Y}dY + \frac{1}{2}\frac{\partial^2 Z}{\partial Y^2}dY^2 \\ &= \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial Y}(\sigma Y dX) + \frac{1}{2}\frac{\partial^2 Z}{\partial Y^2}\sigma^2 Y^2 dt \end{aligned}$$
but
$$\begin{aligned} \frac{\partial Z}{\partial t} &= 0 \\ \frac{\partial Z}{\partial Y} &= \frac{1}{Y} \\ \frac{\partial^2 Z}{\partial Y^2} &= -\frac{1}{Y^2} \end{aligned}$$

Solving SDEs

- Thus

$$
\begin{aligned}
dZ &= \sigma dX - \frac{1}{2}\sigma^2 dt \\
&= -\frac{1}{2}\sigma^2 dt + \sigma dX
\end{aligned}
$$

- But we know the solution to this SDE from above,

$$Z(t) = Z(0) - \frac{1}{2}\sigma^2 t + \sigma X(t)$$

and so replacing $Z(t)$ by its definition gives us:

$$\ln Y(t) = \ln Y(0) - \frac{1}{2}\sigma^2 t + \sigma X(t)$$

or

$$Y(t) = Y(0)\exp\left(-\frac{1}{2}\sigma^2 t + \sigma X(t)\right)$$

Solving SDEs: stock price

- Now, let's return to our financial applications, consider our model of the stock price:

$$dS(t) = \mu S(t)dt + \sigma S(t)dX(t)$$

and again introduce $Z = \ln(S)$ and as before use Ito's lemma:

$$\begin{aligned} dZ &= \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial S}dS + \frac{1}{2}\frac{\partial^2 Z}{\partial S^2}dS^2 \\ &= \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial S}(\mu S dt + \sigma S dX) + \frac{1}{2}\frac{\partial^2 Z}{\partial S^2}\sigma^2 S^2 dt \end{aligned}$$

but

$$\begin{aligned} \frac{\partial Z}{\partial t} &= 0 \\ \frac{\partial Z}{\partial S} &= \frac{1}{S} \\ \frac{\partial^2 Z}{\partial S^2} &= -\frac{1}{S^2} \end{aligned}$$

Solving SDEs: stock price

- Thus

$$
\begin{aligned}
dZ &= \mu dt + \sigma dX - \frac{1}{2}\sigma^2 dt \\
&= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dX
\end{aligned}
$$

- But we know the solution to this SDE from above,

$$Z(t) = Z(0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma X(t)$$

and so replacing $Z(t)$ by its definition gives us:

$$\ln S(t) = \ln S(0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma X(t)$$

or

$$S(t) = S(0) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma X(t)\right)$$

- Compare the appearance of the $\frac{1}{2}\sigma^2$ term to our heuristic work on Brownian motion in lecture 7.

QUIZ

Consider $dY = \mu dt + \sigma dX(t)$,

- What process is followed by $S = e^Y$

Solving SDEs: CEV processes

- As with ODEs and PDEs, most SDEs cannot b easily solved.
- For example, one popular extension to the GBM model is to use a Constant Elasticity of Variance model (CEV). This is one of the class of **local volatility** models that can reproduce volatility smiles. Here, the stock price follows:

$$dS = \mu S^{\alpha} dt + \sigma S^{\beta} dX$$

  where $\beta \neq 0, 1$.
- To solve this SDE we would like to make the coefficients of the drift and diffusion terms constant (that's the only one we can solve at the moment!).
- For a general function $F(S)$ the volatility (dX) term in dF is $F'(S)\sigma S^{\beta}$ and so for this to have a constant coefficient then

$$F(S) = \frac{S^{1-\beta}}{1-\beta}$$

Solving SDEs: CEV processes

- Upon applying Ito's lemma we get

$$dF = \left(\mu S^{\alpha-\beta} - \frac{\beta}{2}\sigma^2 S^{\beta-1} \right) dt + \sigma dX$$

- The drift term will only be a constant if $\alpha = \beta = 1$ but this is the lognormal case we have already seen.
- So, in general we cannot solve this SDE directly.

Ito's lemma for functions of two stochastic processes

- We have two stochastic processes

$$dS_1 = a_1(S_1, S_2, t)dt + b_1(S_1, S_2, t)dX_1$$
$$dS_2 = a_2(S_1, S_2, t)dt + b_2(S_1, S_2, t)dX_2$$

then for a general function $F(S_1, S_2, t)$, Ito's lemma states

$$\begin{aligned} dF = & \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S_1}dS_1 + \frac{\partial F}{\partial S_2}dS_2 \\ & + \frac{1}{2}\frac{\partial^2 F}{\partial S_1^2}b_1^2(S_1, S_2, t)dt + \frac{1}{2}\frac{\partial^2 F}{\partial S_2^2}b_2^2(S_1, S_2, t)dt \\ & + \frac{\partial^2 F}{\partial S_1 \partial S_2}\rho b_1(S_1, S_2, t)b_2(S_1, S_2, t)dt \end{aligned}$$

Deriving this using rules of thumb

- We again have

$$dS_1 = a_1(S_1, S_2, t)dt + b_1(S_1, S_2, t)dX_1$$
$$dS_2 = a_2(S_1, S_2, t)dt + b_2(S_1, S_2, t)dX_2$$

- Doing a Taylor expansion and using rules of thumb gives

$$\begin{aligned} dF &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S_1}dS_1 + \frac{\partial F}{\partial S_2}dS_2 \\ &\quad + \frac{1}{2}\frac{\partial^2 F}{\partial S_1^2}(dS_1)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial S_2^2}(dS_2)^2 + \frac{\partial^2 F}{\partial S_1 \partial S_2}dS_1 dS_2 \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S_1}dS_1 + \frac{\partial F}{\partial S_2}dS_2 \\ &\quad + \frac{1}{2}\frac{\partial^2 F}{\partial S_1^2}b_1^2(S_1, S_2, t)dt + \frac{1}{2}\frac{\partial^2 F}{\partial S_2^2}b_2^2(S_1, S_2, t)dt \\ &\quad + \frac{\partial^2 F}{\partial S_1 \partial S_2}\rho b_1(S_1, S_2, t)b_2(S_1, S_2, t)dt \end{aligned}$$

Product rule

- Now consider two stochastic processes driven by the same Brownian motion:

$$\begin{aligned} dY &= \mu dt + \sigma dX \\ dZ &= \nu dt + \eta dX \end{aligned}$$

then via Ito's lemma we find that

$$d(YZ) = YdZ + ZdY + \sigma\eta dt$$

again different from deterministic calculus.

- If they are drivcen by correlated Brownian motions,

$$\begin{aligned} dY &= \mu dt + \sigma dX_1 \\ dZ &= \nu dt + \eta dX_2 \end{aligned}$$

such that $E[dX_1 dX_2] = \rho dt$

Product rule

- Then

$$d(YZ) = YdZ + ZdY + \rho\sigma\eta dt$$

- If, however, the Brownian motion terms are independent of each other, $E[dX_1 dX_2] = 0dt$ then we get the standard deterministic product rule:

$$d(YZ) = YdZ + ZdY$$

Overview

- We now have a more rigorous model for the movement of stock prices. By considering a stochastic process as an Ito integral it is possible to write a model of stock price movements as:

$$dS(t) = \mu(S(t), t)dt + \sigma(S(t), t)dX(t)$$

- We also noted that this modeling fits in well with forming portfolios or trading strategies with shares as μ and σ are determined now rather than at the subsequent time period.
- We have also introduced the most important result from stochastic calculus: Ito's lemma, which allows us to write the function of a stochastic process as a stochastic process itself. It can be viewed as the stochastic equivalent of Taylor's series.
- This will be essential for derivative pricing.
- To apply Ito's lemma the rule of thumb is to perform a Taylor expansion and replace $(dX)^2$ by dt and remove terms of $O(dt^{3/2})$.

Aside: Ito's lemma

- There is a rigorous proof for Ito's lemma which is well beyond the scope of the course (and you will not find in any books on your reading list), here we start by looking at two alternative arguments.
- First recall our Taylor series expansion on F (which possible even for stochastic variables) to yield:

$$dF \approx \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 + \dots$$

- Now integrate between t and t + dt to get:

$$\begin{aligned} F(X(t+\delta t), t+\delta t) - F(X(t), t) &= \int_t^{t+\delta t} \frac{\partial F}{\partial X} dX(\tau) + \int_t^{t+\delta t} \frac{\partial F}{\partial t} d\tau \\ &+ \frac{1}{2}\int_t^{t+\delta t} \frac{\partial^2 F}{\partial X^2}(dX(\tau))^2 + \dots \end{aligned}$$

- where the derivatives are known values, calculated at time t and we have the seen the third integral earlier

Aside: Ito's lemma

- We can simplify the first two integrals:

$$\int_t^{t+\delta t} \frac{\partial F}{\partial X} dX(\tau) = \frac{\partial F}{\partial X}(X(t+\delta t) - X(t))$$

and

$$\int_t^{t+\delta t} \frac{\partial F}{\partial t} d\tau = \frac{\partial F}{\partial t}\delta t$$

- And, from earlier, the third integral converges (via the weak law of large numbers) to

$$\frac{1}{2}\frac{\partial^2 F}{\partial X^2}\int_t^{t+\delta t} (dX(\tau))^2 = \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\delta t$$

Aside: Ito's lemma

- Note that the heuristic '$(dX)^2 = dt$'also explains why we don't need to consider anymore terms as $dXdt = O(dt^{3/2})$ and so can be ignored for small changes in time dt.
- Thus we can write:

$$
\begin{aligned}
F(X(t+\delta t), t+\delta t) - F(X(t), t) &= \int_t^{t+\delta t} \frac{\partial F}{\partial X} dX(\tau) + \int_t^{t+\delta t} \frac{\partial F}{\partial t} d\tau + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\delta t \\
&= \frac{\partial F}{\partial X}(X(t+\delta t) - X(t)) + \frac{\partial F}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\delta t
\end{aligned}
$$

or in more familiar notation, Ito's lemma for Brownian motion:

$$
dF = \frac{\partial F}{\partial X} dX + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial X^2} dt
$$

Aside: Ito's lemma

- The key step is to justify that '$(dX)^2 = dt$ '. To consider how we may alternatively argue this again consider the same fixed interval of time only now consider the variance of $(dX)^2$ recalling that

$$\sum_{j=1}^{n} E[(X(t_j) - X(t_{j-1}))^2] = \delta t$$

and so

$$\sum_{j=1}^{n} Var[(X(t_j) - X(t_{j-1}))^2] = \sum_{j=1}^{n} 2\left[\frac{\delta t^2}{n^2}\right] = \frac{2\delta t^2}{n}$$

so as n becomes large the variance shrinks and so this sum tends to a deterministic limit of dt.

- Note that this is slightly more rigorous than the previous argument, but still not quite the full story.