FIN 514: Problem Set #6

Due on Wednesday, May 2, 2018

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Problem 1

(a) By Ito's product rule, $dY(t)$ satisfies the following equation.

$$\begin{aligned} dY(t) &= B_P(t)dS(t) + S(t)dB_P(t) + dB_P(t)dS(t) \\ &= B_P(t)[\mu S(t)dt + \sigma S(t)dX(t)] + r_P S(t)B_P(t)dt \\ &= (\mu + r_P)Y(t)dt + \sigma Y(t)dX(t) \end{aligned}$$

In order to find martingale measure with respect to $B(t)$ as a numeraire, dynamics of $Y(t)/B(t)$ is derived as follows.

$$\begin{aligned} d\left(\frac{Y(t)}{B(t)}\right) &= Y(t)d\left(\frac{1}{B(t)}\right) + \frac{1}{B(t)}dY(t) + dY(t)d\left(\frac{1}{B(t)}\right) \\ d\left(\frac{1}{B(t)}\right) &= -\frac{1}{B^2(t)}dB(t) \\ &= -\frac{1}{B^2(t)}rB(t)dt = -r\frac{1}{B(t)}dt \\ \Rightarrow d\left(\frac{Y(t)}{B(t)}\right) &= Y(t)\left(-r\frac{1}{B(t)}dt\right) + \frac{1}{B(t)}[(\mu + r_P)Y(t)dt + \sigma Y(t)dX(t)] \\ &= (\mu + r_P - r)\frac{Y(t)}{B(t)}dt + \sigma\frac{Y(t)}{B(t)}dX(t) \end{aligned}$$

By Girsanov's theorem, there exists a probability measure such that $\widetilde{X}(t) = X(t) + \int_0^t \frac{\mu + r_P - r}{\sigma}ds$ is a brownian motion under the measure. Therefore, by plugging $dX(t) = d\widetilde{X}(t) - \frac{\mu+r_P-r}{\sigma}dt$ into the equation above, then $d\left(\frac{Y(t)}{B(t)}\right)$ becomes $\sigma\frac{Y(t)}{B(t)}d\widetilde{X}(t)$, hence becomes martingale because there is no drift. Therefore, from the perspective of U.S dollar investor, under risk-neutral measure, $dX(t) = d\widetilde{X}(t) - \frac{\mu+r_P-r}{\sigma}dt$. By plugging it into dynamics of $Y(t)$, we can find dynamics of the U.S price of a GBP bond under risk-neutral measure as follows.

$$\begin{aligned} dY(t) &= (\mu + r_P)Y(t)dt + \sigma Y(t)dX(t) \\ &= (\mu + r_P)Y(t)dt + \sigma Y(t)\left[d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt\right] \\ &= rY(t)dt + \sigma Y(t)d\widetilde{X}(t) \end{aligned}$$

And it is consistent with the fact that expected return of every tradable asset is risk-free rate under risk-neutral measure.

(b) By plugging $dX(t) = d\widetilde{X}(t) - \frac{\mu+r_P-r}{\sigma}dt$ into dynamics of $S(t)$, we can find dynamics of U.S. dollar price of a British pound under risk-neutral probability as follows.

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dX(t) \\ &= \mu S(t)dt + \sigma S(t)\left[d\widetilde{X}(t) - \frac{\mu + r_P - r}{\sigma}dt\right] \\ &= (r - r_P)S(t)dt + \sigma S(t)d\widetilde{X}(t) \end{aligned}$$

(c) Unlike the assumption of ordinary Black-Scholes-Merton formula, since expected return of underlying asset has changed from r to $r - r_P$, formula for call option should be changed to following equation.

$$\begin{aligned}
&e^{-rT}[S_0 e^{(r-r_P)T} N(d_1) - KN(d_2)] \\
&d_1 = \frac{\log(S_0/K) + (r - r_P + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
&d_2 = d_1 - \sigma\sqrt{T}
\end{aligned}$$

Problem 2

(a) By Ito's product rule, dynamics of $B(t)/S(t)$ is as follows.

$$\begin{aligned}
d\left(\frac{B(t)}{S(t)}\right) &= d\left(\frac{1}{S(t)}\right)B(t) + \frac{1}{S(t)}dB(t) + d\left(\frac{1}{S(t)}\right)dB(t) \\
d\left(\frac{1}{S(t)}\right) &= -\frac{1}{S^2(t)}dS(t) + \frac{1}{2} \times 2 \times \frac{1}{S^3(t)}(dS(t))^2 \\
&= -\frac{1}{S^2(t)}[(\mu - d)S(t)dt + \sigma S(t)dX(t)] + \frac{1}{S^3(t)}\sigma^2 S^2(t)dt \\
&= [-(\mu - d) + \sigma^2]\frac{1}{S^2(t)}dt - \sigma\frac{1}{S(t)}dX(t) \\
\Rightarrow d\left(\frac{B(t)}{S(t)}\right) &= \frac{B(t)}{S(t)}[-(\mu - d) + \sigma^2]dt - \sigma\frac{B(t)}{S(t)}dX(t) + r\frac{B(t)}{S(t)}dt \\
&= [r - (\mu - d) + \sigma^2]\frac{B(t)}{S(t)}dt - \sigma\frac{B(t)}{S(t)}dX(t)
\end{aligned}$$

(b) By Girsanov's theorem, there exists a probability measure such that $\widetilde{X}(t) = X(t) - \int_0^t \frac{r-(\mu-d)+\sigma^2}{\sigma}ds$ is a brownian motion under the measure. Under the measure, process $d\left(\frac{B(t)}{S(t)}\right)$ changes as follows.

$$\begin{aligned}
d\left(\frac{B(t)}{S(t)}\right) &= [r - (\mu - d) + \sigma^2]\frac{B(t)}{S(t)}dt - \sigma\frac{B(t)}{S(t)}dX(t) \\
&= [r - (\mu - d) + \sigma^2]\frac{B(t)}{S(t)}dt - \sigma\frac{B(t)}{S(t)}\left[d\widetilde{X}(t) + \frac{r - (\mu - d) + \sigma^2}{\sigma}dt\right] \\
&= -\sigma\frac{B(t)}{S(t)}d\widetilde{X}(t)
\end{aligned}$$

Therefore, under the measure, $\frac{B(t)}{S(t)}$ is a martingale since there is no drift term in dynamics. Plugging $dX(t) = d\widetilde{X}(t) + \frac{r-(\mu-d)+\sigma^2}{\sigma}dt$ into the process of $S(t)$, dynamics of $S(t)$ under martingale measure with respect to $S(t)$ as a numeraire is as follows.

$$\begin{aligned}
dS(t) &= (\mu - d)S(t)dt + \sigma S(t)dX(t) \\
&= (\mu - d)S(t)dt + \sigma S(t)\left[d\widetilde{X}(t) + \frac{r - (\mu - d) + \sigma^2}{\sigma}dt\right] \\
&= (r + \sigma^2)S(t)dt + \sigma S(t)d\widetilde{X}(t)
\end{aligned}$$

(c) Let $\eta(t) = \mathrm{E}_t^Q[V(T)/S(T)]$. Since conditional expectation is always a martingale under corresponding probability measure, $\eta(t)$ is a **Q** - martingale. Then by martingale representation theorem, there exists a unique process $\phi(t)$ such that $d\eta(t) = \phi(t)d(B(t)/S(t))$. Let us construct a portfolio $\Pi(t)$ such that $\Pi(t) = \psi(t)S(t) + \phi(t)B(t)$, where $\psi(t) = \eta(t) - \phi(t)B(t)/S(t)$. Then $\Pi(t) = \eta(t)S(t)$ for all t. If $\Pi(t)$ is self-financing, then since there is no intermediate cash flow, and $\Pi(T) = \eta(T)S(T) = V(T)$, by no arbitrage principle, value of the option at t must be equal to $\Pi(t)$. Therefore, it needs to figure out whether $\Pi(t)$ is self-financing or not. In order to check it, dynamics of $\Pi(t)$ is derived as follows.

$$\begin{aligned}
d\Pi(t) &= S(t)d\eta(t) + \eta(t)dS(t) + dS(t)d\eta(t) \\
&= \phi(t)S(t)d\left(\frac{B(t)}{S(t)}\right) + \left(\psi(t) + \phi(t)\frac{B(t)}{S(t)}\right)dS(t) + \phi(t)dS(t)d\left(\frac{B(t)}{S(t)}\right) \\
&= \phi(t)\left(S(t)d\left(\frac{B(t)}{S(t)}\right) + \frac{B(t)}{S(t)}dS(t) + dS(t)d\left(\frac{B(t)}{S(t)}\right)\right) + \psi(t)dS(t) \\
&= \phi(t)d\left(S(t)\frac{B(t)}{S(t)}\right) + \psi(t)dS(t) \\
&= \psi(t)dS(t) + \phi(t)dB(t)
\end{aligned}$$

From the equation above, we can find out that $\Pi(t)$ is a self-financing strategy. Therefore, option value $V(t)$ must equal to $\Pi(t) = \eta(t)S(t) = S(t)\mathrm{E}_t^Q[V(T)/S(T)]$, which is represented as follows.

$$\begin{aligned}
V(t) &= S(t)\mathrm{E}_t^Q\left[\frac{V(T)}{S(T)}\right] \\
&= S(t)\mathrm{E}_t^Q\left[\frac{\max(S(T)-K,0)}{S(T)}\right] \\
&= S(t)\mathrm{E}_t^Q\left[\max\left(1-\frac{K}{S(T)},0\right)\right]
\end{aligned}$$

Where K is strike price of the option, and Q is a probability measure in which $B(t)/S(t)$ is a martingale.

(d) In order to evaluate call option value, we need to derive the solution of SDE $dS(t) = (r+\sigma^2)S(t)dt + \sigma S(t)d\widetilde{X}(t)$ first. The solution is derived as follows.

$$\begin{aligned}
d\log S(t) &= \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S^2(t)}(dS(t))^2 \\
&= \left(r + \frac{1}{2}\sigma^2\right)dt + \sigma d\widetilde{X}(t) \\
\Rightarrow S(T) &= S(t)\exp\left[\left(r+\frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\phi\right] \\
\phi &\sim N(0,1)
\end{aligned}$$

Plugging the result of equation above, we can evaluate call option value at time t as follows.

$$\begin{aligned}
V(t) &= S(t)\mathrm{E}_t^Q\left[\max\left(1-\frac{K}{S(T)},0\right)\right] \\
&= S(t)\mathrm{E}_t^Q\left[\max\left(1-\frac{K}{S(t)\exp[(r+\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\phi]},0\right)\right]
\end{aligned}$$

Since $1 - \frac{K}{S(t)\exp[(r+\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\phi]} \geq 0$ is equivalent to $\phi \geq \frac{\log(K/S(t)-(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \equiv L$, and ϕ

follows standard normal distribution, call option value can be calculated as follows.

$$\begin{aligned}
V(t) &= S(t)\int_L^\infty \left(1-\frac{K}{S(t)\exp[(r+\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}x]}\right)\frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}x^2}dx \\
&= S(t)\int_L^\infty \frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}x^2}dx - K\int_L^\infty \frac{1}{\sqrt{2\pi}}e^{-(r+\frac{1}{2}\sigma^2)(T-t)-\sigma\sqrt{T-t}x-\frac{1}{2}x^2}dx \\
&= S(t)(1-N(L)) - Ke^{-r(T-t)}\int_L^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x+\sigma\sqrt{T-t})^2}dx \\
&= S(t)(1-N(L)) - Ke^{-r(T-t)}\int_{L+\sigma\sqrt{T-t}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy \\
&= S(t)(1-N(L)) - Ke^{-r(T-t)}(1-N(L+\sigma\sqrt{T-t})) \\
&= S(t)N(-L) - Ke^{-r(T-t)}N(-L-\sigma\sqrt{T-t}) \\
&= S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)
\end{aligned}$$

$$\begin{aligned}
y &= x+\sigma\sqrt{T-t} \\
d_1 &= \frac{\log(S_t/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1-\sigma\sqrt{T-t}
\end{aligned}$$

Which is the desired solution. It is not easier than when $B(t)$ was numeraire because it also has some integration and transformation in evaluating procedure.

Problem 3

(a) Using Ito's lemma, dynamics of $\log B_e$ is as follows.

$$\begin{aligned}
d\log B_e &= \frac{1}{B_e}dB_e \\
&= \frac{1}{B_e}r_eB_edt \\
&= r_edt
\end{aligned}$$

Since $B_e(0)=0$, $B_e(T)$ is evaluated as $B_e(0)e^{r_eT}=e^{r_eT}$.

(b) By Ito's product rule, dynamics of Z_e and Z_g is derived as follows.

$$\begin{aligned}
dZ_e &= d(eB_e) = B_ede + edB_e + dB_ede \\
&= B_e(\mu_eedt+\sigma_eedX_2) + er_eB_edt \\
&= (\mu_e+r_e)Z_edt + \sigma_eZ_edX_2 \\
dZ_g &= d(gB_g) = B_gdg + gdB_g + dB_gdg \\
&= B_g(\mu_ggdt+\sigma_ggdX_1) + gr_gB_gdt \\
&= (\mu_g+r_g)Z_gdt + \sigma_gZ_gdX_1
\end{aligned}$$

(c) By Ito's product rule, dynamics of X is derived as follows.

$$
\begin{aligned}
dX &= d\left(\frac{g}{e}\right) = \frac{1}{e}dg + gd\left(\frac{1}{e}\right) + dgd\left(\frac{1}{e}\right) \\
d\left(\frac{1}{e}\right) &= -\frac{1}{e^2}de + 2 \times \frac{1}{2} \times \frac{1}{e^3}(de)^2 \\
&= -\frac{1}{e^2}(\mu_e edt + \sigma_e edX_2) + \frac{1}{e^3}\sigma_e^2 e^2 dt \\
&= (-\mu_e + \sigma_e^2)\frac{1}{e}dt - \sigma_e \frac{1}{e}dX_2 \\
\Rightarrow dX &= \frac{1}{e}(\mu_g gdt + \sigma_g gdX_1) + g[(-\mu_e + \sigma_e^2)\frac{1}{e}dt - \sigma_e \frac{1}{e}dX_2] + (\mu_g gdt + \sigma_g gdX_1)[(-\mu_e + \sigma_e^2)\frac{1}{e}dt - \sigma_e \frac{1}{e}dX_2] \\
&= (\mu_g - \mu_e + \sigma_e^2 - \rho\sigma_e\sigma_g)Xdt + X(\sigma_g dX_1 - \sigma_e dX_2)
\end{aligned}
$$

Let $dX_3 = \frac{\sigma_g dX_1 - \sigma_e dX_2}{\sqrt{\sigma_g^2 - 2\sigma_g\sigma_e + \sigma_e^2}}$. Then by Levy's theorem, X_3 is a brownian motion. Let $\sigma_X^2 = \sigma_g^2 - 2\sigma_g\sigma_e + \sigma_e^2$, then dX can be represented as follows.

$$
dX = (\mu_g - \mu_e + \sigma_e^2 - \rho\sigma_e\sigma_g)Xdt + \sigma_X X dX_3
$$

(d) By Ito's product rule, dynamics of Y is derived as follows.

$$
\begin{aligned}
dY &= d\left(\frac{Z_g}{Z_e}\right) = Z_g d\left(\frac{1}{Z_e}\right) + \frac{1}{Z_e}dZ_g + dZ_g d\left(\frac{1}{Z_e}\right) \\
d\left(\frac{1}{Z_e}\right) &= -\frac{1}{Z_e^2}dZ_e + \frac{1}{Z_e^3}(dZ_e)^2 \\
&= -\frac{1}{Z_e}[(\mu_e + r_e)Z_e dt + \sigma_e Z_e dX_2] + \frac{1}{Z_e^3}\sigma_e^2 Z_e^2 dt \\
&= [-(\mu_e + r_e) + \sigma_e^2]\frac{1}{Z_e}dt - \sigma_e \frac{1}{Z_e}dX_2 \\
\Rightarrow dY &= \frac{Z_g}{Z_e}([-(\mu_e + r_e) + \sigma_e^2]dt - \sigma_e dX_2) \\
&\quad + \frac{Z_g}{Z_e}[(\mu_g + r_g)dt + \sigma_g dX_1] \\
&\quad + \frac{Z_g}{Z_e}([-(\mu_e + r_e) + \sigma_e^2]dt - \sigma_e dX_2)[(\mu_g + r_g)dt + \sigma_g dX_1] \\
&= [(\mu_g + r_g) - (\mu_e + r_e) + \sigma_e^2 - \rho\sigma_e\sigma_g]Ydt + \sigma_X Y dX_3
\end{aligned}
$$

(e) By fundamental theorem of finance, if market is complete, there exists a martingale measure such that Y is a martingale under the measure. Therefore, under $\mathbf{Q}$, there is no drift term in dY so that Y is a martingale.

(f) By Girsanov's theorem, if we choose $\widetilde{X}_3(t)$ such that $\widetilde{X}_3(t) = X_3(t) + \int_0^t \frac{(\mu_g + r_g) - (\mu_e + r_e) + \sigma_e^2 - \rho\sigma_e\sigma_g}{\sigma_X}ds$, then there exists a probability measure $\mathbf{Q}$ so that $\widetilde{X}_3$ is a brownian motion under $\mathbf{Q}$. Plugging $dX_3 = d\widetilde{X}_3 - \frac{(\mu_g + r_g) - (\mu_e + r_e) + \sigma_e^2 - \rho\sigma_e\sigma_g}{\sigma_X}dt$ into dY, it becomes to $dY = \sigma_X Y d\widetilde{X}_3$, which is a martingale.

(g) Under $\mathbf{Q}$, $\widetilde{X}_3(t) = X_3(t) + \int_0^t \frac{(\mu_g + r_g) - (\mu_e + r_e) + \sigma_e^2 - \rho\sigma_e\sigma_g}{\sigma_X}ds$ is a brownian motion. Plugging $dX_3 = d\widetilde{X}_3 -$

$\frac{(\mu_g+r_g)-(\mu_e+r_e)+\sigma_e^2-\rho\sigma_e\sigma_g}{\sigma_X}dt$ into dX, dynamics of X is derived as follows.

$$\begin{aligned}
dX &= (\mu_g - \mu_e + \sigma_e^2 - \rho\sigma_e\sigma_g)Xdt + \sigma_X X dX_3 \\
&= (\mu_g - \mu_e + \sigma_e^2 - \rho\sigma_e\sigma_g)Xdt + \sigma_X X(d\widetilde{X}_3 - \frac{(\mu_g + r_g) - (\mu_e + r_e) + \sigma_e^2 - \rho\sigma_e\sigma_g}{\sigma_X}dt) \\
&= (r_e - r_g)Xdt + \sigma_X X d\widetilde{X}_3
\end{aligned}$$

(h) Let $\eta(t) = \mathrm{E}_t^Q[V(T)/Z_e(T)]$. Then $\eta(t)$ is a $\mathbf{Q}$ - martingale. Since $Y = Z_g/Z_e$ is also a martingale under $\mathbf{Q}$, by martingale representation theorem, there exists a unique process $\phi(t)$ such that $d\eta(t) = \phi(t)dY(t)$. Then, let us construct a portfolio $\Pi(t)$ such that $\Pi(t) = \phi(t)Z_g(t) + \psi(t)Z_e(t)$, where $\psi(t) = \eta(t) - \phi(t)Z_g(t)/Z_e(t)$ so that $\Pi(t) = \eta(t)Z_e(t)$ for all t. Since $\Pi(T) = \eta(T)Z_e(T) = V(T)$, if the strategy is self-financing, by no arbitrage principle, $V(t)$ must be equal to $\eta(t)Z_e(t) = Z_e(t)\mathrm{E}_t^Q[V(T)/Z_e(T)]$. Therefore, let us figure out whether the strategy is self-fiancing or not by deriving dynamics of $\Pi(t)$ as follows.

$$\begin{aligned}
d\Pi(t) &= d(\eta(t)Z_e(t)) \\
&= \eta(t)dZ_e(t) + Z_e(t)d\eta(t) + dZ_e(t)d\eta(t) \\
&= \left(\psi(t) + \phi(t)\frac{Z_g(t)}{Z_e(t)}\right)dZ_e(t) + Z_e(t)\left(\phi(t)d\left(\frac{Z_g(t)}{Z_e(t)}\right)\right) + \phi(t)d\left(\frac{Z_g(t)}{Z_e(t)}\right)dZ_e(t) \\
&= \phi(t)\left[\frac{Z_g(t)}{Z_e(t)}dZ_e(t) + Z_e(t)d\left(\frac{Z_g(t)}{Z_e(t)}\right) + d\left(\frac{Z_g(t)}{Z_e(t)}\right)dZ_e(t)\right] + \psi(t)dZ_e(t) \\
&= \phi(t)d\left(Z_e(t)\frac{Z_g(t)}{Z_e(t)}\right) + \psi(t)dZ_e(t) \\
&= \phi(t)dZ_g(t) + \psi(t)dZ_e(t)
\end{aligned}$$

From the equation above, it can be found that $\Pi(t)$ is a self-financing portfolio. Therefore, as mentioned above, value of the option $V(t)$ must be equal to $\Pi(t)$, which is $Z_e(t)\mathrm{E}_t^Q[V(T)/Z_e(T)]$. Since $Z_e = eB_e$, and $V(T) = 1000 \times \max[g_T - e_T, 0]$, $V(t)$ can also be represented as follows.

$$\begin{aligned}
V(t) &= Z_e(t)\mathrm{E}_t^Q\left[\frac{V(T)}{Z_e(T)}\right] \\
&= e_t B_e(t) \times 1000 \times \max\left[\frac{g_T - e_T}{e_T B_e(T)}, 0\right] \\
&= e_t e^{-r_e(T-t)} \times 1000 \times \max[X(T) - 1, 0]
\end{aligned}$$

(i) By analogy to the Black-Scholes-Merton formula, $V(g, e, 0)$ is represented as follows.

$$\begin{aligned}
V(0) &= e_0 e^{-r_e T} \times 1000 \times [X_0 e^{(r_e - r_g)T} N(d_1) - N(d_2)] \\
&= e_0 e^{-r_e T} \times 1000 \times \left[\left(\frac{g_0}{e_0} \right) e^{(r_e - r_g)T} N(d_1) - N(d_2) \right] \\
&= 1000 e^{-r_e T} [g_0 e^{(r_e - r_g)T} N(d_1) - e_0 N(d_2)] \\
d_1 &= \frac{\log X_0 + (r_e - r_g + \frac{1}{2}\sigma_X^2)T}{\sigma_X \sqrt{T}} \\
&= \frac{\log(g_0/e_0) + (r_e - r_g + \frac{1}{2}\sigma_X^2)T}{\sigma_X \sqrt{T}} \\
d_2 &= d_1 - \sigma_X \sqrt{T} \\
\sigma_X^2 &= \sigma_g^2 - 2\sigma_g \sigma_e + \sigma_e^2
\end{aligned}$$