

# **FIN 514: Problem Set #4**

Due on Wednesday, March 7, 2018

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## Problem 1

a. By Ito's lemma,

$$\begin{aligned}
 df(X) &= \frac{\partial f}{\partial X} dX + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} (dX)^2 \\
 &= 2X(t)dX(t) + \frac{1}{2} \times 2dt \\
 \Rightarrow X(t)dX(t) &= \frac{1}{2} df(X) - \frac{1}{2} dt \\
 \Rightarrow \int_0^t X(\tau) d\tau &= \frac{1}{2} \int_0^t dX^2(\tau) + \int_0^t \frac{1}{2} d\tau \\
 &= \frac{1}{2} X^2(t) + \frac{1}{2} t
 \end{aligned}$$

b. Let  $g(t, X) = tX(t)$ . Then by Ito's lemma,

$$\begin{aligned}
 dg(t, X) &= \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial X} dX + \frac{1}{2} \frac{\partial^2 g}{\partial X^2} (dX)^2 \\
 &= X(t)dt + t dX(t) \\
 \Rightarrow t dX(t) &= d(tX(t)) - X(t)dt \\
 \Rightarrow \int_0^t \tau dX(\tau) &= tX(t) - \int_0^t X(\tau) d\tau
 \end{aligned}$$

c. Let  $h(X) = X^3(t)$ . Then by Ito's lemma,

$$\begin{aligned}
 dh(X) &= \frac{\partial h}{\partial X} dX + \frac{1}{2} \frac{\partial^2 h}{\partial X^2} (dX)^2 \\
 &= 3X^2(t)dX(t) + \frac{1}{2} 6X(t)dt \\
 &= 3X^2(t)dX(t) + 3X(t)dt \\
 \Rightarrow X^2(t)dX(t) &= \frac{1}{3} d(X^3(t)) - X(t)dt \\
 \Rightarrow \int_0^t X^2(\tau) dX(\tau) &= \frac{1}{3} X^3(t) - \int_0^t X(\tau) d\tau
 \end{aligned}$$

## Problem 2

By Ito's lemma,

$$\begin{aligned}
 dF &= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial S} dS + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} (dS)^2 \\
 &= -(r - \delta)Fdt + \frac{F}{S} dS(t)
 \end{aligned}$$

Since  $dS(t) = (\mu - \delta)S(t)dt + \sigma S(t)dX(t)$ ,

$$\begin{aligned}
 dF &= -(r - \delta)Fdt + \frac{F}{S} ((\mu - \delta)S(t)dt + \sigma S(t)dX(t)) \\
 &= -(r - \delta)Fdt + (\mu - \delta)Fdt + \sigma FdX(t) \\
 &= (\mu - r)Fdt + \sigma FdX(t)
 \end{aligned}$$

### Problem 3

a. By Ito's product rule,

$$\begin{aligned} dY &= d\left(\frac{S_2}{S_1}\right) \\ &= \frac{1}{S_1}dS_2 + S_2d\left(\frac{1}{S_1}\right) + dS_2d\left(\frac{1}{S_1}\right) \end{aligned}$$

By Ito's lemma,

$$\begin{aligned} d\left(\frac{1}{S_1}\right) &= -\frac{1}{S_1^2}dS_1 + \frac{1}{2} \times 2(dS_1)^2 \\ &= -\frac{1}{S_1^2}(\mu_1 S_1 dt + \sigma_1 S_1 dX_1) + \frac{1}{S_1^3} \sigma_1^2 S_1^2 dt \\ &= -\frac{\mu_1}{S_1} dt - \frac{\sigma_1}{S_1} dX_1 + \frac{\sigma_1^2}{S_1} dt \\ &= \frac{-\mu_1 + \sigma_1^2}{S_1} dt - \frac{\sigma_1}{S_1} dX_1 \end{aligned}$$

Therefore,

$$\begin{aligned} dY &= \frac{1}{S_1}(\mu_2 S_2 dt + \sigma_2 S_2 dX_2) + S_2 \left( \frac{-\mu_1 + \sigma_1^2}{S_1} dt - \frac{\sigma_1}{S_1} dX_1 \right) + (\mu_2 S_2 dt + \sigma_2 S_2 dX_2) \left( \frac{-\mu_1 + \sigma_1^2}{S_1} dt - \frac{\sigma_1}{S_1} dX_1 \right) \\ &= \mu_2 Y dt + \sigma_2 Y dX_2 + (-\mu_1 + \sigma_1^2) Y dt - \sigma_1 Y dX_1 - \sigma_1 \sigma_2 \rho Y dt \quad (\because dX_1 dX_2 = \rho dt) \end{aligned}$$

Let  $\sigma_3 X_3 = \sigma_1 X_1 - \sigma_2 X_2$ , where  $\sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ . Since  $X_1$  and  $X_2$  have expected value zero,  $E[X_3]$  also equal to zero. Furthermore,  $\text{Var}[X_3] = \text{Var}[\sigma_1 X_1 - \sigma_2 X_2]/\sigma_3^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)t/\sigma_3^2 = t$ . Since  $X_1$  and  $X_2$  are normally distributed and have independent increments,  $X_3$  is also normally distributed and has independent increments, so  $X_3$  is a Brownian motion. Therefore,  $\sigma_3 X_3 = \sigma_1 X_1 - \sigma_2 X_2$ . Finally  $dY$  can be represented as follows.

$$\begin{aligned} dY &= (\mu_2 - \mu_1 + \sigma_1^2 - \sigma_1 \sigma_2 \rho) Y dt + \sigma_3 Y dX_3 \\ &= \mu_Y Y dt + \sigma_Y Y dt dX_{3t} \\ \text{where } \mu_Y &= \mu_2 - \mu_1 + \sigma_1^2 - \sigma_1 \sigma_2 \rho, \sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2} \end{aligned}$$

b. By Ito's product rule,

$$\begin{aligned} dZ_t &= d(Y_t B_{Et}) \\ &= B_{Et} dY_t + Y_t dB_{Et} + dY_t dB_{Et} \\ &= B_{Et}(\mu_Y Y_t dt + \sigma_Y Y_t dX_{3t}) + Y_t(r_E B_{Et} dt), \quad dB_{Et} = r_E B_{Et} dt \quad \text{because the bond is riskless.} \\ &= (\mu_Y + r_E) Z_t dt + \sigma_Y Z_t dX_{3t} \\ &= (\mu_Z) Z_t dt + \sigma_Y Z_t dX_{3t} \\ \text{where } \mu_Z &= \mu_Y + r_E. \end{aligned}$$

## Problem 4

a. By Ito's product rule,

$$\begin{aligned}
 dY &= d(S_1 S_2) \\
 &= S_1 dS_2 + S_2 dS_1 + dS_1 dS_2 \\
 &= S_1(\mu_2 S_2 dt + \sigma_1 S_2 dX_2) + S_2(\mu_1 S_1 dt + \sigma_1 S_1 dX_1) + (\mu_1 S_1 dt + \sigma_1 S_1 dX_1)(\mu_2 S_2 dt + \sigma_2 S_2 dX_2) \\
 &= (\mu_1 + \mu_2 + \sigma_1 \sigma_2 \rho) Y dt + Y(\sigma_1 dX_1 + \sigma_2 dX_2)
 \end{aligned}$$

Let  $X_{3t} = \frac{\sigma_1 X_1 + \sigma_2 X_2}{\sigma_3}$ , where  $\sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1 \sigma_2 \rho}$ , then since  $E[X_{1t}] = E[X_{2t}] = 0$ ,  $E[X_{3t}] = 0$ , and  $\text{Var}[X_{3t}] = \frac{(\sigma_1^2 + \sigma_2^2 + 2\sigma_1 \sigma_2 \rho)t}{\sigma_3^2} = t$ . Since  $X_{1t}$  and  $X_{2t}$  are normally distributed and have independent increments,  $X_3$  is also normally distributed and has independent increments. Therefore,  $X_{3t}$  is a Brownian motion. Finally,  $dY$  can be represented as follows.

$$\begin{aligned}
 dY &= (\mu_1 + \mu_2 + \sigma_1 \sigma_2 \rho) Y dt + \sigma_3 Y dX_3 \\
 &= \mu_Y Y dt + \sigma_Y Y dX_3 \\
 \text{where } \mu_Y &= \mu_1 + \mu_2 + \sigma_1 \sigma_2 \rho, \sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1 \sigma_2 \rho}
 \end{aligned}$$

b. Since  $Y = S_1 S_2$ , the payoff of call option is represented as  $\max(Y^{\frac{1}{2}} - K, 0)$ .

c. By Ito's lemma,

$$\begin{aligned}
 dZ &= \frac{\partial Z}{\partial Y} dY + \frac{1}{2} \frac{\partial^2 Z}{\partial Y^2} (dY)^2 \\
 &= \frac{1}{2} Y^{-\frac{1}{2}} (\mu_Y Y dt + \sigma_Y Y dX_3) + \frac{1}{2} \left(-\frac{1}{4}\right) Y^{-\frac{3}{2}} (\sigma_Y^2 Y^2 dt) \\
 &= \frac{1}{2} [\mu_Y Y^{\frac{1}{2}} dt + \sigma_Y Y^{\frac{1}{2}} dX_3 - \frac{1}{4} Y^{\frac{1}{2}} \sigma_Y^2 dt] \\
 &= \frac{1}{2} \left[ Z \left( \mu_Y - \frac{1}{4} \sigma_Y^2 \right) dt + \sigma_Y Z dX_3 \right]
 \end{aligned}$$

d. By Ito's lemma,

$$\begin{aligned}
 d \log Z &= \frac{1}{2} [(\mu_Y - \frac{1}{2} \sigma_Y^2) dt + \sigma_Y dX_3] \\
 \Rightarrow \log Z_T &= \log Z_0 + \frac{1}{2} [(\mu_Y - \frac{1}{2} \sigma_Y^2) T + \sigma_Y X_{3T}] \\
 \Rightarrow \frac{Z_T}{Z_0} &= \exp \left( \frac{1}{2} (\mu_Y - \frac{1}{2} \sigma_Y^2) T + \sigma_Y X_{3T} \right) \\
 \Rightarrow E \left[ \frac{Z_T}{Z_0} \right] &= \frac{1}{2} (\mu_Y - \frac{1}{2} \sigma_Y^2) T + \frac{1}{4} \sigma_Y^2 T \\
 &= \frac{1}{2} \mu_Y T
 \end{aligned}$$

Therefore, expected return of the geometric average of two stocks is  $\frac{1}{2} \mu_Y T$ .

**Problem 5**

a. By Ito's lemma,

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2 \\ &= \left( \frac{\partial V}{\partial t} + (\mu - \delta) S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dX \end{aligned}$$

b. If the value  $\mu$  equals to  $r$ , drift term of  $dV$  becomes equal to left hand side of the pde.

c. Since  $\frac{dV}{V}$  can be interpreted to rate of return of option,  $E[\frac{dV}{V}]$  can be interpreted as expected return of the option. If  $\mu = r$ , then  $E[\frac{dV}{V}] = (\frac{\partial V}{\partial t} + (r - \delta) S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}) / V$ , which equals  $r$  regarding the pde. Therefore, setting  $\mu$  equals to  $r$  includes an assumption that expected rate of return of option is risk-free rate.

d. If  $\mu = r$ , by Ito's lemma,

$$\begin{aligned} dF &= -(r - \delta) F dt + \frac{F}{S} ((r - \delta) S(t) dt + \sigma S(t) dX(t)) \\ &= -(r - \delta) F dt + (r - \delta) F dt + \sigma F dX(t) \\ &= \sigma F dX(t) \end{aligned}$$

It does not have a drift term.