Fin 514: Financial Engineering II

Lecture 2: Recap on binomial pricing

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Outline

- The most useful technique for valuing *most* types of options is a numerical technique called the binomial model.
- It is based on a simplified process where we limit the number of states that the stock price can move to over the next time period.
- Development of this technique will also show us how to understand some key concepts in option pricing such as delta hedging, self-financing, Arrow-Debreu securities, and risk-neutral probabilities.
- many of you have seen most of this material before so we will progress swiftly.

Our problem

- Let C_t be the value of the call option at time t. We know that at expiration date T the value C_T is

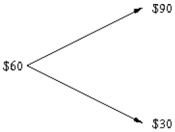
$$C_T = \max[S_T - X, 0]$$

What is the value of the option prior to expiration? That is, what is the (valuation) function V that gives the price of the call option

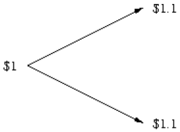
$$C_t = V(S_t, t)$$

An example

- Let's suppose that the current price of a stock is \$60 and that over the next period the stock price will increase by 50% or decrease by 50%:

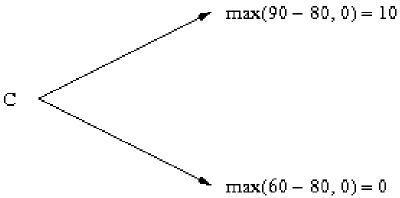


- The interest rate is 10% per period (i.e $1 + R = 1.1$ or $e^r = 1.1$), so that a risk-free investment is given as follows:



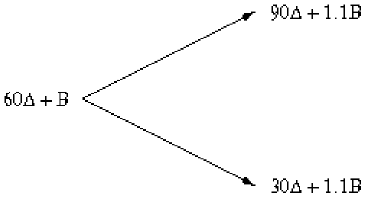
An example

- Finally a one-period European call option with exercise price $K = 80$ is described by the following:



Replicating portfolio

- Consider a portfolio with Δ shares of stock and B dollars in a risk-free investment, then:
  $60\Delta + B$ is the initial value
  $90\Delta + 1.1B$ is the value if the stock goes up
  $30\Delta + 1.1B$ is the value if the stock goes down
- If we represent this using our binomial 'tree' (or lattice) then we have



QUIZ

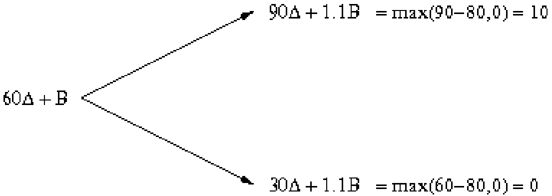
What should be true about the value if the replicating portfolio and the value of the option?

- Portfolio $>$ option.
- Portfolio $=$ option.
- Portfolio $<$ option.
- It will depend upon the state of the world.

Why?

Replicating portfolio

- Choose Δ and B so that the payoffs of the replicating portfolio match the payoffs of the call option:



- This gives us a system of two equations:

$$\begin{aligned} 90\Delta + 1.1B &= 10 \\ 30\Delta + 1.1B &= 0 \end{aligned}$$

Replicating portfolio

- So, we have

$$\begin{aligned} 90\Delta + 1.1B &= 10 \\ 30\Delta + 1.1B &= 0 \end{aligned}$$

- Solving these equations yields:

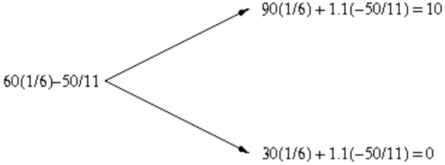
$$\begin{aligned} \Delta &= \frac{10-0}{90-30} = \frac{1}{6} \\ B &= -\frac{50}{11} \end{aligned}$$

- Thus, the current value of the call option is current value of replicating portfolio is:

$$60\Delta + B = 60\left(\frac{1}{6}\right) - \frac{50}{11} = \frac{60}{11}$$

Review

- We have found a portfolio ($\Delta = 1/6, B = -50/11$) such that the payoffs of the portfolio match those of the call:



- The value of the call option must be equal to the value of the replicating portfolio:

$$C = 60\left(\frac{1}{6}\right) - \frac{50}{11} = \frac{60}{11}$$

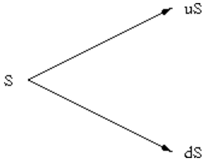
QUIZ

Where do the probabilities of up and down movements appear in these calculations?

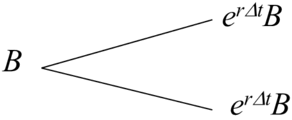
- They do not.
- They are implicit in the values of Δ and B.
- They are used to calculate the expected option value.

Generalize the example

- Stock price is S, and if there are no dividends during the next period of length Δt stock price will increase to uS or decrease to dS:

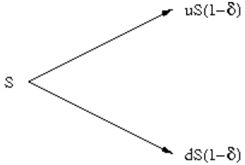


- Period is of length Δt, continuously compounded interest rate is r. Thus B dollars in bond grows to

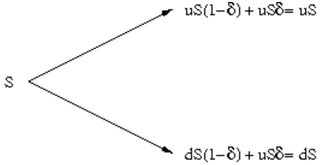


Dividends

- If there are dividends, the stock movement is:



- Dividend of $uS\delta$ or $dS\delta$ is paid at end of period (time Δt) to person who owns stock at beginning (time 0). Thus total return is:



Generalize the example

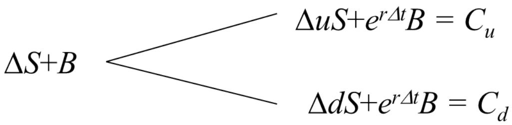
- One-period European call with strike K:

$$
\begin{array}{lll}
 & \nearrow & C_u = \max(uS(1-\delta)-K,\ 0) \\
C & & \\
 & \searrow & C_d = \max(dS(1-\delta)-K,\ 0)
\end{array}
$$

- Dividends are paid to person who owns stock at time 0, which explains why the call payoffs are based on $uS(1-\delta)$ and $dS(1-\delta)$.

Replicating portfolio

- Choose Δ and B so that the payoffs of the replicating portfolio match the payoffs of the call option:



- Result is a system of 2 equations:

$$\begin{aligned} \Delta uS + Be^{r\Delta t} &= C_u \\ \Delta dS + Be^{r\Delta t} &= C_d \end{aligned}$$

Replicating portfolio

- Solving these equations yields:

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

$$B = e^{-r\Delta t}\frac{uC_d - dC_u}{u-d}$$

- Current value of the call option is the current value of the replicating portfolio:

$$C = \Delta S + B = e^{-r\Delta t}\left(C_u \frac{e^{r\Delta t} - d}{u-d} + C_d \frac{u - e^{r\Delta t}}{u-d}\right)$$

The probabilistic interpretation

- Now define:

$$q = \frac{e^{r\Delta t} - d}{u - d}, \quad 1 - q = \frac{u - e^{r\Delta t}}{u - d}$$

- Formula on previous page becomes

$$\begin{aligned} C &= \Delta S + B = e^{-r\Delta t}\left(C_u \frac{e^{r\Delta t} - d}{u - d} + C_d \frac{u - e^{r\Delta t}}{u - d}\right) \\ &= e^{-r\Delta t}\left[qC_u + (1-q)C_d\right] \end{aligned}$$

- Here the option price is the expected value computed using risk-neutral probabilities (under the risk neutral measure), discounted at riskless rate.

QUIZ

When is the following condition satisfied? Are they realistic constraints

- $0 \leq q \leq 1$

As before

- The coefficients q and $1-q$ are our risk-neutral probabilities risk-neutral probabilities.
- Using them, over a period of length Δt, 1 + expected return on stock is
$$\frac{E[S_{\Delta t}]}{S} = \frac{quS + (1-q)dS}{S} = e^{r\Delta t}$$
thus the expected return on stock = riskless return

Risk neutral probabilities

- The coefficients q and $1 - q$ are called 'risk-neutral' probabilities.
  - Such probabilities always exist, if there are no arbitrage opportunities (This can be proved using theorems about the existence of solutions to systems of linear inequalities, see later classes!)
  - They are not necessarily the actual or 'physical' probabilities.
  - They do not necessarily correspond to the beliefs of any actual investor.
  - They are probabilities in the sense that they satisfy the minimal mathematical requirements of probabilities.

Interpreting the risk neutral probabilities

- We argued that call price was $60/11$ using a replication argument:
    - All investors would agree on the price of $60/11$, regardless of their risk aversion and beliefs.
    - Thus, given the stock price investors' risk aversion and beliefs are irrelevant.
    - Thus, we can assume whatever beliefs are convenient.
- It is convenient to assume investors are risk-neutral as then the expected return on option is the riskless rate and its price is the expected cash flow, discounted at riskless rate.

Interpreting the risk neutral probabilities

- So, if investors are risk-neutral the expected return on option is equal to riskless rate and so the price is the expected cash flow, discounted at riskless rate.
- That is, if investors are risk-neutral:

$$\begin{aligned}\frac{60}{11} &= \frac{0.6(10)+0.4(0)}{1.1} \\ &= \frac{\text{expected payoff}}{1.1}\end{aligned}$$

- In general we can view the general option price, V as

$$V = \frac{E^Q[Payoff]}{1+r_f} = \frac{E[payoff]}{1+r_{true}}$$

where Q denotes a risk-neutral expectation, r_f is the risk-free rate and r_{true} is the real, risky, discount rate.

QUIZ

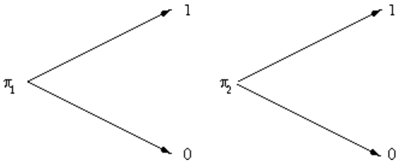
For **Call** options, what do you think will be the general relation between r_{true} and r_f?

- $r_{true} > r_f$
- $r_{true} = r_f$
- $r_{true} < r_f$

What does this imply about the real world expected payoff?

A second interpretation

- Let's introduce two elementary (or primitive) securities:

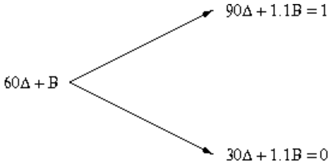


where π_1 and π_2 are the current prices of these securities.

- We will call these securities "state contingent claims". They are also referred to as **Arrow-Debreu securities**.

In detail

- Choose Δ and B so payoffs of the replicating portfolio match the payoffs of state-contingent claim:



- This gives us system of 2 equations:

$$\begin{aligned} 90\Delta + 1.1B &= 1 \\ 30\Delta + 1.1B &= 0 \end{aligned}$$

In detail

- Solving these equations yields:

$$\begin{aligned} \Delta &= \frac{1-0}{90-30} = \frac{1}{60} \\ B &= -\frac{5}{11} \end{aligned}$$

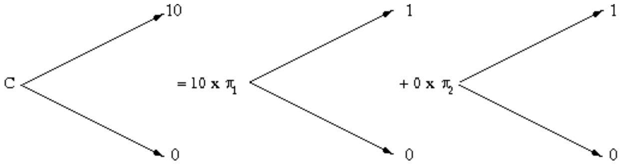
- Thus, π_1 is the current value of replicating portfolio is:

$$\pi_1 = 60\Delta + B = 60\left(\frac{1}{60}\right) - \frac{5}{11} = \frac{6}{11}$$

- Similar analysis leads to conclusion $\pi_2 = 4/11$.

Why are π_1 and π_2 useful?

- Option is a combination of 10 of the first state-contingent claim and 0 of the second.

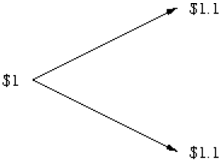


- Therefore our call option is worth

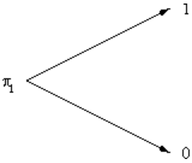
$$C = 10\pi_1 + 0\pi_2$$

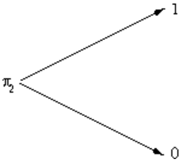
Why are π_1 and π_2 useful?

- This is true for *all* securities. The bond:



is a combination of 1.1 of the first and 1.1 of the second





- This gives us that

$$1 = 1.1\pi_1 + 1.1\pi_2$$

Why are π_1 and π_2 useful?

- The prices of each security can be expressed in terms of the π's:

$$
\begin{aligned}
C &= 10\pi_1 + 0\pi_2 \\
&= 10\frac{6}{11} + 0\frac{4}{11} \\
&= \frac{60}{11} \\
60 &= 90\pi_1 + 30\pi_2 \\
&= 90\frac{6}{11} + 30\frac{4}{11} \\
&= 60 \\
1 &= 1.1\pi_1 + 1.1\pi_2 \\
&= 1.1\frac{6}{11} + 1.1\frac{4}{11} \\
&= 1
\end{aligned}
$$

Remark

- This is actually nothing new. Consider bonds with cash flows:

Current Price	Time 1	Time 2
90	100	
97	10	110

- From these bonds, conclude that discount factors are 0.9 and 0.8, because

$$\begin{aligned} 90 &= 0.9(100) \\ 97 &= 0.9(10) + 0.8(110) \end{aligned}$$

- Use these discount factors to price other bonds, swaps, etc.

Remark

- Now instead of bonds with cash flows at different dates, we have stock and bond with values in different states:

Current Price	State 1	State 2
60	90	30
1	1.1	1.1

- From these prices, conclude that π_1 and π_2 are are $6/11$ and $4/11$, because

$$\begin{aligned} 60 &= (6/11)90 + (4/11)(30) \\ 1 &= (6/11)1.1 + (4/11)1.1 \end{aligned}$$

- Use π_1 and π_2 to price options, etc.

QUIZ

What is the relationship between the π values and the q values?

- $q = \pi_1$
- $q = \pi_1 e^{r\Delta t}$
- $q = \pi_1 e^{-r\Delta t}$

Link back to risk-neutral probabilities

- The π's are very useful if we know them. But how do we find and use the π's?
- Let's rescale the π's by defining:

$$q = 1.1\pi_1$$
$$(1 - q) = 1.1\pi_2$$

(in general, we rescale them as $q_i = e^{r\Delta t}\pi_i$)

- Notice that the q's are just rescaled prices (the π's are prices)

Rewrite our eq.'s in terms of the q's

Now let, $q = q_1$ and $(1 - q) = q_2$. Using $q_i = 1.1\pi_1$ (or $\pi_i = q_i/1.1$) then we have that

$$
\begin{aligned}
C &= 10\frac{q_1}{1.1} + 0\frac{q_2}{1.1} \\
&= 10\frac{6}{11} + 0\frac{4}{11} \\
&= \frac{60}{11} \\
60 &= 90\frac{q_1}{1.1} + 30\frac{q_2}{1.1} \\
&= 90\frac{6}{11} + 30\frac{4}{11} \\
&= 60 \\
1 &= 1.1\frac{q_1}{1.1} + 1.1\frac{q_2}{1.1} \\
&= 1.1\frac{6}{11} + 1.1\frac{4}{11} \\
&= 1
\end{aligned}
$$

QUIZ

What do you think will be the general relation between the true r_{stock} and r_{option}?

- $r_{stock} > r_{option}$
- $r_{stock} = r_{option}$
- $r_{stock} < r_{option}$
- It will depend upon the option terms

Link back to risk-neutral probabilities

- The q's are, again, the probabilities that make the expected returns on all assets equal to the riskless rate:
- For the stock:

$$\begin{aligned} 1 + E[\text{expected rate of return}] &= \frac{0.6(90) + 0.4(30)}{60} \\ &= 1.1 \end{aligned}$$

- For the option:

$$\begin{aligned} 1 + E[\text{expected rate of return}] &= \frac{0.6(10) + 0.4(0)}{60/11} \\ &= 1.1 \end{aligned}$$

Our procedure for option valuation

- Choose probabilities so expected rate of return on the underlying asset is equal to the riskless rate. Calculate the expected payoff of the option under this assumption, e.g.

$$\text{expected payoff} = 0.6(10) + 0.4(0)$$

- Discount (at riskless rate) to get the option price:

$$\frac{60}{11} = \frac{0.6(10) + 0.4(0)}{1.1}$$

An interpretation of the procedure

- Using the approach

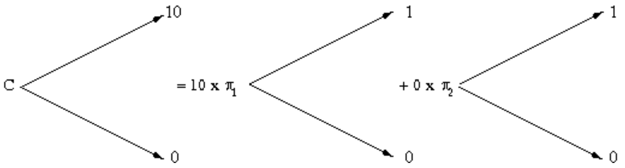
$$C = \frac{q_1}{1.1}(10) + \frac{q_2}{1.1}(0) = \frac{0.6}{1.1}(10) + \frac{0.4}{1.1}(0) = \frac{60}{11}$$

- Is equivalent to using (recall $\pi = \pi/1.1$)

$$C = \pi_1(10) + \pi_2(0) = \frac{0.6}{1.1}(10) + \frac{0.4}{1.1}(0) = \frac{60}{11}$$

An interpretation...

But to use the equation $C = 10\pi_1 + 0\pi_2$ is to interpret the option as portfolio of state-contingent claims:



General technique

- Our procedure for option valuation:
  1. Compute expected value using risk-neutral probabilities.
  2. Discount at the riskless rate.
- In terms of the equation, where V is the value of any derivative:

$$
\begin{aligned}
V &= \Delta S + B = e^{-r\Delta t}\left(V_u \frac{e^{r\Delta t} - d}{u - d} + V_d \frac{u - e^{r\Delta t}}{u - d}\right) \\
&= e^{-r\Delta t}\left[qV_u + (1 - q)V_d\right]
\end{aligned}
$$

where the first term is the risk-neutral discounting and the second expression is the risk-neutral discounting. We will also adapt this approach to derive the Black-Scholes equation.

Tree construction

- We will wish to match our binomial distribution to our perceived true distribution of stock returns. If we believe that returns $r_{\Delta t} = \ln(S_{\Delta t}/S_0)$ are normally distributed (!) then stock prices are *lognormally distributed*.
- In the risk-neutral world the mean (μ) and variance (ν) of a normally distributed $r_{\Delta t}$ are:

$$\begin{aligned} \mu &= \left(r - \frac{1}{2}\sigma^2\right)\Delta t \\ \nu &= \sigma^2 \Delta t \end{aligned}$$

  where σ is the volatility of the stock returns and r is the risk-free rate.
- We have two equations and three unknowns q, u and d, an additional choice/constraint (Joshi, p 57) gives us one possible choice of u and d:

$$u = e^{r\Delta t + \sigma\sqrt{\Delta t}}, \quad d = e^{r\Delta t - \sigma\sqrt{\Delta t}}$$

QUIZ

According to the binomial set up, what is $E[S_{\Delta t}]$?

- $S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t\right)$
- $S_0 \exp\left(r\Delta t\right)$
- $S_0 \exp\left(\left(r + \frac{1}{2}\sigma^2\right)\Delta t\right)$

QUIZ

What do you think the relationship between the volatility in the real world σ_{rw} and the risk-neutral volatility σ_{rn} is?

- $\sigma_{rw} > \sigma_{rn}$
- $\sigma_{rw} = \sigma_{rn}$
- $\sigma_{rw} < \sigma_{rn}$
- It depends.

Tree construction

- The $\sigma\sqrt{\Delta t}$ term ensures that the volatility of the stock price determines the size of the jumps and that the lognormal volatility is matched (**for small time steps**).
- Interestingly we can choose any 'first term', such as

$$
\begin{aligned}
u &= e^{r\Delta t+\sigma\sqrt{\Delta t}}, & d &= e^{r\Delta t-\sigma\sqrt{\Delta t}} \\
u &= e^{\sigma\sqrt{\Delta t}}, & d &= e^{-\sigma\sqrt{\Delta t}} \\
u &= e^{\left(r-\frac{1}{2}\sigma^2\right)\Delta t+\sigma\sqrt{\Delta t}}, & d &= e^{\left(r-\frac{1}{2}\sigma^2\right)\Delta t-\sigma\sqrt{\Delta t}}
\end{aligned}
$$

as the risk-neutral probabilities are always given by

$$
q = \frac{e^{r\Delta t} - d}{u - d}, \quad 1 - q = \frac{u - e^{r\Delta t}}{u - d}
$$

then these adjust to ensure that the expected return is correct, thus matching the first two moments.

Tree construction

- The tree is then

$$
S \;\begin{array}{l} \nearrow \; Se^{r\Delta t+\sigma\sqrt{\Delta t}} = uS \\[2em] \searrow \; Se^{r\Delta t-\sigma\sqrt{\Delta t}} = dS \end{array}
$$

Tree construction

- If $\Delta t = 1$, r $= 0.08$ (8%), no dividends, $\sigma = 0.3$ (30%), and S $=$ 41, we have

$$
41 \begin{cases} 41e^{0.08+0.3\sqrt{1}} = 59.954 \\[2ex] 41e^{0.08-0.3\sqrt{1}} = 32.903 \end{cases}
$$

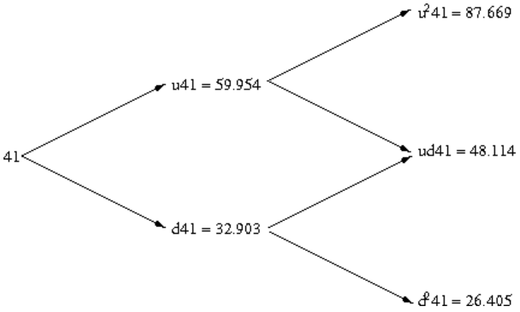
- Here $u = 1.462285$, $d = 0.802519$.

QUIZ

Can you think of any difficulties in extending the tree to more than one period?

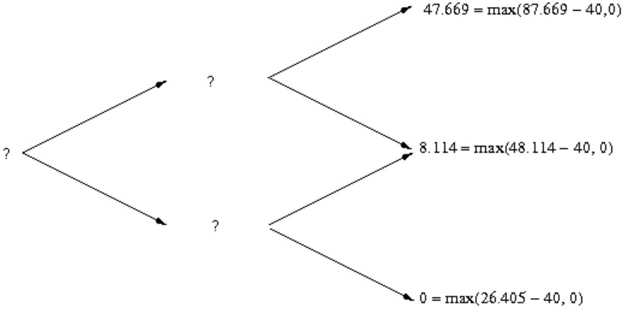
Two or more periods

- If an option has two periods until expiration, we make a two-period tree:



Computing call option price

- Suppose strike $K = 40$. Make a two-period tree for the option, and start at the end:

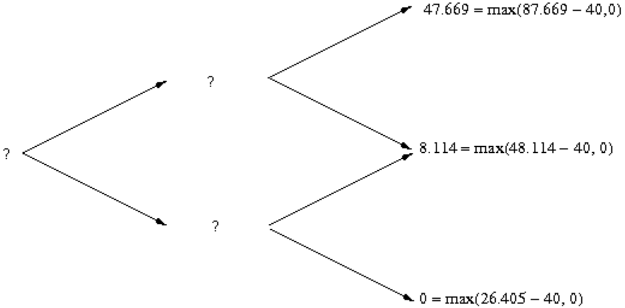


Computing call option price

- Risk-neutral probabilities are:

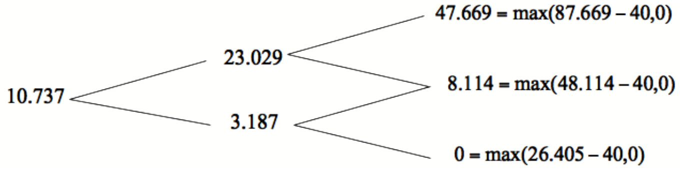
$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.425557, \quad 1 - q = \frac{u - e^{r\Delta t}}{u - d} = 0.574443$$

- Then use $C = e^{-r\Delta t}\left[qC_u + (1-q)C_d\right]$



Computing call option price

- Then

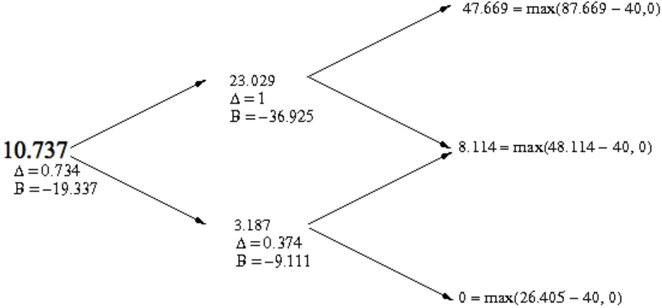


- The first equation uses information from the up and down states , and then the other calculations proceed in a similar way:

$$
\begin{aligned}
23.029 &= e^{-0.08}\left[0.425557 \times 47.669 + 0.574443 \times 8.114\right] \\
3.187 &= e^{-0.08}\left[0.425557 \times 8.114 + 0.574443 \times 0\right] \\
10.737 &= e^{-0.08}\left[0.425557 \times 23.029 + 0.574443 \times 3.187\right]
\end{aligned}
$$

Computing call option price

- An alternative approach would be to compute replicating portfolio at each node:



where at the point where the option is worth 3.187, this portfolio will replicate cash flows in the up and down states starting from this point.

QUIZ

Will the portfolio formed at time 0 replicate the option payoff at $t = 2$ in all states with no future cash flows?

- Yes, always.
- No.
- Yes, if you adjust the portfolio at time 1.

Replicating portfolio

- How did I compute Δs, B's on previous slide?

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

$$B = e^{-r\Delta t}\frac{uC_d - dC_u}{u-d}$$

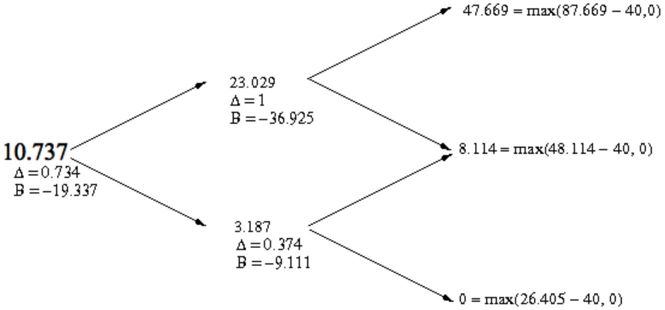
- Current value of option is current value of replicating portfolio:

$$C = \Delta S + B = e^{-r\Delta t}\left(C_u\frac{e^{r\Delta t}-d}{u-d} + C_d\frac{u-e^{r\Delta t}}{u-d}\right)$$

Delta calculation

- To compute Δ, use

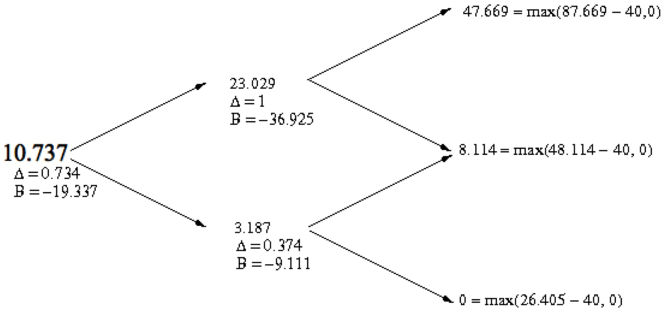
$$\Delta = \frac{C_u - C_d}{S(u-d)} = \frac{8.114 - 0}{48.114 - 26.405} = 0.374$$



B calculation

- To compute B, use

$$\begin{aligned} C &= \Delta S + B \\ 3.187 &= 0.374(32.903) + B \\ B &= -9.111 \end{aligned}$$



Full replication: First period

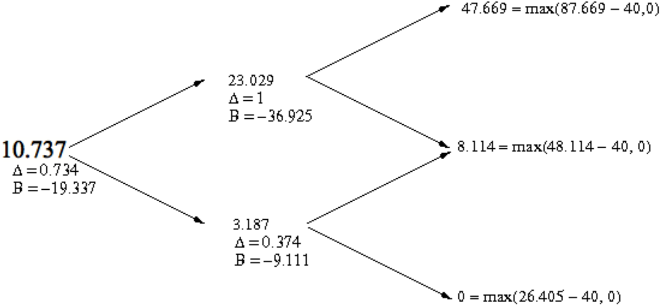
- At time 0 then

$$\begin{aligned} 10.737 &= \Delta(41) - B \\ &= 0.734(41) - 19.337 \end{aligned}$$

- At $t = 1$, up-state and down state

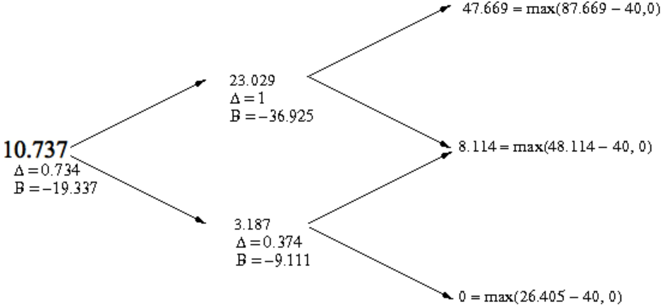
$$23.029 = 0.734(59.954) - 19.337e^{0.08}$$

$$3.187 = 0.734(32.903) - 19.337e^{0.08}$$



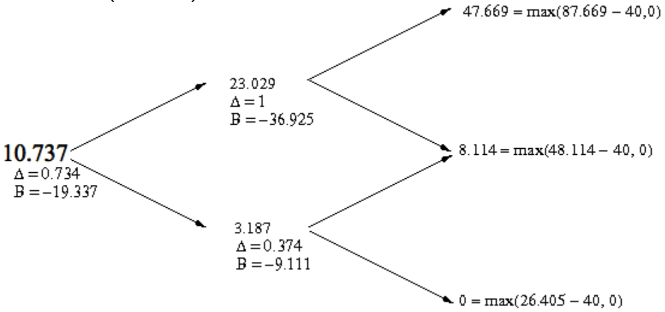
Full replication: Rebalance at $t = 1$, up-state

- In the upstate we arrive with portfolio of 0.734 shares and $-19.337e^{0.08}$ in riskless assets, worth $23.029 = 0.734(59.954) - 19.337e^{0.08}$.
- Leave this node with portfolio of 1 share and -36.925 in riskless assets, worth $23.029 = 1(59.954) - 36.925$.
- That is, at this node we increase our borrowing, and use the money to buy shares



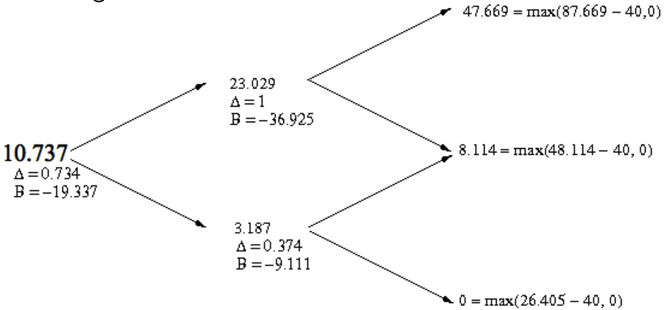
Full replication: Moving through to time 2

- We leave time 1, up-state, with portfolio of 1 share and -36.925 in riskless assets, worth $23.029 = 1(59.954) - 36.925$.
- With another up move this grows to $47.669 = 1(87.669) - 36.925e^{0.08}$, with a down move this goes to $8.114 = 1(48.114) - 36.925e^{0.08}$



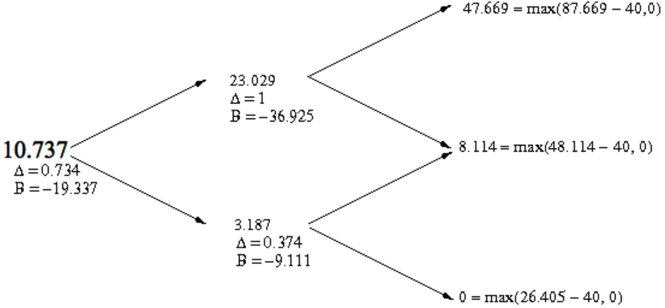
Full replication: Rebalance at $t = 1$, down-state

- In the downstate we arrive with portfolio of 0.734 shares and $-19.337e^{0.08}$ in riskless assets, worth $3.187 = 0.734(32.903) - 19.337e^{0.08}$.
- Leave this node with portfolio of 0.374 share and -9.111 in riskless assets, worth $3.187 = 0.374(32.903) - 9.111$.
- That is, at this node we sell shares and use the money to reduce our borrowing



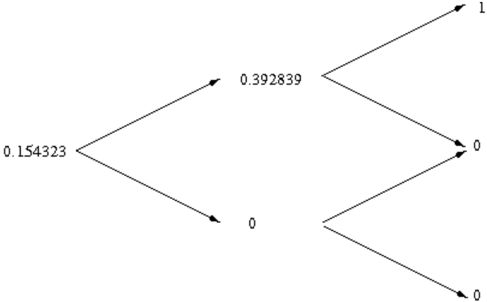
Full replication: Moving through to time 2

- We leave time 1, down-state, with portfolio of 0.374 shares and -9.111 in riskless assets, worth $3.187 = 0.374(32.903) - 9.111$.
- With an up move this grows to $8.114 = 0.374(48.114) - 9.111e^{0.08}$, with a down move this goes to $0 = 0.374(26.405) - 9.111e^{0.08}$



State contingent claims

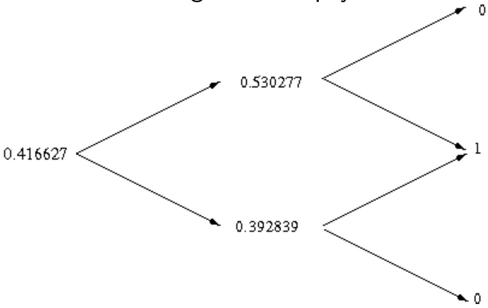
- The first contingent claim pays out 1 in state *uu* and is valued as



- The values at earlier time periods can be computed using a replication argument.

State contingent claims

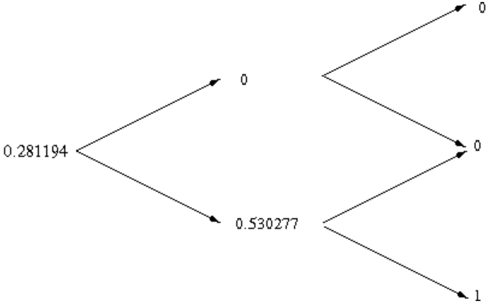
- The second contingent claim pays out 1 in state *ud* and is valued as



- The values at earlier time periods can be computed using a replication argument.

State contingent claims

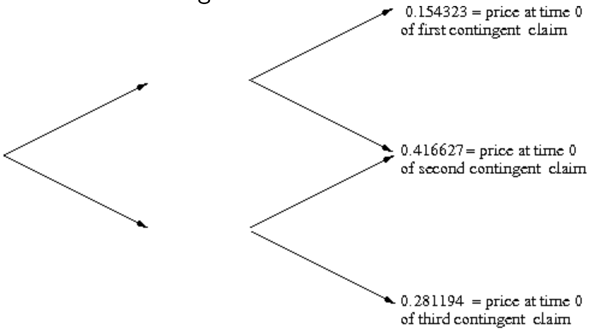
- The third contingent claim pays out 1 in state *ud* and is valued as



- The values at earlier time periods can be computed using a replication argument.

Interpretation in terms of state-contingent claims

- Let's make a different diagram



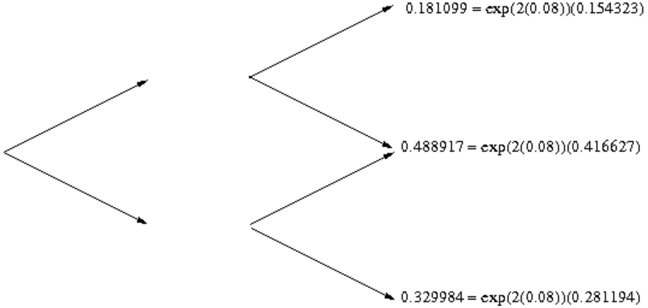
Interpretation in terms of state-contingent claims

- Option payoff $=$ 47.669 of 1st cont. claim, plus 8.114 of second, plus 0 of third.
- Option value at time 0 $= 10.737 = 47.669(0.154323) + 8.114(0.416627) + 0(0.281194)$.

$$\begin{array}{rcl} \text{Value of this payoff at Time 0} & & \text{Option payoff at time 2} \\ 47.669(0.154323) & \longleftrightarrow & 47.669 = 47.669 \times 1 \\ 8.114(0.416627) & \longleftrightarrow & 8.114 = 8.114 \times 1 \\ 0(0.281194) & \longleftrightarrow & 0 = 0 \times 1 \end{array}$$

Interpretation in terms of risk-neutral probabilities

- Risk-neutral probabilities are state prices scaled by interest factor $e^{2(0.08)}$.



- Option value at time 0:

$$
\begin{aligned}
10.737 &= 47.669(0.154323) + 8.114(0.416627) + 0(0.281194) \\
&= e^{-2(0.08)}[47.669(0.181099) + 8.114(0.488917) + 0(0.329984)]
\end{aligned}
$$

QUIZ

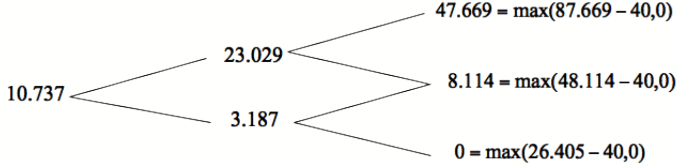
- How else could you calculate the risk-neutral probabilities?

Interpretation of what we have done

- These are some of most important ideas in finance. If there are no arbitrage opportunities, then:
- Option value is value of the of replicating portfolio
- Option value is value of portfolio of state-contingent claims
- If there is a riskless asset, option value can be written in terms of risk-neutral probabilities

American option pricing

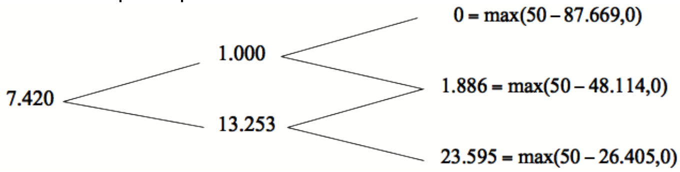
- Our procedure was illustrated using a call option



- But the fact that it was a call was used only in filling in the terminal values.

American option pricing

- For a European put with a strike of 50:



- Fill in final nodes using rule for put option
- Then move through tree using:

$$P = e^{-r\Delta t}\left[qP_u + (1-q)P_d\right]$$

American option pricing

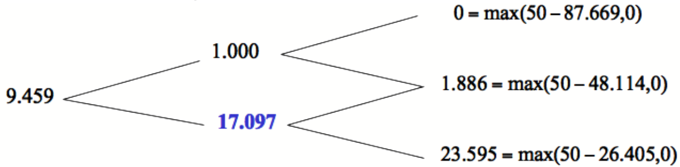
- For an American put with a strike of 50:
- Fill in final nodes using rule for put option
- Then move through tree using:

$$P = \max\left(e^{-r\Delta t}\left[qP_u + (1-q)P_d\right], K - S\right)$$

  where the first term is the value if the option is not exercised and the second is the exercise value.
- This amounts to checking whether exercise is optimal at every node of the tree.

American option pricing

- For an American put with a strike of 50:



- Example: At time 1, 'down' we have

$$
\begin{aligned}
e^{-r\Delta t}\left[qP_u + (1-q)P_d\right] &= e^{-0.08}\left[0.42556(1.886) + 0.57444(23.595)\right] \\
&= 13.253
\end{aligned}
$$

The exercise value is $K - S = 50 - 32.903 = 17.097$. Early exercise is optimal here, and the value is 17.097

Outline

- We have seen all of the key features of binomial pricing.
- We know how to construct the tree to match with desired properties.
- We can value European and American options, on stocks potentially paying dividends.
- We have seen the idea of risk-neutral probabilities and primitive securities.
- We have seen how to replicate the payoffs and how to roll this replication forward over more than one period.