### Problem Set 4

Total 15 points

Due Wednesday, 7 March 2018 before 11:59pm

*Instructions*: This is an **individual assignment**. Handwritten solutions are acceptable here, but please submit a scanned or photographed copy to COMPASS. Of course, you can also typeset your solutions (word, latex etc.) them and upload them electronically.

*(b) The file name should be: PSn.FAMILYNAME1Initial1.xls, for example my solution to Problem set 1 would have the file name PS1.WIDDICKSM.xls.*

*(c) Please make sure that your solutions are well-organized and clear, with appropriate text, explanations, and formatting. If I cannot figure out what you did, then what you did is wrong.*

**1.** **(2 points total) Practicing using Ito’s lemma and the Ito integral,** (X is a Brownian motion in the questions below)

* 1. (1 point) By considering the process followed by *f(X)* = , please show that

.

*Hint*: Use Itô’s lemma to compute the differential of, and then rearrange.

* 1. (0.5 points) Use a similar approach to show that

.

* 1. (0.5 points) and

.

**2.** **(2 points) Process followed by the Futures (or Forward) Price.**

Let *X* be a Brownian motion. The price of a stock follows the process



with initial condition , where  is the expected rate of return,is the dividend yield, and **> 0 is the volatility. Consider a function

,

where *T* is a constant. What is the process followed by *F*? (That is, what is d*F*?) make sure that you have no “S” values in you final SDE.

*Remark*:  is the futures price of a futures contract maturing at time *T*, but the question does not require that you know this.

1. **(4 points) A practical example of using Ito’s Lemma**

Let  be the U.S. dollar price of a UK Pound (GBP), and let  be the U.S. dollar price of a Euro. These exchange rates follow the processes



where *X1* and *X3* are independent Brownian motions. The US dollar, Euro, and GBP interest rates are **constants** *r*, *rE*, and *rP*, respectively. There are risk-free bonds in each of the currencies, *B*, *BP*, and *BE.*

Let *SEt* and *SPt* denote the US dollar prices of the Euro bond and the GBP bond respectively.

Consider a European option expiring at time *T* giving the owner to right to exchange GBP for Euros; this option has a payoff in USD of $1000

1. Consider the process for , which is the price of Euros () in terms of GBP (). Find *dYt*. Show that it is possible to write  in terms of a new Brownian motion , such that .

*Hint: Here you can combine two Brownian motions to create a new one, for example we can write 3X3(t)= aX1(t)+ bX2(t) where 3 = √(a2+b2+2ab), and you can check that E[X3] = 0 and Var[X3] = t.*

1. Now consider the process followed by Zt = YtBEt which is the GBP price of a Euro risk-free Bond. Find *dZ*t. Show that it is possible to write  in terms of the same Brownian motion , such that .

**4. (4 points) Ratios and products work very well with GBM….**

Suppose that the prices of two (non-dividend paying) stocks and a risk-free asset follow the processes:

dS1 = 1S1dt + 1S1dX1

dS2 = 2S2dt + 2S2dX2

dB = rBdt

where *X1* and *X2* are correlated Brownian motions with correlation . Additionally, 1, 2, 1, 2, r are positive constants.

1. Consider the process for, which is the product of the two stocks, Y = S1S2. Find dY. Show that it is possible to write dY in terms of a new Brownian motion X3.

**Now, consider a geometric average call option where the payoff is V(S1, S2, T) = max((S1S2)1/2 - K, 0)**

1. Write down this payoff in terms of Y.
2. Determine the stochastic process followed by Z = Y1/2 .

1. Using your answer to c) determine the real world expected return of the geometric average of two stocks?

**5**. **(total 3 points) Black-Scholes equation and expected returns**

Suppose that the price of a dividend paying stock follows the process

d*S*(*t*) = (*S(t)*d*t* + *S(t)*d*X*(*t*),

the price of a risk-free bond or money-market account follows d*B*(*t*) = *rB*(*t*)d*t*, and that there is an option on the stock with price *V*(*t*) = *V*(*S*(*t*), *t*). If the stock price follows the process above, the Black-Scholes pde is,

(a) (1 point) Use Ito’s lemma to find the dynamics of the option price, i.e. find d*V*. (Your answer should be in terms of the partial derivatives ∂*V*/∂*t*, ∂*V*/∂*S*, and ∂2*V*/∂*S*2.)

(b) (1 point) Compare the drift of *V* that you computed in part (a) to the left-hand side of the pde above. What is the value of  that makes the drift of *V* equal to the left-hand side of the pde?

(c) (0.5 points) Setting  equal to this value also amounts to making an assumption about the expected rate of return on the option (assume that the option value *V*(*t*) > 0, so the expected rate of return on the option is well defined). What is this assumption?

*Hint: In part (c) it might help to divide the process dS(t) = (S(t)dt + S(t)dX(t) by S(t), yielding dS(t)/S(t) = (dt + dX(t) and then recall that the drift can be interpreted as the expected change in the process. Thus  is the expected rate of return on the stock. Similarly, the expected rate of return on the option is the drift of the option price, divided by its value V.*

(d) (0.5 points) Now return to the scenario in Q2, where. If we set the drift of the stock equal to the drift from parts (b) and (c) above then what is the process followed by F, (i.e what is dF) now?