### Problem Set 4

Solutions

**1.** a) By considering , please show that

.

(*X* is a Brownian motion.)

***Solution*: Applying Itô’s formula to *f*(*t*) =*X*2(*t*):**

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**Integrating both sides, recalling that *X*(0) = 0, and re-arranging the terms we find:**

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1. Use a similar approach to show that

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***Solution*: Choose *f = tX*. Applying Itô’s formula,**

****

**Integrating both sides, recalling that *X*(0) = 0, and re-arranging the terms we find**

****

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1. and

.

***Solution*: Choose *f =* (1/3)*X*(*t*)3. Applying Itô’s lemma:**

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**Integrating both sides, recalling that *X*(0) = 0, and re-arranging terms, we find:**

****

**2.** Let *X* be a Brownian motion. The price of a stock follows the process



with initial condition , where ,  is the expected rate of return,is the dividend yield, and **> 0 is the volatility. Consider a function

,

where *T* is a constant. What is the process followed by *F*? (That is, what is d*F*?).

*Remark*:  is the futures price of a futures contract maturing at time *T*, but the question does not require that you know this.

***Solution*: Using Itô’s formula, we have**

**,  and .**

**Then**

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****

**where the fourth equality simplifies the result by using the fact that *F*(*t*) = *e*(*r–*)(*T–t*)*S*(*t*). The initial condition is *F*(0) = e(*r*)(*Tt*)*S*(0).**

1. **(4 points) A practical example of using Ito’s Lemma**

Let  be the U.S. dollar price of a UK Pound (GBP), and let  be the U.S. dollar price of a Euro. These exchange rates follow the processes



where *X1* and *X3* are independent Brownian motions. The US dollar, Euro, and GBP interest rates are **constants** *r*, *rE*, and *rP*, respectively. There are risk-free bonds in each of the currencies, *B*, *BP*, and *BE.*

Let *SEt* and *SPt* denote the US dollar prices of the Euro bond and the GBP bond respectively.

Consider a European option expiring at time *T* giving the owner to right to exchange GBP for Euros; this option has a payoff in USD of $1000

***Solution*: We need the derivatives**

**, , ,, , , and .**

**The process for *Y* is then**

****

**where , , and *X* is a new Brownian motion defined by .**

1. Now consider the process followed by Zt = YtBEt which is the GBP price of a Euro risk-free Bond. Find *dZ*t. Show that it is possible to write  in terms of the same Brownian motion , such that .

[2 points]

**Solution:**

**4. (4 points) Ratios and products work very well with GBM….**

Suppose that the prices of two (non-dividend paying) stocks and a risk-free asset follow the processes:

dS1 = 1S1dt + 1S1dX1

dS2 = 2S2dt + 2S2dX2

dB = rBdt

where *X1* and *X2* are correlated Brownian motions with correlation . Additionally, 1, 2, 1, 2, r are positive constants.

1. Consider the process for, which is the product of the two stocks, Y = S1S2. Find dY. Show that it is possible to write dY in terms of a new Brownian motion X3.

[2 points]

**Solution**

**dY = 3Ydt + 3YdX3**

**where**

**3 =  + 2+ 12**

**3 = √(12+22+212)**

**Now, consider a geometric average call option where the payoff is V(S1, S2, T) = max((S1S2)1/2 - K, 0)**

1. Write down this payoff in terms of Y.

(1 point)

**Solution**

**V(Y,T) = max(Y1/2 – K, 0)**

1. Determine the stochastic process followed by Z = Y1/2 .

(2 points)

**Solution**

**dZ = ½(3 – ¼32)Zdt + ½3dX3**

1. Using your answer to c) determine the real world expected return of the geometric average of two stocks?
2. point)

**Solution**

**½(3 – ¼32)**

**5**. Suppose that the price of a dividend paying stock follows the process

d*S*(*t*) = (*S(t)*d*t* + *S(t)*d*X*(*t*),

the price of a risk-free bond or money-market account follows d*B*(*t*) = *rB*(*t*)d*t*, and that there is an option on the stock with price *V*(*t*) = *V*(*S*(*t*), *t*). If the stock price follows the process above, the Black-Scholes pde is,

Please do the following:

(a) (1 point) Use Ito’s lemma to find the dynamics of the option price, i.e. find d*V*. (Your answer should be in terms of the partial derivatives ∂*V*/∂*t*, ∂*V*/∂*S*, and ∂2*V*/∂*S*2.)

***Solution*: We have done this in class. We have  and. From Itô’s formula,**

****

**or**

****

**.**

(b) (1 point) Compare the drift of *V* that you computed in part (a) to the left-hand side of the pde above. Is there a value of  that will make the drift of *V* equal to the left-hand side of the pde? If so, what is it?

***Solution*: From part (a), the drift of the option price *V* is**

**.**

**Comparing this to the left-hand side of the Black-Scholes pde above,**

**,**

**so if  then the drift of the option price *V* equals the left-hand side of the pde.**

(c) (0.5 points) Setting  equal to this value also amounts to making an assumption about the expected rate of return on the option (assume that the option value *V*(*t*) > 0, so the expected rate of return on the option is well defined). What is this assumption?

*Hint: In part (c) it might help to divide the process dS(t) = (S(t)dt + S(t)dX(t) by S(t), yielding dS(t)/S(t) = (dt + dX(t) and then recall that the drift can be interpreted as the expected change in the process. Thus  is the expected rate of return on the stock. Similarly, the expected rate of return on the option is the drift of the option price, divided by its value V.*

***Solution*: If we assume, then the stochastic process for the stock price becomes . Dividing both sides by *S*(*t*), we have . The interpretation of this is that if the expected rate of return on the stock is the risk-free interest rate (*r - )*.**

**The stochastic differential equation describing the movement of the option price becomes**

**.**

**Dividing both sides by *V*(*t*),**

**.**

**Similarly, dividing both sides of the pde by *V*(*t*)**

**.**

**Thus ifthe expected rate of return on the option price is also the risk-free rate *r*.**

(d) (0.5 points) Now return to the scenario in Q2, where. If we set the drift of the stock equal to the drift from parts (b) and (c) above then what is the process followed by F, (i.e what is dF) now?

***Solution*: From above** **Now,  = r, and so this becomes**



**i.e F is driftless under risk-neutral probabilities, or F is a martingale (see later!).**