### Problem Set 5

Total 15 points

Due Tuesday, 13 March 2017 before 11:59pm

**SOLUTIONS WILL BE POSTED IMMEDIATELY AFTER THIS DEADLINE**

*Instructions. This is an individual assignment. Handwritten solutions are acceptable here, please either scan them to submit or hand them to me in class on Tuesday. Of course, you can also typset your solutions and upload them electronically.*

*(a) The file name should be: PSn.FAMILYNAME1Initial1.xls, for example my solution to Problem set 1 would have the file name PS1.WIDDICKSM.xls.*

*(b) Please make sure that your solutions are well-organized and clear, with appropriate text, explanations, and formatting. If I cannot figure out what you did, then what you did is wrong.*

**1. (1 point)** Suppose that there are a stock and a bond/money-market account with price processes

where, , and  are positive constants. What is the pde that option prices must satisfy?

*Hint*: Note that the question does not ask you to derive the pde. You may if you want.

**2. (3 points)** Suppose that the price of a stock that pays no dividends follows the process (SDE)

where ** and ** are constants. The interest rate *r* is also constant.

Consider a hypothetical derivative known as a “log contract” with a payment at time *T* given by *V*(*S*, *T*) = ln(*S*(*T*)/*S*(0)). Let *V*(*S*, *t*) denote the value of the log contract at time *t*.

(a) What partial differential equation is satisfied by the function *V*? What is the terminal boundary condition? (It is not necessary to derive the p.d.e.)

(b) What is the value of the log contract at some general time t, i.e what is *V(S, t)* for a given *S*?

*Hint: You will need to solve the SDE for S to answer this question (we did this in Lecture 8)*

**3. (2 points)** Let *B* be a $-money market account, *BK* be a Korean Won (KRW)-denominated money market account and *S* be the price of Samsung stock in KRW, the exchange rate (or the value of KRW in dollars) is *e*. These prices follow the processes

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where  and X2 are Brownian motions under the (real world) ***P*** measure and where the correlation, **, the risk-free rates, *r* and *r*K, the drifts, , e and standard deviation, ** and e,terms are constants.

1. What is the process followed by the dollar value of the Samsung stock price SD = eS (I would not combine dX1 and dX2 here)
2. What is the process followed by the dollar value of the KRW money market account BKD = eBK

**3** **(4 points)** Let  be the U.S. dollar price of a UK Pound (GBP), and let  be the U.S. dollar price of a Euro. These exchange rates follow the processes



where *X1* and *X3* are independent Brownian motions. The US dollar, Euro, and GBP interest rates are **constants** *r*, *rE*, and *rP*, respectively. There are risk-free bonds in each of the currencies, *B*, *BP*, and *BE.*

Let *SEt* and *SPt* denote the US dollar prices of the Euro bond and the GBP bond respectively.

Consider a European option expiring at time *T* giving the owner to right to exchange GBP for Euros; this option has a payoff in USD of $1000

1. Consider the process for , which is the price of Euros () in terms of GBP (). Find *dYt*. Show that it is possible to write  in terms of a new Brownian motion , such that .

**[Use your answer from PS4 here]**

1. Now consider the process followed by Zt = YtBEt which is the GBP price of a Euro risk-free Bond. Find *dZ*t. Show that it is possible to write  in terms of the same Brownian motion , such that .

**[Use your answer from PS4 here]**

1. Now consider the GBP value of the exchange option, What is the value of U at maturity, i.e what is U(Y, T)?
2. Now, construct a hedging portfolio that is of the following form:



However, first divide all of the values by *S1t.* (this ensures that all the values are denominated in GBP and there are *U* and *Z* terms in the portfolio). Now, consider the case where and . Show that and derive the partial differential equation for U(Y, t).

1. By analogy to the Black-Scholes equation, or otherwise, write down the USD value of the exchange option .

**4 (4 points)** Suppose that the prices of two (non-dividend paying) stocks and a risk-free asset follow the processes:

dS1 = 1S1dt + 1S1dX1

dS2 = 2S2dt + 2S2dX2

dB = rBdt

where *X1* and *X2* are correlated Brownian motions with correlation . Additionally, 1, 2, 1, 2, r are positive constants.

1. Consider the process for, which is the product of the two stocks, Y = S1S2. Find dY. Show that it is possible to write dY in terms of a new Brownian motion X3.

**[Use your answer from PS4 here]**

**Now, consider a geometric average call option where the payoff is V(S1, S2, T) = max((S1S2)1/2 - K, 0)**

1. Write down this payoff in terms of Y.

**[Use your answer from PS4 here]**

1. Determine the stochastic process followed by Z = Y1/2 .

**[Use your answer from PS4 here]**

1. Now, construct a hedging portfolio that is of the following form:

 = -V + Z + B

where we (heroically) assume we can hold a portfolio with value equal to the geometric average of the two stocks. Choose so that  = 0 and determine the value of . Show that the PDE followed by V(Z, t) is simply the Black-Scholes PDE in Z.

1. What is the *risk-neutral* expected return of the geometric average of two stock?
2. Now write down the value of the geometric average call option, V(S1, S2, t)
3. As the correlation between the two stocks increases what happens to the value of this option? Explain your answer.