

Equilibrium Asset Returns

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Introduction

- We analyze the Intertemporal Capital Asset Pricing Model (ICAPM) of Robert Merton (1973).
- The standard single-period CAPM holds when investment opportunities are constant, but with changing investment opportunities, a multi-beta ICAPM results.
- Breeden (1979) shows that this multi-beta ICAPM can be written as a single “consumption beta” CAPM (CCAPM).
- We also analyze the general equilibrium production economy model of (Cox, Ingersoll and Ross, 1985) that is useful for pricing contingent claims.

ICAPM Model Assumptions

- Individuals can trade in a risk-free asset paying rate of return of $r(t)$ and in n risky assets whose rates of return are

$$dS_i(t) / S_i(t) = \mu_i(x, t) dt + \sigma_i(x, t) dz_i \quad (1)$$

where $i = 1, \dots, n$, and $(\sigma_i dz_i)(\sigma_j dz_j) = \sigma_{ij} dt$.

- The k state variables follow the process:

$$dx_i = a_i(x, t) dt + b_i(x, t) d\zeta_i \quad (2)$$

where $i = 1, \dots, k$, and $(b_i d\zeta_i)(b_j d\zeta_j) = b_{ij} dt$ and $(\sigma_i dz_i)(b_j d\zeta_j) = \phi_{ij} dt$.

Constant Investment Opportunities

- When r and the μ_i 's, σ_i 's, and σ_{ij} 's are constants, we showed previously that it is optimal for all individuals to choose the risky assets in the relative proportions

$$\delta_k = \frac{\sum_{j=1}^n \nu_{kj}(\mu_j - r)}{\sum_{i=1}^n \sum_{j=1}^n \nu_{ij}(\mu_j - r)} \quad (3)$$

- This “single” risky asset portfolio's mean and variance is

$$\mu \equiv \sum_{i=1}^n \delta_i \mu_i \quad \sigma^2 \equiv \sum_{i=1}^n \sum_{j=1}^n \delta_i \delta_j \sigma_{ij}. \quad (4)$$

CAPM Equilibrium

- Similar to our derivation of the single-period CAPM, we argue that in equilibrium this common risky-asset portfolio must be the market portfolio; that is, $\mu = \mu_m$ and $\sigma^2 = \sigma_m^2$.
- Moreover, the continuous-time market portfolio is exactly the same as that implied by the single-period CAPM.
- Thus, asset returns in this continuous-time environment satisfy the same relationship as the single-period CAPM:

$$\mu_i - r = \beta_i (\mu_m - r), \quad i = 1, \dots, n \quad (5)$$

where $\beta_i = \sigma_{im} / \sigma_m^2$.

CAPM Implications

- Thus, the constant investment opportunity set assumption replicates the standard, single-period CAPM.
- Yet, rather than asset returns being *normally* distributed as in the single-period CAPM, the ICAPM has asset returns that are *lognormally* distributed.
- While the standard CAPM results continue to hold for this more realistic intertemporal environment, the assumptions of a constant risk-free rate and unchanging asset return means and variances are untenable.
- Clearly, interest rates vary over time, as do the volatilities of assets such as common stocks.

Stochastic Investment Opportunities

- For a single state variable, x , the system of n equations that an individual's portfolio weights satisfy are:

$$0 = -A(\mu_i - r) + \sum_{j=1}^n \sigma_{ij} \omega_j^* W - H \phi_i, \quad i = 1, \dots, n \quad (6)$$

where $A = -J_W / J_{WW} = -U_C / [U_{CC} (\partial C / \partial W)]$ and $H = -J_{Wx} / J_{WW} = -(\partial C / \partial x) / (\partial C / \partial W)$.

- Rewrite (6) in matrix form, and use the superscript p to denote the values for the p^{th} person (individual):

$$A^p (\boldsymbol{\mu} - r\mathbf{e}) = \Omega \boldsymbol{\omega}^p W^p - H^p \boldsymbol{\phi} \quad (7)$$

where $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)'$, \mathbf{e} is an n -dimensional vector of ones, $\boldsymbol{\omega}^p = (\omega_1^p, \dots, \omega_n^p)'$ and $\boldsymbol{\phi} = (\phi_1, \dots, \phi_n)'$.

Aggregate Asset Demands

- Sum (7) across all persons and divide by $\sum_p A^p$:

$$\mu - r\mathbf{e} = a\Omega\alpha - h\phi \quad (8)$$

where $a \equiv \sum_p W^p / \sum_p A^p$, $h \equiv \sum_p H^p / \sum_p A^p$, and $\alpha \equiv \sum_p \omega^p W^p / \sum_p W^p$ is the average investment in each asset. These must be the market weights.

- The i^{th} row (i^{th} asset excess return) of equation (8) is

$$\mu_i - r = a\sigma_{im} - h\phi_i \quad (9)$$

- Pre-multiply (8) by α' to obtain

$$\mu_m - r = a\sigma_m^2 - h\sigma_{mx} \quad (10)$$

where $\sigma_{mx} = \alpha'\phi$ is the covariance between the market portfolio and the state variable, x .

Portfolio that Best Hedges the State Variable

- Define $\eta \equiv \frac{\Omega^{-1}\phi}{e'\Omega^{-1}\phi}$, which are the weights of the portfolio that best hedge changes in the state variable, x .
- The expected excess return on this portfolio is found by pre-multiplying (8) by η' :

$$\mu_\eta - r = a\sigma_{\eta m} - h\sigma_{\eta x} \quad (11)$$

where $\sigma_{\eta m}$ and $\sigma_{\eta x}$ are the hedge portfolio's covariances with the market portfolio and the state variable, respectively.

- Solving the two linear equations (10) and (11) for a and h , and substituting them back into equation (9) gives:

$$\mu_i - r = \frac{\sigma_{im}\sigma_{\eta x} - \phi_i\sigma_{m\eta}}{\sigma_m^2\sigma_{\eta x} - \sigma_{mx}\sigma_{m\eta}} (\mu_m - r) + \frac{\phi_i\sigma_m^2 - \sigma_{im}\sigma_{mx}}{\sigma_m^2\sigma_{\eta x} - \sigma_{mx}\sigma_{m\eta}} (\mu_\eta - r) \quad (12)$$

- It can be shown that (12) is equivalent to

$$\begin{aligned}\mu_i - r &= \frac{\sigma_{im}\sigma_\eta^2 - \sigma_{i\eta}\sigma_{m\eta}}{\sigma_m^2\sigma_\eta^2 - \sigma_{m\eta}^2} (\mu_m - r) + \frac{\sigma_{i\eta}\sigma_m^2 - \sigma_{im}\sigma_{m\eta}}{\sigma_\eta^2\sigma_m^2 - \sigma_{m\eta}^2} (\mu_\eta - r) \\ &\equiv \beta_i^m (\mu_m - r) + \beta_i^\eta (\mu_\eta - r)\end{aligned}\quad (13)$$

where $\sigma_{i\eta}$ is the covariance between the return on asset i and that of the hedge portfolio.

- $\sigma_{i\eta} = 0$ iff $\phi_i = 0$. If x is uncorrelated with the market so that $\sigma_{m\eta} = 0$, equation (13) simplifies to

$$\mu_i - r = \frac{\sigma_{im}}{\sigma_m^2} (\mu_m - r) + \frac{\sigma_{i\eta}}{\sigma_\eta^2} (\mu_\eta - r) \quad (14)$$

- Equation (13) generalizes to multiple state variables with an additional “beta” for each state (c.f., APT).

Extension to State-Dependent Utility

- If an individual's utility is affected by the state of economy, so that $U(C_t, x_t, t)$, the form of the first order conditions for consumption (C_t) and the portfolio weights (ω_i) remain unchanged and equation (13) continues to hold.
- The only change is the interpretation of H , the hedging coefficient. Taking the total derivative of envelope condition $J_W = U_C$:

$$J_{Wx} = U_{CC} \frac{\partial C}{\partial x} + U_{Cx} \quad (15)$$

so that

$$H = -\frac{\partial C / \partial x}{\partial C / \partial W} - \frac{U_{Cx}}{U_{CC} \frac{\partial C}{\partial W}} \quad (16)$$

implying that portfolio holdings minimize the variance of marginal utility.

Breeden's Consumption CAPM

- Substitute in for A^p and H^p in equation (7) for the case of k state variables and rearrange to obtain:

$$-\frac{U_C^p}{U_{CC}^p C_W^p} (\mu - re) = \Omega \omega^p W^p + \Phi \mathbf{C}_x^p / C_W^p \quad (17)$$

where $C_W^p = \partial C^p / \partial W^p$, $\mathbf{C}_x^p = \left(\frac{\partial C^p}{\partial x_1} \dots \frac{\partial C^p}{\partial x_k} \right)'$, and Φ is the $n \times k$ asset return - state variable covariance matrix whose i, j^{th} element is ϕ_{ij} . Pre-multiplying (17) by C_W^p :

$$-\frac{U_C^p}{U_{CC}^p} (\mu - re) = \Omega_{\mathbf{W}^p} C_W^p + \Phi \mathbf{C}_x^p \quad (18)$$

where $\Omega_{\mathbf{W}^p}$ is $n \times 1$ vector of covariances between asset returns and investor p 's wealth.

Covariance between Asset Returns and Consumption

- Using Itô's lemma, individual p 's optimal consumption, $C^p(W^p, \mathbf{x}, t)$, follows a process whose stochastic terms for dC^p are

$$C_W^p (\omega_1^p W^p \sigma_1 dz_1 + \dots + \omega_n^p W^p \sigma_n dz_n) + (b_1 d\zeta_1 \quad b_2 d\zeta_2 \dots b_k d\zeta_k) \mathbf{C}_x^p \quad (19)$$

- The covariance of asset returns with changes in individual p 's consumption are given by calculating the covariance between each asset (having stochastic term $\sigma_i dz_i$) with the terms given in (19). Denoting $\Omega_{\mathbf{C}^p}$ as this $n \times 1$ vector of covariances:

$$\Omega_{\mathbf{C}^p} = \Omega_{W^p} C_W^p + \Phi \mathbf{C}_x^p \quad (20)$$

Consumption Covariance

- The right-hand sides of (20) and (18) are equal, implying:

$$\Omega_{C^p} = - \frac{U_C^p}{U_{CC}^p} (\mu - r\mathbf{e}) \quad (21)$$

- Define C as aggregate consumption per unit time and define \mathcal{F} as an aggregate rate of risk tolerance where

$$\mathcal{F} \equiv \sum_p - \frac{U_C^p}{U_{CC}^p} \quad (22)$$

- Then aggregate (21) over all individuals to obtain

$$\mu - r\mathbf{e} = \mathcal{F}^{-1} \Omega_C \quad (23)$$

where Ω_C is the $n \times 1$ vector of covariances between asset returns and changes in aggregate consumption.

Excess Expected Returns on Risky Assets

- Multiply and divide the right-hand side of (23) by current aggregate consumption to obtain:

$$\mu - r\mathbf{e} = (\mathcal{F}/C)^{-1} \Omega_{\ln \mathbf{c}} \quad (24)$$

where $\Omega_{\ln \mathbf{c}}$ is the $n \times 1$ vector of covariances between asset returns and changes in log consumption growth.

- Let a portfolio, m , have weights ω^m that pre-multiply (24):

$$\mu_m - r = (\mathcal{F}/C)^{-1} \sigma_{m, \ln C} \quad (25)$$

where portfolio m 's expected return and covariance with consumption growth is μ_m and $\sigma_{m, \ln C}$. Portfolio m may or may not be the market portfolio.

Consumption CAPM

- Use (25) to substitute for $(\mathcal{F}/C)^{-1}$ in (24):

$$\begin{aligned}\mu - re &= (\Omega_{\ln \mathbf{C}} / \sigma_{m, \ln C}) (\mu_m - r) \\ &= (\beta_{\mathbf{C}} / \beta_{mC}) (\mu_m - r)\end{aligned}\tag{26}$$

where $\beta_{\mathbf{C}}$ and β_{mC} are the “consumption betas” of all asset returns and portfolio m ’s return.

- The consumption beta for any asset is defined as:

$$\beta_{iC} = \text{cov}(dS_i/S_i, d \ln C) / \text{var}(d \ln C)\tag{27}$$

- Equation (26) says that the ratio of expected excess returns on any two assets or portfolios of assets is equal to the ratio of their betas measured relative to aggregate consumption.
- Hence, the risk of a security’s return can be summarized by a single consumption beta.

A Cox, Ingersoll, and Ross Production Economy

- The ICAPM and CCAPM are not general equilibrium models since they take the form of equilibrium asset price processes as given.
- However, their asset price process can be justified based on the Cox, Ingersoll, and Ross (1985) continuous-time, production economy model.
- The model assumes that there is a single good that can be either consumed or invested. This “capital-consumption” good can be invested in any of n different risky technologies that produce an instantaneous change in the amount of this good.

Model Assumptions

- If an amount η_i is physically invested in technology i , then the proportional change in the good produced is

$$\frac{d\eta_i(t)}{\eta_i(t)} = \mu_i(x, t) dt + \sigma_i(x, t) dz_i, \quad i = 1, \dots, n \quad (28)$$

where $(\sigma_i dz_i)(\sigma_j dz_j) = \sigma_{ij} dt$. The rate of change in the invested good produced has expected value and standard deviation of μ_i and σ_i .

- Note that each technology displays “constant returns to scale” and μ_i and σ_i can vary with time and with a $k \times 1$ vector of state variables, $x(t)$.
- Thus, the economy’s technologies for transforming consumption into more consumption reflect changing (physical) investment opportunities.

Model Assumptions cont'd

- The i^{th} state variable is assumed to follow the process

$$dx_i = a_i(x, t) dt + b_i(x, t) d\zeta_i \quad (29)$$

where $i = 1, \dots, k$, and $(b_i d\zeta_i)(b_j d\zeta_j) = b_{ij} dt$ and $(\sigma_i dz_i)(b_j d\zeta_j) = \phi_{ij} dt$.

- If each technology is owned by an individual firm, financed entirely by shareholders' equity, then the rate of return on shareholders' equity of firm i , $dS_i(t)/S_i(t)$, equals the proportional change in the value of the firm's physical assets (capital), $d\eta_i(t)/\eta_i(t)$.
- Here, $dS_i(t)/S_i(t) = d\eta_i(t)/\eta_i(t)$ equals the instantaneous dividend yield where dividends come in the form of a physical capital-consumption good.

Model Assumptions cont'd

- CIR's specification allows one to solve for the equilibrium prices of securities other than those represented by the n risky technologies.
- This is done by imagining there to be other securities that have zero net supplies.
- Assuming all individuals are identical in preferences and wealth, this amounts to the riskless investment having a zero supply in the economy, so that r is really a “shadow” riskless rate.
- Yet, this rate would be consistent, in equilibrium, with the specification of the economy's other technologies.

Solving for the Equilibrium Riskless Rate

- An individual's consumption and portfolio choice problem allocates savings to firms investing in the n technologies.
- An equilibrium is defined as a set of interest rate, consumption, and portfolio weight processes $\{r, C^*, \omega_1^*, \dots, \omega_n^*\}$ such that the individual's first order conditions hold and markets clear: $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \geq 0$.
- Since the capital-consumption good is physically invested in the technologies, the constraint against short-selling, $\omega_i \geq 0$, applies.
- Because, in equilibrium, the representative individuals do not borrow or lend, the situation is exactly as if a riskless asset did not exist.

Optimization without a Riskless Asset

- Consider the individual's problem as before except that the process for wealth excludes a risk-free asset:

$$\max_{C_s, \{\omega_{i,s}\}, \forall s, i} E_t \left[\int_t^T U(C_s, s) ds + B(W_T, T) \right] \quad (30)$$

subject to

$$dW = \sum_{i=1}^n \omega_i W \mu_i dt - C_t dt + \sum_{i=1}^n \omega_i W \sigma_i dz_i \quad (31)$$

and $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \geq 0$. The first order condition for consumption is

$$0 = \frac{\partial U(C^*, t)}{\partial C} - \frac{\partial J(W, x, t)}{\partial W} \quad (32)$$

Optimal Portfolio Weights with Short Sale Constraints

- Let λ be the Lagrange multiplier for $\sum_{i=1}^n \omega_i = 1$. Then the first order conditions for ω_i is

$$\begin{aligned}\Psi_i &\equiv \frac{\partial J}{\partial W} \mu_i W + \frac{\partial^2 J}{\partial W^2} \sum_{j=1}^n \sigma_{ij} \omega_j^* W^2 + \sum_{j=1}^k \frac{\partial^2 J}{\partial x_j \partial W} \phi_{ij} W - \lambda \leq 0 \\ 0 &= \Psi_i \omega_i^* \quad i = 1, \dots, n\end{aligned}\quad (33)$$

- Kuhn-Tucker conditions (33) imply that if $\Psi_i < 0$, then $\omega_i^* = 0$, so that i^{th} technology is not employed.
- Assuming (28) and (29) are such that all technologies are employed so $\Psi_i = 0 \forall i$, then the solution is

$$\omega_i^* = -\frac{J_W}{J_{WW} W} \sum_{j=1}^n \nu_{ij} \mu_j - \sum_{m=1}^k \sum_{j=1}^n \frac{J_{Wx_m}}{J_{WW} W} \nu_{ij} \phi_{jm} + \frac{\lambda}{J_{WW} W^2} \sum_{j=1}^n \nu_{ij} \quad (34)$$

Equilibrium Portfolio Allocation

- Using matrix notation, (34) becomes

$$\omega^* = \frac{A}{W} \Omega^{-1} \mu - \frac{A\lambda}{J_W W^2} \Omega^{-1} \mathbf{e} + \sum_{j=1}^k \frac{H_j}{W} \Omega^{-1} \phi_j \quad (35)$$

where $A = -J_W / J_{WW}$, $H_j = -J_{Wx_j} / J_{WW}$, and $\phi_j = (\phi_{1j}, \dots, \phi_{nj})'$.

- The weights are a linear combination of $k + 2$ portfolios.
- The first two portfolios are mean-variance efficient portfolios: $\Omega^{-1} \mu$ is the portfolio on the efficient frontier that is tangent to a line drawn from the origin (a zero interest rate) while $\Omega^{-1} \mathbf{e}$ is the global minimum variance portfolio.
- The last k portfolios, $\Omega^{-1} \phi_j$, $j = 1, \dots, k$, hedge against changes in the technological investment opportunities.

Equilibrium Consumption and Portfolio Weights

- The proportions of these $k + 2$ portfolios chosen depend on the individual's utility.
- An exact solution is found in the usual manner of substituting (35) and $C^* = G(J_W)$ into the Bellman equation.
- For specific functional forms, a value for the indirect utility function, $J(W, x, t)$ can be derived. This, along with the restriction $\sum_{i=1}^n \omega_i = 1$, determines the specific optimal consumption and portfolio weights.
- Since in the CIR economy the riskless asset is in zero net supply, portfolio weights in (35) must be those chosen by the representative individual even if offered the opportunity to borrow or lend at rate r .

Portfolio Allocation with a Riskless Asset

- Recall that portfolio weights for the case of including a risk-free asset are

$$\omega^* = \frac{A}{W} \Omega^{-1} (\mu - r\mathbf{e}) + \sum_{j=1}^k \frac{H_j}{W} \Omega^{-1} \phi_j, \quad i = 1, \dots, n \quad (36)$$

- (35) and (36) are identical when $r = \lambda / (J_W W)$. Hence,

$$\begin{aligned} r &= \frac{\lambda}{W J_W} \\ &= \omega^{*'} \mu - \frac{W}{A} \omega^{*'} \Omega \omega^* + \sum_{j=1}^k \frac{H_j}{A} \omega^{*'} \phi_j \end{aligned} \quad (37)$$

- (37) is the same as (10) extended to k state variables.
- Hence, the ICAPM and CCAPM hold for the CIR economy.

Valuing Contingent Claims in Zero Net Supply

- The CIR model also can be used to find the equilibrium “shadow prices” of other contingent claims.
- Suppose the payoff of a zero-net-supply contingent claim depends on wealth, time, and the state variables, so that $P(W, t, \{x_i\})$. Itô's lemma implies

$$dP = uPdt + P_W W \sum_{i=1}^n \omega_i^* \sigma_i dz_i + \sum_{i=1}^k P_{x_i} b_i d\zeta_i \quad (38)$$

where

$$\begin{aligned} uP = & P_W (W\omega^{*\prime}\mu - C) + \sum_{i=1}^k P_{x_i} a_i + P_t + \frac{P_{WW} W^2}{2} \omega^{*\prime} \Omega \omega^* \\ & + \sum_{i=1}^k P_{Wx_i} W\omega^{*\prime} \phi_i + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k P_{x_i x_j} b_{ij} \end{aligned} \quad (39)$$

Expected Return under ICAPM

- The ICAPM relation (9) extended to k states is

$$u = r + \frac{W}{A} \text{Cov}(dP/P, dW/W) - \sum_{i=1}^k \frac{H_i}{A} \text{Cov}(dP/P, dx_i) \quad (40)$$

- Noting that the representative agent's wealth is the market portfolio:

$$\begin{aligned} uP &= rP + \frac{1}{A} \text{Cov}(dP, dW) - \sum_{i=1}^k \frac{H_i}{A} \text{Cov}(dP, dx_i) \\ &= rP + \frac{1}{A} \left(P_W W^2 \omega^{*'} \Omega \omega^* + \sum_{i=1}^k P_{x_i} W \omega^{*'} \phi_i \right) \\ &\quad - \sum_{i=1}^k \frac{H_i}{A} \left(P_W W \omega^{*'} \phi_i + \sum_{j=1}^k P_{x_j} b_{ij} \right) \end{aligned} \quad (41)$$

Equilibrium Partial Differential Equation

- Equating (39) and (41) and recalling the value of the equilibrium risk-free rate in (37), we obtain a partial differential equation for the contingent claim's value:

$$\begin{aligned} 0 = & \frac{P_{WW} W^2}{2} \omega^{*'} \Omega \omega^* + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k P_{x_i x_j} b_{ij} + \\ & \sum_{i=1}^k P_{W x_i} W \omega^{*'} \phi_i + P_W (rW - C) + P_t \\ & \sum_{i=1}^k P_{x_i} \left[a_i - \frac{W}{A} \omega^{*'} \phi_i + \sum_{j=1}^k \frac{H_j b_{ij}}{A} \right] - rP \quad (42) \end{aligned}$$

An Example with Log Utility

- Let $U(C_s, s) = e^{-\rho s} \ln(C_s)$ and $B(W_T, T) = e^{-\rho T} \ln(W_T)$.
- Then we previously showed that $J(W, x, t) = d(t) e^{-\rho t} \ln(W_t) + F(x, t)$ where $d(t) = \frac{1}{\rho} [1 - (1 - \rho) e^{-\rho(T-t)}]$, so that $C_t^* = W_t/d(t)$.
- Since $J_{Wx_i} = 0$, $H_i = 0$, and $A = W$, the portfolio weights in (35) are

$$\omega^* = \Omega^{-1} (\mu - re) \quad (43)$$

where $r = \lambda / (J_W W)$.

Market Portfolio Weights

- Market clearing requires $\mathbf{e}'\boldsymbol{\omega}^* = 1$, and solving for $r(t)$:

$$r = \frac{\mathbf{e}'\Omega^{-1}\boldsymbol{\mu} - 1}{\mathbf{e}'\Omega^{-1}\mathbf{e}} \quad (44)$$

- Substituting (44) into (43), the equilibrium portfolio weights are:

$$\boldsymbol{\omega}^* = \Omega^{-1} \left[\boldsymbol{\mu} - \left(\frac{\mathbf{e}'\Omega^{-1}\boldsymbol{\mu} - 1}{\mathbf{e}'\Omega^{-1}\mathbf{e}} \right) \mathbf{e} \right] \quad (45)$$

Square Root Process State Variable

- Assume that a single state variable, $x(t)$, affects all production processes:

$$d\eta_i/\eta_i = \hat{\mu}_i x dt + \hat{\sigma}_i \sqrt{x} dz_i, \quad i = 1, \dots, n \quad (46)$$

where $\hat{\mu}_i$ and $\hat{\sigma}_i$ are assumed to be constants and the state variable follows the *square root process*

$$dx = (a_0 + a_1 x) dt + b_0 \sqrt{x} d\zeta \quad (47)$$

where $dz_i d\zeta = \rho_i dt$.

- If $a_0 > 0$ and $a_1 < 0$, x is a nonnegative, mean-reverting random variable.

Equilibrium Interest Rate Process

- Write the technologies' $n \times 1$ vector of expected rates of return as $\mu = \hat{\mu}x$ and their $n \times n$ matrix of rate of return covariances as $\Omega = \hat{\Omega}x$.
- Then from (44), the equilibrium interest rate is

$$r = \frac{\mathbf{e}'\hat{\Omega}^{-1}\hat{\mu} - 1}{\mathbf{e}'\hat{\Omega}^{-1}\mathbf{e}}x = \alpha x \quad (48)$$

where $\alpha \equiv (\mathbf{e}'\hat{\Omega}^{-1}\hat{\mu} - 1) / \mathbf{e}'\hat{\Omega}^{-1}\mathbf{e}$ is a constant.

- Thus, the risk-free rate follows the “square root” process:

$$dr = \alpha dx = \kappa(\bar{r} - r)dt + \sigma\sqrt{r}d\zeta \quad (49)$$

where $\kappa \equiv -a_1 > 0$, $\bar{r} \equiv -\alpha a_0 / a_1 > 0$, and $\sigma \equiv b_0\sqrt{\alpha}$.

- When $2\kappa\bar{r} \geq \sigma^2$, if $r(t) > 0$ it remains positive at all future dates, as would a nominal interest rate.

Valuing Default-Free Discount Bonds

- Consider the price of a default-free bond that matures at $T \geq t$.
- P_W , P_{WW} , and P_{Wx} in (42) are zero, and since $r = \alpha x$, the bond's price can be written as $P(r, t, T)$. The PDE (42) becomes

$$\frac{\sigma^2 r}{2} P_{rr} + [\kappa(\bar{r} - r) - \psi r] P_r - rP + P_t = 0 \quad (50)$$

where $\psi \equiv \hat{\omega}' \hat{\phi}$. $\hat{\omega}$ equals the right-hand side of equation (45) but with μ replaced by $\hat{\mu}$ and Ω replaced by $\hat{\Omega}$, while $\hat{\phi}$ is an $n \times 1$ vector whose i^{th} element is $\sigma \hat{\sigma}_i \rho_i$.

- $\psi r = \omega^{*'} \phi$ is the covariance of interest rate changes with (market) wealth, or the interest rate's "beta".

Solution for Bond Prices

- Solving PDE (50) subject to $P(r, T, T) = 1$ leads to

$$P(r, t, T) = A(\tau) e^{-B(\tau)r} \quad (51)$$

where $\tau = T - t$, $\theta \equiv \sqrt{(\kappa + \psi)^2 + 2\sigma^2}$,

$$A(\tau) \equiv \left[\frac{2\theta e^{(\theta + \kappa + \psi)\frac{\tau}{2}}}{(\theta + \kappa + \psi)(e^{\theta\tau} - 1) + 2\theta} \right]^{2\kappa\bar{r}/\sigma^2} \quad (52)$$

$$B(\tau) \equiv \frac{2(e^{\theta\tau} - 1)}{(\theta + \kappa + \psi)(e^{\theta\tau} - 1) + 2\theta} \quad (53)$$

- Note that the bond price is derived from an equilibrium model of preferences and technologies rather than the absence of arbitrage (c.f., Vasicek).

Equilibrium Bond Price Process

- Note that Itô's lemma implies the bond price follows

$$\begin{aligned}dP &= P_r dr + \frac{1}{2} P_{rr} \sigma^2 r dt + P_t dt \\&= \left(\frac{1}{2} P_{rr} \sigma^2 r + P_r [\kappa (\bar{r} - r)] + P_t \right) dt + P_r \sigma \sqrt{r} d\zeta\end{aligned}\tag{54}$$

- From (50) $\frac{1}{2} P_{rr} \sigma^2 r + P_r [\kappa (\bar{r} - r)] + P_t = rP + \psi r P_r$, so that

$$\begin{aligned}dP/P &= r \left(1 + \psi \frac{P_r}{P} \right) dt + \frac{P_r}{P} \sigma \sqrt{r} d\zeta \\&= r (1 - \psi B(\tau)) dt - B(\tau) \sigma \sqrt{r} d\zeta\end{aligned}\tag{55}$$

where from (51) we substituted $P_r/P = -B(\tau)$.

Equilibrium Market Price of Interest Rate Risk

- Hence,

$$\frac{\mu_p(r, \tau) - r}{\sigma_p(r, \tau)} = \frac{-\psi r B(\tau)}{\sigma \sqrt{r} B(\tau)} = -\frac{\psi \sqrt{r}}{\sigma} \quad (56)$$

- Thus we see that the market price of interest rate risk is proportional to the square root of the interest rate.
- When $\psi < 0$, which occurs when the interest rate is negatively correlated with the return on the market portfolio (and bond prices are positively correlated with the market portfolio), bonds will carry a positive risk premium.

Summary

- The Merton ICAPM shows that when investment opportunities are constant, the expected returns on assets satisfy the single-period CAPM
- In general, an asset's risk premia include the asset's covariances with asset portfolios that hedge against changes in investment opportunities.
- The multi-beta ICAPM can be simplified to a single consumption beta CCAPM.
- The Cox, Ingersoll, and Ross model shows the ICAPM results are consistent with a general equilibrium production economy.
- The model also is used to derive the equilibrium interest rate and the shadow prices of securities in zero net supply.