FIN 591: Homework #3

Due on Wednesday, April 11, 2018

Wanbae Park

Problem 1

a. Since the final payoff of P is 1, using continuous-time version stochastic discount factor, $P_t(\tau)$ is derived as follows.

$$\begin{aligned} P_t(\tau) &= \mathrm{E}_t\left[\frac{U_c(C_{t+\tau,t+\tau})}{U_c(C_t,t)} \times 1\right] \\ &= \mathrm{E}_t\left[\frac{e^{-\phi(t+\tau)}C_{t+\tau}^{\gamma-1}}{e^{\phi t}C_t^{\gamma-1}}\right] \\ &= \mathrm{E}_t\left[e^{-\phi\tau}\frac{C_{t+\tau}^{\gamma-1}}{C_t^{\gamma-1}}\right] \end{aligned} \tag{1}$$

b. From $P_t(\tau) = \mathrm{E}_t\left[\frac{e^{-\phi(t+\tau)}C_{t+\tau}^{\gamma-1}}{e^{\phi t}C_t^{\gamma-1}}\right]$, we can find that process M_t is equal to $e^{-\phi t}C_t^{\gamma-1}$. Therefore, using Ito's lemma, dynamics of M_t can be derived as equation (2).

$$\begin{aligned} dM_t &= -\phi e^{-\phi t}C^{\gamma-1}dt + e^{-\phi t}(\gamma-1)C^{\gamma-2}C[(\mu_c-\lambda k)dt + \sigma_c dZ_c] \\ &\quad + \frac{1}{2}e^{-\phi t}(\gamma-1)(\gamma-2)C^2C^{\gamma-3}\sigma_c^2 dt + [e^{-\phi t}(YC)^{\gamma-1} - e^{-\phi t}C^{\gamma-1}]dq \\ &= [-\phi + (\gamma-1)(\mu_c-\lambda k) + \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2]Mdt + (\gamma-1)\sigma_c M dZ_c + (Y^{\gamma-1}-1)Mdq \end{aligned} \tag{2}$$

c. Since $\mathrm{E}\left[\frac{dM}{M}\right] = -rdt$, the following equation holds.

$$\begin{aligned} r &= -\mathrm{E}\left[\frac{dM}{M}\right] \\ &= \phi - (\gamma-1)(\mu_c-\lambda k) - \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2 - \lambda \mathrm{E}[e^{(\gamma-1)\log Y} - 1] \\ &= \phi - (\gamma-1)(\mu_c-\lambda k) - \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2 - \lambda(e^{(\gamma-1)\alpha + \frac{1}{2}(\gamma-1)^2\delta^2} - 1) \end{aligned} \tag{3}$$

Since μ_c, k, λ are constant, instantaneous risk free rate is constant.

d. Since an asset price is equal to sum of discounted future payoffs, assuming some regularity conditions hold, S_t is represented as follows.

$$\begin{aligned} S_t &= \mathrm{E}_t\left[\int_t^\infty \frac{M_s}{M_t}D_s ds\right] \\ &= \mathrm{E}_t\left[\int_t^\infty \frac{e^{-\phi s}C_s^{\gamma-1}}{e^{-\phi t}C_t^{\gamma-1}}D_s ds\right] \\ \Rightarrow \frac{S_t}{D_t} &= \mathrm{E}_t\left[\int_t^\infty e^{-\phi(s-t)}\left(\frac{C_s}{C_t}\right)^{\gamma-1}\left(\frac{D_s}{D_t}\right)ds\right] \\ &= \mathrm{E}_t\left[\int_t^\infty e^{-\phi(s-t)+(\gamma-1)\log(C_s/C_t)+\log(D_s/D_t)}ds\right] \\ &= \int_t^\infty \mathrm{E}_t\left[e^{-\phi(s-t)+(\gamma-1)\log(C_s/C_t)+\log(D_s/D_t)}ds\right] \end{aligned} \tag{4}$$

Considering the process of C_t, $\mathrm{E}_t[e^{(\gamma-1)\log(C_s/C_t)}]$ is calculated as follows.

$$\begin{aligned} \mathrm{E}_t[e^{(\gamma-1)\log(C_s/C_t)}] &= \mathrm{E}_t\left[e^{(\gamma-1)(\mu_c-\frac{1}{2}\sigma_c^2-\lambda k)(s-t)+\frac{1}{2}(\gamma-1)^2\sigma_c^2(s-t)+(\gamma-1)\log y(s,t)}\right] \\ y(s,t) &= \prod_{i=s}^{t} Y_i \end{aligned} \tag{5}$$

Problem 1 continued on next page...

Since $\log y(s,t) = \sum_{i=s}^{t} \log Y_i$, and $\log Y_i$'s are independently and identically distributed as $N(\alpha, \delta^2)$, $\log(C_s/C_t)$ is also normally distributed, and its expected value from equation (5) is calculated as follows.

$$\mathrm{E}_t[e^{(\gamma-1)\log(C_s/C_t)}] = e^{(\gamma-1)(\mu_c-\frac{1}{2}\sigma_c^2-\lambda k)(s-t)+\frac{1}{2}(\gamma-1)^2\sigma_c^2(s-t)+(\gamma-1)(\alpha+\frac{1}{2}\delta^2)(s-t)} \tag{6}$$

Applying the result from equation (6) and considering the correlation between z_d and z_c is ρ, $\frac{S_t}{D_t}$ from equation (4) is solved as follows.

$$\begin{aligned}
\frac{S_t}{D_t} &= \int_t^{\infty} e^{-(s-t)[\phi+(1-\gamma)(\mu_c-\frac{1}{2}\sigma_c^2-\lambda k)+\frac{1}{2}(1-\gamma)^2\sigma_c^2+(1-\gamma)(\alpha+\frac{1}{2}\delta^2)-\mu_d+(1-\gamma)\rho\sigma_c\sigma_d]}ds \\
&= -\frac{1}{A}e^{-(s-t)A}\bigg|_t^{\infty} = \frac{1}{A} \\
A &= \phi + (1-\gamma)(\mu_c - \frac{1}{2}\sigma_c^2 - \lambda k) + \frac{1}{2}(1-\gamma)^2\sigma_c^2 + (1-\gamma)(\alpha + \frac{1}{2}\delta^2) - \mu_d + (1-\gamma)\rho\sigma_c\sigma_d
\end{aligned} \tag{7}$$

Problem 2

a. Considering the process of risky asset price, intertemporal budget constraint is derived as follows.

$$\begin{aligned}
dW &= \omega\frac{dS}{S} + (1-\omega)rdt - Cdt \\
&= (\omega(\mu - \lambda k - r)W + rW - C)dt + \sigma W dZ + \omega(Y-1)W dq
\end{aligned} \tag{8}$$

b. Investors maximize $\mathrm{E}_0[\int_0^T e^{-\phi t}u(C_t)dt]$, subject to $dW = (\omega(\mu-\lambda k-r)W+rW)dt+\sigma W dZ+\omega(Y-1)W dq$. Let $J(W,s) = \max_{C,\omega} \mathrm{E}_s[\int_0^T e^{-\phi t}u(C_t)dt]$. Then the following equation follows.

$$\begin{aligned}
J(W,0) &= \max_{C,\omega} \mathrm{E}_0\left[\int_0^{\Delta t} e^{-\phi t}u(C_t)dt + J(W,\Delta t)\right] \\
&= \max_{C,\omega} \mathrm{E}_0[u(C_0)\Delta t + J(W,0) + J_W(\omega(\mu-\lambda k-r)W + rW - C)\Delta t \\
&+ \frac{1}{2}\omega^2\sigma^2 J_{WW}\Delta t + (J(\omega(Y-1)W,0) - J(W,0))dq]
\end{aligned} \tag{9}$$

Letting $\Delta t \to 0$, equation (9) becomes equation (10), and it is Bellman equation.

$$0 = \max_{C,\omega}[u(c_0) + J_W(\omega(\mu-\lambda k-r)W + rW - C) + \frac{1}{2}\omega^2\sigma^2 J_{WW} + \lambda(\mathrm{E}[J(\omega(Y-1)W,0)] - J(W,0))] \tag{10}$$

c. Applying first order condition to equation (10), the following equation holds.

$$\begin{aligned}
u_c &= J_W \\
J_W(\mu-\lambda k-r)W + \omega\sigma^2 J_{WW} + \frac{\partial}{\partial\omega}(\lambda(\mathrm{E}[J(\omega(Y-1)W,0)] - J(W,0))]) &= 0
\end{aligned} \tag{11}$$