# FIN 591: Homework #3

Due on Wednesday, April 11, 2018

**Wanbae Park**

# Problem 1

a. Since the final payoff of $P$ is 1, using continuous-time version stochastic discount factor, $P_t(\tau)$ is derived as follows.

$$\begin{aligned} P_t(\tau) &= \mathrm{E}_t\left[\frac{U_c(C_{t+\tau,t+\tau})}{U_c(C_t,t)} \times 1\right] \\ &= \mathrm{E}_t\left[\frac{e^{-\phi(t+\tau)}C_{t+\tau}^{\gamma-1}}{e^{\phi t}C_t^{\gamma-1}}\right] \\ &= \mathrm{E}_t\left[e^{-\phi\tau}\frac{C_{t+\tau}^{\gamma-1}}{C_t^{\gamma-1}}\right] \end{aligned} \tag{1}$$

b. From $P_t(\tau) = \mathrm{E}_t\left[\frac{e^{-\phi(t+\tau)}C_{t+\tau}^{\gamma-1}}{e^{\phi t}C_t^{\gamma-1}}\right]$, we can find that process $M_t$ is equal to $e^{-\phi t}C_t^{\gamma-1}$. Therefore, using Ito's lemma, dynamics of $M_t$ can be derived as equation (2).

$$\begin{aligned} dM_t &= -\phi e^{-\phi t}C^{\gamma-1}dt + e^{-\phi t}(\gamma-1)C^{\gamma-2}C[(\mu_c - \lambda k)dt + \sigma_c dZ_c] \\ &\quad + \frac{1}{2}e^{-\phi t}(\gamma-1)(\gamma-2)C^2C^{\gamma-3}\sigma_c^2 dt + [e^{-\phi t}(YC)^{\gamma-1} - e^{-\phi t}C^{\gamma-1}]dq \\ &= [-\phi + (\gamma-1)(\mu_c - \lambda k) + \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2]Mdt + (\gamma-1)\sigma_c M dZ_c + (Y^{\gamma-1}-1)Mdq \end{aligned} \tag{2}$$

c. Since $\mathrm{E}\left[\frac{dM}{M}\right] = -rdt$, the following equation holds.

$$\begin{aligned} r &= -\mathrm{E}\left[\frac{dM}{M}\right] \\ &= \phi - (\gamma-1)(\mu_c - \lambda k) - \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2 - \lambda\mathrm{E}[e^{(\gamma-1)\log Y} - 1] \\ &= \phi - (\gamma-1)(\mu_c - \lambda k) - \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2 - \lambda(e^{(\gamma-1)\alpha + \frac{1}{2}(\gamma-1)^2\delta^2} - 1) \end{aligned} \tag{3}$$

Since $\mu_c, k, \lambda$ are constant, instantaneous risk free rate is constant.

d.