FIN 591: Homework #1

Due on Monday, February 5, 2018

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Problem 1

a. *(Short selling restriction)* If an investor cannot sell short risky asset, the amount of investment in risky asset should be nonnegative. Therefore, the maximization problem is stated as the following equation.

$$\begin{aligned} &\max_A \mathrm{E}[U(\widetilde{W})] = \max_A \mathrm{E}[U(W_0(1+r_f) + A(\widetilde{r} - r_f))] \\ &\text{subject to } A \geq 0 \end{aligned} \tag{1}$$

The maximization problem (1) is equivalent to

$$\max_A \mathrm{E}[U(W_0(1+r_f) + A(\widetilde{r} - r_f))] + \lambda A \tag{2}$$

By applying Kuhn-Tucker conditions, the value of A maximizing expected utility must must satisfy the following conditions.

$$\begin{aligned} \mathrm{E}[U'(\widetilde{W})(\widetilde{r} - r_f)] + \lambda = 0 \\ A \geq 0 \\ A\mathrm{E}[U'(\widetilde{W})(\widetilde{r} - r_f)] = 0 \end{aligned} \tag{3}$$

b. *(Restriction of riskless borrowing)* A restriction of riskless borrowing implies that the dollar amount of investment in riskless asset should be nonnegative. It leads to the following maximization problem.

$$\begin{aligned} &\max_A \mathrm{E}[U(\widetilde{W})] = \max_A \mathrm{E}[U(W_0(1+r_f) + A(\widetilde{r} - r_f))] \\ &\text{subject to } W_0 - A \geq 0 \end{aligned} \tag{4}$$

From the analogy of problem 1.a, the value of A which maximizes expected utility must satisfy the following first order conditions.

$$\begin{aligned} \mathrm{E}[U'(\widetilde{W})(\widetilde{r} - r_f)] + \lambda = 0 \\ W_0 - A \geq 0 \\ (W_0 - A)\mathrm{E}[U'(\widetilde{W})(\widetilde{r} - r_f)] = 0 \end{aligned} \tag{5}$$

Problem 2

It does not necessary that two frontier portfolios which creates other efficient portfolio must be efficient. To explain it, let us consider a simple economy. In the economy, there are two assets, whose return and variance is denoted as r_i, σ_i^2, $i = 1, 2$, respectively. Let ρ denote the correlation coefficient between r_1 and r_2. Now, construct a portfolio such that $r_p = w_1 r_1 + (1 - w_1) r_2$. In this case, the expected return of the portfolio is $w_1 \mathrm{E}[r_1] + (1 - w_2)\mathrm{E}[r_2]$, which is exactly on the straight line connecting $\mathrm{E}[r_1]$ to $\mathrm{E}[r_2]$ because it is just a weighted average of expected returns. However, since $\mathrm{Var}[r_p] = \mathrm{Var}[w_1 r_1 + (1 - w_1) r_2] = w_1^2 \sigma_1^2 + (1 - w_1)^2 \sigma_2^2 + \rho \sigma_1 \sigma_2$, variance of portfolio can be greater or less than weighted average of individual variances, depending on correlation coefficient, ρ. Therefore, if we properly choose portfolios which have negative correlation, we can create a portfolio which is on slightly left to the straight line which connects $(\mathrm{E}[r_1], \sigma_1)$ and $(\mathrm{E}[r_2], \sigma_2)$ on $(\mathrm{E}[r], \sigma)$-space. (Actually, it justifies diversification effect.) It gives an intuition that we can create minimum variance portfolio(or similar portfolios) by appropriately choosing an efficient portfolio and an inefficient portfolio. Since minimum variance portfolio is also an efficient portfolio, it is possible to generate an efficient portfolio using two portfolios even if they are not both efficient.

Problem 3

Problem 4

a. Since risk-free asset does exist, the efficient frontier is a straight line whose intercept is risk-free rate. In addition, it is tangent to the frontier including only risky assets. (Even if there are only two risky assets, there exist a curve-shaped frontier by two-fund separation theorem.) Because Sharpe ratio is a slope of frontier, it is maximized when the frontier is efficient. Therefore, in order to derive maximum Sharpe ratio, it suffices to derive (r_t, σ_t), which is expected return and volatility of tangency portfolio. Since the sum of weight of risky asset on tangency portfolio is equal to 1, the following equation holds.

$$\begin{aligned} e'w &= e'[\lambda V^{-1}(\overline{R} - R_f e)] = 1 \\ &\Rightarrow \lambda = [e'V^{-1}(\overline{R} - R_f e)]^{-1} \end{aligned} \tag{6}$$

$$\overline{R}_t = R_f + \frac{(\overline{R} - R_f e)'V^{-1}(\overline{R} - R_f e)}{e'V^{-1}(\overline{R} - R_f e)} \tag{7}$$

$$\begin{aligned} \sigma_t^2 &= w'Vw \\ &= \lambda^2(\overline{R} - R_f e)'V^{-1}(\overline{R} - R_f e) \\ &= \frac{(\overline{R} - R_f e)'V^{-1}(\overline{R} - R_f e)}{(e'V^{-1}(\overline{R} - R_f e))^2} \\ \Rightarrow \sigma_t &= \frac{\sqrt{\overline{R} - R_f e)'V^{-1}(\overline{R} - R_f e)}}{e'V^{-1}(\overline{R} - R_f e)} \end{aligned} \tag{8}$$

Combining equation (7) and (8), Sharpe ratio of the portfolio is calculated as

$$\begin{aligned} \frac{\overline{R}_t - R_f}{\sigma_t^2} &= \frac{(\overline{R} - R_f e)'V^{-1}(\overline{R} - R_f e)}{e'V^{-1}(\overline{R} - R_f e)} / \frac{\sqrt{(\overline{R} - R_f e)'V^{-1}(\overline{R} - R_f e)}}{e'V^{-1}(\overline{R} - R_f e)} \\ &= \sqrt{(\overline{R} - R_f e)'V^{-1}(\overline{R} - R_f e)} \end{aligned} \tag{9}$$

Plug $\overline{R} - R_f e = (\overline{R}_1 - R_f, \overline{R}_2 - R_f)'$ and $V^{-1} = \begin{bmatrix} 1/\sigma_1^2 & -1/\sigma_1\sigma_2 \\ -1/\sigma_1\sigma_2 & 1/\sigma_2^2 \end{bmatrix} /(1-\rho^2)$ into equation (9), we can get the following result which is desired.

$$\frac{\overline{R}_t - R_f}{\sigma_t^2} = \frac{s_1 - s_2}{\sqrt{1-\rho^2}} \tag{10}$$

b.