

# **FIN 591: Homework #2**

Due on Wednesday, February 28, 2018

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## Problem 1

a.

b.

## Problem 2

a. Under the optimal choice,  $U_C(C_{T-1}, T-1) = E_{T-1}[B_W(W_T, T)R_{T-1}]$  holds. If we plug given utility and bequest function to the equation, the following equation holds.

$$\begin{aligned}\delta^{T-1}C_{T-1}^{\gamma-1} &= E_{T-1}[\delta^T W_T^{\gamma-1} R_{T-1}] \\ \Rightarrow \delta^{T-1}C_{T-1}^{\gamma-1} &= \delta^T S_{T-1}^{\gamma-1} E[R_{T-1}^\gamma] \quad \text{where } W_T = S_{T-1}R_{T-1}, S_{T-1} = W_{T-1} - C_{T-1}\end{aligned}\quad (1)$$

Therefore, if we rearrange the equation (1), the optimal consumption at time  $T-1$ ,  $C_{T-1}^*$  can be obtained as follows.

$$C_{T-1}^* = \frac{\delta^{\frac{1}{\gamma-1}} E_{T-1}[R_{T-1}^\gamma]^{\frac{1}{\gamma-1}}}{1 - \delta^{\frac{1}{\gamma-1}} E_{T-1}[R_{T-1}^\gamma]^{\frac{1}{\gamma-1}}} W_{T-1} \quad (2)$$

Another condition under optimal choice is  $E_{T-1}[B_W(W_T, T)(R_{i,T-1} - R_f)] = 0$  for  $i = 1, 2, 3, \dots, n$ . Therefore, the following equation holds.

$$\begin{aligned}E_{T-1}[\delta^T W_T^{\gamma-1} R_{i,T-1}] &= R_f E_{T-1}[\delta^T W_T^{\gamma-1}] \\ \Rightarrow E_{T-1}[(S_{T-1}R_{T-1})^{\gamma-1} R_{i,T-1}] &= R_f E_{T-1}[\delta^T (S_{T-1}R_{T-1})^{\gamma-1}] \\ \Rightarrow E_{T-1}[R_{T-1}^{\gamma-1} R_{i,T-1}] &= R_f E_{T-1}[R_{T-1}^{\gamma-1}]\end{aligned}\quad (3)$$

b. Let  $\delta^{\frac{1}{\gamma-1}} E_{T-1}[R_{T-1}^\gamma]^{\frac{1}{\gamma-1}} = a$ . Then  $C_{T-1}^* = \frac{a}{1+a} W_{T-1}$ . Since  $J(W_{T-1}, T-1) = U(C_{T-1}^*, T-1) = E_{T-1}[B(W_T, T)]$ ,  $J(W_{T-1}, T-1)$  can be represented as follows.

$$\begin{aligned}J(W_{T-1}, T-1) &= \frac{\delta^{T-1} C_{T-1}^{*\gamma}}{\gamma} + E_{T-1}\left[\frac{\delta^T W_T^\gamma}{\gamma}\right] \\ &= \frac{\delta^{T-1}}{\gamma} \left(\frac{a}{1+a} W_{T-1}\right)^\gamma + \frac{\delta^T}{\gamma} E_{T-1}\left[\left((1 - \frac{a}{1+a}) W_{T-1} R_{T-1}\right)^\gamma\right] \\ &= \frac{\delta^{T-1}}{\gamma} \left(\left(\frac{a}{1+a}\right)^\gamma W_{T-1}^\gamma + \delta \left(\frac{1}{1+a}\right)^\gamma W_{T-1}^\gamma E_{T-1}[R_{T-1}^\gamma]\right) \\ &= \frac{\delta^{T-1}}{\gamma} \left(\frac{1}{1+a}\right)^\gamma (a^\gamma + \delta E_{T-1}[R_{T-1}^\gamma]) W_{T-1}^\gamma\end{aligned}\quad (4)$$

c. Let  $\frac{1}{1+a} (a^\gamma + \delta E_{T-1}[R_{T-1}^\gamma]) = b$ . Then  $J(W_{T-1}, T-1)$  can be represented as  $\frac{\delta^{T-1}}{\gamma} b W_{T-1}^\gamma$ . Since under optimal choice,  $U_C(C_{T-2}, T-2) = E_{T-2}[J_W(W_{T-1}, T-1)R_{T-2}]$  holds, the following equation must hold.

$$\begin{aligned}\delta^{T-2} C_{T-2}^{\gamma-1} &= E_{T-2}[\delta^{T-1} b W_{T-1}^{\gamma-1} R_{T-2}] \\ &= \delta^{T-1} b E_{T-2}[S_{T-2}^{\gamma-1} R_{T-2}^\gamma] \\ &= \delta^{T-1} b (W_{T-2} - C_{T-2})^{\gamma-1} E_{T-2}[R_{T-2}^\gamma] \\ C_{T-2} &= \delta b (W_{T-2} - C_{T-2}) E_{T-2}[R_{T-2}^\gamma]^{\frac{1}{\gamma-1}}\end{aligned}\quad (5)$$

By rearranging the terms in equation (5), we can get an explicit form of  $C_{T-2}^*$  as follows.

$$\begin{aligned} C_{T-2}^* &= \frac{\delta b E_{T-2}[R_{T-2}^\gamma]^{\frac{1}{\gamma-1}}}{1 + \delta b E_{T-2}[R_{T-2}^\gamma]^{\frac{1}{\gamma-1}}} W_{T-2} \\ &= \frac{c}{1+c} W_{T-2} \quad \text{where } c = \delta b E_{T-2}[R_{T-2}^\gamma]^{\frac{1}{\gamma-1}} \end{aligned} \quad (6)$$

Another optimal condition is  $E_{T-2}[R_{i,T-2} J_W(W_{T-1}, T-1)] = R_f E_{T-2}[J_W(W_{T-1}, T-1)]$ . Therefore, the following equations hold.

$$\begin{aligned} E_{T-2}[R_{i,T-2} \delta^{T-1} b W_{T-1}^{\gamma-1}] &= R_f E_{T-2}[\delta^{T-1} b W_{T-1}^{\gamma-1}] \\ \Rightarrow E_{T-2}[R_{i,T-2} W_{T-1}^{\gamma-1}] &= R_f E_{T-2}[W_{T-1}^{\gamma-1}] \\ \Rightarrow E_{T-2}[R_{i,T-2} (S_{T-2} R_{T-2})^{\gamma-1}] &= R_f E_{T-2}[(S_{T-2} R_{T-2})^{\gamma-1}] \\ \Rightarrow E_{T-2}[R_{i,T-2} R_{T-2}^{\gamma-1}] &= R_f E_{T-2}[R_{T-2}^{\gamma-1}] \end{aligned} \quad (7)$$

d.