

# **FIN 591: Homework #4**

Due on Monday, April 30, 2018

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## Problem 1

Assuming  $\theta = 1$ ,  $V_t$  is represented as follows.

$$V_t = [C_t^\gamma + \delta E_t[V_{t+1}^\gamma]]^{\frac{1}{\gamma}} \quad (1)$$

Solving the difference equation iteratively, we can solve  $V_t$  as follows.

$$\begin{aligned} V_t &= [C_t^\gamma + \delta E_t[V_{t+1}^\gamma]]^{\frac{1}{\gamma}} \\ &= [C_t^\gamma + \delta E_t[\{C_{t+1}^\gamma + \delta E_{t+1}[V_{t+2}^\gamma]^{\gamma \times \frac{1}{\gamma}}\}]]^{\frac{1}{\gamma}} \\ &= [C_t^\gamma + \delta E_t[C_{t+1}^\gamma + \delta E_{t+1}[V_{t+2}^\gamma]]^{\frac{1}{\gamma}} \\ &= [C_t^\gamma + \delta E_t[C_{t+1}^\gamma] + \delta^2 E_t[V_{t+2}^\gamma]]^{\frac{1}{\gamma}} \end{aligned} \quad (2)$$

Using this procedure iteratively,  $V_t$  is derived as follows.

$$\begin{aligned} V_t &= [C_t^\gamma + \delta E_t[C_{t+1}^\gamma] + \delta^2 E_t[C_{t+2}^\gamma] + \delta^3 E_t[C_{t+3}^\gamma] + \delta^4 E_t[C_{t+4}^\gamma] + \dots]^{\frac{1}{\gamma}} \\ &= \sum_{i=t}^{\infty} \delta^i E_t[C_i^\gamma]^{\frac{1}{\gamma}} \end{aligned} \quad (3)$$

Considering the discrete-time version of power utility  $U(C_t, t) = \delta^t \times C_t^\gamma / \gamma$ ,  $\gamma V_t^\gamma$  implies a lifetime utility function.

## Problem 2

Since  $m_{t,t+1} = \frac{\partial V_t}{\partial C_{t+1}} / \frac{\partial V_t}{\partial C_t}$ , stochastic discount factor of this case can be derived as follows.

$$\frac{\partial V_t}{\partial C_t} = \frac{\theta}{\gamma} [C_t^{\frac{\gamma}{\theta}} + \delta E_t[V_{t+1}^\gamma]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\gamma}{\theta} C_t^{\frac{\gamma}{\theta}-1} \quad (4)$$

$$\frac{\partial V_t}{\partial C_{t+1}} = \frac{\theta}{\gamma} [C_t^{\frac{\gamma}{\theta}} + \delta E_t[V_{t+1}^\gamma]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\delta}{\theta} E_t[V_{t+1}^\gamma]^{\frac{1}{\theta}-1} \times \gamma V_{t+1}^{\gamma-1} \times \frac{\partial V_{t+1}}{\partial C_{t+1}} \quad (5)$$

Since  $V_{t+1} = [C_{t+1}^{\frac{\gamma}{\theta}} + \delta E_{t+1}[V_{t+2}^\gamma]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}}$ ,  $\frac{\partial V_{t+1}}{\partial C_{t+1}}$  is derived as follows.

$$\begin{aligned} \frac{\partial V_{t+1}}{\partial C_{t+1}} &= \frac{\theta}{\gamma} [C_{t+1}^{\frac{\gamma}{\theta}} + \delta E_{t+1}[V_{t+2}^\gamma]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\gamma}{\theta} C_{t+1}^{\frac{\gamma}{\theta}-1} \\ &= \frac{\theta}{\gamma} [C_{t+1}^{\frac{\gamma}{\theta}} + \delta E_{t+1}[V_{t+2}^\gamma]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}(\frac{\theta-\gamma}{\theta})} \times \frac{\gamma}{\theta} C_{t+1}^{\frac{\gamma}{\theta}-1} \\ &= \frac{\theta}{\gamma} V_{t+1}^{(1-\frac{\gamma}{\theta})} \times \frac{\gamma}{\theta} C_{t+1}^{\frac{\gamma}{\theta}-1} \\ &= V_{t+1}^{(1-\frac{\gamma}{\theta})} \times C_{t+1}^{\frac{\gamma}{\theta}-1} \end{aligned} \quad (6)$$

Therefore, plugging equation (6) to equation (5), equation (7) holds.

$$\begin{aligned} \frac{\partial V_t}{\partial C_{t+1}} &= \frac{\theta}{\gamma} [C_t^{\frac{\gamma}{\theta}} + \delta E_t[V_{t+1}^\gamma]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\delta}{\theta} E_t[V_{t+1}^\gamma]^{\frac{1}{\theta}-1} \times \gamma V_{t+1}^{\gamma-1} \times V_{t+1}^{(1-\frac{\gamma}{\theta})} \times C_{t+1}^{\frac{\gamma}{\theta}-1} \\ &= \frac{\theta}{\gamma} [C_t^{\frac{\gamma}{\theta}} + \delta E_t[V_{t+1}^\gamma]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\delta}{\theta} E_t[V_{t+1}^\gamma]^{\frac{1}{\gamma}(\frac{\gamma}{\theta}-\gamma)} \times \gamma V_{t+1}^{\gamma-\frac{\gamma}{\theta}} \times C_{t+1}^{\frac{\gamma}{\theta}-1} \end{aligned} \quad (7)$$

Combining equation (7) and (4),  $m_{t,t+1}$  is derived as follows.

$$\begin{aligned}
 m_{t,t+1} &= \frac{\partial V_t}{\partial C_{t+1}} / \frac{\partial V_t}{\partial C_t} \\
 &= \frac{\frac{\theta}{\gamma} [C_t^{\frac{\gamma}{\theta}} + \delta E_t[V_{t+1}^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\delta}{\theta} E_t[V_{t+1}^{\frac{1}{\theta}}]^{\frac{1}{\gamma}(\frac{\gamma}{\theta}-\gamma)} \times \gamma V_{t+1}^{\gamma-\frac{\gamma}{\theta}} \times C_{t+1}^{\frac{\gamma}{\theta}-1}}{\frac{\theta}{\gamma} [C_t^{\frac{\gamma}{\theta}} + \delta E_t[V_{t+1}^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\gamma}{\theta} C_t^{\frac{\gamma}{\theta}-1}} \\
 &= \delta \left( \frac{C_{t+1}}{C_t} \right)^{\frac{\gamma}{\theta}-1} \left( \frac{V_{t+1}}{E_t[V_{t+1}]^{\frac{1}{\gamma}}} \right)^{\gamma-\frac{\gamma}{\theta}}
 \end{aligned} \tag{8}$$

Since  $\theta = \gamma/(1 - \frac{1}{\varepsilon})$ , equation (8) is represented as follows.

$$m_{t,t+1} = \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\varepsilon}} \left( \frac{V_{t+1}}{E_t[V_{t+1}]^{\frac{1}{\gamma}}} \right)^{\gamma-(1-\frac{1}{\varepsilon})} \tag{9}$$

Which is the desired solution.