Multiperiod Discrete-Time Consumption and Portfolio Choice

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Introduction

The previous single period consumption and portfolio choice problem is now extended to determining consumption and portfolio choices over a multiple period planning horizon.

- We now consider expected utility maximization over *many* periods.
- Dynamic programming results in a recursive solution.
- It allows us to transform multiperiod decisions into multiple *single*-period decisions.
- By deriving individual asset demands, we build the foundation for an intertemporal general equilibrium asset pricing model.

Assumptions and Notation

- An individual makes decisions at the start of each unit-length period during a T-period planning horizon. Let the initial date be 0.
- Denote consumption at date t as C_t, $t = 0, ..., T-1$, and a terminal bequest as W_T, where W_t indicates the individual's level of wealth at date t.
- Expected utility is assumed to be *time-separable*, or *additively separable*:

$$E_0 \left[\Upsilon \left(C_0, C_1, ..., C_{T-1}, W_T \right) \right] = E_0 \left[\sum_{t=0}^{T-1} U \left(C_t, t \right) + B \left(W_T, T \right) \right] \tag{1}$$

where U and B are increasing, concave functions of consumption and wealth, respectively.

Wealth Dynamics

- Let date t wage income be y_t. Date t savings of $S_t \equiv (W_t + y_t - C_t)$ is allocated between n risky assets and the risk-free asset, returning R_{it} and R_{ft} respectively over the interval t to $t+1$.
- The evolution of the individual's tangible wealth is

$$\begin{aligned} W_{t+1} &= (W_t + y_t - C_t)\left(R_{ft} + \sum_{i=1}^{n} \omega_{it}\left(R_{it} - R_{ft}\right)\right) \quad (2) \\ &= S_t R_t \end{aligned}$$

where ω_{it} is the portfolio weight of risky asset i and $R_t \equiv R_{ft} + \sum_{i=1}^{n} \omega_{it}\left(R_{it} - R_{ft}\right)$ is the total portfolio return from date t to $t+1$.

Multiperiod Decisions

Sequence of Individual's Consumption and Portfolio Choices

Information Variables

W_0	W_1	$\cdots$	W_t	$\cdots$	W_{T-1}	
y_0	y_1	$\cdots$	y_t	$\cdots$	y_{T-1}	
R_{f0}	R_{f1}	$\cdots$	R_{ft}	$\cdots$	$R_{f,T-1}$	
$F_{R_{f\tau}} \mid I_0$	$F_{R_{f\tau}} \mid I_1$	$\cdots$	$F_{R_{f\tau}} \mid I_t$	$\cdots$	$F_{R_{i,T-1}} \mid I_{T-1}$	
$F_{y_\tau} \mid I_0$	$F_{y_\tau} \mid I_1$	$\cdots$	$F_{y_\tau} \mid I_t$	$\cdots$		

| $F_{R_{i\tau}} \mid I_0$ | $F_{R_{i\tau}} \mid I_1$ | $\cdots$ | $F_{R_{i\tau}} \mid I_t$ | $\cdots$ | | |

Decisions

C_0	C_1	$\cdots$	C_t	$\cdots$	C_{T-1}	
$\{\omega_{i0}\}$	$\{\omega_{i1}\}$	$\cdots$	$\{\omega_{it}\}$	$\cdots$	$\{\omega_{i,T-1}\}$	
0	1	...	t	...	T-1	T
						Date

Solving the Multiperiod Model

- Define $J(W_t, t)$ as the derived utility of wealth function:

$$J(W_t, I_t, t) \equiv \max_{C_s, \{\omega_{is}\}, \forall s, i} E_t \left[\sum_{s=t}^{T-1} U(C_s, s) + B(W_T, T) \right] \quad (3)$$

  where "max" means to choose the decisions C_s and $\{\omega_{is}\}$ for $s = t, t+1, ..., T-1$ and $i = 1, ..., n$ so as to maximize the expected value of the term in brackets.

- J is a function of current wealth and information up until and including date t, but not current or future decisions.
- We solve this problem with backward dynamic programming, first at $T-1$, then at $T-2$, all the way back to 0.

Final Period Solution

- From the definition of J, at time T we have

$$J(W_T, T) = E_T[B(W_T, T)] = B(W_T, T) \tag{4}$$

- Working backwards, consider the individual's problem at date $T-1$ with a single period left in the planning horizon.

$$\begin{aligned} J(W_{T-1}, T-1) &= \max_{C_{T-1}, \{\omega_{i,T-1}\}} E_{T-1}[U(C_{T-1}, T-1) + B(W_T, T)] \\ &= \max_{C_{T-1}, \{\omega_{i,T-1}\}} U(C_{T-1}, T-1) + E_{T-1}[B(W_T, T)] \end{aligned} \tag{5}$$

- To clarify how W_T depends on C_{T-1} and $\{\omega_{i,T-1}\}$, substitute equation (2) for $t = T-1$ into equation (5):

$$J(W_{T-1}, T-1) = \max_{C_{T-1}, \{\omega_{i,T-1}\}} U(C_{T-1}, T-1) + E_{T-1}[B(S_{T-1}R_{T-1}, T)] \tag{6}$$

where recall that $S_{T-1} \equiv W_{T-1} + y_{T-1} - C_{T-1}$ and $R_{T-1} \equiv R_{f,T-1} + \sum_{i=1}^{n} \omega_{i,T-1}(R_{i,T-1} - R_{f,T-1})$.

$T-1$ Solution

- Differentiate with respect to C_{T-1} and $\{\omega_{i,T-1}\}$ and set the results to zero:

$$U_C\left(C_{T-1}, T-1\right) - E_{T-1}\left[B_W\left(W_T, T\right) R_{T-1}\right] = 0 \tag{7}$$

$$E_{T-1}\left[B_W\left(W_T, T\right)\left(R_{i,T-1} - R_{f,T-1}\right)\right] = 0, \;\; i = 1, ..., n \tag{8}$$

where the subscripts on U and B denote partial differentiation.
- Note $\partial B\left(W_T, T\right)/\partial C_{T-1} = B_W \partial W_T/\partial C_{T-1} = B_W \partial\left(S_{T-1}R_{T-1}\right)/\partial C_{T-1} = -B_W R_{T-1}$ since S_{T-1} depends on C_{T-1}.
- Using (8), we see that (7) can be rewritten

$$\begin{aligned} & U_C\left(C_{T-1}, T-1\right) \\ = \;& E_{T-1}\left[B_W\left(W_T, T\right)\left(R_{f,T-1} + \sum_{i=1}^{n} \omega_{i,T-1}\left(R_{i,T-1} - R_{f,T-1}\right)\right)\right] \\ = \;& R_{f,T-1} E_{T-1}\left[B_W\left(W_T, T\right)\right] \end{aligned} \tag{9}$$

$T-1$ Solution cont'd

- Substitute the optimal decisions C^*_{T-1} and $\omega^*_{i,T-1}$ back into (6) and differentiate totally with respect to W_{T-1}:

$$
\begin{aligned}
J_W &= U_C \frac{\partial C^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left[B_{W_T} \cdot \left(\frac{dW_T}{dW_{T-1}} \right) \right] \\
&= U_C \frac{\partial C^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left[B_{W_T} \cdot \left(\frac{\partial W_T}{\partial W_{T-1}} + \sum_{i=1}^{n} \frac{\partial W_T}{\partial \omega^*_{i,T-1}} \frac{\partial \omega^*_{i,T-1}}{\partial W_{T-1}} \right.\right. \\
&\quad \left.\left. + \frac{\partial W_T}{\partial C^*_{T-1}} \frac{\partial C^*_{T-1}}{\partial W_{T-1}} \right) \right] \\
&= U_C \frac{\partial C^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left[B_{W_T} \cdot \left(\sum_{i=1}^{n} \left[R_{i,T-1} - R_{f,T-1} \right] S_{T-1} \frac{\partial \omega^*_{i,T-1}}{\partial W_{T-1}} \right.\right. \\
&\quad \left.\left. + R_{T-1} \left(1 - \frac{\partial C^*_{T-1}}{\partial W_{T-1}} \right) \right) \right] \qquad (10)
\end{aligned}
$$

$T-1$ Solution cont'd

- Using the first-order condition (8), $E_{T-1}\left[B_{W_T}\left(R_{i,T-1}-R_{f,T-1}\right)\right]=0$, as well as (9), $U_C=E_{T-1}\left[B_{W_T}R_T\right]$, (10) simplifies to

$$\begin{aligned} J_W &= U_C\frac{\partial C^*_{T-1}}{\partial W_{T-1}}-E_{T-1}\left[B_{W_T}R_T\right]\frac{\partial C^*_{T-1}}{\partial W_{T-1}}+E_{T-1}\left[B_{W_T}R_T\right] \\ &= E_{T-1}\left[B_{W_T}R_T\right] \end{aligned}$$

- Using (9) once again, this can be rewritten as

$$J_W\left(W_{T-1},T-1\right)=U_C\left(C^*_{T-1},T-1\right) \tag{11}$$

- This "envelope condition" says that the individual's optimal policy equates her marginal utility of current consumption, U_C, to her marginal utility of wealth (future consumption).

$T-2$ Solution (Deriving the Bellman Equation)

- Next, we solve the individual's problem at $T-2$:

$$J\left(W_{T-2}, T-2\right) = \max_{C_{T-2}, \{\omega_{i,T-2}\}} U\left(C_{T-2}, T-2\right)$$

$$+E_{T-2}\left[U\left(C_{T-1}, T-1\right) + B\left(W_T, T\right)\right] \tag{12}$$

- We optimize over C_{T-2} and $\{\omega_{i,T-2}\}$ for a function that includes $U\left(C_{T-1}, T-1\right) + B\left(W_T, T\right)$ which depend on future decisions C_{T-1} and $\{\omega_{i,T-1}\}$.
- The *Principle of Optimality* informs us how to do this:

An optimal set of decisions has the property that given an initial decision, the remaining decisions must be optimal with respect to the outcome that results from the initial decision.

Deriving the Bellman Equation cont'd

- The "max" in (12) is over all remaining decisions, but the Principle of Optimality says that whatever decision is made in period $T-2$, given the outcome, the remaining decisions (for period $T-1$) must be optimal. In other words:

$$\max_{\{(T-2),(T-1)\}}(Y) = \max_{\{T-2\}}\left[\max_{\{T-1,|(T-2)\}}(Y)\right] \tag{13}$$

- This principle allows us to rewrite (12) as

$$\begin{aligned} J\left(W_{T-2}, T-2\right) \quad = \quad & \max_{C_{T-2},\{\omega_{i,T-2}\}} \left\{U\left(C_{T-2}, T-2\right) + \right. \\ & \left. E_{T-2}\left[\max_{C_{T-1},\{\omega_{i,T-1}\}} E_{T-1}\left[U\left(C_{T-1}, T-1\right) + B\left(W_T, T\right)\right]\right]\right\} \end{aligned} \tag{14}$$

Deriving the Bellman Equation cont'd

- Then, using the definition of $J(W_{T-1}, T-1)$ from (5), equation (14) can be rewritten as

$$J(W_{T-2}, T-2) = \max_{C_{T-2}, \{\omega_{i,T-2}\}} U(C_{T-2}, T-2) + E_{T-2}[J(W_{T-1}, T-1)] \tag{15}$$

- The recursive condition (15) is the Bellman (1957) equation.
- The only difference between a one-period problem (5) and this is that (15) replaces the known function of wealth next period, B, with another (known in principle) function of wealth next period, J.
- Yet, the solution to (15) is of the same form as that for (5).

The General Solution

- Thus, the optimality conditions for (15) are

$$\begin{aligned} U_C\left(C^*_{T-2}, T-2\right) &= E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) R_{T-2}\right] \\ &= R_{f,T-2} E_{T-2}\left[J_W\left(W_{T-1}, T-1\right)\right] \\ &= J_W\left(W_{T-2}, T-2\right) \qquad (16) \end{aligned}$$

  where the second line is implied by the FOC:

$$\begin{aligned} &E_{T-2}\left[R_{i,T-2} J_W\left(W_{T-1}, T-1\right)\right] = \\ &R_{f,T-2} E_{T-2}\left[J_W\left(W_{T-1}, T-1\right)\right] \quad \forall i \qquad (17) \end{aligned}$$

- From the preceding pattern, inductive reasoning implies that for any $t = 0, 1, ..., T-1$, we have the Bellman equation:

$$J\left(W_t, t\right) = \max_{C_t, \{\omega_{i,t}\}} U\left(C_t, t\right) + E_t\left[J\left(W_{t+1}, t+1\right)\right] \qquad (18)$$

The General Solution cont'd

and, therefore, the date t optimality conditions are

$$\begin{aligned} U_C\left(C_t^*, t\right) &= E_t\left[J_W\left(W_{t+1}, t+1\right) R_t\right] \\ &= R_{f,t} E_t\left[J_W\left(W_{t+1}, t+1\right)\right] \\ &= J_W\left(W_t, t\right) \end{aligned} \tag{19}$$

$$E_t\left[R_{i,t} J_W\left(W_{t+1}, t+1\right)\right] = R_{f,t} E_t\left[J_W\left(W_{t+1}, t+1\right)\right], \;\; i = 1, ..., n \tag{20}$$

- Equations (19) and (20) equate the marginal utilities of consumption and wealth and set portfolio weights to equate all assets' expected marginal utility-weighted asset returns.
- These conditions depend on future investment opportunities $(R_{i,t+j}, R_{f,t+j}, j \geq 1)$, income flows, y_{t+j}, and states of the world affecting utilities $(U(\cdot, t+j))$.

The General Solution cont'd

- Solving this system involves starting from the end of the planning horizon and dynamically programing backwards toward the present.

Step	Action
1	Construct $J(W_T, T)$.
2	Solve for C^*_{T-1} and $\{\omega_{i,T-1}\}$, $i = 1, ..., n$.
3	Substitute decisions in step 2 to construct $J(W_{T-1}, T-1)$.
4	Solve for C^*_{T-2} and $\{\omega_{i,T-2}\}$, $i = 1, ..., n$.
5	Substitute decisions in step 4 to construct $J(W_{T-2}, T-2)$.
6	Repeat steps 4 and 5 for date $T-3$.
7	Repeat step 6 for all prior dates until date 0 is reached.

The General Solution cont'd

- By following this recursive procedure, we find that the optimal policy will be of the form

$$C_t^* = g\left[W_t, y_t, I_t, t\right] \tag{21}$$

$$\omega_{it}^* = h\left[W_t, y_t, I_t, t\right] \tag{22}$$

- Deriving analytical expressions for the functions g and h is not always possible, in which case numerical solutions satisfying the first-order conditions at each date can be computed.
- Next we consider an example with analytical solutions.

Log Utility

- Assume $U(C_t, t) \equiv \delta^t \ln[C_t]$, $B(W_T, T) \equiv \delta^T \ln[W_T]$, and $y_t \equiv 0 \ \forall \ t$, where $\delta = \frac{1}{1+\rho}$. At date $T-1$, using condition (7):

$$\begin{aligned} U_C(C_{T-1}, T-1) &= E_{T-1}[B_W(W_T, T) R_{T-1}] \qquad (23) \\ \delta^{T-1} \frac{1}{C_{T-1}} &= E_{T-1}\left[\delta^T \frac{R_{T-1}}{W_T}\right] = E_{T-1}\left[\delta^T \frac{R_{T-1}}{S_{T-1} R_{T-1}}\right] \\ &= \frac{\delta^T}{S_{T-1}} = \frac{\delta^T}{W_{T-1} - C_{T-1}} \end{aligned}$$

or

$$C^*_{T-1} = \frac{1}{1+\delta} W_{T-1} \qquad (24)$$

- Consumption for this log utility investor is a fixed proportion of wealth and independent of investment opportunities.

FOCs

- Conditions (8) imply

$$
\begin{aligned}
E_{T-1}\left[B_{W_T} R_{i,T-1}\right] &= R_{f,T-1} E_{T-1}\left[B_{W_T}\right], \ \ i = 1, ..., n \\
\delta^T E_{T-1}\left[\frac{R_{i,T-1}}{S_{T-1} R_{T-1}}\right] &= \delta^T R_{f,T-1} E_{T-1}\left[\frac{1}{S_{T-1} R_{T-1}}\right] \\
E_{T-1}\left[\frac{R_{i,T-1}}{R_{T-1}}\right] &= R_{f,T-1} E_{T-1}\left[\frac{1}{R_{T-1}}\right] \qquad (25)
\end{aligned}
$$

- Moreover, with log utility (25) equals unity, since from (9):

$$
\begin{aligned}
U_C\left(C_{T-1}, T-1\right) &= R_{f,T-1} E_{T-1}\left[B_W\left(W_T, T\right)\right] \\
\frac{\delta^{T-1}}{C^*_{T-1}} &= R_{f,T-1} E_{T-1}\left[\delta^T \frac{1}{S_{T-1} R_{T-1}}\right] \\
1 &= \frac{\delta C^*_{T-1} R_{f,T-1}}{W_{T-1} - C^*_{T-1}} E_{T-1}\left[\frac{1}{R_{T-1}}\right] \\
1 &= R_{f,T-1} E_{T-1}\left[\frac{1}{R_{T-1}}\right] \qquad (26)
\end{aligned}
$$

FOCs cont'd

- Here we substituted for C^*_{T-1} using (24) in the third to fourth line of (26).
- While we would need to make specific assumptions regarding the distribution of asset returns in order to derive the portfolio weights $\{\omega^*_{i,T-1}\}$ satisfying (25), note that the conditions in (25) are rather special in that they do not depend on W_{T-1}, C_{T-1}, or δ, but only on the particular distribution of asset returns.
- Next we solve for $J\left(W_{T-1}, T-1\right)$ by substituting in the date $T-1$ optimal consumption and portfolio rules into the individual's objective function.
- Denoting $R^*_t \equiv R_{f,t} + \sum_{i=1}^n \omega^*_{it}\left(R_{it} - R_{ft}\right)$ as the individual's total optimal portfolio return, we have

FOCs cont'd

$$\begin{aligned} J\left(W_{T-1}, T-1\right) &= \delta^{T-1} \ln\left[C_{T-1}^*\right] + \delta^T E_{T-1}\left[\ln\left[R_{T-1}^*\left(W_{T-1} - C_{T-1}^*\right)\right]\right] \\ &= \delta^{T-1}\left(-\ln\left[1+\delta\right] + \ln\left[W_{T-1}\right]\right) + \\ &\quad \delta^T\left(E_{T-1}\left[\ln\left[R_{T-1}^*\right]\right] + \ln\left[\frac{\delta}{1+\delta}\right] + \ln\left[W_{T-1}\right]\right) \\ &= \delta^{T-1}\left[\left(1+\delta\right)\ln\left[W_{T-1}\right] + H_{T-1}\right] \end{aligned} \tag{27}$$

where $H_{T-1} \equiv -\ln\left[1+\delta\right] + \delta \ln\left[\frac{\delta}{1+\delta}\right] + \delta E_{T-1}\left[\ln\left[R_{T-1}^*\right]\right]$.

- Notably, from (25) $\omega_{i,T-1}^*$ does not depend on W_{T-1}, and therefore R_{T-1}^* and H_{T-1} do not depend on W_{T-1}.
- At time $T-2$, from equation (15) we have

FOCs cont'd

$$
\begin{aligned}
J\left(W_{T-2}, T-2\right) &= \max_{C_{T-2},\{\omega_{i,T-2}\}} U\left(C_{T-2}, T-2\right) + E_{T-2}\left[J\left(W_{T-1}, T-1\right)\right] \\
&= \max_{C_{T-2},\{\omega_{i,T-2}\}} \delta^{T-2} \ln\left[C_{T-2}\right] \\
&\quad + \delta^{T-1} E_{T-2}\left[(1+\delta)\ln\left[W_{T-1}\right] + H_{T-1}\right] \qquad (28)
\end{aligned}
$$

- Using (16), the optimality condition for consumption is

$$
\begin{aligned}
U_C\left(C^*_{T-2}, T-2\right) &= E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) R_{T-2}\right] \\
\frac{\delta^{T-2}}{C_{T-2}} &= (1+\delta)\,\delta^{T-1} E_{T-2}\left[\frac{R_{T-2}}{S_{T-2} R_{T-2}}\right] \\
&= \frac{(1+\delta)\,\delta^{T-1}}{W_{T-2} - C_{T-2}} \qquad (29)
\end{aligned}
$$

or

FOCs cont'd

$$C^*_{T-2} = \frac{1}{1+\delta+\delta^2} W_{T-2} \tag{30}$$

- Using (17), the optimality conditions for $\{\omega^*_{i,T-2}\}$ turn out to be of the same form as at $T-1$:

$$E_{T-2}\left[\frac{R_{i,T-2}}{R^*_{T-2}}\right] = R_{f,T-2}E_{T-2}\left[\frac{1}{R^*_{T-2}}\right], \ i=1,...,n \tag{31}$$

and, as in the case of $T-1$, equation (31) equals unity:

$$\begin{aligned} \frac{\delta^{T-2}}{C^*_{T-2}} &= R_{f,T-2}\delta^{T-1}E_{T-2}\left[\frac{1+\delta}{S_{T-2}R_{T-2}}\right] \\ 1 &= \frac{\delta(1+\delta)C^*_{T-2}R_{f,T-2}}{W_{T-2}-C^*_{T-2}}E_{T-2}\left[\frac{1}{R_{T-2}}\right] \\ 1 &= R_{f,T-2}E_{T-2}\left[\frac{1}{R_{T-2}}\right] \end{aligned} \tag{32}$$

- Recognizing the above pattern, the optimal consumption and portfolio rules for any prior date, t, are

$$C_t^* = \frac{1}{1 + \delta + ... + \delta^{T-t}} W_t = \frac{1 - \delta}{1 - \delta^{T-t+1}} W_t \tag{33}$$

(using the definition of a geometric sum) and

$$E_t \left[\frac{R_{i,t}}{R_t^*} \right] = R_{ft} E_t \left[\frac{1}{R_t^*} \right] = 1, \;\; i = 1, ..., n \tag{34}$$

- Hence, optimal consumption and portfolio rules are separable for a log utility individual.
- The consumption-savings decision does not depend on the distribution of asset returns, and optimal portfolio weights depend on the distribution of one-period returns (*myopic behavior*).

Summary

- Backward, stochastic dynamic program allows one to solve for optimal multi-period consumption and portfolio choices.
- With log utility and no labor income, optimal consumption is a fixed proportion of wealth.
- In the same setting, optimal portfolio choices only depend on the current period's distribution of returns.
- These last two results do not hold, in general, with other utility/wealth specifications.