Time-Inseparable Utility

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Introduction

- We consider two types of lifetime utility functions that are not time separable: **habit persistence** and **recursive utility**.
- Habit persistence utility allows past consumption to play a role in determining current utility.
- Two examples are the "internal" habit model of Constantinides (1990) and the "external" habit model of Campbell and Cochrane (1999).
- Recursive utility makes current utility depend on expected values of future utility, and we study a continuous-time version of the model by Obstfeld (1994).

Constantinides' Internal Habit Model Assumptions

- Constantinides' internal habit formation model derives a representative individual's consumption and portfolio choices in a simple production economy.
- **Technology**: A single capital-consumption good can be invested in up to two different technologies. The first is a risk-free technology whose output, B_t, follows the process

$$dB/B = r\,dt \tag{1}$$

- The second is a risky technology whose output, η_t, satisfies

$$d\eta/\eta = \mu\,dt + \sigma\,dz \tag{2}$$

where r, μ, and σ are constants so there are constant investment opportunities.

Assumptions (continued)

- **Preferences**:Representative agents have date t consumption of C_t and maximize

$$E_0 \left[\int_0^{\infty} e^{-\rho t} u\left(\widehat{C}_t\right)\, dt \right] \tag{3}$$

where $u\left(\widehat{C}_t\right) = \widehat{C}_t^{\gamma}/\gamma$, $\gamma < 1$, $\widehat{C}_t = C_t - bx_t$, and

$$x_t \equiv e^{-at} x_0 + \int_0^t e^{-a(t-s)} C_s \, ds \tag{4}$$

- x_t is an exponentially weighted sum of past consumption.
- $b = 0$ is time-separable constant relative risk aversion utility, while $b < 0$ implies "consumption durability."
- When $b > 0$, bx_t is "subsistence" or "habit" consumption with $\widehat{C}_t = C_t - bx_t$ referred to as "surplus" consumption.

Additional Parametric Assumptions

- Let W_0 be the initial wealth of the representative individual.
- The additional parametric assumptions are made:

$$
\begin{aligned}
W_0 &> \frac{bx_0}{r+a-b} > 0 && (5) \\
r+a &> b > 0 && (6) \\
\rho - \gamma r - \frac{\gamma(\mu-r)^2}{2(1-\gamma)\sigma^2} &> 0 && (7) \\
0 &\leq m \equiv \frac{\mu-r}{(1-\gamma)\sigma^2} \leq 1 && (8)
\end{aligned}
$$

Reasons for Assumptions

- Conditions (5) and (6) ensure that an admissible (feasible) consumption and portfolio choice strategy exists that enables $C_t > bx_t$.
- To see this, note that the individual's wealth dynamics are

$$dW = \{[(\mu - r)\omega_t + r]W - C_t\} \, dt + \sigma \omega_t W \, dz \qquad (9)$$

  where the risky technology weight satisfies $0 \leq \omega_t \leq 1$.
- Now if $\omega_t = 0$ for all t and consumption equals a fixed proportion of wealth, $C_t = (r + a - b)W_t$, then

$$dW = \{rW - (r + a - b)W\} \, dt = (b - a) \, Wdt \qquad (10)$$

- Equation (10) implies

$$W_t = W_0 e^{(b-a)t} > 0 \qquad (11)$$

Reasons for Assumptions

- This implies $C_t = (r + a - b)\, W_0\, e^{(b-a)t} > 0$ and

$$
\begin{aligned}
C_t - bx_t &= (r+a-b) W_0 e^{(b-a)t} \\
&\quad - b\left[e^{-at} x_0 + \int_0^t e^{-a(t-s)} (r+a-b)\, W_0\, e^{(b-a)s}\, ds \right] \\
&= (r+a-b)\, W_0\, e^{(b-a)t} \\
&\quad - \left[e^{-at} bx_0 + b(r+a-b) W_0 e^{-at} \int_0^t e^{bs}\, ds \right] \\
&= (r+a-b)\, W_0\, e^{(b-a)t} \\
&\quad - \left[e^{-at} bx_0 + (r+a-b) W_0 e^{-at} (e^{bt} - 1) \right] \\
&= e^{-at} \left[(r+a-b) W_0 - bx_0 \right]
\end{aligned}
\tag{12}
$$

which is greater than zero by assumption (5).

Reasons for Assumptions

- Condition (7) is a transversality condition that ensures that if the individual follows an optimal policy, the expected utility of consumption over an infinite horizon is finite.
- Condition (8) ensures that the individual chooses a nonnegative amount of wealth in the risky and risk-free technologies.
- Note that $m \equiv \frac{\mu - r}{(1-\gamma)\sigma^2}$ is the optimal risky-asset portfolio weight for the time-separable, constant relative-risk-aversion case.

Consumption and Porfolio Choices

- The individual's maximization problem is

$$\max_{\{C_s,\omega_s\}} E_t\left[\int_t^{\infty} e^{-\rho s}\frac{[C_s-bx_s]^{\gamma}}{\gamma}\,ds\right] \equiv e^{-\rho t}J(W_t,\,x_t) \quad (13)$$

  subject to (4) and (9).

- Given the infinite horizon, we can simplify the indirect utility function $\widehat{J}\left(W_t,x_t,t\right)=e^{-\rho t}J(W_t,\,x_t)$.
- Note from (4) that the dynamics of $x\left(t\right)$ are:

$$dx/dt = -ae^{-at}x_0 + C_t - a\int_0^t e^{-a(t-s)}C_s\,ds, \quad \text{or} \quad (14)$$

$$dx = \left(C_t - ax_t\right)\,dt \quad (15)$$

Bellman Equation

- The Bellman equation is then

$$
\begin{aligned}
0 &= \max_{\{C_t,\omega_t\}} \left\{ U(C_t,\, x_t,\, t) + L[e^{-\rho t} J] \right\} \\
&= \max_{\{C_t,\omega_t\}} \Big\{ e^{-\rho t} \gamma^{-1} (C_t - b x_t)^{\gamma} \\
&\quad + e^{-\rho t} J_W [((\mu - r)\omega_t + r) W - C_t] \\
&\quad + \frac{1}{2} e^{-\rho t} J_{WW} \sigma^2 \omega_t^2 W^2 \; + \; e^{-\rho t} J_x \left(C_t - a x_t \right) - \rho e^{-\rho t} J \Big\}
\end{aligned} \tag{16}
$$

First Order Conditions

- The first-order condition with respect to C_t is

$$\begin{aligned} (C_t - bx_t)^{\gamma-1} &= J_W - J_x, \quad \text{or} \\ C_t &= bx_t + [J_W - J_x]^{\frac{1}{\gamma-1}} \end{aligned} \tag{17}$$

- The first-order condition with respect to ω_t is

$$\begin{aligned} (\mu - r) W J_W + \omega_t \sigma^2 W^2 J_{WW} &= 0, \quad \text{or} \\ \omega_t &= -\frac{J_W}{J_{WW} W} \frac{\mu - r}{\sigma^2} \end{aligned} \tag{18}$$

Equilibrium Partial Differential Equation

- Substituting (17) and (18) back into (16):

$$0 = \frac{1-\gamma}{\gamma}\left[J_W - J_x\right]^{\frac{\gamma}{\gamma-1}} - \frac{J_W^2}{J_{WW}}\frac{(\mu-r)^2}{2\sigma^2} + (rW - bx)J_W + (b-a)xJ_x - \rho J \tag{19}$$

- When $a = b = x = 0$, we saw that $J(W) = kW^\gamma$, so that $u = C^\gamma/\gamma$, $u_c = J_W$, and

$$C_t^* = (\gamma k)^{\frac{1}{(\gamma-1)}} W_t = W_t\left[\rho - r\gamma - \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)\frac{(\mu-r)^2}{\sigma^2}\right] / (1-\gamma) \tag{20}$$

and

$$\omega_t^* = m \tag{21}$$

Solution for Derived Utility of Wealth

- For the time-inseparable case, we try the form

$$J(W, x) = k_0[W + k_1 x]^\gamma \tag{22}$$

- Substituting into (19) and setting the coefficients on x and W equal to zero, we find

$$k_0 = \frac{(r + a - b)h^{\gamma-1}}{(r + a)\gamma} \tag{23}$$

where

$$h \equiv \frac{r + a - b}{(r + a)(1 - \gamma)} \left[\rho - \gamma r - \frac{\gamma(\mu - r)^2}{2(1 - \gamma)\sigma^2} \right] > 0 \tag{24}$$

and

$$k_1 = -\frac{b}{r + a - b} < 0. \tag{25}$$

Optimal Consumption and Portfolio Weights

- Given the solution for J, (17) and (18) imply

$$C_t{}^* = bx_t + h\left[W_t - \frac{bx_t}{r+a-b}\right] \tag{26}$$

and

$$\omega_t^* = m\left[1 - \frac{bx_t/W_t}{r+a-b}\right] \tag{27}$$

- Since $r+a>b$, so that $\omega_t^* < m$, agents invest less in the risky asset and wealth has lower volatility compared to the time-separable case.

Dynamics of Consumption

- Consider the dynamics of the term $\left[W_t - \frac{bx_t}{r+a-b}\right]$ in C_t^*:

$$d\left[W_t - \frac{bx_t}{r+a-b}\right] = \{[(\mu - r)\omega_t^* + r]W_t - C_t^* \qquad (28)$$

$$-b\frac{C_t^* - ax_t}{r+a-b}\Big\} dt + \sigma\omega_t^* W_t dz$$

- Substituting in for ω_t^* and C_t^*, one obtains

$$d\left[W_t - \frac{bx_t}{r+a-b}\right] = \left[W_t - \frac{bx_t}{r+a-b}\right][ndt + m\sigma dz] \qquad (29)$$

where

$$n \equiv \frac{r-\rho}{1-\gamma} + \frac{(\mu - r)^2(2-\gamma)}{2(1-\gamma)^2\sigma^2} \qquad (30)$$

Equilibrium Consumption Growth

- Using (29) and (26), one can show

$$\frac{dC_t}{C_t} = \left[n + b - \frac{(n+a)bx_t}{C_t} \right] dt + \left(\frac{C_t - bx_t}{C_t} \right) m\sigma dz \quad (31)$$

- From the term $\left(\frac{C_t - bx_t}{C_t} \right) m\sigma \, dz$, consumption growth is smoother than in the case of no habit persistence.
- For a given equity (risky-asset) risk premium, this can imply a relatively smooth consumption path, even though risk aversion, γ, may not be high in magnitude.
- Recall the Hansen-Jagannathan bound for the time-separable case

$$\left| \frac{\mu - r}{\sigma} \right| \leq (1 - \gamma)\, \sigma_c \quad (32)$$

Hansen-Jagannathan Bound

- For the current habit persistence case, from (31):

$$\begin{aligned} \sigma_{c,t} &= \left(\frac{C_t - bx_t}{C_t}\right) m\sigma \\ &= \left(\frac{\widehat{C}_t}{C_t}\right)\left[\frac{\mu - r}{(1-\gamma)\,\sigma^2}\right]\sigma \end{aligned} \tag{33}$$

- Define the *surplus consumption ratio* $S_t \equiv \widehat{C}_t / C_t$ and rearrange (33):

$$\frac{\mu - r}{\sigma} = \frac{(1-\gamma)\,\sigma_{c,t}}{S_t} \tag{34}$$

- Since $S_t \equiv \frac{C_t - bx_t}{C_t} < 1$ habit persistence may help reconcile the empirical violation of the H-J bound.

The Campbell-Cochrane External Habit Model

- This model has "keeping up with the Joneses" preferences and makes the following assumptions.

- **Technology**: There is a discrete-time endowment economy where date t aggregate consumption - output, C_t, follows the lognormal process:

$$\ln\left(C_{t+1}\right) - \ln\left(C_t\right) = g + \nu_{t+1} \tag{35}$$

where $v_{t+1} \sim N\left(0, \sigma^2\right)$ and is independently distributed.

Preferences

- **Preferences**: A representative individual maximizes

$$E_0 \left[\sum_{t=0}^{\infty} \delta^t \frac{(C_t - X_t)^\gamma - 1}{\gamma} \right] \tag{36}$$

where $\gamma < 1$ and X_t denotes the "habit level" that is related to the surplus consumption ratio, $S_t \equiv \frac{C_t - X_t}{C_t}$, where

$$\ln(S_{t+1}) = (1-\phi)\ln\left(\overline{S}\right) + \phi \ln(S_t) + \lambda(S_t)\nu_{t+1} \tag{37}$$

and where $\lambda(S_t)$ is the *sensitivity function*

$$\lambda(S_t) = \frac{1}{\overline{S}}\sqrt{1 - 2\left[\ln(S_t) - \ln\left(\overline{S}\right)\right]} - 1 \tag{38}$$

and

$$\overline{S} = \sigma\sqrt{\frac{1-\gamma}{1-\phi}} \tag{39}$$

Concept of External Habit

- In Constantinides (1990) an individual's habit depends on her own past consumption, so that when choosing C_t she takes into account how it will affect her future utility.
- In Campbell and Cochrane (1999) an individual's habit depends on everyone else's current and past consumption, so that when choosing C_t she views X_t as exogenous.
- The external habit assumption simplifies the agent's decision making because habit is an exogenous state variable that depends on aggregate, not the individual's, consumption.

Equilibrium Asset Prices

- The individual's marginal utility of consumption is

$$u_c\left(C_t, X_t\right) = \left(C_t - X_t\right)^{\gamma-1} = C_t^{\gamma-1} S_t^{\gamma-1} \tag{40}$$

and the representative agent's stochastic discount factor is

$$m_{t,t+1} = \delta \frac{u_c\left(C_{t+1}, X_{t+1}\right)}{u_c\left(C_t, X_t\right)} = \delta \left(\frac{C_{t+1}}{C_t}\right)^{\gamma-1} \left(\frac{S_{t+1}}{S_t}\right)^{\gamma-1} \tag{41}$$

Risk-free Interest Rate

- Let $r = -\ln\left(E_t\left[m_{t,t+1}\right]\right)$ be the continuously compounded risk-free rate between dates t and $t+1$:

$$\begin{aligned} r &= -\ln\left(\delta E_t\left[e^{-(1-\gamma)\ln(C_{t+1}/C_t)-(1-\gamma)\ln(S_{t+1}/S_t)}\right]\right) \qquad (42) \\ &= -\ln\left(\delta e^{-(1-\gamma)E_t[\ln(C_{t+1}/C_t)]-(1-\gamma)E_t[\ln(S_{t+1}/S_t)]}\right. \\ &\qquad \left.\times e^{\frac{1}{2}(1-\gamma)^2 Var_t[\ln(C_{t+1}/C_t)+\ln(S_{t+1}/S_t)]}\right) \\ &= -\ln(\delta) + (1-\gamma)g + (1-\gamma)(1-\phi)\left(\ln\overline{S} - \ln S_t\right) \\ &\qquad -\frac{1}{2}(1-\gamma)^2\sigma^2\left[1+\lambda(S_t)\right]^2 \end{aligned}$$

- Substituting in for $\lambda(S_t)$ shows that the rate is constant:

$$r = -\ln(\delta) + (1-\gamma)g - \frac{1}{2}(1-\gamma)(1-\phi) \qquad (43)$$

Price of Market Portfolio

- Aggregate consumption equals the economy's aggregate dividends (output) paid by the market portfolio. Therefore,

$$P_t = E_t \left[m_{t,t+1} \left(C_{t+1} + P_{t+1} \right) \right] \tag{44}$$

- The price-dividend ratio for the market portfolio is:

$$\begin{aligned} \frac{P_t}{C_t} &= E_t \left[m_{t,t+1} \frac{C_{t+1}}{C_t} \left(1 + \frac{P_{t+1}}{C_{t+1}} \right) \right] \\ &= \delta E_t \left[\left(\frac{S_{t+1}}{S_t} \right)^{\gamma - 1} \left(\frac{C_{t+1}}{C_t} \right)^{\gamma} \left(1 + \frac{P_{t+1}}{C_{t+1}} \right) \right] \end{aligned} \tag{45}$$

Solution

- Solve forward this difference equation by repeatedly updating and substituting for P_{t+i}/C_{t+i} to obtain:

$$\frac{P_t}{C_t} = E_t\left[\sum_{i=1}^{\infty} \delta^i \left(\frac{S_{t+i}}{S_t}\right)^{\gamma-1} \left(\frac{C_{t+i}}{C_t}\right)^{\gamma}\right] \tag{46}$$

- The solution is computed numerically by simulating the lognormal processes for C_t and S_t, noting that S_{t+1}/S_t depends on the current level of S_t.
- P_t/C_t varies only with S_t, so that the portfolio's expected returns and volatility are also functions of S_t.

Coefficient of Relative Risk Aversion

- Note that the coefficient of relative risk aversion is

$$-\frac{C_t u_{cc}}{u_c} = \frac{1-\gamma}{S_t} \tag{47}$$

- As shown earlier, when consumption is lognormally distributed the H-J bound is approximately

$$\left| \frac{E\left[r_i\right] - r}{\sigma_{r_i}} \right| \leq -\frac{C_t u_{cc}}{u_c} \sigma_c = \frac{(1-\gamma)\,\sigma_c}{S_t} \tag{48}$$

which is similar to Constantinides' internal habit model except, here, σ_c is a constant and $E\left[r_i\right]$ and σ_{r_i} are time-varying functions of S_t.

Model's Match to Data

- The coefficient of relative risk aversion is relatively high when S_t is relatively low, such as during a recession.
- Moreover, the model predicts that the equity risk premium increases during a recession (when $-\frac{C_t u_{cc}}{u_c}$ is high), which seems to be a phenomenon of the postwar U.S. stock market.
- Campbell and Cochrane calibrate the model to U.S. consumption and stock market data and, due to the nonlinear form for S_t, have more success in describing actual asset returns.

Recursive Utility

- Recursive utility is forward looking, and was developed by Kreps and Porteus (1978) and Epstein and Zin (1989).
- We will follow Duffie and Epstein (1992) and study the continuous-time limit of recursive utility.
- Recall that time-separable utility takes the form

$$V_t = E_t \left[\int_t^T U(C_s, s)\, ds \right] \tag{49}$$

where $U(C_s, s)$ is often specified $U(C_s, s) = e^{-\rho(s-t)} u(C_s)$.

- Recursive utility, however, takes the form

$$V_t = E_t \left[\int_t^T f(C_s, V_s)\, ds \right] \tag{50}$$

where f is known as an *aggregator function*.

Features of Recursive Utility

- Utility (50) is recursive since current lifetime utility, V_t, depends on expected values of future lifetime utility, V_s, $s > t$.
- When f has appropriate properties, Duffie and Epstein (1992) show that a Bellman-type equation can be used to derive optimal consumption and portfolio choices.
- We consider a form of recursive utility that generalizes power (CRRA) utility.
- Unlike CRRA where the elasticity of intertemporal substitution, ϵ, must equal the inverse of the coefficient of relative risk aversion, $1/\left(1-\gamma\right)$, recursive utility distinguishes ϵ (an intertemporal consumption-savings choice concept) from $(1-\gamma)$ (an atemporal asset risk choice concept).

Assumptions of the Obstfeld Model

- **Technology**: There is a production economy where a capital-consumption good can be invested in two different technologies. The first is a risk-free technology whose output, B_t, follows the process

$$dB/B = rdt \tag{51}$$

- The second is a risky technology whose output, η_t, follows the process

$$d\eta/\eta = \mu dt + \sigma dz \tag{52}$$

- Since r, μ, and σ are constants, there are constant investment opportunities.

Recursive Preferences

- **Preferences**: Representative, infinitely-lived agents maximize

$$V_t = E_t \int_t^\infty f\left(C_s, V_s\right) ds \tag{53}$$

where f, the aggregator function, is given by

$$f(C_s, V_s) = \rho \frac{C_s^{1-\frac{1}{\epsilon}} - \left[\gamma V_s\right]^{\frac{\epsilon-1}{\epsilon\gamma}}}{\left(1-\frac{1}{\epsilon}\right)\left[\gamma V_s\right]^{\frac{\epsilon-1}{\epsilon\gamma}-1}} \tag{54}$$

- $\rho > 0$ is the agent's rate of time preference; $\epsilon > 0$ is the elasticity of intertemporal substitution; and $1-\gamma > 0$ is the coefficient of relative risk aversion. When $\epsilon = 1/\left(1-\gamma\right)$, (53) and (54) are (ordinally) equivalent to

$$V_t = E_t \int_t^\infty e^{-\rho s} \frac{C_s^\gamma}{\gamma} ds \tag{55}$$

Derived Utility of Wealth

- If ω_t is the weight invested in the risky asset (technology), wealth satisfies

$$dW = [\omega(\mu - r)W + rW - C]\,dt + \omega\sigma W dz \tag{56}$$

- Define $J(W_t)$ as the maximized lifetime utility at date t:

$$\begin{aligned} J(W_t) &= \max_{\{C_s,\omega_s\}} E_t \int_t^\infty f(C_s, V_s)\,ds \\ &= \max_{\{C_s,\omega_s\}} E_t \int_t^\infty f(C_s, J(W_s))\,ds \end{aligned} \tag{57}$$

- Due to the infinite horizon problem with constant investment opportunities, $f(C, V)$ is not an explicit function of calendar time and the only state variable is W.

Bellman Equation

- The Bellman equation is

$$0 = \max_{\{C_t,\omega_t\}} f\left[C_t, J\left(W_t\right)\right] + L\left[J\left(W_t\right)\right] \tag{58}$$

or

$$\begin{aligned} 0 &= \max_{\{C_t,\omega_t\}} f\left[C, J\left(W\right)\right] + J_W\left[\omega\left(\mu - r\right)W + rW - C\right] \\ &\quad + \frac{1}{2} J_{WW}\omega^2\sigma^2W^2 \\ &= \max_{\{C_t,\omega_t\}} \rho \frac{C^{1-\frac{1}{\epsilon}} - \left[\gamma J\right]^{\frac{\epsilon-1}{\epsilon\gamma}}}{\left(1-\frac{1}{\epsilon}\right)\left[\gamma J\right]^{\frac{\epsilon-1}{\epsilon\gamma}-1}} + J_W\left[\omega\left(\mu - r\right)W + rW - C\right] \\ &\quad + \frac{1}{2} J_{WW}\omega^2\sigma^2W^2 \end{aligned} \tag{59}$$

First-Order Conditions

- The first-order condition with respect to C_t is

$$\rho \frac{C^{-\frac{1}{\epsilon}}}{[\gamma J]^{\frac{\epsilon-1}{\epsilon\gamma}-1}} - J_W = 0 \tag{60}$$

or

$$C = \left(\frac{J_W}{\rho}\right)^{-\epsilon} [\gamma J]^{\frac{1-\epsilon}{\gamma}+\epsilon} \tag{61}$$

- The first-order condition with respect to ω_t is

$$J_W \left(\mu - r\right) W + J_{WW} \omega \sigma^2 W^2 = 0 \tag{62}$$

or

$$\omega = -\frac{J_W}{J_{WW} W} \frac{\mu - r}{\sigma^2} \tag{63}$$

Equilibrium Partial Differential Equation

- Substituting the optimal values for C and ω into (59):

$$\rho \frac{\left(\frac{J_W}{\rho}\right)^{1-\epsilon} [\gamma J]^{(\epsilon-1)\left[1-\frac{\epsilon-1}{\epsilon\gamma}\right]} - [\gamma J]^{\frac{1-\epsilon}{\epsilon\gamma}}}{\left(1-\frac{1}{\epsilon}\right)[\gamma J]^{\frac{\epsilon-1}{\epsilon\gamma}-1}} \tag{64}$$

$$+J_W\left[-\frac{J_W}{J_{WW}}\frac{(\mu-r)^2}{\sigma^2} + rW - \left(\frac{J_W}{\rho}\right)^{-\epsilon}[\gamma J]^{\frac{1-\epsilon}{\gamma}+\epsilon}\right]$$

$$+\frac{1}{2}\frac{J_W^2}{J_{WW}}\frac{(\mu-r)^2}{\sigma^2} = 0$$

Equilibrium Partial Differential Equation (continued)

- Simplifying, one obtains:

$$\frac{\epsilon\rho}{\epsilon-1}\left[\left(\frac{J_W}{\rho}\right)^{-\epsilon}[\gamma J]^{\frac{1-\epsilon}{\gamma}+\epsilon}-\gamma J\right] \tag{65}$$

$$+J_W\left[-\frac{J_W}{J_{WW}}\frac{(\mu-r)^2}{\sigma^2}+rW-\left(\frac{J_W}{\rho}\right)^{-\epsilon}[\gamma J]^{\frac{1-\epsilon}{\gamma}+\epsilon}\right]$$

$$+\frac{1}{2}\frac{J_W^2}{J_{WW}}\frac{(\mu-r)^2}{\sigma^2}=0$$

Solution

- Guessing a solution of the form $J(W) = (aW)^\gamma / \gamma$ and substituting into (65), one finds that $a = \alpha^{1/(1-\epsilon)}$ where

$$\alpha \equiv \rho^{-\epsilon} \left(\epsilon\rho + (1-\epsilon) \left[r + \frac{(\mu - r)^2}{2(1-\gamma)\sigma^2} \right] \right) \tag{66}$$

- In turn, substituting this value for J into (61), one obtains

$$\begin{aligned} C &= \alpha\rho^{\epsilon} W \qquad (67) \\ &= \left(\epsilon\rho + (1-\epsilon) \left[r + \frac{(\mu - r)^2}{2(1-\gamma)\sigma^2} \right] \right) W \end{aligned}$$

and the optimal portfolio weight of the risky asset is

$$\omega = \frac{\mu - r}{(1-\gamma)\sigma^2} \tag{68}$$

Results

- Bhamra and Uppal (2003) show that when investment opportunities are stochastic, the portfolio weight, ω, can depend on both γ and ϵ.
- Note when $\epsilon = 1/\left(1-\gamma\right)$, equation (67) equals $C = \frac{\gamma}{1-\gamma}\left[\frac{\rho}{\gamma} - r - \frac{(\mu - r)^2}{2(1-\gamma)\sigma^2}\right] W$ derived earlier for the CRRA case.
- For an infinite horizon solution to exist, C_t in (67) must be positive, requiring $\rho > \frac{\epsilon - 1}{\epsilon}\left(r + \left[\mu - r\right]^2 / \left[2\left(1-\gamma\right)\sigma^2\right]\right)$, which occurs when ϵ is small.
- For example, when $\rho > 0$, the inequality is satisfied when $\epsilon < 1$.

Optimal Consumption

- For C^* in (67), the term $r + [\mu - r]^2 / \left[2(1-\gamma)\sigma^2\right]$ can be rewritten by substituting $\omega = (\mu - r) / \left[(1-\gamma)\sigma^2\right]$:

$$r + \frac{(\mu - r)^2}{2(1-\gamma)\sigma^2} = r + \omega \frac{\mu - r}{2} \tag{69}$$

- An increase in (69) raises (*reduces*) C when $\epsilon < 1$ (ϵ>*1*).
- The intuition is that when $\epsilon < 1$, the income effect from an improvement in investment opportunities dominates the substitution effect, so that consumption rises and savings fall.
- The reverse occurs when $\epsilon > 1$: the substitution effect dominates the income effect and savings rise.

Wealth Dynamics

- Assuming $0 < \omega < 1$ and substituting (67) and (68) into (56), wealth follows the geometric Brownian motion:

$$
\begin{aligned}
\frac{dW}{W} &= \left[\omega^* \left(\mu - r\right) + r - \alpha\rho^{\epsilon}\right] dt + \omega^* \sigma dz \qquad (70) \\
&= \left[\frac{\left(\mu - r\right)^2}{\left(1-\gamma\right)\sigma^2} + r - \epsilon\rho - \left(1-\epsilon\right)\left(r + \frac{\left(\mu - r\right)^2}{2\left(1-\gamma\right)\sigma^2}\right)\right] dt \\
&\quad + \frac{\mu - r}{\left(1-\gamma\right)\sigma} dz \\
&= \left[\epsilon\left(r + \frac{\left(\mu - r\right)^2}{2\left(1-\gamma\right)\sigma^2} - \rho\right) + \frac{\left(\mu - r\right)^2}{2\left(1-\gamma\right)\sigma^2}\right] dt \\
&\quad + \frac{\mu - r}{\left(1-\gamma\right)\sigma} dz
\end{aligned}
$$

Economic Growth

- Note that since $C = \alpha\rho^{\epsilon}W$, then dC/C has the same drift and volatility as wealth in (70).
- Thus, $d\ln C$ has a volatility, σ_c, and a mean, g_c, equal to

$$\sigma_c = \frac{\mu - r}{(1-\gamma)\sigma} \tag{71}$$

and

$$\begin{aligned} g_c &= \epsilon\left(r + \frac{(\mu - r)^2}{2(1-\gamma)\sigma^2} - \rho\right) + \frac{(\mu - r)^2}{2(1-\gamma)\sigma^2} - \frac{1}{2}\sigma_c^2 \\ &= \epsilon\left(r + \frac{(\mu - r)^2}{2(1-\gamma)\sigma^2} - \rho\right) - \frac{\gamma(\mu - r)^2}{2(1-\gamma)^2\sigma^2} \end{aligned} \tag{72}$$

Comparative Statics

- From (72), if $r + [\mu - r]^2 / \left[2(1-\gamma)\sigma^2\right] > \rho$, growth rises with ϵ as individuals save more.
- The squared Sharpe ratio, $[\mu - r]^2 / \sigma^2$ is a measure of the attractiveness of the risky asset, and the sign of $\partial g_c / \partial \left([\mu - r]^2 / \sigma^2\right)$ equals the sign of $\epsilon - \gamma / (1-\gamma)$.
- For the CRRA case of $\epsilon = 1/(1-\gamma)$, the derivative is positive, so that $\partial g_c / \partial \mu > 0$ and $\partial g_c / \partial \sigma < 0$.
- In general, $\partial g_c / \partial \left([\mu - r]^2 / \sigma^2\right) < 0$ if $\epsilon < \gamma / (1-\gamma)$ since from (68) as agents invest more in the faster-growing risky asset they also raise C_t (and reduce savings) when $\epsilon < 1$.
- Thus, when $\epsilon < \gamma / (1-\gamma)$, less savings dominates the portfolio effect and the economy grows more slowly.

Financial Market Globalization

- Obstfeld points out that the integration of global financial markets that allows residents to hold risky foreign, as well as domestic, investments increases diversification and effectively reduces individuals' portfolio variance, σ^2.

- The model predicts that if $\epsilon > \gamma / (1 - \gamma)$, financial market integration causes countries to grow faster.

- This recursive utility model does not help in explaining the equity premium puzzle since, from (71), the risky-asset Sharpe ratio, $(\mu - r) / \sigma$, equals $(1 - \gamma) \sigma_c$, the same form as with time-separable utility.

Risk-Free Rate Puzzle

- Recursive utility might explain the risk-free rate puzzle: substitute (71) into (72) and solve for r:

$$r = \rho + \frac{g_c}{\epsilon} - \left[1 - \gamma - \frac{\gamma}{\epsilon}\right] \frac{\sigma_c^2}{2} \tag{73}$$

- Recall that when $\epsilon = 1/(1-\gamma)$ we have

$$r = \rho + (1-\gamma) g_c - (1-\gamma)^2 \frac{\sigma_c^2}{2} \tag{74}$$

- Empirically, $g_c \approx 0.018$ is large relative to $\sigma_c^2/2 \approx 0.03^2/2 = 0.00045$, so the net effect of higher risk aversion, $1-\gamma$, needed to fit the equity risk premium implies too high a risk-free rate in (74).
- (73) may circumvent this problem because g_c is divided by ϵ.

Estimating the Elasticity of Intertemporal Substitution

- From (70) and (72), if the risky-asset Sharpe ratio, $(\mu - r)/\sigma$, is independent of the level of the real interest rate, r, then ϵ can be estimated by regressing consumption growth, $d \ln C$, on the real interest rate, r.
- Tests using aggregate consumption data find that ϵ is small, often close to zero.
- Other tests based on disaggregated consumption data find higher estimates for ϵ, often around 1.
- A value of $\epsilon = 1$ makes r independent of γ and, assuming ρ is small, could produce a reasonable value for r.

Summary

- For utility with habit persistence, the standard coefficient of relative risk aversion, $1-\gamma$, is transformed to $(1-\gamma)/S_t$ where $S_t < 1$ is the surplus consumption ratio.
- These models may imply aversion to holding risky assets sufficient to justify a high equity risk premium.
- Recursive utility distinguishes between an individual's level of risk aversion and his elasticity of intertemporal substitution.
- Such utility might allow a high equity risk premium and a low risk-free interest rate that is present in historical data.