Asset Pricing with Differential Information

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Introduction

- We consider environments where individuals have different private information regarding an asset's future payoff or value.
- The Sanford Grossman (1976) model shows how individuals' information affects their demands for an asset and the asset's equilibrium price.
- It examines two equilibria: one that is "competitive" but not fully rational and another that is fully-revealing and rational.
- An extension of the model that allows for asset supply shifts leads to a partially-revealing equilibrium.

Introduction (continued)

- We also present the seminal market microstructure model by Albert "Pete" Kyle (1985).
- The model environment analyzes trading where one individual, the "insider," has private information and trades with lesser-informed agents composed of a market maker and "noise" traders.
- The model solves for the strategic trading behavior of the insider and the pricing policy of the market maker, showing that the equilibrium asset price is partially revealing of the insider's private information.
- The model also provides a theoretical framework for determining bid-ask spreads and the market impact of trades.

Grossman Model Assumptions

- The model considers how rational individuals learn about others' private information from the risky asset's price, a concept known as "price discovery."
- **Assets:** This is a single-period portfolio choice problem. At the beginning of the period, traders can choose between a risk-free asset, which pays a known end-of-period return (1 plus the interest rate) of R_f, and a risky asset that has a beginning-of-period price of P_0 per share and an end-of-period random payoff (price) of $\widetilde{P}_1$ per share.
- The unconditional distribution of $\widetilde{P}_1$ is assumed to be normally distributed as $N(m, \sigma^2)$.
- The aggregate supply of shares of the risky asset is fixed at $\bar{X}$, but the risk-free asset is in perfectly elastic supply.

Assumptions (continued)

- **Trader Wealth and Preferences**: There are n different traders with the i^{th} trader having beginning-of-period wealth W_{0i}.
- Trader i maximizes end-of-period utility of the form

$$U_i(\tilde{W}_{1i}) = -e^{-a_i \tilde{W}_{1i}}, \quad a_i > 0 \tag{1}$$

- **Trader Information**: At the beginning of the period, the i^{th} trader observes the signal y_i:

$$\tilde{y}_i = \tilde{P}_1 + \tilde{\epsilon}_i \tag{2}$$

where $\tilde{\epsilon}_i \sim N(0, \sigma_i^2)$ and is independent of $\tilde{P}_1$.

Individuals' Asset Demands

- If X_i is the number of shares of the risky asset chosen by the i^{th} trader at the beginning of the period, then

$$\tilde{W}_{1i} = R_f W_{0i} + \left[\tilde{P}_1 - R_f P_0 \right] X_i \tag{3}$$

- Denote I_i as the information available to the i^{th} trader at the beginning of the period. The trader's maximization problem is

$$\max_{X_i} E \left[U_i(\tilde{W}_{1i}) \mid I_i \right] = \max_{X_i} E \left[-e^{-a_i \left(R_f W_{0i} + \left[\tilde{P}_1 - R_f P_0 \right] X_i \right)} \mid I_i \right] \tag{4}$$

which is equivalent to

$$\max_{X_i} \left\{ E \left[\tilde{W}_{1i} \mid I_i \right] - \frac{1}{2} a_i \operatorname{Var} \left[\tilde{W}_{1i} \mid I_i \right] \right\} \tag{5}$$

Individual's Risky Asset Demand

- Hence, the maximization problem (5) can be written

$$\max_{X_i} \left\{ X_i \left(E\left[\tilde{P}_1 \mid I_i \right] - R_f P_0 \right) - \frac{1}{2} a_i X_i^2 \operatorname{Var}\left[\tilde{P}_1 \mid I_i \right] \right\} \tag{6}$$

- The first-order condition with respect to X_i leads to:

$$X_i = \frac{E\left[\tilde{P}_1 \mid I_i \right] - R_f P_0}{a_i \operatorname{Var}\left[\tilde{P}_1 \mid I_i \right]} \tag{7}$$

- Equation (7) indicates that demand is increasing expected excess return but declining in variance and risk aversion.

A Competitive Equilibrium

- Consider an equilibrium where each trader uses information on the unconditional distribution of $\tilde{P}_1$ and his private signal, y_i, so that $I_i = \{y_i\}$.
- Bayes rule and the fact that $\tilde{P}_1$ and $\tilde{y}_i$ are jointly normally distributed with a squared correlation $\rho_i^2 \equiv \dfrac{\sigma^2}{\sigma^2 + \sigma_i^2}$ implies

$$\begin{aligned} E\left[\tilde{P}_1 \mid I_i\right] &= m + \rho_i^2 (y_i - m) \\ \mathrm{Var}\left[\tilde{P}_1 \mid I_i\right] &= \sigma^2 (1 - \rho_i^2) \end{aligned} \tag{8}$$

- Substituting these into (7), we have

$$X_i = \frac{m + \rho_i^2 (y_i - m) - R_f P_0}{a_i \sigma^2 (1 - \rho_i^2)} \tag{9}$$

Market Clearing

- (9) shows demand is increasing in the precision of the signal (the closer is ρ_i to 1, that is, the lower is σ_i).
- Market clearing requires aggregate demands equal supply:

$$\begin{aligned} \bar{X} &= \sum_{i=1}^{n} \left[\frac{m + \rho_i^2 (y_i - m) - R_f P_0}{a_i \sigma^2 (1 - \rho_i^2)} \right] \quad (10) \\ &= \sum_{i=1}^{n} \left[\frac{m + \rho_i^2 (y_i - m)}{a_i \sigma^2 (1 - \rho_i^2)} \right] - \sum_{i=1}^{n} \left[\frac{R_f P_0}{a_i \sigma^2 (1 - \rho_i^2)} \right] \end{aligned}$$

- Solving (10) for the equilibrium price leads to

$$P_0 = \frac{1}{R_f} \left[\sum_{i=1}^{n} \frac{m + \rho_i^2 (y_i - m)}{a_i \sigma^2 (1 - \rho_i^2)} - \bar{X} \right] \Bigg/ \left[\sum_{i=1}^{n} \frac{1}{a_i \sigma^2 (1 - \rho_i^2)} \right] \quad (11)$$

Competitive Equilibrium Implications

- P_0 reflects a weighted average of conditional expectations of $\tilde{P}_1$, where the weight on the i^{th} trader's conditional expectation, $m + \rho_i^2 (y_i - m)$, is

$$\frac{1}{a_i \, \sigma^2 \left(1 - \rho_i^2\right)} \Bigg/ \left[\sum_{i=1}^{n} \frac{1}{a_i \, \sigma^2 \left(1 - \rho_i^2\right)} \right] \tag{12}$$

- Trader i trades more aggressively the more precise his signal (higher ρ_i) and the lower his risk aversion a_i.
- But P_0 in (11) is not a fully rational equilibrium price because the solution neglects that traders might learn about other traders' signals from P_0 itself (price discovery).
- If traders formulate their demand according to (9), then a trader could learn from P_0 in (11) and would have an incentive to change her demand, negating the equilibrium.

A Rational Expectations Equilibrium

- A rational expectations equilibrium implies $I_i = \{y_i, P_0^*(y)\}$ where $y \equiv (y_1 \;\; y_2 \;\; ... \;\; y_n)$ and $P_0^*(y)$ is the rational expectations equilibrium price. Thus,

$$\bar{X} = \sum_{i=1}^{n} \left[\frac{E\left[\tilde{P}_1 \mid y_i, \, P_0^*(y) \right] - R_f \, P_0^*(y)}{a_i \, \mathrm{Var}\left[\tilde{P}_1 \mid y_i, \, P_0^*(y) \right]} \right] \tag{13}$$

- We next show that a rational expectations equilibrium exists when the ϵ_i's are independent and have the same variance, $\sigma_i^2 = \sigma_\epsilon^2$, for $i = 1, ..., n$.

Rational Expectations Equilibrium Price

- **Theorem**: There exists a rational expectations equilibrium with $P_0^*(y)$ given by

$$P_0^*(y) \;=\; \frac{1-\rho^2}{R_f} m + \frac{\rho^2}{R_f}\bar{y} - \frac{\sigma^2\left(1-\rho^2\right)}{R_f \sum_{i=1}^{n}\frac{1}{a_i}}\bar{X} \qquad (14)$$

where $\bar{y} \equiv \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} y_i$ and $\rho^2 \equiv \dfrac{\sigma^2}{\sigma^2 + \frac{\sigma_\epsilon^2}{n}}$.

- Proof: In (14), $P_0^*(y)$ is a linear function of $\bar{y}$ with coefficient of ρ^2/R_f. Thus, if a trader observes $P_0^*(y)$, it can be inverted to infer $\bar{y}$. Since all signals have equal precision, $\bar{y}$ is a sufficient statistic for the information contained in all signals and has the same precision as a single signal with variance $\frac{\sigma_\epsilon^2}{n}$.

Proof (continued)

- Now if each trader's demand equals (9) but where y_i is replaced with $\bar{y}$ and ρ_i is replaced with ρ, then aggregating them to $\bar{X}$ leads to (10), consistent with the assumption that traders can invert $P_0^*(y)$ to find $\bar{y}$. QED

- Note that $\bar{y}$ reflected in $P_0^*(y)$ is superior to any private signal, y_i, making y_i redundant. The equilibrium would be the same if all traders received the same signal, $\bar{y} \sim N(m, \sigma^2 + \frac{\sigma_\epsilon^2}{n})$ or if they all decided to share their private signals prior to trading.

- Thus $P_0^*(y)$ is *fully revealing*: strong-form market efficiency.

Robustness of Results

- Note that the equilibrium breaks down if each trader i needed to pay a tiny cost, c, to obtain y_i since there is not personal benefit from y_i (Grossman and Stiglitz, 1980).
- To benefit, there must be additional uncertainty so that other agents cannot infer y_i perfectly.

Asymmetric Information, Trading, and Markets

- The Kyle (1985) market microstructure model considers a security market where traders submit "market" orders.
- The market maker cannot tell whether an order is from an "insider" who is trading on private information, or from an individual whose order is non-information related, also know as a "noise" or "liquidity" trader.
- The model shows that the equilibrium is one where the security price only partially reveals the insider's private information, and the insider profits, on average, at the expense of the liquidity traders.

Kyle Model Assumptions

- **Asset Return Distribution**: At the beginning of the model's single period, agents trade in shares of a risky asset. Each share has a random end-of-period liquidation value of $\tilde{\nu} \sim N\left(p_0, \sigma_v^2\right)$.

- **Liquidity Traders**: *Noise traders* have exogenous needs to trade. As a group, they submit a "market" order to buy $\tilde{u}$ shares of the asset, where $\tilde{u} \sim N\left(0, \sigma_u^2\right)$ and is independent of $\tilde{\nu}$.

Assumptions (continued)

- **Better-Informed Traders**: A single risk-neutral *insider* knows with perfect certainty the realized end-of-period value of the risky security $\tilde{v}$ (but not $\tilde{u}$) and submits a market order of size x that maximizes his expected end-of-period profits.

- **Competitive Market Maker**: The single risk-neutral *market maker* receives the total market orders $\widetilde{u} + \widetilde{x}$ but cannot distinguish what parts were submitted by noise traders or by the insider. The competitive market maker takes the position $-\left(\widetilde{u} + \widetilde{x}\right)$ and sets the market price, p, so that his end-of-period profit is expected to be zero.

Market Maker Objective

- The market maker observes $u + x$ and then sets p to make his end-of-period profits, $-(\tilde{\nu} - p)(u + x)$, have an expectation of zero, implying
$$p = E\left[\tilde{\nu} \mid u + x\right] \tag{15}$$

- The more positive (*negative*) is $u + x$, the more likely x is high (*low*) because the insider knows ν is above (*below*) p_0. Thus, the market maker's *pricing rule* is a function of $x + u$, that is, $P(x + u)$.

Insider Objective and Equilibrium

- The insider chooses x to maximize his expected end-of-period profits, $\tilde{\pi}$, given knowledge of ν and the market maker's pricing rule:

$$\max_{x} E\left[\tilde{\pi} \mid \nu\right] = \max_{x} E\left[\left(\nu - P\left(x + \tilde{u}\right)\right) x \mid \nu\right] \tag{16}$$

- An equilibrium is a fixed point where each agent's actual behavior (e.g., pricing rule or trading strategy) is that which is expected by the other.

Insider Trading Strategy

- We hypothesize that the market maker sets a linear pricing rule $P(x+u) = \mu + \lambda(x+u)$ and show later that this is optimal. The insider's problem becomes

$$\begin{aligned} & \max_x E\left[(\nu - P(x+\tilde{u}))\,x \mid \nu\right] \qquad (17) \\ = \; & \max_x E\left[(\nu - \mu - \lambda(x+\tilde{u}))\,x \mid \nu\right] \\ = \; & \max_x (\nu - \mu - \lambda x)\,x, \text{ since } E[\tilde{u}] = 0 \end{aligned}$$

- The solution to (17) is

$$x = \alpha + \beta\nu \qquad (18)$$

where $\alpha = -\frac{\mu}{2\lambda}$ and $\beta = \frac{1}{2\lambda}$.

Market Maker's Pricing Strategy

- The competitive market maker sets $p = E\left[\tilde{\nu} \mid u + x\right]$, where according to (18) $x = \alpha + \beta\tilde{\nu}$. For the market maker, $\tilde{\nu}$ and $y \equiv \tilde{u} + x = \tilde{u} + \alpha + \beta\tilde{\nu}$ are jointly normally distributed, so that the maximum likelihood estimate of $E\left[\tilde{\nu} \mid y\right]$ is linear in y, say $P(y) = E\left[\tilde{\nu} \mid y\right] = \mu + \lambda y$. This (least squares) estimator minimizes

$$
\begin{aligned}
E\left[\left(\tilde{\nu} - P(y)\right)^2\right] &= E\left[\left(\tilde{\nu} - \mu - \lambda y\right)^2\right] \qquad (19) \\
&= E\left[\left(\tilde{\nu} - \mu - \lambda\left(\tilde{u} + \alpha + \beta\tilde{\nu}\right)\right)^2\right]
\end{aligned}
$$

- Thus, μ and λ minimize

$$
\min_{\mu,\lambda} E\left[\left(\tilde{\nu}\left(1 - \lambda\beta\right) - \lambda\tilde{u} - \mu - \lambda\alpha\right)^2\right] \qquad (20)
$$

Market Maker's Pricing Strategy (continued)

- Recall that $E[\nu] = p_0$, $E\left[(\nu - p_0)^2\right] = \sigma_v^2$, $E[u] = 0$, $E\left[u^2\right] = \sigma_u^2$, and $E[u\nu] = 0$, so that (20) is equivalent to

$$\min_{\mu,\lambda} (1-\lambda\beta)^2 \left(\sigma_v^2 + p_0^2\right) + (\mu + \lambda\alpha)^2 + \lambda^2\sigma_u^2 - 2(\mu+\lambda\alpha)(1-\lambda\beta)\, p_0 \tag{21}$$

- The first-order conditions with respect to μ and λ are

$$\mu = -\lambda\alpha + p_0 (1 - \lambda\beta) \tag{22}$$

$$\begin{aligned} 0 \;=\; & -2\beta(1-\lambda\beta)\left(\sigma_v^2 + p_0^2\right) + 2\alpha(\mu + \lambda\alpha) + 2\lambda\sigma_u^2 \\ & -2p_0\left[-\beta(\mu+\lambda\alpha) + \alpha(1-\lambda\beta)\right] \end{aligned} \tag{23}$$

- Substituting $\mu + \lambda\alpha = p_0(1-\lambda\beta)$ from (22) into (23), (23) is

$$\lambda = \frac{\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2} \tag{24}$$

Equilibrium Price and Insider Order

- Substituting in $\alpha = -\frac{\mu}{2\lambda}$ and $\beta = \frac{1}{2\lambda}$ in (22) and (24):

$$\mu = p_0 \tag{25}$$

$$\lambda = \frac{1}{2}\frac{\sigma_v}{\sigma_u} \tag{26}$$

- In summary, the equilibrium price is

$$p = p_0 + \frac{1}{2}\frac{\sigma_v}{\sigma_u}\left(\tilde{u} + \tilde{x}\right) \tag{27}$$

where the equilibrium order submitted by the insider is

$$x = \frac{\sigma_u}{\sigma_v}\left(\tilde{\nu} - p_0\right) \tag{28}$$

Analysis of the Results

- From (28), the greater the volatility (amount) of noise trading (camouflage), σ_u, the larger is x for a given deviation of ν from its unconditional mean since it becomes more difficult for the market maker to extract the "signal" of insider trading from the noise.

- Note that if equation (28) is substituted into (27):

$$\begin{aligned} p &= p_0 + \tfrac{1}{2}\frac{\sigma_v}{\sigma_u}\tilde{u} + \tfrac{1}{2}\left(\tilde{\nu} - p_0\right) \\ &= \tfrac{1}{2}\left(\frac{\sigma_v}{\sigma_u}\tilde{u} + p_0 + \tilde{\nu}\right) \end{aligned} \tag{29}$$

- The price is partially revealing since only one half of the insider's private information, $\frac{1}{2}\ \tilde{\nu}$, is reflected in the price.

Insider Profits

- Using (28) and (29), the insider's expected profits are:

$$E\left[\tilde{\pi}\right] = E\left[x\left(\nu - p\right)\right] = E\left[\frac{\sigma_u}{\sigma_v}\left(\tilde{\nu} - p_0\right)\tfrac{1}{2}\left(\nu - p_0 - \frac{\sigma_v}{\sigma_u}\tilde{u}\right)\right] \tag{30}$$

- Conditional on knowing ν, the insider expects profits of

$$E\left[\tilde{\pi} \mid \nu\right] = \tfrac{1}{2}\frac{\sigma_u}{\sigma_v}\left(\nu - p_0\right)^2 \tag{31}$$

- Unconditional on knowing $\tilde{\nu}$, the insider expects a profit of

$$E\left[\tilde{\pi}\right] = \tfrac{1}{2}\frac{\sigma_u}{\sigma_v}E\left[\left(\tilde{\nu} - p_0\right)^2\right] = \tfrac{1}{2}\sigma_u\sigma_v \tag{32}$$

- Note that the insider's expected profits equal the noise traders' expected losses: on average noise traders' buy (*sell*) orders are executed at a higher (*lower*) price than p_0.

Liquidity: Kyle's lambda

- $\lambda = \frac{1}{2}\frac{\sigma_v}{\sigma_u}$, the amount that the market maker raises the price when the total order flow, $(u + x)$, goes up by 1, relates to the security's bid-ask spread.
- Conversely, $1/\lambda = 2\frac{\sigma_u}{\sigma_v}$, is a measure of the "depth" or "liquidity" of the market.
- The more noise traders relative to the value of insider information, the less the market maker adjusts the price in response to a given order, since the likelihood of the order being that of a noise trader, rather than an insider, is greater.
- The more noise traders (greater is σ_u), the greater is $E[\tilde{\pi}]$ and *total* expected noise trader losses, but the expected loss per *individual* noise trader falls with σ_u.

Summary

- An investor's private information, along with risk aversion, affects her demand for a risky asset, thereby affecting the asset's equilibrium price.
- More subtly, a rational investor also learns about the private information of others through the asset price itself, and this price discovery affects investors' equilibrium demands.
- When non-information-based "noise" factors affect the supply or demand for an asset, investors cannot perfectly infer others' private information.
- The greater the likelihood of trading due to private information, the larger will be a security's bid-ask spread.
- Hence, a security's liquidity is determined by the degree of noise (non-information-based) trading relative to insider (private-information-based) trading.