FIN 591: Homework #3

Due on Wednesday, April 11, 2018

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Problem 1

a. Since the final payoff of P is 1, using continuous-time version stochastic discount factor, $P_t(\tau)$ is derived as follows.

$$\begin{aligned} P_t(\tau) &= \mathrm{E}_t\left[\frac{U_c(C_{t+\tau,t+\tau})}{U_c(C_t,t)}\times 1\right] \\ &= \mathrm{E}_t\left[\frac{e^{-\phi(t+\tau)}C_{t+\tau}^{\gamma-1}}{e^{\phi t}C_t^{\gamma-1}}\right] \\ &= \mathrm{E}_t\left[e^{-\phi\tau}\frac{C_{t+\tau}^{\gamma-1}}{C_t^{\gamma-1}}\right] \end{aligned} \tag{1}$$

b. From $P_t(\tau) = \mathrm{E}_t\left[\frac{e^{-\phi(t+\tau)}C_{t+\tau}^{\gamma-1}}{e^{\phi t}C_t^{\gamma-1}}\right]$, we can find that process M_t is equal to $e^{-\phi t}C_t^{\gamma-1}$. Therefore, using Ito's lemma, dynamics of M_t can be derived as equation (2).

$$\begin{aligned} dM_t &= -\phi e^{-\phi t}C^{\gamma-1}dt + e^{-\phi t}(\gamma-1)C^{\gamma-2}C[(\mu_c-\lambda k)dt+\sigma_c dZ_c] \\ &\quad + \frac{1}{2}e^{-\phi t}(\gamma-1)(\gamma-2)C^2C^{\gamma-3}\sigma_c^2dt + [e^{-\phi t}(YC)^{\gamma-1} - e^{-\phi t}C^{\gamma-1}]dq \\ &= [-\phi + (\gamma-1)(\mu_c-\lambda k) + \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2]Mdt + (\gamma-1)\sigma_c M dZ_c + (Y^{\gamma-1}-1)Mdq \end{aligned} \tag{2}$$

c. Since $\mathrm{E}\left[\frac{dM}{M}\right] = -rdt$, the following equation holds.

$$\begin{aligned} r &= -\mathrm{E}\left[\frac{dM}{M}\right] \\ &= \phi - (\gamma-1)(\mu_c-\lambda k) - \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2 - \lambda\mathrm{E}[e^{(\gamma-1)\log Y}-1] \\ &= \phi - (\gamma-1)(\mu_c-\lambda k) - \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2 - \lambda(e^{(\gamma-1)\alpha+\frac{1}{2}(\gamma-1)^2\delta^2}-1) \end{aligned} \tag{3}$$

Since μ_c, k, λ are constant, instantaneous risk free rate is constant.

d. Since an asset price is equal to sum of discounted future payoffs, assuming some regularity conditions hold, S_t is represented as follows.

$$\begin{aligned} S_t &= \mathrm{E}_t\left[\int_t^\infty \frac{M_s}{M_t}D_s ds\right] \\ &= \mathrm{E}_t\left[\int_t^\infty \frac{e^{-\phi s}C_s^{\gamma-1}}{e^{-\phi t}C_t^{\gamma-1}}D_s ds\right] \\ \Rightarrow \frac{S_t}{D_t} &= \mathrm{E}_t\left[\int_t^\infty e^{-\phi(s-t)}\left(\frac{C_s}{C_t}\right)^{\gamma-1}\left(\frac{D_s}{D_t}\right)ds\right] \\ &= \mathrm{E}_t\left[\int_t^\infty e^{-\phi(s-t)+(\gamma-1)\log(C_s/C_t)+\log(D_s/D_t)}ds\right] \\ &= \int_t^\infty \mathrm{E}_t\left[e^{-\phi(s-t)+(\gamma-1)\log(C_s/C_t)+\log(D_s/D_t)}ds\right] \end{aligned} \tag{4}$$

Considering the process of C_t, $\mathrm{E}_t[e^{(\gamma-1)\log(C_s/C_t)}]$ is calculated as follows.

$$\begin{aligned} \mathrm{E}_t[e^{(\gamma-1)\log(C_s/C_t)}] &= \mathrm{E}_t\left[e^{(\gamma-1)(\mu_c-\frac{1}{2}\sigma_c^2-\lambda k)(s-t)+\frac{1}{2}(\gamma-1)^2\sigma_c^2(s-t)+(\gamma-1)\log y(s,t)}\right] \\ y(s,t) &= \prod_{i=s}^{t} Y_i \end{aligned} \tag{5}$$

Since $\log y(s,t) = \sum_{i=s}^{t} \log Y_i$, and $\log Y_i$'s are independently and identically distributed as $N(\alpha, \delta^2)$, $\log(C_s/C_t)$ is also normally distributed, and its expected value from equation (5) is calculated as follows.

$$\mathrm{E}_t[e^{(\gamma-1)\log(C_s/C_t)}] = e^{(\gamma-1)(\mu_c - \frac{1}{2}\sigma_c^2 - \lambda k)(s-t) + \frac{1}{2}(\gamma-1)^2\sigma_c^2(s-t) + (\gamma-1)\alpha(s-t) + \frac{1}{2}(\gamma-1)^2\delta^2(s-t)} \tag{6}$$

Applying the result from equation (6) and considering the correlation between z_d and z_c is ρ, $\frac{S_t}{D_t}$ from equation (4) is solved as follows.

$$\begin{aligned}
\frac{S_t}{D_t} &= \int_t^{\infty} e^{-(s-t)[\phi + (1-\gamma)(\mu_c - \frac{1}{2}\sigma_c^2 - \lambda k) + \frac{1}{2}(1-\gamma)^2\sigma_c^2 + (1-\gamma)\alpha + \frac{1}{2}(1-\gamma)^2\delta^2 - \mu_d + (1-\gamma)\rho\sigma_c\sigma_d]} ds \\
&= -\frac{1}{A} e^{-(s-t)A}\Big|_t^{\infty} = \frac{1}{A} \\
A &= \phi + (1-\gamma)(\mu_c - \frac{1}{2}\sigma_c^2 - \lambda k) + \frac{1}{2}(1-\gamma)^2\sigma_c^2 + (1-\gamma)\alpha + \frac{1}{2}(1-\gamma)^2\delta^2 - \mu_d + (1-\gamma)\rho\sigma_c\sigma_d
\end{aligned} \tag{7}$$

Problem 2

a. Considering the process of risky asset price, intertemporal budget constraint is derived as follows.

$$\begin{aligned}
dW &= \omega_t \frac{dS}{S} + (1-\omega_t) r dt - C_t dt \\
&= (\omega_t(\mu - \lambda k - r)W_t + rW_t - C_t)dt + \sigma\omega_t W_t dz + \omega_t W_t (Y_t - 1) dq
\end{aligned} \tag{8}$$

b. Investors maximize $\mathrm{E}_0[\int_0^T e^{-\phi t} u(C_t) dt]$, subject to the equation (8).
Let $J(W_t, t) = \max_{C_t, \omega_t} \mathrm{E}_t[\int_t^T e^{-\phi s} u(C_s) ds]$. Then the following equation follows.

$$\begin{aligned}
J(W_t, t) &= \max_{C_t, \omega_t} \mathrm{E}_t \left[\int_t^{t+\Delta t} e^{-\phi s} u(C_s) ds + J(W_{t+\Delta t}, t + \Delta t) \right] \\
&= \max_{C_t, \omega_t} \mathrm{E}_t [u(C_t)\Delta t + J(W_t, t) + J_t \Delta t + J_W(\omega_t(\mu - \lambda k - r)W_t + rW_t - C_t)\Delta t \\
&+ \frac{1}{2}\omega_t^2 W_t^2 \sigma^2 J_{WW} \Delta t + (J(1 + \omega_t W_t(Y_t - 1), t) - J(W_t, t))dq], \quad (t \in [0, T])
\end{aligned} \tag{9}$$

Letting $\Delta t \to 0$, equation (9) becomes equation (10), and it is the Bellman equation.

$$\begin{aligned}
0 &= \max_{C,\omega} [u(C_t) + L(J)] \\
L(J) &= J_t + J_W(\omega_t W_t(\mu - \lambda k - r) + rW_t - C_t) + \frac{1}{2}\omega_t^2 W_t^2 \sigma^2 J_{WW} + \lambda \mathrm{E}_t[J(1 + \omega_t W_t(Y_t - 1), t) - J(W_t, t)]
\end{aligned} \tag{10}$$

c. Applying first order condition to equation (10), the following equation holds.

$$\begin{aligned}
&u_C = J_W \\
&J_W W_t(\mu - \lambda k - r) + \omega_t W_t^2 \sigma^2 J_{WW} + \lambda \mathrm{E}_t[W_t(Y_t - 1) J_W(1 + \omega_t W_t(Y_t - 1), t)] = 0
\end{aligned} \tag{11}$$