FIN 591: Homework #1

Due on Monday, February 5, 2018

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Problem 1

a. *(Short selling restriction)* If an investor cannot sell short risky asset, the amount of investment in risky asset should be nonnegative. Therefore, the maximization problem is stated as the following equation.

$$\begin{aligned} &\max_A \mathrm{E}[U(\widetilde{W})] = \max_A \mathrm{E}[U(W_0(1+r_f) + A(\widetilde{r} - r_f))] \\ &\text{subject to } A \geq 0 \end{aligned} \tag{1}$$

The maximization problem (1) is equivalent to

$$\max_A \mathrm{E}[U(W_0(1+r_f) + A(\widetilde{r} - r_f))] + \lambda A \tag{2}$$

By applying Kuhn-Tucker conditions, the value of A maximizing expected utility must must satisfy the following conditions.

$$\begin{aligned} \mathrm{E}[U'(\widetilde{W})(\widetilde{r} - r_f)] + \lambda = 0 \\ A \geq 0 \\ A\mathrm{E}[U'(\widetilde{W})(\widetilde{r} - r_f)] = 0 \end{aligned} \tag{3}$$

b. *(Restriction of riskless borrowing)* A restriction of riskless borrowing implies that the dollar amount of investment in riskless asset should be nonnegative. It leads to the following maximization problem.

$$\begin{aligned} &\max_A \mathrm{E}[U(\widetilde{W})] = \max_A \mathrm{E}[U(W_0(1+r_f) + A(\widetilde{r} - r_f))] \\ &\text{subject to } W_0 - A \geq 0 \end{aligned} \tag{4}$$

From the analogy of problem 1.a, the value of A which maximizes expected utility must satisfy the following first order conditions.

$$\begin{aligned} \mathrm{E}[U'(\widetilde{W})(\widetilde{r} - r_f)] + \lambda = 0 \\ W_0 - A \geq 0 \\ (W_0 - A)\mathrm{E}[U'(\widetilde{W})(\widetilde{r} - r_f)] = 0 \end{aligned} \tag{5}$$

Problem 2

It is not necessary that two frontier portfolios which creates other efficient portfolio must be efficient. Although there is no risk-free asset, any portfolio can be generated by using market portfolio and zero beta portfolio. Since it is already known that zero beta portfolio is an inefficient portfolio, it is not necessary that two frontier portfolio must be efficient. Additionally, it can be explained intuitively. Let us consider a simple economy. In the economy, there are two assets, whose return and variance is denoted as r_i, σ_i^2, $i = 1, 2$, respectively. Let ρ denote the correlation coefficient between r_1 and r_2. Now, construct a portfolio such that $r_p = w_1 r_1 + (1 - w_1) r_2$. In this case, the expected return of the portfolio is $w_1 \mathrm{E}[r_1] + (1 - w_2)\mathrm{E}[r_2]$, which is exactly on the straight line connecting $\mathrm{E}[r_1]$ to $\mathrm{E}[r_2]$ because it is just a weighted average of expected

Problem 2 continued on next page. . .

returns. However, since $\mathrm{Var}[r_p] = \mathrm{Var}[w_1 r_1 + (1-w_1) r_2] = w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + \rho\sigma_1\sigma_2$, variance of portfolio can be greater or less than weighted average of individual variances, depending on correlation coefficient, ρ. Therefore, if we properly choose portfolios which have negative correlation, we can create a portfolio which is on slightly left to the straight line which connects $(\mathrm{E}[r_1], \sigma_1)$ and $(\mathrm{E}[r_2], \sigma_2)$ on $(\mathrm{E}[r], \sigma)$-space. (Actually, it justifies diversification effect.) It gives an intuition that we can create minimum variance portfolio(or similar portfolios) by appropriately choosing an efficient portfolio and an inefficient portfolio. Since minimum variance portfolio is also an efficient portfolio, it is possible to generate an efficient portfolio using two portfolios even if they are not both efficient.

Problem 3

Return used to mean-variance analysis and CAPM should be nominal rate. It is because risk-free rate is adjusted(subtracted) in both mean-variance analysis and CAPM. Since risk-free rate already contain information of inflation(i.e. risk-free rate increases when inflation occurs, and decreases when deflation occurs), if we use real rate for mean-variance analysis or CAPM, inflation is "over-adjusted" in the analysis, and therefore it goes wrong. Hence, nominal returns should be used.

Problem 4

a. Let x and y denote weights of risky asset 1 and 2, respectively. Then since the expected return and variance of portfolio is $\overline{R}_p = x\overline{R}_1 + y\overline{R}_2 + (1-x-y)R_f$, $\sigma_p^2 = x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho$, respectively, the Sharpe ratio of portfolio can be represented as follows.

$$\begin{aligned} s_e &= \frac{\overline{R}_p - R_f}{\sigma_p} \\ &= \frac{x\overline{R}_1 + y\overline{R}_2 - (x+y)R_f}{\sqrt{x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho}} \\ &= \frac{x\sigma_1 s_1 + y\sigma_2 s_2}{\sqrt{x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho}} \end{aligned} \tag{6}$$

Let $A_1 = x\sigma_1 s_1 + y\sigma_2 s_2$ and $A_2 = (x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-\frac{1}{2}}$. Then s_e can be represented as A_1A_2. By taking derivative to s_e with respect to x and y, we can obtain first order conditions as follows.

$$\begin{aligned} \frac{\partial S_e}{\partial x} &= \frac{\partial A_1}{\partial x}A_2 + \frac{\partial A_2}{\partial x}A_1 \\ \frac{\partial S_e}{\partial y} &= \frac{\partial A_1}{\partial y}A_2 + \frac{\partial A_2}{\partial y}A_1 \end{aligned} \tag{7}$$

$$
\begin{aligned}
\frac{\partial A_1}{\partial x} &= \sigma_1 s_1 \\
\frac{\partial A_1}{\partial y} &= \sigma_2 s_2 \\
\frac{\partial A_2}{\partial x} &= -\frac{1}{2}(x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-\frac{3}{2}}(2x\sigma_1^2 + 2y\sigma_1\sigma_2\rho) \\
&= -(x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-\frac{3}{2}}(x\sigma_1^2 + y\sigma_1\sigma_2\rho) \\
\frac{\partial A_2}{\partial y} &= -\frac{1}{2}(x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-\frac{3}{2}}(2y\sigma_2^2 + 2x\sigma_1\sigma_2\rho) \\
&= -(x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-\frac{3}{2}}(y\sigma_2^2 + x\sigma_1\sigma_2\rho)
\end{aligned} \tag{8}
$$

Plugging the results of equation (8) into equation (7) we can get equations as follows.

$$
\begin{aligned}
\frac{\partial S_e}{\partial x} &= \sigma_1 s_1(x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-\frac{1}{2}} - (x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-\frac{3}{2}}(x\sigma_1^2 + y\sigma_1\sigma_2\rho)(x\sigma_1 s_1 + y\sigma_2 s_2) \\
&= (x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-\frac{1}{2}}[\sigma_1 s_1 - (x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-1}(x\sigma_1^2 + y\sigma_1\sigma_2\rho)(x\sigma_1 s_1 + y\sigma_2 s_2)] \\
&= 0 \\
\frac{\partial S_e}{\partial y} &= \sigma_2 s_2(x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-\frac{1}{2}} - (x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-\frac{3}{2}}(y\sigma_2^2 + x\sigma_1\sigma_2\rho)(x\sigma_1 s_1 + y\sigma_2 s_2) \\
&= (x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-\frac{1}{2}}[\sigma_2 s_2 - (x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-1}(y\sigma_2^2 + x\sigma_1\sigma_2\rho)(x\sigma_1 s_1 + y\sigma_2 s_2) \\
&= 0
\end{aligned} \tag{9}
$$

Since we assume that $x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho \neq 0$, equation (9) is reduced to equation (10).

$$
\begin{aligned}
\sigma_1 s_1 - (x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-1}(x\sigma_1^2 + y\sigma_1\sigma_2\rho)(x\sigma_1 s_1 + y\sigma_2 s_2) = 0 \\
\sigma_2 s_2 - (x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho)^{-1}(y\sigma_2^2 + x\sigma_1\sigma_2\rho)(x\sigma_1 s_1 + y\sigma_2 s_2) = 0
\end{aligned} \tag{10}
$$

Multiplying $x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho$ to both sides of equation (10), we obtain (11) and (12).

$$
\begin{aligned}
&\sigma_1 s_1(x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho) - (x\sigma_1^2 + y\sigma_1\sigma_2\rho)(x\sigma_1 s_1 + y\sigma_2 s_2) \\
&= y^2\sigma_1\sigma_2^2(s_1 - s_2\rho) - xy\sigma_1^2\sigma_2(s_2 - s_1\rho) = 0
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&\sigma_2 s_2(x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\sigma_1\sigma_2\rho) - (y\sigma_2^2 + x\sigma_1\sigma_2\rho)(x\sigma_1 s_1 + y\sigma_2 s_2) \\
&= x^2\sigma_1^2\sigma_2(s_2 - s_1\rho) - xy\sigma_1\sigma_2^2(s_1 - s_2\rho) = 0
\end{aligned} \tag{12}
$$

By adding (11) to (12) we can obtain (13).

$$
\begin{aligned}
&(x^2\sigma_1^2\sigma_2 - xy\sigma_1^2\sigma_2)(s_2 - s_1\rho) + (y^2\sigma_1\sigma_2^2 - xy\sigma_1\sigma_2^2)(s_1 - s_2\rho) \\
&= x\sigma_1^2\sigma_2(x - y)(s_2 - s_1\rho) + y\sigma_1\sigma_2^2(y - x)(s_1 - s_2\rho) \\
&= \sigma_1\sigma_2(x - y)[x\sigma_1(s_2 - s_1\rho) - y\sigma_2(s_1 - s_2\rho)] = 0
\end{aligned} \tag{13}
$$

Assuming $x \neq y$, equation (13) reduces to (14), which indicates optimal weight on risky asset 1.

$$
\begin{aligned}
&x\sigma_1(s_2 - s_1\rho) - y\sigma_2(s_1 - s_2\rho) = 0 \\
&\Rightarrow x^* = \frac{y\sigma_2(s_1 - s_2\rho)}{\sigma_1(s_2 - s_1\rho)}
\end{aligned} \tag{14}
$$

Plug x^* into equation (6), we can get the maximum Sharpe ratio as follows.

$$\begin{aligned}
s_e^* &= \frac{x^*\sigma_1 s_1 + y\sigma_2 s_2}{\sqrt{x^{*2}\sigma_1^2 + y^2\sigma_2^2 + 2x^*y\sigma_1\sigma_2\rho}} \\
x^*\sigma_1 s_1 + y\sigma_2 s_2 &= \frac{y\sigma_2(s_1 - s_2\rho)}{\sigma_1(s_2 - s_1\rho)}\sigma_1 s_1 + y\sigma_2 s_2 \\
&= y\sigma_2\frac{s_1 - s_2\rho}{s_2 - s_1\rho} + s_2 \quad (numerator) \\
\sqrt{x^{*2}\sigma_1^2 + y^2\sigma_2^2 + 2x^*y\sigma_1\sigma_2\rho} &= y\sigma_2\sqrt{(\frac{s_1 - s_2\rho}{s_2 - s_1\rho})^2 + 2\frac{s_1 - s_2\rho}{s_2 - s_1\rho} + 1} \quad (denominator) \\
\Rightarrow s_e^* &= \frac{s_1^2 - 2s_1s_2\rho + s_2^2}{[(s_1 - s_2\rho)^2 + 2(s_1 - s_2\rho)(s_2 - s_1\rho)\rho + (s_2 - s_1\rho)^2]^{\frac{1}{2}}}
\end{aligned} \tag{15}$$

b. From 4.a, we derived that maximum Sharpe ratio is attained if $x = \frac{y\sigma_2(s_1 - s_2\rho)}{\sigma_1(s_2 - s_1\rho)}$. Therefore, since σ_1 and σ_2 are both positive, in order to achieve maximum Sharpe ratio under constraints that $x > 0$ and $y > 0$, it is necessary that $\frac{s_1 - s_2\rho}{s_2 - s_1\rho}$ should be positive.