FIN 591: Homework #2

Due on Wednesday, February 28, 2018

Wanbae Park

Problem 1

a.

b.

Problem 2

a. Under the optimal choice, $U_C(C_{T-1}, T-1) = \mathrm{E}_{T-1}[B_W(W_T, T)R_{T-1}]$ holds. If we plug given utility and bequest function to the equation, the following equation holds.

$$\begin{aligned} \delta^{T-1} C_{T-1}^{\gamma-1} &= \mathrm{E}_{T-1}[\delta^T W_T^{\gamma-1} R_{T-1}] \\ \Rightarrow \delta^{T-1} C_{T-1}^{\gamma-1} &= \delta^T S_{T-1}^{\gamma-1} \mathrm{E}[R_{T-1}^{\gamma}] \quad \text{where } W_T = S_{T-1}R_{T-1}, S_{T-1} = W_{T-1} - C_{T-1} \end{aligned} \tag{1}$$

Therefore, if we rearrange the equation (1), the optimal consumption at time $T-1$, C_{T-1}^* can be obtained as follows.

$$C_{T-1}^* = \frac{\delta^{\frac{1}{\gamma-1}} \mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}}}{1 - \delta^{\frac{1}{\gamma-1}} \mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}}} W_{T-1} \tag{2}$$

Another condition under optimal choice is $\mathrm{E}_{T-1}[B_W(W_T, T)(R_{i,T-1} - R_f)] = 0$ for $i = 1, 2, 3, \ldots, n$. Therefore, the following equation holds.

$$\begin{aligned} &\mathrm{E}_{T-1}[\delta^T W_T^{\gamma-1} R_{i,T-1}] = R_f \mathrm{E}_{T-1}[\delta^T W_T^{\gamma-1}] \\ &\Rightarrow \mathrm{E}_{T-1}[(S_{T-1}R_{T-1})^{\gamma-1} R_{i,T-1}] = R_f \mathrm{E}_{T-1}[\delta^T (S_{T-1}R_{T-1})^{\gamma-1}] \\ &\Rightarrow \mathrm{E}_{T-1}[R_{T-1}^{\gamma-1} R_{i,T-1}] = R_f \mathrm{E}_{T-1}[R_{T-1}^{\gamma-1}] \end{aligned} \tag{3}$$

b. Let $\delta^{\frac{1}{\gamma-1}} \mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}} = a$. Then $C_{T-1}^* = \frac{a}{1+a} W_{T-1}$. Since $J(W_{T-1}, T-1) = U(C_{T-1}^*, T-1) = \mathrm{E}_{T-1}[B(W_T, T)]$, $J(W_{T-1}, T-1)$ can be represented as follows.

$$\begin{aligned} J(W_{T-1}, T-1) &= \frac{\delta^{T-1} C_{T-1}^{*\gamma}}{\gamma} + \mathrm{E}_{T-1}[\frac{\delta^T W_T^{\gamma}}{\gamma}] \\ &= \frac{\delta^{T-1}}{\gamma} (\frac{a}{1+a} W_{T-1})^{\gamma} + \frac{\delta^T}{\gamma} \mathrm{E}_{T-1}[((1 - \frac{a}{1+a}) W_{T-1} R_{T-1})^{\gamma}] \\ &= \frac{\delta^{T-1}}{\gamma} ((\frac{a}{1+a})^{\gamma} W_{T-1}^{\gamma} + \delta (\frac{1}{1+a})^{\gamma} W_{T-1}^{\gamma} \mathrm{E}_{T-1}[R_{T-1}^{\gamma}]) \end{aligned} \tag{4}$$

c.

d.