FIN 591: Homework #4

Due on Monday, April 30, 2018

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Problem 1

Assuming $\theta = 1$, V_t is represented as follows.

$$V_t = [C_t^{\gamma} + \delta E_t[V_{t+1}^{\gamma}]]^{\frac{1}{\gamma}} \tag{1}$$

Solving the difference equation iteratively, we can solve V_t as follows.

$$\begin{aligned} V_t &= [C_t^{\gamma} + \delta \mathrm{E}_t[V_{t+1}^{\gamma}]]^{\frac{1}{\gamma}} \\ &= [C_t^{\gamma} + \delta \mathrm{E}_t[\{C_{t+1}^{\gamma} + \delta \mathrm{E}_{t+1}[V_{t+2}^{\gamma}]^{\gamma \times \frac{1}{\gamma}}\}]]^{\frac{1}{\gamma}} \\ &= [C_t^{\gamma} + \delta \mathrm{E}_t[C_{t+1}^{\gamma} + \delta \mathrm{E}_{t+1}[V_{t+2}^{\gamma}]]^{\frac{1}{\gamma}} \\ &= [C_t^{\gamma} + \delta \mathrm{E}_t[C_{t+1}^{\gamma}] + \delta^2 \mathrm{E}_t[V_{t+2}^{\gamma}]]^{\frac{1}{\gamma}} \end{aligned} \tag{2}$$

Using this procedure iteratively, V_t is derived as follows.

$$\begin{aligned} V_t &= [C_t^{\gamma} + \delta \mathrm{E}_t[C_{t+1}^{\gamma}] + \delta^2 \mathrm{E}_t[C_{t+2}^{\gamma}] + \delta^3 \mathrm{E}_t[C_{t+3}^{\gamma}] + \delta^4 \mathrm{E}_t[C_{t+4}^{\gamma}] + \dots]^{\frac{1}{\gamma}} \\ &= \sum_{i=t}^{\infty} \delta^i \mathrm{E}_t[C_i^{\gamma}]^{\frac{1}{\gamma}} \end{aligned} \tag{3}$$

Considering the discrete-time version of power utility $U(C_t, t) = \delta^t \times C_t^{\gamma}/\gamma$, γV_t^{γ} implies a lifetime utility function.

Problem 2

Since $m_{t,t+1} = \frac{\partial V_t}{\partial C_{t+1}} / \frac{\partial V_t}{\partial C_t}$, stochastic discount factor of this case can be derived as follows.

$$\frac{\partial V_t}{\partial C_t} = \frac{\theta}{\gamma}[C_t^{\frac{\gamma}{\theta}} + \delta \mathrm{E}_t[V_{t+1}^{\gamma}]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\gamma}{\theta} C_t^{\frac{\gamma}{\theta}-1} \tag{4}$$

$$\frac{\partial V_t}{\partial C_{t+1}} = \frac{\theta}{\gamma}[C_t^{\frac{\gamma}{\theta}} + \delta \mathrm{E}_t[V_{t+1}^{\gamma}]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\delta}{\theta} \mathrm{E}_t[V_{t+1}^{\gamma}]^{\frac{1}{\theta}-1} \times \gamma V_{t+1}^{\gamma-1} \times \frac{\partial V_{t+1}}{\partial C_{t+1}} \tag{5}$$

Since $V_{t+1} = [C_{t+1}^{\frac{\gamma}{\theta}} + \delta \mathrm{E}_{t+1}[V_{t+2}^{\gamma}]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}}$, $\frac{\partial V_{t+1}}{\partial C_{t+1}}$ is derived as follows.

$$\begin{aligned} \frac{\partial V_{t+1}}{\partial C_{t+1}} &= \frac{\theta}{\gamma}[C_{t+1}^{\frac{\gamma}{\theta}} + \delta \mathrm{E}_{t+1}[V_{t+2}^{\gamma}]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\gamma}{\theta} C_{t+1}^{\frac{\gamma}{\theta}-1} \\ &= \frac{\theta}{\gamma}[C_{t+1}^{\frac{\gamma}{\theta}} + \delta \mathrm{E}_{t+1}[V_{t+2}^{\gamma}]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}(\frac{\theta-\gamma}{\theta})} \times \frac{\gamma}{\theta} C_{t+1}^{\frac{\gamma}{\theta}-1} \\ &= \frac{\theta}{\gamma} V_{t+1}^{(1-\frac{\gamma}{\theta})} \times \frac{\gamma}{\theta} C_{t+1}^{\frac{\gamma}{\theta}-1} \\ &= V_{t+1}^{(1-\frac{\gamma}{\theta})} \times C_{t+1}^{\frac{\gamma}{\theta}-1} \end{aligned} \tag{6}$$

Therefore, plugging equation (6) to equation (5), equation (7) holds.

$$\begin{aligned} \frac{\partial V_t}{\partial C_{t+1}} &= \frac{\theta}{\gamma}[C_t^{\frac{\gamma}{\theta}} + \delta \mathrm{E}_t[V_{t+1}^{\gamma}]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\delta}{\theta} \mathrm{E}_t[V_{t+1}^{\gamma}]^{\frac{1}{\theta}-1} \times \gamma V_{t+1}^{\gamma-1} \times V_{t+1}^{(1-\frac{\gamma}{\theta})} \times C_{t+1}^{\frac{\gamma}{\theta}-1} \\ &= \frac{\theta}{\gamma}[C_t^{\frac{\gamma}{\theta}} + \delta \mathrm{E}_t[V_{t+1}^{\gamma}]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\delta}{\theta} \mathrm{E}_t[V_{t+1}^{\gamma}]^{\frac{1}{\gamma}(\frac{\gamma}{\theta}-\gamma)} \times \gamma V_{t+1}^{\gamma-\frac{\gamma}{\theta}} \times C_{t+1}^{\frac{\gamma}{\theta}-1} \end{aligned} \tag{7}$$

Problem 2 continued on next page. . .

Combining equation (7) and (4), $m_{t,t+1}$ is derived as follows.

$$\begin{aligned}
m_{t,t+1} &= \frac{\partial V_t}{\partial C_{t+1}} \Big/ \frac{\partial V_t}{\partial C_t} \\
&= \frac{\frac{\theta}{\gamma}[C_t^{\frac{\gamma}{\theta}} + \delta \mathrm{E}_t[V_{t+1}^{\gamma}]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\delta}{\theta}\mathrm{E}_t[V_{t+1}^{\gamma}]^{\frac{1}{\gamma}(\frac{\gamma}{\theta}-\gamma)} \times \gamma V_{t+1}^{\gamma-\frac{\gamma}{\theta}} \times C_{t+1}^{\frac{\gamma}{\theta}-1}}{\frac{\theta}{\gamma}[C_t^{\frac{\gamma}{\theta}} + \delta \mathrm{E}_t[V_{t+1}^{\gamma}]^{\frac{1}{\theta}}]^{\frac{\theta}{\gamma}-1} \times \frac{\gamma}{\theta} C_t^{\frac{\gamma}{\theta}-1}} \qquad (8) \\
&= \delta \left(\frac{C_{t+1}}{C_t}\right)^{\frac{\gamma}{\theta}-1} \left(\frac{V_{t+1}}{E_t[V_{t+1}^{\gamma}]^{\frac{1}{\gamma}}}\right)^{\gamma-\frac{\gamma}{\theta}}
\end{aligned}$$

Since $\theta = \gamma/(1-\frac{1}{\varepsilon})$, equation (8) is represented as follows.

$$m_{t,t+1} = \delta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\varepsilon}} \left(\frac{V_{t+1}}{E_t[V_{t+1}^{\gamma}]^{\frac{1}{\gamma}}}\right)^{\gamma-(1-\frac{1}{\varepsilon})} \qquad (9)$$

Which is the desired solution.