FIN 591: Homework #3

Due on Wednesday, April 11, 2018

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Problem 1

a. Since the final payoff of P is 1, using continuous-time version stochastic discount factor, $P_t(\tau)$ is derived as follows.

$$\begin{aligned} P_t(\tau) &= \mathrm{E}_t\left[\frac{U_c(C_{t+\tau,t+\tau})}{U_c(C_t,t)}\times 1\right] \\ &= \mathrm{E}_t\left[\frac{e^{-\phi(t+\tau)}C_{t+\tau}^{\gamma-1}}{e^{\phi t}C_t^{\gamma-1}}\right] \\ &= \mathrm{E}_t\left[e^{-\phi\tau}\frac{C_{t+\tau}^{\gamma-1}}{C_t^{\gamma-1}}\right] \end{aligned} \tag{1}$$

b. From $P_t(\tau) = \mathrm{E}_t\left[\frac{e^{-\phi(t+\tau)}C_{t+\tau}^{\gamma-1}}{e^{\phi t}C_t^{\gamma-1}}\right]$, we can find that process M_t is equal to $e^{-\phi t}C_t^{\gamma-1}$. Therefore, using Ito's lemma, dynamics of M_t can be derived as equation (2).

$$\begin{aligned} dM_t &= -\phi e^{-\phi t}C^{\gamma-1}dt + e^{-\phi t}(\gamma-1)C^{\gamma-2}C[(\mu_c-\lambda k)dt + \sigma_c dZ_c] \\ &+ \frac{1}{2}e^{-\phi t}(\gamma-1)(\gamma-2)C^2C^{\gamma-3}\sigma_c^2 dt + [e^{-\phi t}(YC)^{\gamma-1} - e^{-\phi t}C^{\gamma-1}]dq \\ &= [-\phi + (\gamma-1)(\mu_c-\lambda k) + \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2]Mdt + (\gamma-1)\sigma_c M dZ_c + (Y^{\gamma-1}-1)Mdq \end{aligned} \tag{2}$$

c. Since $\mathrm{E}\left[\frac{dM}{M}\right] = -rdt$, the following equation holds.

$$\begin{aligned} r &= -\mathrm{E}\left[\frac{dM}{M}\right] \\ &= \phi - (\gamma-1)(\mu_c-\lambda k) - \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2 - \lambda\mathrm{E}[e^{(\gamma-1)\log Y} - 1] \\ &= \phi - (\gamma-1)(\mu_c-\lambda k) - \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2 - \lambda(e^{(\gamma-1)\alpha + \frac{1}{2}(\gamma-1)^2\delta^2} - 1) \end{aligned} \tag{3}$$

Since μ_c, k, λ are constant, instantaneous risk free rate is constant.

d.

Problem 2

a. Considering the process of risky asset price, intertemporal budget constraint is derived as follows.

$$\begin{aligned} dW &= \omega\frac{dS}{S} + (1-\omega)rdt - Cdt \\ &= (\omega(\mu-\lambda k - r)W + rW - C)dt + \sigma W dZ + \omega(Y-1)Wdq \end{aligned} \tag{4}$$

b. Investors maximize $\mathrm{E}_0[\int_0^T e^{-\phi t}u(C_t)dt]$, subject to $dW = (\omega(\mu-\lambda k-r)W + rW)dt + \sigma W dZ + \omega(Y-1)Wdq$. Let $J(W,s) = \max_{C,\omega}\mathrm{E}_s[\int_0^T e^{-\phi t}u(C_t)dt]$. Then the following equation follows.

$$\begin{aligned} J(W,0) &= \max_{C,\omega}\mathrm{E}_0\left[\int_0^{\Delta t} e^{-\phi t}u(C_t)dt + J(W,\Delta t)\right] \\ &= \max_{C,\omega}\mathrm{E}_0[u(C_0)\Delta t + J(W,0) + J_W(\omega(\mu-\lambda k-r)W + rW - C)\Delta t \\ &+ \frac{1}{2}\omega^2\sigma^2 J_{WW}\Delta t + (J(\omega(Y-1)W,0) - J(W,0))dq] \end{aligned} \tag{5}$$

Problem 2 continued on next page. . .

Letting $\Delta t \to 0$, equation (5) becomes equation (6), and it is Bellman equation.

$$0 = \max_{C,\omega}[u(c_0) + J_W(\omega(\mu - \lambda k - r)W + rW - C) + \frac{1}{2}\omega^2\sigma^2 J_{WW} + \lambda(\mathrm{E}[J(\omega(Y-1)W, 0)] - J(W, 0))] \tag{6}$$

c. Applying first order condition to equation (6), the following equation holds.

$$\begin{aligned} u_c &= J_W \\ J_W(\mu - \lambda k - r)W + \omega\sigma^2 J_{WW} + \frac{\partial}{\partial\omega}(\lambda(\mathrm{E}[J(\omega(Y-1)W, 0)] - J(W, 0))]) &= 0 \end{aligned} \tag{7}$$