FIN 591: Homework #2

Due on Wednesday, February 28, 2018

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Problem 1

a.

b.

Problem 2

a. Under the optimal choice, $U_C(C_{T-1}, T-1) = \mathrm{E}_{T-1}[B_W(W_T, T)R_{T-1}]$ holds. If we plug given utility and bequest function to the equation, the following equation holds.

$$\begin{aligned} \delta^{T-1}C_{T-1}^{\gamma-1} &= \mathrm{E}_{T-1}[\delta^T W_T^{\gamma-1} R_{T-1}] \\ \Rightarrow \delta^{T-1}C_{T-1}^{\gamma-1} &= \delta^T S_{T-1}^{\gamma-1}\mathrm{E}[R_{T-1}^{\gamma}] \quad \text{where } W_T = S_{T-1}R_{T-1}, S_{T-1} = W_{T-1} - C_{T-1} \end{aligned} \tag{1}$$

Therefore, if we rearrange the equation (1), the optimal consumption at time $T-1$, C_{T-1}^* can be obtained as follows.

$$C_{T-1}^* = \frac{\delta^{\frac{1}{\gamma-1}}\mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}}}{1 - \delta^{\frac{1}{\gamma-1}}\mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}}} W_{T-1} \tag{2}$$

Another condition under optimal choice is $\mathrm{E}_{T-1}[B_W(W_T, T)(R_{i,T-1} - R_f)] = 0$ for $i = 1, 2, 3, \ldots, n$. Therefore, the following equation holds.

$$\begin{aligned} &\mathrm{E}_{T-1}[\delta^T W_T^{\gamma-1} R_{i,T-1}] = R_f \mathrm{E}_{T-1}[\delta^T W_T^{\gamma-1}] \\ &\Rightarrow \mathrm{E}_{T-1}[(S_{T-1}R_{T-1})^{\gamma-1}R_{i,T-1}] = R_f\mathrm{E}_{T-1}[\delta^T(S_{T-1}R_{T-1})^{\gamma-1}] \\ &\Rightarrow \mathrm{E}_{T-1}[R_{T-1}^{\gamma-1}R_{i,T-1}] = R_f\mathrm{E}_{T-1}[R_{T-1}^{\gamma-1}] \end{aligned} \tag{3}$$

b. Let $\delta^{\frac{1}{\gamma-1}}\mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}} = a$. Then $C_{T-1}^* = \frac{a}{1+a}W_{T-1}$. Since $J(W_{T-1}, T-1) = U(C_{T-1}^*, T-1) = \mathrm{E}_{T-1}[B(W_T, T)]$, $J(W_{T-1}, T-1)$ can be represented as follows.

$$\begin{aligned} J(W_{T-1}, T-1) &= \frac{\delta^{T-1}C_{T-1}^{*\gamma}}{\gamma} + \mathrm{E}_{T-1}[\frac{\delta^T W_T^{\gamma}}{\gamma}] \\ &= \frac{\delta^{T-1}}{\gamma}(\frac{a}{1+a}W_{T-1})^{\gamma} + \frac{\delta^T}{\gamma}\mathrm{E}_{T-1}[((1 - \frac{a}{1+a})W_{T-1}R_{T-1})^{\gamma}] \\ &= \frac{\delta^{T-1}}{\gamma}((\frac{a}{1+a})^{\gamma}W_{T-1}^{\gamma} + \delta(\frac{1}{1+a})^{\gamma}W_{T-1}^{\gamma}\mathrm{E}_{T-1}[R_{T-1}^{\gamma}]) \\ &= \frac{\delta^{T-1}}{\gamma}(\frac{1}{1+a})^{\gamma}(a^{\gamma} + \delta\mathrm{E}_{T-1}[R_{T-1}^{\gamma}])W_{T-1}^{\gamma} \end{aligned} \tag{4}$$

c. Let $\frac{1}{1+a}^{\gamma}(a^{\gamma} + \delta\mathrm{E}_{T-1}[R_{T-1}^{\gamma}]) = b$. Then $J(W_{T-1}, T-1)$ can be represented as $\frac{\delta^{T-1}}{\gamma}bW_{T-1}^{\gamma}$. Since under optimal choice, $U_C(C_{T-2}, T-2) = \mathrm{E}_{T-2}[J_W(W_{T-1}, T-1)R_{T-2}]$ holds, the following equation must hold.

$$\begin{aligned} \delta^{T-2}C_{T-2}^{\gamma-1} &= \mathrm{E}_{T-2}[\delta^{T-1}bW_{T-1}^{\gamma-1}R_{T-2}] \\ &= \delta^{T-1}b\mathrm{E}_{T-2}[S_{T-2}^{\gamma-1}R_{T-2}^{\gamma}] \\ &= \delta^{T-1}b(W_{T-2} - C_{T-2})^{\gamma-1}\mathrm{E}_{T-2}[R_{T-2}^{\gamma}] \\ C_{T-2} &= \delta b(W_{T-2} - C_{T-2})\mathrm{E}_{T-2}[R_{T-2}]^{\frac{1}{\gamma-1}} \end{aligned} \tag{5}$$

Problem 2 continued on next page. . .

By rearranging the terms in equation (5), we can get an explicit form of C^*_{T-2} as follows.

$$\begin{aligned} C^*_{T-2} &= \frac{\delta b \mathrm{E}_{T-2}[R^{\gamma}_{T-2}]^{\frac{1}{\gamma-1}}}{1+\delta b \mathrm{E}_{T-2}[R^{\gamma}_{T-2}]^{\frac{1}{\gamma-1}}} W_{T-2} \\ &= \frac{c}{1+c} W_{T-2} \quad \text{where } c = \delta b \mathrm{E}_{T-2}[R^{\gamma}_{T-2}]^{\frac{1}{\gamma-1}} \end{aligned} \tag{6}$$

Another optimal condition is $\mathrm{E}_{T-2}[R_{i,T-2} J_W(W_{T-1}, T-1)] = R_f \mathrm{E}_{T-2}[J_W(W_{T-1}, T-1)]$. Therefore, the following equations hold.

$$\begin{aligned} &\mathrm{E}_{T-2}[R_{i,T-2}\delta^{T-1} b W^{\gamma-1}_{T-1}] = R_f \mathrm{E}_{T-2}[\delta^{T-1} b W^{\gamma-1}_{T-1}] \\ &\Rightarrow \mathrm{E}_{T-2}[R_{i,T-2} W^{\gamma-1}_{T-1}] = R_f \mathrm{E}_{T-2}[W^{\gamma-1}_{T-1}] \\ &\Rightarrow \mathrm{E}_{T-2}[R_{i,T-2}(S_{T-2}R_{T-2})^{\gamma-1}] = R_f \mathrm{E}_{T-2}[(S_{T-2}R_{T-2})^{\gamma-1}] \\ &\Rightarrow \mathrm{E}_{T-2}[R_{i,T-2} R^{\gamma-1}_{T-2}] = R_f \mathrm{E}_{T-2}[R^{\gamma-1}_{T-2}] \end{aligned} \tag{7}$$

d. Since $J(W_{T-2}, T-2) = U(C^*_{T-2}, T-2) + \mathrm{E}_{T-2}[J(W_{T-1}, T-1)]$, the following equation holds.

$$\begin{aligned} J(W_{T-2}, T-2) &= \frac{\delta^{T-2} C^{*\gamma}_{T-2}}{\gamma} + \mathrm{E}_{T-2}[\frac{\delta^{T-1}}{\gamma} b W^{\gamma}_{T-1}] \\ &= \frac{\delta^{T-2}}{\gamma}(\frac{c}{1+c})^{\gamma} W^{\gamma}_{T-2} + \mathrm{E}_{T-2}[\frac{\delta^{T-1}}{\gamma} b (W_{T-2} - C_{T-2})^{\gamma} R^{\gamma}_{T-2}] \\ &= \frac{\delta^{T-2}}{\gamma}(\frac{c}{1+c})^{\gamma} W^{\gamma}_{T-2} + \mathrm{E}_{T-2}[\frac{\delta^{T-1}}{\gamma} b (\frac{1}{1+c})^{\gamma} W^{\gamma}_{T-2} R^{\gamma}_{T-2}] \\ &= \frac{\delta^{T-2}}{\gamma}(\frac{1}{1+c})^{\gamma}((c^{\gamma}+\delta b) W^{\gamma}_{T-2} \mathrm{E}_{T-2}[R^{\gamma}_{T-2}]) \\ &= \frac{\delta^{T-2}}{\gamma} d W^{\gamma}_{T-2} \quad \text{where } d = (\frac{1}{1+c})^{\gamma}(c^{\gamma}+\delta b)\mathrm{E}_{T-2}[R^{\gamma}_{T-2}] \end{aligned} \tag{8}$$

From the pattern, the optimal consumption at $T-t$ is kW_{T-t} for some constant k, and the optimal portfolio weight satisfies $\mathrm{E}_{T-t}[R_{i,T-t} R^{\gamma-1}_{T-t}] = R_f \mathrm{E}_{T-t}[R^{\gamma-1}_{T-t}]$.