FIN 591: Homework #2

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Problem 1

a. Since there is only one risky asset, consumption at date 1, C_1 is represented as $C_1 = y_1 + (W_0 + y_0 - C_0)(R_f + W(R - R_f))$. Let $S_0 = W_0 + y_0 + C_0$. Then since $y_1 \sim N(\bar{y}, \sigma_y^2)$ and $R \sim N(\bar{R}, \sigma^2)$, $\mathrm{E}[C_1] = \bar{y} + S_0(R_f + W(\bar{R} - R_f))$ and $\mathrm{Var}[C_1] = \sigma_y^2 + S_0^2 W^2 \sigma^2 + 2S_0\rho\sigma\sigma_y W$. Therefore, expected utility satisfy equation (1).

$$\begin{aligned} & e^{-bC_0} - \delta \mathrm{E}[e^{-bC_1}] \\ =& e^{-bC_0} - \delta \exp(-b(\bar{y} + S_0(R_f + W(\bar{R} - R_f))) + \frac{1}{2}b^2(\sigma_y^2 + S_0^2 W^2 \sigma^2 + 2S_0\rho\sigma\sigma_y W)) \end{aligned} \tag{1}$$

Since e^{-bC_0} does not depend on W, $\max_W e^{-bC_0} - \delta\mathrm{E}[e^{-bC_1}]$ is equivalent to minimize $-b(\bar{y} + S_0(R_f + W(\bar{R} - R_f))) + \frac{1}{2}b^2(\sigma_y^2 + S_0^2 W^2\sigma^2 + 2S_0\rho\sigma\sigma_y W)$. Applying first order condition leads to equation (2).

$$\begin{aligned} & -bS_0(\bar{R} - R_f) + \frac{1}{2}b^2(2S_0^2\sigma^2 W + 2S_0\rho\sigma\sigma_y) = 0 \\ & \Rightarrow -(\bar{R} - R_f) + bS_0\sigma^2 W + b\rho\sigma\sigma_y = 0 \\ & \Rightarrow W = \frac{\bar{R} - R_f - b\rho\sigma\sigma_y}{bS_0\sigma^2} \end{aligned} \tag{2}$$

The equation (2) shows that the weight on risky asset is negatively correlated with volatility of future wage income if correlation between return of risky asset and future wage income is positive and vice versa. Furthermore, the weight increases if correlation between return of risky asset and future wage income decreases, and the weight decreases if the correlation increases.

b. Applying first order condition on expected utility, equation (3) and (4) are obtained.

$$\begin{aligned} & U'(C_0) - \delta\mathrm{E}[U'(C_1)(R_f + W(R - R_f))] \\ & = be^{-bC_0} - \delta\mathrm{E}[be^{-bC_1}(R_f + W(R - R_f))] = 0 \\ & \Rightarrow e^{-bC_0} - \delta\mathrm{E}[e^{-bC_1}(R_f + W(R - R_f))] = 0 \end{aligned} \tag{3}$$

$$\begin{aligned} & \delta\mathrm{E}[U'(C_1)(W_0 + y_0 - C_0)(R - R_f)] \\ & = \delta\mathrm{E}[be^{-bC_1}(W_0 + y_0 - C_0)(R - R_f)] = 0 \\ & \Rightarrow \delta\mathrm{E}[e^{-bC_1}(W_0 + y_0 - C_0)(R - R_f)] = 0 \\ & \Rightarrow \mathrm{E}[e^{-bC_1}R] = R_f\mathrm{E}[e^{-bC_1}] \end{aligned} \tag{4}$$

Plugging (4) in (3), equation (5) is obtained. Therefore, the optimal choice should satisfy (5).

$$e^{-bC_0} = R_f\delta\mathrm{E}[e^{-bC_1}] \tag{5}$$

By using the equation (5), optimal consumption at time 0 should satisfy the following equation.

$$\begin{aligned} C_0 &= -b\log(R_f\delta\mathrm{E}[e^{-bC_1}]) \\ &= -b\log(R_f\delta) - b\log\mathrm{E}[e^{-bC_1}] \end{aligned} \tag{6}$$

By (1), $\log \mathrm{E}[e^{-bC_1}]$ is represented as $-b(\bar{y}+S_0(R_f+W(\bar{R}-R_f)))+\frac{1}{2}b^2(\sigma_y^2+S_0^2W^2\sigma^2+2S_0\rho\sigma\sigma_yW)$. Applying $S_0=W_0+y_0+C_0$, the following equation holds.

$$\begin{aligned}
&\log\mathrm{E}[e^{-bC_1}]\\
&\quad=-b(\bar{y}+(W_0+y_0-C_0)(R_f+W(\bar{R}-R_f)))\\
&\qquad+\frac{1}{2}b^2(\sigma_y^2+(W_0+y_0-C_0)^2W^2\sigma^2+2(W_0+y_0-C_0)\rho\sigma\sigma_yW)\\
&\quad=-b\bar{y}+\frac{1}{2}b^2\sigma_y^2+\frac{1}{2}b^2(W_0+y_0)W^2\sigma^2-b(W_0+y_0)(R_f+W(\bar{R}-R_f))+b^2(W_0+y_0)\rho\sigma\sigma_y\\
&\qquad-\frac{1}{2}b^2W^2\sigma^2C_0-b^2\rho\sigma\sigma_yC_0+b(R_f+W(\bar{R}-R_f))C_0\\
&\quad=A+BC_0\\
&\quad\text{where } A=-b\bar{y}+\frac{1}{2}b^2\sigma_y^2+\frac{1}{2}b^2(W_0+y_0)W^2\sigma^2-b(W_0+y_0)(R_f+W(\bar{R}-R_f))+b^2(W_0+y_0)\rho\sigma\sigma_y\\
&\quad B=-\frac{1}{2}b^2W^2\sigma^2-b^2\rho\sigma\sigma_y+b(R_f+W(\bar{R}-R_f))
\end{aligned} \tag{7}$$

Since $C_0=-b\log(R_f\delta)-b\log\mathrm{E}[e^{-bC_1}]=-b\log(R_f\delta)-b(A+BC_0)$, $C_0=\frac{-b\log(R_f\delta)-bA}{1+bB}$.

Problem 2

a. Under the optimal choice, $U_C(C_{T-1},T-1)=\mathrm{E}_{T-1}[B_W(W_T,T)R_{T-1}]$ holds. If we plug given utility and bequest function to the equation, the following equation holds.

$$\begin{aligned}
&\delta^{T-1}C_{T-1}^{\gamma-1}=\mathrm{E}_{T-1}[\delta^TW_T^{\gamma-1}R_{T-1}]\\
&\Rightarrow\delta^{T-1}C_{T-1}^{\gamma-1}=\delta^TS_{T-1}^{\gamma-1}\mathrm{E}[R_{T-1}^{\gamma}]\quad\text{where } W_T=S_{T-1}R_{T-1},S_{T-1}=W_{T-1}-C_{T-1}
\end{aligned} \tag{8}$$

Therefore, if we rearrange the equation (8), the optimal consumption at time $T-1$, C_{T-1}^* can be obtained as follows.

$$C_{T-1}^*=\frac{\delta^{\frac{1}{\gamma-1}}\mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}}}{1-\delta^{\frac{1}{\gamma-1}}\mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}}}W_{T-1} \tag{9}$$

Another condition under optimal choice is $\mathrm{E}_{T-1}[B_W(W_T,T)(R_{i,T-1}-R_f)]=0$ for $i=1,2,3,\ldots,n$. Therefore, the following equation holds.

$$\begin{aligned}
&\mathrm{E}_{T-1}[\delta^TW_T^{\gamma-1}R_{i,T-1}]=R_f\mathrm{E}_{T-1}[\delta^TW_T^{\gamma-1}]\\
&\Rightarrow\mathrm{E}_{T-1}[(S_{T-1}R_{T-1})^{\gamma-1}R_{i,T-1}]=R_f\mathrm{E}_{T-1}[\delta^T(S_{T-1}R_{T-1})^{\gamma-1}]\\
&\Rightarrow\mathrm{E}_{T-1}[R_{T-1}^{\gamma-1}R_{i,T-1}]=R_f\mathrm{E}_{T-1}[R_{T-1}^{\gamma-1}]
\end{aligned} \tag{10}$$

b. Let $\delta^{\frac{1}{\gamma-1}}\mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}}=a$. Then $C_{T-1}^*=\frac{a}{1+a}W_{T-1}$. Since $J(W_{T-1},T-1)=U(C_{T-1}^*,T-1)=$

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$E_{T-1}[B(W_T,T)]$, $J(W_{T-1},T-1)$ can be represented as follows.

$$\begin{aligned}J(W_{T-1},T-1) &= \frac{\delta^{T-1}C^{*\gamma}_{T-1}}{\gamma} + E_{T-1}[\frac{\delta^T W_T^\gamma}{\gamma}] \\ &= \frac{\delta^{T-1}}{\gamma}(\frac{a}{1+a}W_{T-1})^\gamma + \frac{\delta^T}{\gamma}E_{T-1}[((1-\frac{a}{1+a})W_{T-1}R_{T-1})^\gamma] \\ &= \frac{\delta^{T-1}}{\gamma}((\frac{a}{1+a})^\gamma W^\gamma_{T-1} + \delta(\frac{1}{1+a})^\gamma W^\gamma_{T-1}E_{T-1}[R^\gamma_{T-1}]) \\ &= \frac{\delta^{T-1}}{\gamma}(\frac{1}{1+a})^\gamma(a^\gamma + \delta E_{T-1}[R^\gamma_{T-1}])W^\gamma_{T-1}\end{aligned} \tag{11}$$

c. Let $\frac{1}{1+a}^\gamma(a^\gamma+\delta E_{T-1}[R^\gamma_{T-1}]) = b$. Then $J(W_{T-1},T-1)$ can be represented as $\frac{\delta^{T-1}}{\gamma}bW^\gamma_{T-1}$. Since under optimal choice, $U_C(C_{T-2},T-2) = E_{T-2}[J_W(W_{T-1},T-1)R_{T-2}]$ holds, the following equation must hold.

$$\begin{aligned}\delta^{T-2}C^{\gamma-1}_{T-2} &= E_{T-2}[\delta^{T-1}bW^{\gamma-1}_{T-1}R_{T-2}] \\ &= \delta^{T-1}bE_{T-2}[S^{\gamma-1}_{T-2}R^\gamma_{T-2}] \\ &= \delta^{T-1}b(W_{T-2}-C_{T-2})^{\gamma-1}E_{T-2}[R^\gamma_{T-2}] \\ C_{T-2} &= \delta b(W_{T-2}-C_{T-2})E_{T-2}[R_{T-2}]^{\frac{1}{\gamma-1}}\end{aligned} \tag{12}$$

By rearranging the terms in equation (12), we can get an explicit form of C^*_{T-2} as follows.

$$\begin{aligned}C^*_{T-2} &= \frac{\delta b E_{T-2}[R^\gamma_{T-2}]^{\frac{1}{\gamma-1}}}{1+\delta b E_{T-2}[R^\gamma_{T-2}]^{\frac{1}{\gamma-1}}}W_{T-2} \\ &= \frac{c}{1+c}W_{T-2} \quad \text{where } c = \delta b E_{T-2}[R^\gamma_{T-2}]^{\frac{1}{\gamma-1}}\end{aligned} \tag{13}$$

Another optimal condition is $E_{T-2}[R_{i,T-2}J_W(W_{T-1},T-1)] = R_f E_{T-2}[J_W(W_{T-1},T-1)]$. Therefore, the following equations hold.

$$\begin{aligned}&E_{T-2}[R_{i,T-2}\delta^{T-1}bW^{\gamma-1}_{T-1}] = R_f E_{T-2}[\delta^{T-1}bW^{\gamma-1}_{T-1}] \\ &\Rightarrow E_{T-2}[R_{i,T-2}W^{\gamma-1}_{T-1}] = R_f E_{T-2}[W^{\gamma-1}_{T-1}] \\ &\Rightarrow E_{T-2}[R_{i,T-2}(S_{T-2}R_{T-2})^{\gamma-1}] = R_f E_{T-2}[(S_{T-2}R_{T-2})^{\gamma-1}] \\ &\Rightarrow E_{T-2}[R_{i,T-2}R^{\gamma-1}_{T-2}] = R_f E_{T-2}[R^{\gamma-1}_{T-2}]\end{aligned} \tag{14}$$

d. Since $J(W_{T-2},T-2) = U(C^*_{T-2},T-2) + E_{T-2}[J(W_{T-1},T-1)]$, the following equation holds.

$$\begin{aligned}J(W_{T-2},T-2) &= \frac{\delta^{T-2}C^{*\gamma}_{T-2}}{\gamma} + E_{T-2}[\frac{\delta^{T-1}}{\gamma}bW^\gamma_{T-1}] \\ &= \frac{\delta^{T-2}}{\gamma}(\frac{c}{1+c})^\gamma W^\gamma_{T-2} + E_{T-2}[\frac{\delta^{T-1}}{\gamma}b(W_{T-2}-C_{T-2})^\gamma R^\gamma_{T-2}] \\ &= \frac{\delta^{T-2}}{\gamma}(\frac{c}{1+c})^\gamma W^\gamma_{T-2} + E_{T-2}[\frac{\delta^{T-1}}{\gamma}b(\frac{1}{1+c})^\gamma W^\gamma_{T-2}R^\gamma_{T-2}] \\ &= \frac{\delta^{T-2}}{\gamma}(\frac{1}{1+c})^\gamma((c^\gamma+\delta b)W^\gamma_{T-2}E_{T-2}[R^\gamma_{T-2}]) \\ &= \frac{\delta^{T-2}}{\gamma}dW^\gamma_{T-2} \quad \text{where } d = (\frac{1}{1+c})^\gamma(c^\gamma+\delta b)E_{T-2}[R^\gamma_{T-2}]\end{aligned} \tag{15}$$

From the pattern, the optimal consumption at $T-t$ is kW_{T-t} for some constant k, and the optimal portfolio weight satisfies $\mathrm{E}_{T-t}[R_{i,T-t}R_{T-t}^{\gamma-1}] = R_f \mathrm{E}_{T-t}[R_{T-t}^{\gamma-1}]$.