Basics of Derivative Pricing

George Pennacchi

University of Illinois

Introduction

- Derivative securities have cashflows that derive from another "underlying" variable, such as an asset price, interest rate, or exchange rate.
- The absence of arbitrage opportunities places restrictions on the derivative's value relative to that of its underlying asset.
- For forward contracts, no-arbitrage considerations alone may lead to an exact pricing formula.
- For options, no-arbitrage restrictions cannot determine an exact price, but only bounds on the option's price.
- An exact option pricing formula requires additional assumptions on the probability distribution of the underlying asset's returns (e.g., binomial).

Forward Contracts on Assets Paying Dividends

- Let $F_{0\tau}$ be the date 0 forward price for exchanging one share of an underlying asset τ periods in the future. This price is agreed to at date 0 but paid at date $\tau > 0$ for delivery at date τ of the asset.
- Hence, the date $\tau > 0$ payoff to the long (*short*) party in this forward contract is $S_\tau - F_{0\tau}$, ($F_{0\tau} - S_\tau$) where S_τ is the date τ spot price of one share of the underlying asset.
- The parties set $F_{0\tau}$ to make the date 0 contract's value equal 0 (no payment at date 0).
- Let $R_f > 1$ be the per-period risk-free return for borrowing or lending over the period from date 0 to date τ, and let D be the date 0 present value of dividends paid by the underlying asset over the period from date 0 to date τ.

Forward Contract Cash Flows

- Consider a long forward contract and the trades that would exactly replicate its date τ payoffs:

Date 0 Trade	**Date 0 Cashflow**	**Date τ Cashflow**
Long Forward Contract	0	$S_\tau - F_{0\tau}$
Replicating Trades		
1) Buy Asset and Sell Dividends	$-S_0 + D$	S_τ
2) Borrow	$R_f^{-\tau} F_{0\tau}$	$-F_{0\tau}$
Net Cashflow	$-S_0 + D + R_f^{-\tau} F_{0\tau}$	$S_\tau - F_{0\tau}$

- In the absence of arbitrage, the cost of the replicating trades equals the zero cost of the long position:

$$S_0 - D - R_f^{-\tau} F_{0\tau} = 0 \tag{1}$$

or

$$F_{0\tau} = (S_0 - D)\, R_f^{\tau} \tag{2}$$

Forward Contract Replication

- If the contract had been initiated at a previous date, say date -1, at the forward price $F_{-1\tau} = X$, then the date 0 value (replacement cost) of the long party's payoff, say f_0, would still be the cost of replicating the two cashflows:

$$f_0 = S_0 - D - R_f^{-\tau} X \tag{3}$$

- The forward price in equation (2) did not require an assumption regarding the random distribution of the underlying asset price, S_τ, because it was a *static* replication strategy.
- Replicating option payoffs will entail, in general, a *dynamic* replication strategy requiring distributional assumptions.

Basic Characteristics of Option Prices

- The owner of a *call* option has the right to buy an asset in the future at a pre-agreed price, called the *exercise* or *strike* price.
- Since the option owner's payoff is always non-negative, this buyer must make an initial payment to the seller.
- A *European* option can be exercised only at the maturity of the option contract.
- Let S_0 and S_τ be the current and maturity date prices per share of the underlying asset, X be the exercise price, and c_t and p_t be the date t prices of European call and put options, respectively.
- Then the maturity values of European call and put options are

$$c_\tau = \max\left[S_\tau - X, 0\right] \tag{4}$$

$$p_\tau = \max\left[X - S_\tau, 0\right] \tag{5}$$

Lower Bounds on European Option Values

- Recall that the long (*short*) party's payoff of a forward contract is $S_\tau - F_{0\tau}$ ($F_{0\tau} - S_\tau$).
- If $F_{0\tau}$ is like an option's strike, X, then assuming $X = F_{0\tau}$ implies the payoff of a call (*put*) option weakly dominates that of a long (*short*) forward.
- Because equation (3) is the current value of a long forward position contract, the European call's value must satisfy

$$c_0 \geq S_0 - D - R_f^{-\tau} X \tag{6}$$

- Furthermore, combining $c_0 \geq 0$ with (6) implies

$$c_0 \geq \max\left[S_0 - D - R_f^{-\tau} X,\ 0\right] \tag{7}$$

- By a similar argument,

$$p_0 \geq \max\left[R_f^{-\tau} X + D - S_0,\ 0\right] \tag{8}$$

Put-Call Parity

- *Put-call parity* links options written on the same underlying, with the same maturity date, and exercise price.

$$c_0 + R_f^{-\tau} X + D \;=\; p_0 + S_0 \tag{9}$$

- Consider forming the following two portfolios at date 0:
  1. Portfolio A = a put option having value p_0 and a share of the underlying asset having value S_0
  2. Portfolio B = a call option having value c_0 and a bond with initial value of $R_f^{-\tau} X + D$

Then at date τ, these two portfolios are worth:

- Portfolio A = $\max\left[X - S_\tau, 0\right] + S_\tau + DR_f^\tau = \max\left[X,\ S_\tau\right] + DR_f^\tau$
- Portfolio B = $\max\left[0,\ S_\tau - X\right] + X + DR_f^\tau = \max\left[X,\ S_\tau\right] + DR_f^\tau$

American Options

- An American option is at least as valuable as its corresponding European option because of its early exercise right.
- Hence if C_0 and P_0, the current values of American options, then $C_0 \geq c_0$ and $P_0 \geq p_0$.
- Some American options' early exercise feature has no value.
- Consider a European call option on a non-dividend-paying asset, and recall that $c_0 \geq S_0 - R_f^{-\tau} X$.
- An American call option on the same asset exercised early is worth $C_0 = S_0 - X < S_0 - R_f^{-\tau} X < c_0$, a contradiction.
- For an American put option, selling the asset immediately and receiving \$$X$ now may be better than receiving \$$X$ at date τ (which has a present value of $R_f^{-\tau} X$). At exercise $P_0 = X - S_0$ may exceed $R_f^{-\tau} X + D - S_0$ if remaining dividends are small.

Binomial Option Pricing

- The no-arbitrage assumption alone cannot determine an exact option price as a function of the underlying asset.
- However, particular distributional assumptions for the underlying asset can allow the option's payoff to be replicated by trading in the underlying asset and a risk-free asset.
- Cox, Ross, and Rubinstein (1979) developed a binomial model to value a European option on a non-dividend-paying stock.
- The model assumes that the current stock price, S, either moves up by a proportion u, or down by a proportion d, each period. The probability of an up move is π.

Binomial Option Pricing cont'd

$$S \begin{array}{l} \nearrow \\ \\ \searrow \end{array} \begin{array}{ll} uS & \text{with probability } \pi \\ & \\ dS & \text{with probability } 1-\pi \end{array} \tag{10}$$

- Let R_f be one plus the risk-free rate for the period, where in the absence of arbitrage $d < R_f < u$.
- Let c equal the current value of a European call option written on the stock and having a strike price of X, so that its payoff at maturity τ equals $\max[0, S_\tau - X]$.
- Thus, *one period prior to maturity*:

Binomial Option Pricing cont'd

$$
c \begin{array}{l} \nearrow \\ \searrow \end{array}
\begin{array}{l}
c_u \equiv \max\left[0, uS - X\right] \ \text{ with probability } \pi \\ \\ \\
c_d \equiv \max\left[0, dS - X\right] \quad \text{ with probability } 1 - \pi
\end{array}
\tag{11}
$$

- To value c, consider a portfolio containing Δ shares of stock and $\$B$ of bonds so that its current value is $\Delta S + B$.
- This portfolio's value evolves over the period as

$$
\Delta S + B \begin{array}{l} \nearrow \\ \searrow \end{array}
\begin{array}{l}
\Delta uS + R_f B \text{ with probability } \pi \\ \\ \\
\Delta dS + R_f B \quad \text{with probability } 1 - \pi
\end{array}
\tag{12}
$$

Binomial Option Pricing cont'd

- With two securities (bond and stock) and two states (up or down), Δ and B can be chosen to replicate the option's payoffs:

$$\Delta uS + R_f B = c_u \tag{13}$$

$$\Delta dS + R_f B = c_d \tag{14}$$

- Solving for Δ and B that satisfy these two equations:

$$\Delta^* = \frac{c_u - c_d}{(u-d)\,S} \tag{15}$$

$$B^* = \frac{uc_d - dc_u}{(u-d)\,R_f} \tag{16}$$

- Hence, a portfolio of Δ^* shares of stock and $\$B^*$ of bonds produces the same cashflow as the call option.

Binomial Option Pricing Example

- Therefore, the absence of arbitrage implies

$$c = \Delta^* S + B^* \tag{17}$$

where Δ^* is the option's *hedge ratio* and B^* is the debt financing that are positive/negative (*negative/positive*) for calls (*puts*).

- *Example*: If $S = \$50$, $u = 2$, $d = .5$, $R_f = 1.25$, and $X = \$50$, then $uS = \$100$, $dS = \$25$, $c_u = \$50$, $c_d = \$0$.
- Therefore,

$$\Delta^* = \frac{50 - 0}{(2 - .5)\,50} = \frac{2}{3}$$

Binomial Option Pricing cont'd

$$B^* = \frac{0 - 25}{(2 - .5)\,1.25} = -\frac{40}{3}$$

so that

$$c = \Delta^* S + B^* = \frac{2}{3}(50) - \frac{40}{3} = \frac{60}{3} = \$20$$

- This option pricing formula can be rewritten:

$$\begin{aligned} c &= \Delta^* S + B^* = \frac{c_u - c_d}{(u - d)} + \frac{uc_d - dc_u}{(u - d)\,R_f} \qquad (18) \\ &= \frac{\left[\frac{R_f - d}{u - d} \max[0, uS - X] + \frac{u - R_f}{u - d} \max[0, dS - X]\right]}{R_f} \end{aligned}$$

which *does not* depend on the stock's up/down probability, π.

Binomial Option Pricing cont'd

- Since the stock's expected rate of return equals $u\pi + d(1-\pi) - 1$, it need not be known or estimated to solve for the no-arbitrage value of the option, c.
- However, we do need to know u and d, the size of the stock's movements per period which determine its *volatility*.
- Note also that we can rewrite c as

$$c = \frac{1}{R_f}\left[\widehat{\pi} c_u + (1-\widehat{\pi})\, c_d\right] \tag{19}$$

  where $\widehat{\pi} \equiv \frac{R_f - d}{u-d}$ is the *risk-neutral* probability of the up state.
- $\widehat{\pi} = \pi$ if individuals are risk-neutral since

$$\left[u\pi + d\,(1-\pi)\right] S = R_f S \tag{20}$$

  which implies that

Binomial Option Pricing cont'd

$$\pi = \frac{R_f - d}{u - d} = \widehat{\pi} \tag{21}$$

so that $\widehat{\pi}$ does equal π under risk neutrality.

- Thus, (19) can be expressed as

$$c_t = \frac{1}{R_f} \widehat{E}\left[c_{t+1}\right] \tag{22}$$

where $\widehat{E}\left[\cdot\right]$ denotes the expectation operator evaluated using the risk-neutral probabilities $\widehat{\pi}$ rather than the true, or physical, probabilities π.

Multiperiod Binomial Option Pricing

- Next, consider the option's value with *two periods prior to maturity*. The stock price process is

$$\begin{array}{ccccc} & & & & u^2S \\ & & uS & \begin{array}{c}\nearrow\\ \searrow\end{array} & \\ S & \begin{array}{c}\nearrow\\ \searrow\end{array} & & & duS \\ & & dS & \begin{array}{c}\nearrow\\ \searrow\end{array} & \\ & & & & d^2S \end{array} \tag{23}$$

so that the option price process is

Multiperiod Binomial Option Pricing cont'd

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & c_{uu} \equiv \max\left[0, u^2 S - X\right] \\
 & c_u \arrow[ur] \arrow[dr] & \\
c \arrow[ur] \arrow[dr] & & c_{du} \equiv \max\left[0, duS - X\right] \\
 & c_d \arrow[ur] \arrow[dr] & \\
 & & c_{dd} \equiv \max\left[0, d^2 S - X\right]
\end{tikzcd}
\tag{24}
$$

- We know how to solve one-period problems:

$$c_u = \frac{\widehat{\pi} c_{uu} + \left(1 - \widehat{\pi}\right) c_{du}}{R_f} \tag{25}$$

$$c_d = \frac{\widehat{\pi} c_{du} + \left(1 - \widehat{\pi}\right) c_{dd}}{R_f} \tag{26}$$

Multiperiod Binomial Option Pricing cont'd

- With two periods to maturity, the next period cashflows of c_u and c_d are replicated by a portfolio of $\Delta^* = \frac{c_u - c_d}{(u-d)S}$ shares of stock and $B^* = \frac{uc_d - dc_u}{(u-d)R_f}$ of bonds. No arbitrage implies

$$c = \Delta^* S + B^* = \frac{1}{R_f}\left[\widehat{\pi} c_u + (1-\widehat{\pi})\, c_d\right] \tag{27}$$

which, as before says that $c_t = \frac{1}{R_f}\widehat{E}\left[c_{t+1}\right]$.

- The market is complete over both the last period and second-to-last periods. Substituting in for c_u and c_d, we have

$$c = \frac{1}{R_f^2}\left[\widehat{\pi}^2 c_{uu} + 2\widehat{\pi}\,(1-\widehat{\pi})\, c_{ud} + (1-\widehat{\pi})^2\, c_{dd}\right]$$

Multiperiod Binomial Option Pricing cont'd

$$= \frac{1}{R_f^2}\left[\widehat{\pi}^2 \max\left[0, u^2 S - X\right] + 2\widehat{\pi}\left(1-\widehat{\pi}\right)\max\left[0, duS - X\right]\right] + \frac{1}{R_f^2}\left[\left(1-\widehat{\pi}\right)^2 \max\left[0, d^2 S - X\right]\right]$$

which says $c_t = \frac{1}{R_f^2}\widehat{E}\left[c_{t+2}\right]$. Note when a market is complete each period, it becomes *dynamically complete*. By appropriate trading in just two assets, payoffs in three states of nature can be replicated.

- Repeating this analysis for any period prior to maturity, we always obtain

$$c = \Delta^* S + B^* = \frac{1}{R_f}\left[\widehat{\pi} c_u + \left(1-\widehat{\pi}\right) c_d\right] \tag{28}$$

Multiperiod Binomial Option Pricing cont'd

- Repeated substitution for c_u, c_d, c_{uu}, c_{ud}, c_{dd}, c_{uuu}, and so on, we obtain the formula, with *n periods prior to maturity*:

$$c = \frac{1}{R_f^n}\left[\sum_{j=0}^{n}\left(\frac{n!}{j!\,(n-j)!}\right)\widehat{\pi}^j\,(1-\widehat{\pi})^{n-j}\max\left[0, u^j d^{n-j} S - X\right]\right] \tag{29}$$

or $c_t = \frac{1}{R_f^n}\widehat{E}\left[c_{t+n}\right]$. Define "$a$" as the minimum number of upward jumps of S for it to exceed X.

- Then for all $j < a$ (out of the money):

$$\max\left[0, u^j d^{n-j} S - X\right] = 0 \tag{30}$$

while for all $j > a$ (in the money):

$$\max\left[0, u^j d^{n-j} S - X\right] = u^j d^{n-j} S - X \tag{31}$$

Multiperiod Binomial Option Pricing cont'd

- Thus, the formula for c can be simplified:

$$c = \frac{1}{R_f^n}\left[\sum_{j=a}^{n}\left(\frac{n!}{j!\,(n-j)!}\right)\widehat{\pi}^j\,(1-\widehat{\pi})^{n-j}\left[u^j d^{n-j} S - X\right]\right] \tag{32}$$

- Breaking up (32) into two terms, we have

$$\begin{aligned} c \quad = \quad & S\left[\sum_{j=a}^{n}\left(\frac{n!}{j!\,(n-j)!}\right)\widehat{\pi}^j\,(1-\widehat{\pi})^{n-j}\left[\frac{u^j d^{n-j}}{R_f^n}\right]\right] \\ & -XR_f^{-n}\left[\sum_{j=a}^{n}\left(\frac{n!}{j!\,(n-j)!}\right)\widehat{\pi}^j\,(1-\widehat{\pi})^{n-j}\right] \end{aligned} \tag{33}$$

The terms in brackets are complementary binomial distribution functions, so that (33) can be written

Multiperiod Binomial Option Pricing cont'd

$$c = S\phi[a; n, \widehat{\pi}'] - XR_f^{-n}\phi[a; n, \widehat{\pi}] \tag{34}$$

where $\widehat{\pi}' \equiv \left(\frac{u}{R_f}\right)\widehat{\pi}$ and $\phi[a; n, \widehat{\pi}]$ is the probability that the sum of n random variables that equal 1 with probability $\widehat{\pi}$ and 0 with probability $1 - \widehat{\pi}$ is $\geq a$.

- For time to maturity τ and per-unit variance σ^2 (depending on u and d), as the number of periods $n \to \infty$, but the length of each period $\frac{\tau}{n} \to 0$, this formula converges to:

$$c = SN(z) - XR_f^{-\tau} N\left(z - \sigma\sqrt{\tau}\right) \tag{35}$$

where $z \equiv \left[\ln\left(\frac{S}{XR_f^{-\tau}}\right) + \frac{1}{2}\sigma^2\tau\right] / (\sigma\sqrt{\tau})$ and $N(\cdot)$ is the cumulative standard normal distribution function.

Summary

- Forward contract payoffs can be replicated using a static trading strategy.
- Option contract payoffs require a dynamic trading strategy.
- A dynamically complete market allows us to use risk-neutral valuation.
- Dynamically complete markets imply replication of payoffs in all future states, but we may need to execute many trades to do so.