Mean-Variance Analysis

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Introduction

- How does one optimally choose among *multiple* risky assets?
- Due to diversification, which depends on assets' return covariances, the attractiveness of an asset when held in a portfolio may differ from its appeal when it is the sole asset held by an investor.
- Hence, the variance and higher moments of a portfolio need to be considered.
- Portfolios that make the optimal tradeoff between portfolio expected return and variance are *mean-variance efficient*.

Mean-Variance Utility

- What assumptions do we need for investors to only care about mean and variance (and not skewness, kurtosis...)?
- Suppose a vN-M maximizer invests initial date 0 wealth, W_0, in a portfolio.
- Let $\widetilde{R}_p$ be the gross random return on this portfolio, so that the individual's end-of-period wealth is $\tilde{W} = W_0 \widetilde{R}_p$.
- We write $U(\tilde{W}) = U\left(W_0 \widetilde{R}_p\right)$ as just $U(\widetilde{R}_p)$, because $\tilde{W}$ is completely determined by $\widetilde{R}_p$.
- Express $U(\widetilde{R}_p)$ by expanding it around the mean $E[\widetilde{R}_p]$:

Taylor Series Approximation of Utility

$$
\begin{aligned}
U(\widetilde{R}_p) &= U\left(E[\widetilde{R}_p]\right) + \left(\widetilde{R}_p - E[\widetilde{R}_p]\right) U'\left(E[\widetilde{R}_p]\right) \\
&\quad + \tfrac{1}{2}\left(\widetilde{R}_p - E[\widetilde{R}_p]\right)^2 U''\left(E[\widetilde{R}_p]\right) + \ldots \\
&\quad + \tfrac{1}{n!}\left(\widetilde{R}_p - E[\widetilde{R}_p]\right)^n U^{(n)}\left(E[\widetilde{R}_p]\right) + \ldots \quad (1)
\end{aligned}
$$

- If the utility function is quadratic, ($U^{(n)} = 0, \ \forall \ n \geq 3$), then the individual's expected utility is

$$
\begin{aligned}
E\left[U(\widetilde{R}_p)\right] &= U\left(E[\widetilde{R}_p]\right) + \tfrac{1}{2}E\left[\left(\widetilde{R}_p - E[\widetilde{R}_p]\right)^2\right] U''\left(E[\widetilde{R}_p]\right) \\
&= U\left(E[\widetilde{R}_p]\right) + \tfrac{1}{2}V[\widetilde{R}_p]U''\left(E[\widetilde{R}_p]\right) \quad (2)
\end{aligned}
$$

Alternative Utilities

- Quadratic utility is problematic: it has a "bliss point" after which utility declines in wealth.
- Suppose, instead, we assume general increasing, concave utility but restrict the probability distribution of the risky assets.
- Claim: If individual assets have a multi-variate normal distribution, utility of wealth depends only on portfolio mean and variance.
- Why? First note that the return on a portfolio is a weighted average (sum) of the returns on the individual assets.
- Because sums of normals are normal, if the joint distributions of individual assets are multivariate normal, then the portfolio return is also normally distributed.

Centered Normal Moments

- Let a random variable, X, be distributed $N\left(\mu, \sigma^2\right)$. Its moment generating function is:

$$m(t) = E(e^{tX}) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \qquad (3)$$

- Centralized (multiply by $\exp(-\mu t)$)

$$cm(t) = \exp\left(\frac{1}{2}\sigma^2 t^2\right) \qquad (4)$$

- Then we have following moments

Centered Normal Moments

$$
\begin{aligned}
E[\widetilde{R}_p - \mu]^1 &= \left. \frac{d \exp\left(\frac{1}{2}\sigma^2 t^2\right)}{dt} \right|_{t=0} = 0 \qquad (5) \\[2em]
E[\widetilde{R}_p - \mu]^2 &= \left. \frac{d^2 \exp\left(\frac{1}{2}\sigma^2 t^2\right)}{dt^2} \right|_{t=0} = \sigma^2 \\[2em]
E[\widetilde{R}_p - \mu]^3 &= \left. \frac{d^3 \exp\left(\frac{1}{2}\sigma^2 t^2\right)}{dt^3} \right|_{t=0} = 0 \\[2em]
E[\widetilde{R}_p - \mu]^4 &= \left. \frac{d^4 \exp\left(\frac{1}{2}\sigma^2 t^2\right)}{dt^4} \right|_{t=0} = 3\sigma^4 \\
&\quad \cdots
\end{aligned}
$$

Normal Distribution of Returns

- So moments are either zero or a function of the variance:
  $E\left[\left(\widetilde{R}_p - E[\widetilde{R}_p]\right)^n\right] = 0$ for n odd, and
  $E\left[\left(\widetilde{R}_p - E[\widetilde{R}_p]\right)^n\right] = \frac{n!}{(n/2)!}\left(\frac{1}{2}V[\widetilde{R}_p]\right)^{n/2}$ for n even.

- Therefore, in this case the individual's expected utility equals

$$
\begin{aligned}
E\left[U(\widetilde{R}_p)\right] &= U\left(E[\widetilde{R}_p]\right) + \tfrac{1}{2}V[\widetilde{R}_p]U''\left(E[\widetilde{R}_p]\right) + 0 + \frac{1}{8}\left(V[\widetilde{R}_p]\right)^2 U''''\left(E[\widetilde{R}_p]\right) \\
&\quad + 0 + ... + \frac{1}{(n/2)!}\left(\frac{1}{2}V[\widetilde{R}_p]\right)^{n/2} U^{(n)}\left(E[\widetilde{R}_p]\right) + ... \quad (6)
\end{aligned}
$$

which depends only on the mean and variance of the portfolio

return.

Caveats

- But is a multivariate normal distribution realistic for asset returns?
- If individual assets and $\widetilde{R}_p$ are normally distributed, the gross return will be negative with positive probility because the normal distribution ranges over the entire real line.
- This is a problem since most assets are limited liability, i.e. $\widetilde{R}_p \geq 0$.
- Later, in a continuous-time context, we can assume asset returns are *instantaneously* normal, which allows them to be log-normally distributed over finite intervals.

Preference for Return Mean and Variance

- Therefore, assume U is a general utility function and asset returns are normally distributed. The portfolio return $\tilde{R}_p$ has normal probability density function $f(R; \bar{R}_p, \sigma_p^2)$, where we define $\bar{R}_p \equiv E[\tilde{R}_p]$ and $\sigma_p^2 \equiv V[\tilde{R}_p]$.

- Expected utility can then be written as

$$E\left[U\left(\widetilde{R}_p\right)\right] = \int_{-\infty}^{\infty} U(R)f(R; \bar{R}_p, \sigma_p^2)dR \qquad (7)$$

- Consider an individual's indifference curves. Define $\widetilde{x} \equiv \frac{\tilde{R}_p - \bar{R}_p}{\sigma_p}$,

$$E\left[U\left(\widetilde{R}_p\right)\right] = \int_{-\infty}^{\infty} U(\bar{R}_p + x\sigma_p)n(x)dx \qquad (8)$$

where $n(x) \equiv f(x; 0, 1)$. ($\widetilde{x}$ is a standardized normal)

Mean vs Variance cont'd

- Taking the partial derivative with respect to $\bar{R}_p$:

$$\frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial \bar{R}_p} = \int_{-\infty}^{\infty} U'n(x)dx > 0 \qquad (9)$$

  since U' is always greater than zero.

- Taking the partial derivative of equation (8) with respect to σ_p^2 and using the chain rule:

$$\frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial \sigma_p^2} = \frac{1}{2\sigma_p}\frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial \sigma_p} = \frac{1}{2\sigma_p}\int_{-\infty}^{\infty} U'xn(x)dx$$

$$(10)$$

Risk and Utility

- While U' is always positive, x ranges between $-\infty$ and $+\infty$. Take the positive and negative pair $+x_i$ and $-x_i$. Then $n(+x_i) = n(-x_i)$. Comparing the integrand of equation (10) for equal absolute realizations of x, we can show

$$
\begin{aligned}
& U'(\bar{R}_p + x_i\sigma_p)x_i n(x_i) + U'(\bar{R}_p - x_i\sigma_p)(-x_i)n(-x_i) \\
=\ & U'(\bar{R}_p + x_i\sigma_p)x_i n(x_i) - U'(\bar{R}_p - x_i\sigma_p)x_i n(x_i) \\
=\ & x_i n(x_i)\left[U'(\bar{R}_p + x_i\sigma_p) - U'(\bar{R}_p - x_i\sigma_p)\right] < 0 \qquad (11)
\end{aligned}
$$

because

$$
U'(\bar{R}_p + x_i\sigma_p) < U'(\bar{R}_p - x_i\sigma_p) \qquad (12)
$$

due to the assumed concavity of U.

Risk and Utility cont'd

- Thus, comparing $U' x_i n(x_i)$ for each positive and negative pair, we conclude that

$$\frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial \sigma_p^2} = \frac{1}{2\sigma_p} \int_{-\infty}^{\infty} U' x n(x) dx < 0 \qquad (13)$$

which is intuitive for risk-averse vN-M individuals.

- An indifference curve is the combinations of $\left(\bar{R}_p, \sigma_p^2\right)$ that satisfy the equation $E\left[U\left(\widetilde{R}_p\right)\right] = \overline{U}$, a constant. Higher $\overline{U}$ denotes greater utility. Taking the derivative

$$dE\left[U\left(\widetilde{R}_p\right)\right] = \frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial \sigma_p^2} d\sigma_p^2 + \frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial \bar{R}_p} d\bar{R}_p = 0$$

$$(14)$$

Mean and Variance Indifference Curve

- Rearranging the terms of $dE\left[U\left(\widetilde{R}_p\right)\right] = 0$, we obtain:

$$\frac{d\bar{R}_p}{d\sigma_p^2} = -\frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial \sigma_p^2} \bigg/ \frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial \bar{R}_p} > 0 \qquad (15)$$

  since we showed $\frac{\partial E[U(\widetilde{R}_p)]}{\partial \sigma_p^2} < 0$ and $\frac{\partial E[U(\widetilde{R}_p)]}{\partial \bar{R}_p} > 0$.

- Hence, each indifference curve is positively sloped in $\left(\bar{R}_p, \sigma_p^2\right)$ space. They cannot intersect because since we showed that utility is increasing in expected portfolio return for a given level of portfolio standard deviation.

Mean and Standard Deviation Indifference Curve

- As an exercise, show that the indifference curve is upward sloping and convex in $\left(\bar{R}_p, \sigma_p \right)$ space:
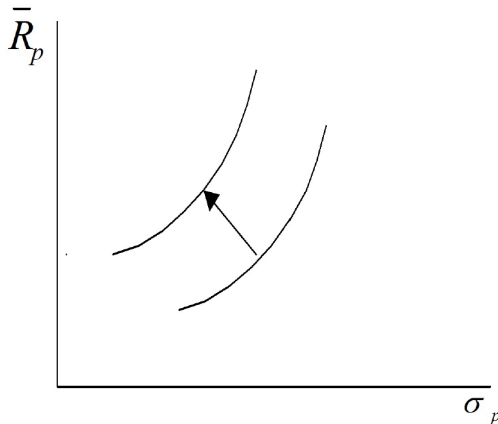
Tangency Portfolios

- The individual's optimal choice of portfolio mean and variance is determined by the point where one of these indifference curves is tangent to the set of means and standard deviations for all feasible portfolios, what we might describe as the "risk versus expected return investment opportunity set."

- This set represents all possible ways of combining various *individual* assets to generate alternative combinations of *portfolio* mean and variance (or standard deviation).

- The set includes inefficient portfolios (those in the interior of the opportunity set) as well as efficient portfolios (those on the "frontier" of the set).

- How can one determine *efficient* portfolios?

Mean/Variance Optimization

- Given the means and covariances of returns for n individual assets, find the portfolio weights that minimize portfolio variance for each level of portfolio expected return (Merton, 1972).
- Let $\bar{R} = (\bar{R}_1 \ \bar{R}_2 \ ... \ \bar{R}_n)'$ be an $n \times 1$ vector of the assets' expected returns, and let V be the $n \times n$ covariance matrix.
- V is assumed to be of full rank. (no redundant assets.)
- Next, let $\omega = (\omega_1 \ \omega_2 \ ... \ \omega_n)'$ be an $n \times 1$ vector of portfolio weights. Then the expected return on the portfolio is

$$\bar{R}_p = \omega'\bar{R} \tag{16}$$

and the variance of the portfolio return is

$$\sigma_p^2 = \omega'V\omega \tag{17}$$

Mean/Variance Optimization cont'd

- The constraint on portfolio weights is $\omega' e = 1$ where e is defined as an $n \times 1$ vector of ones.
- A frontier portfolio minimizes the portfolio's variance subject to the constraints that the portfolio's expected return equals $\overline{R}_p$ and the portfolio's weights sum to one:

$$\min_{\omega} \tfrac{1}{2}\omega' V \omega + \lambda \left[\overline{R}_p - \omega'\bar{R} \right] + \gamma[1 - \omega' e] \qquad (18)$$

- The first-order conditions with respect to ω, λ, and γ, are

$$\begin{aligned} V\omega - \lambda\bar{R} - \gamma e &= 0 & (19) \\ \overline{R}_p - \omega'\bar{R} &= 0 & (20) \\ 1 - \omega' e &= 0 & (21) \end{aligned}$$

Mean/Variance Optimization cont'd

- Solving (19) for ω^*, the portfolio weights are

$$\omega^* = \lambda V^{-1}\bar{R} + \gamma V^{-1}e \qquad (22)$$

- Pre-multiplying equation (22) by $\bar{R}'$ and e' respectively:

$$\overline{R}_p = \bar{R}'\omega^* = \lambda \bar{R}'V^{-1}\bar{R} + \gamma \bar{R}'V^{-1}e \qquad (23)$$
$$1 = e'\omega^* = \lambda e'V^{-1}\bar{R} + \gamma e'V^{-1}e \qquad (24)$$

- Solving equations (23) and (24) for λ and γ:

$$\lambda = \frac{\delta\overline{R}_p - \alpha}{\varsigma\delta - \alpha^2} \qquad (25)$$

$$\gamma = \frac{\varsigma - \alpha\overline{R}_p}{\varsigma\delta - \alpha^2} \qquad (26)$$

Mean/Variance Optimization cont'd

- Here $\alpha \equiv= e'V^{-1}\bar{R}$, $\varsigma \equiv \bar{R}'V^{-1}\bar{R}$, and $\delta \equiv e'V^{-1}e$ are scalars.
- The denominators $\varsigma\delta - \alpha^2$ are positive. Since V is positive definite, so is V^{-1}. Therefore, the quadratic form $\left(\alpha\bar{R} - \varsigma e\right)' V^{-1} \left(\alpha\bar{R} - \varsigma e\right) = \alpha^2\varsigma - 2\alpha^2\varsigma + \varsigma^2\delta = \varsigma\left(\varsigma\delta - \alpha^2\right)$ is positive.
- But since $\varsigma \equiv \bar{R}'V^{-1}\bar{R}$ is a positive quadratic form, then $\left(\varsigma\delta - \alpha^2\right)$ must also be positive.
- Substituting for λ and γ in equation (22), we have

$$\omega^* = \frac{\delta\bar{R}_p - \alpha}{\varsigma\delta - \alpha^2} V^{-1}\bar{R} + \frac{\varsigma - \alpha\bar{R}_p}{\varsigma\delta - \alpha^2} V^{-1}e \qquad (27)$$

Mean/Variance Optimization cont'd

- Collecting terms in $\overline{R}_p$, the portfolio weights are:

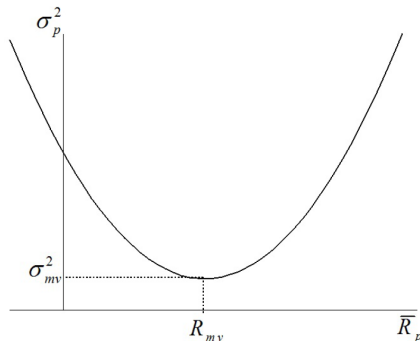$$\omega^* = a + b\overline{R}_p \qquad (28)$$

where $a \equiv \dfrac{\varsigma V^{-1} e - \alpha V^{-1} \bar{R}}{\varsigma \delta - \alpha^2}$ and $b \equiv \dfrac{\delta V^{-1} \bar{R} - \alpha V^{-1} e}{\varsigma \delta - \alpha^2}$.

- Based on these weights, the minimized portfolio variance for given $\overline{R}_p$ is

$$
\begin{aligned}
\sigma_p^2 &= \omega^{*\prime} V \omega^* = (a + b\overline{R}_p)' V (a + b\overline{R}_p) \qquad (29) \\
&= \frac{\delta \overline{R}_p^2 - 2\alpha \overline{R}_p + \varsigma}{\varsigma \delta - \alpha^2} \\
&= \frac{1}{\delta} + \frac{\delta \left(\overline{R}_p - \frac{\alpha}{\delta}\right)^2}{\varsigma \delta - \alpha^2}
\end{aligned}
$$

Mean/Variance Frontier

- Equation (29) is a parabola in σ_p^2, $\overline{R}_p$ space with its minimum at $\overline{R}_p = R_{mv} \equiv \frac{\alpha}{\delta} = \frac{\bar{R}' V^{-1} e}{e' V^{-1} e}$ and $\sigma_{mv}^2 \equiv \frac{1}{\delta} = \frac{1}{e' V^{-1} e}$.

Mean/Variance Optimization cont'd

- Substituting $\overline{R}_p = \frac{\alpha}{\delta}$ into equation (27) and multiplying by $\frac{\delta}{\delta}$ shows that this minimum variance portfolio has weights $\omega_{mv} = \frac{1}{\delta} V^{-1} e = V^{-1} e / \left(e' V^{-1} e \right)$.

- An investor whose utility is increasing in expected portfolio return and is decreasing in portfolio variance would never choose a portfolio having $\overline{R}_p < R_{mv}$.

- Hence, the *efficient* portfolio frontier is represented only by the region $\overline{R}_p \geq R_{mv}$.

- Next, let us plot the frontier in $\sigma_p, \overline{R}_p$ space by taking the square root of both sides of equation (29):
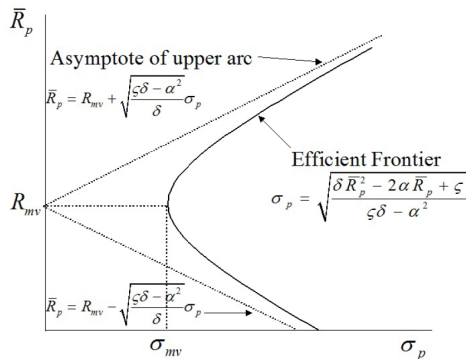
Asymptotes

$$\sigma_p = \sqrt{\frac{1}{\delta} + \frac{\delta \left(\overline{R}_p - \frac{\alpha}{\delta} \right)^2}{\varsigma \delta - \alpha^2}}$$

which is a hyperbola in σ_p, $\overline{R}_p$ space. Differentiating, this hyperbola's slope can be written as

$$\frac{\partial \overline{R}_p}{\partial \sigma_p} = \frac{\varsigma \delta - \alpha^2}{\delta \left(\overline{R}_p - \frac{\alpha}{\delta} \right)} \sigma_p \qquad (30)$$

- The hyperbola's efficient (*inefficient*) upper (*lower*) arc asymptotes to the straight line $\overline{R}_p = R_{mv} + \sqrt{\frac{\varsigma \delta - \alpha^2}{\delta}} \sigma_p$ ($\overline{R}_p = R_{mv} - \sqrt{\frac{\varsigma \delta - \alpha^2}{\delta}} \sigma_p$).

Efficient Frontier

Two Fund Separation

- We now state and prove a fundamental result:

Theorem

Every portfolio on the mean-variance frontier can be replicated by a combination of any two frontier portfolios; and an individual will be indifferent between choosing among the n financial assets, or choosing a combination of just two frontier portfolios.

- The implication is that if a security market offered two mutual or "exchange-traded" funds, each invested in a different frontier portfolio, any mean-variance investor could replicate his optimal portfolio by appropriately dividing his wealth between only these two funds. (He may have to short one.)

Two Fund Separation: Proof

- Proof: Let $\bar{R}_{1p}$, $\bar{R}_{2p}$ and $\bar{R}_{3p}$ be the expected returns on three frontier portfolios. Invest a proportion of wealth, x, in portfolio 1 and the remainder, $(1-x)$, in portfolio 2 such that:

$$\bar{R}_{3p} = x\bar{R}_{1p} + (1-x)\bar{R}_{2p} \tag{31}$$

- Recall that the weights of frontier portfolios 1 and 2 are $\omega^1 = a + b\bar{R}_{1p}$ and $\omega^2 = a + b\bar{R}_{2p}$, respectively. Hence, Portfolio 3's weights are

$$
\begin{aligned}
x\omega^1 + (1-x)\omega^2 &= x(a + b\bar{R}_{1p}) + (1-x)(a + b\bar{R}_{2p}) \tag{32} \\
&= a + b(x\bar{R}_{1p} + (1-x)\bar{R}_{2p}) \\
&= a + b\bar{R}_{3p} = \omega^3
\end{aligned}
$$

which shows it is also a frontier portfolio.

Zero Covariance Portfolios

- Frontier portfolios have another property. Except for the minimum variance portfolio, ω_{mv} , for each frontier portfolio there is another frontier portfolio with which its returns have zero covariance:

$$
\begin{aligned}
\omega^{1\prime} V \omega^2 &= (a + b\overline{R}_{1p})' V (a + b\overline{R}_{2p}) \qquad (33) \\
&= \frac{1}{\delta} + \frac{\delta}{\varsigma\delta - \alpha^2}\left(\overline{R}_{1p} - \frac{\alpha}{\delta}\right)\left(\overline{R}_{2p} - \frac{\alpha}{\delta}\right)
\end{aligned}
$$

- Equating this to zero and solving for $\overline{R}_{2p}$ in terms of $R_{mv} \equiv \frac{\alpha}{\delta}$,

$$
\begin{aligned}
\overline{R}_{2p} &= \frac{\alpha}{\delta} - \frac{\varsigma\delta - \alpha^2}{\delta^2\left(\overline{R}_{1p} - \frac{\alpha}{\delta}\right)} \qquad (34) \\
&= R_{mv} - \frac{\varsigma\delta - \alpha^2}{\delta^2\left(\overline{R}_{1p} - R_{mv}\right)}
\end{aligned}
$$

Zero Covariance cont'd

- Note that if $\left(\overline{R}_{1p} - R_{mv}\right) > 0$ so that frontier portfolio ω^1 is efficient, then by (34) $\overline{R}_{2p} < R_{mv}$: frontier portfolio ω^2 is inefficient.

- We can determine the relative locations of these zero covariance portfolios by noting that in σ_p, $\overline{R}_p$ space, a line tangent to the frontier at the point $\left(\sigma_{1p}, \overline{R}_{1p}\right)$ is of the form

$$\overline{R}_p = \overline{R}_0 + \frac{\partial \overline{R}_p}{\partial \sigma_p} \bigg|_{\sigma_p = \sigma_{1p}} \sigma_p \qquad (35)$$

where $\frac{\partial \overline{R}_p}{\partial \sigma_p} \bigg|_{\sigma_p = \sigma_{1p}}$ is the slope of the hyperbola at point $\left(\sigma_{1p}, \overline{R}_{1p}\right)$ and $\overline{R}_0$ is the tangent line's intercept at $\sigma_p = 0$.
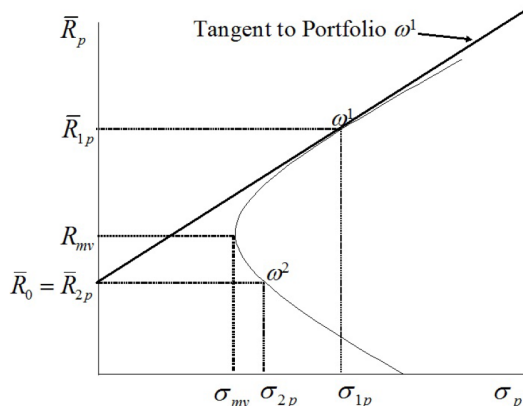
Zero Covariance cont'd

- Using (30) and (29), we can solve for $\overline{R}_0$ by evaluating (35) at the point $\left(\sigma_{1p}, \overline{R}_{1p}\right)$:

$$
\begin{aligned}
\overline{R}_0 &= \overline{R}_{1p} - \frac{\partial \overline{R}_p}{\partial \sigma_p}\Big|_{\sigma_p = \sigma_{1p}} \sigma_{1p} = \overline{R}_{1p} - \frac{\varsigma\delta - \alpha^2}{\delta\left(\overline{R}_{1p} - \frac{\alpha}{\delta}\right)}\sigma_{1p}\sigma_{1p} \\
&= \overline{R}_{1p} - \frac{\varsigma\delta - \alpha^2}{\delta\left(\overline{R}_{1p} - \frac{\alpha}{\delta}\right)}\left[\frac{1}{\delta} + \frac{\delta\left(\overline{R}_{1p} - \frac{\alpha}{\delta}\right)^2}{\varsigma\delta - \alpha^2}\right] \\
&= \frac{\alpha}{\delta} - \frac{\varsigma\delta - \alpha^2}{\delta^2\left(\overline{R}_{1p} - \frac{\alpha}{\delta}\right)} \\
&= \overline{R}_{2p}
\end{aligned}
\tag{36}
$$

The intercept of the line tangent to ω^1 is the expected return of its zero-covariance counterpart, ω^2.

Zero Covariance cont'd

Efficient Frontier with a Riskless Asset

- Assume there is a riskless asset with return R_f (Tobin, 1958).
- Now, the constraint $\omega'e = 1$ does not apply because $1 - \omega'e$ is the portfolio proportion invested in the riskless asset. However, we can now write expected return on the portfolio as

$$\bar{R}_p = R_f + \omega'(\bar{R} - R_f e) \qquad (37)$$

- The variance of the return on the portfolio is still $\omega'V\omega$. Thus, the individual's optimization problem is changed to:

$$\min_{\omega} \tfrac{1}{2}\omega'V\omega + \lambda \left\{ \overline{R}_p - \left[R_f + \omega'(\bar{R} - R_f e) \right] \right\} \qquad (38)$$

Similar to the previous derivation, the solution to the first order conditions is

$$\omega^* = \lambda V^{-1}(\bar{R} - R_f e) \qquad (39)$$

Efficient Frontier with a Riskless Asset

- Here $\lambda \equiv \dfrac{\overline{R}_p - R_f}{\left(\bar{R} - R_f e\right)' V^{-1}(\bar{R} - R_f e)} = \dfrac{\overline{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2}$, and the variance of the frontier portfolio in terms of ω^* is
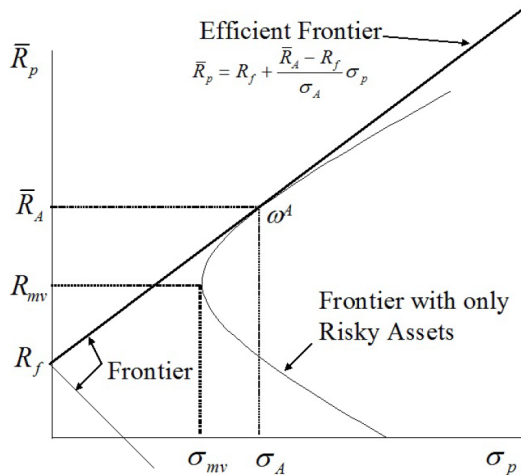
$$
\begin{aligned}
\sigma_p^2 &= \omega^{*\prime} V \omega^* = \frac{\overline{R}_p - R_f}{\left(\bar{R} - R_f e\right)' V^{-1}(\bar{R} - R_f e)} (\bar{R} - R_f e)' V^{-1} V \times \\
&\qquad \frac{\overline{R}_p - R_f}{\left(\bar{R} - R_f e\right)' V^{-1}(\bar{R} - R_f e)} V^{-1}(\bar{R} - R_f e) \\
&= \frac{(\overline{R}_p - R_f)^2}{\left(\bar{R} - R_f e\right)' V^{-1}(\bar{R} - R_f e)} = \frac{(\overline{R}_p - R_f)^2}{\varsigma - 2\alpha R_f + \delta R_f^2}
\end{aligned}
\tag{40}
$$

- Taking the square root of (40) and rearranging:

$$
\overline{R}_p = R_f \pm \left(\varsigma - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}} \sigma_p
\tag{41}
$$

which indicates that the frontier is now *linear* in σ_p, $\overline{R}_p$ space.

Efficient Frontier with a Riskless Asset

$$\bar{R}_p$$

Efficient Frontier

$$\bar{R}_p = R_f + \frac{\bar{R}_A - R_f}{\sigma_A} \sigma_p$$

$$\bar{R}_A \qquad \qquad \omega^A$$

$$R_{mv}$$

Frontier with only
Risky Assets

$$R_f$$ Frontier

$$\sigma_{mv} \quad \sigma_A \qquad \qquad \sigma_p$$

Two Fund Separation: $R_f < R_{mv}$

- When $R_f \neq R_{mv} \equiv \frac{\alpha}{\delta}$, an even stronger separation principle obtains: any frontier portfolio can be replicated with one portfolio that is located on the "risky asset only" frontier and another portfolio that holds only the riskless asset.

- Let us prove this result for the case $R_f < R_{mv}$. We assert that the efficient frontier line $\overline{R}_p = R_f + \left(\varsigma - 2\alpha R_f + \delta R_f^2 \right)^{\frac{1}{2}} \sigma_p$ can be replicated by a portfolio consisting of only the riskless asset and a portfolio on the risky-asset-only frontier that is determined by a straight line tangent to this frontier whose intercept is R_f.

- If we show that the slope of this tangent is $\left(\varsigma - 2\alpha R_f + \delta R_f^2 \right)^{\frac{1}{2}}$, the assertion is proved.

Two Fund Separation: $R_f < R_{mv}$

- Let $\overline{R}_A$ and σ_A be the expected return and standard deviation of return, respectively, of this tangency portfolio. Then the results of (34) and (35) allow us to write the tangent's slope as

$$
\begin{aligned}
Slope \equiv \frac{\overline{R}_A - R_f}{\sigma_A} &= \left[\frac{\alpha}{\delta} - \frac{\varsigma\delta - \alpha^2}{\delta^2 \left(R_f - \frac{\alpha}{\delta}\right)} - R_f \right] / \sigma_A \\
&= \left[\frac{2\alpha R_f - \varsigma - \delta R_f^2}{\delta \left(R_f - \frac{\alpha}{\delta}\right)} \right] / \sigma_A \qquad (42)
\end{aligned}
$$

- Furthermore, we can use (29) and (34) to write

$$
\sigma_A^2 = \frac{1}{\delta} + \frac{\delta \left(\overline{R}_A - \frac{\alpha}{\delta}\right)^2}{\varsigma\delta - \alpha^2}
$$

Two Fund Separation: $R_f < R_{mv}$ cont'd

- We then substitute (34) where $\overline{R}_{1p} = R_f$ for $\overline{R}_A$

$$
\begin{aligned}
\sigma_A^2 &= \frac{1}{\delta} + \frac{\varsigma\delta - \alpha^2}{\delta^3 \left(R_f - \frac{\alpha}{\delta}\right)^2} \\
&= \frac{\delta R_f^2 - 2\alpha R_f + \varsigma}{\delta^2 \left(R_f - \frac{\alpha}{\delta}\right)^2}
\end{aligned}
\tag{43}
$$

- Substituting the square root of (43) into (42):

$$
\begin{aligned}
\frac{\overline{R}_A - R_f}{\sigma_A} &= \left[\frac{2\alpha R_f - \varsigma - \delta R_f^2}{\delta \left(R_f - \frac{\alpha}{\delta}\right)}\right] \frac{-\delta \left(R_f - \frac{\alpha}{\delta}\right)}{\left(\delta R_f^2 - 2\alpha R_f + \varsigma\right)^{\frac{1}{2}}} \tag{44} \\
&= \left(\delta R_f^2 - 2\alpha R_f + \varsigma\right)^{\frac{1}{2}}
\end{aligned}
$$

which is the desired result.

An Important Separation Result

- This result implies that all investors choose to hold risky assets in the same relative proportions given by the tangency portfolio ω^A. Investors differ only in the proportion of wealth allocated to this portfolio versus the risk-free asset.
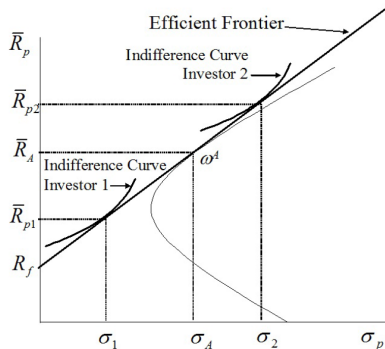
Level of Risk-free Return

- $R_f < R_{mv}$ is required for asset market equilibrium. If

  $R_f > R_{mv}$, the efficient frontier
  $\overline{R}_p = R_f + \left(\varsigma - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}} \sigma_p$ is always above the
  risky-asset-only frontier, implying the investor short-sells the
  tangency portfolio on the inefficient risky asset frontier and
  invests the proceeds in the risk-free asset.

Level of Risk-free Return

- If $R_f = R_{mv}$ the portfolio frontier is given by the asymptotes of the risky frontier. Setting $R_f = R_{mv}$ in (39) and premultiplying by e:

$$\omega^* = \frac{\overline{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2} V^{-1}(\bar{R} - R_f e) \qquad (45)$$

$$e'\omega^* = e'V^{-1}(\bar{R} - \frac{\alpha}{\delta}e)\frac{\overline{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2}$$

$$e'\omega^* = (\alpha - \frac{\alpha}{\delta}\delta)\frac{\overline{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2} = 0$$

which shows that total wealth is invested in the risk-free asset. However, the investor also holds a risky, but zero net wealth, position in risky assets by short-selling particular risky assets to finance long positions in other risky assets.

Example with Negative Exponential Utility

- Given a specific utility function and normally distributed asset returns, optimal portfolio weights can be derived directly by maximizing expected utility:

$$U(\tilde{W}) = -e^{-b\tilde{W}} \tag{46}$$

  where b is the individual's coefficient of absolute risk aversion.

- Now define $b_r \equiv bW_0$, which is the individual's coefficient of relative risk aversion at initial wealth W_0. Equation (46) can be rewritten:

$$U(\tilde{W}) = -e^{-b_r \tilde{W}/W_0} = -e^{-b_r \tilde{R}_p} \tag{47}$$

  where $\tilde{R}_p$ is the total return (one plus the rate of return) on the portfolio.

Example with Negative Exponential Utility cont'd

- We still have n risky assets and R_f as before. Now recall the properties of the lognormal distribution. If $\tilde{x}$ is a normally distributed random variable, for example, $\tilde{x} \sim N(\mu, \sigma^2)$, then $\tilde{z} = e^{\tilde{x}}$ is lognormally distributed. The expected value of $\tilde{z}$ is

$$E[\tilde{z}] = e^{\mu + \frac{1}{2}\sigma^2} \tag{48}$$

- From (47), we see that if $\tilde{R}_p = R_f + \omega'(\tilde{R} - R_f e)$ is normally distributed, then $U\left(\tilde{W}\right)$ is lognormally distributed. Using equation (48), we have

$$E\left[U\left(\tilde{W}\right)\right] = -e^{-b_r\left[R_f + \omega'(\bar{R} - R_f e)\right] + \frac{1}{2}b_r^2 \omega' V \omega} \tag{49}$$

- The individual chooses portfolio weights to maximize expected utility:

Example with Negative Exponential Utility cont'd

$$\max_{\omega} E\left[U\left(\widetilde{W}\right)\right] = \max_{\omega} - e^{-b_r\left[R_f + \omega'(\bar{R} - R_f e)\right] + \frac{1}{2}b_r^2 \omega' V \omega} \qquad (50)$$

- Since expected utility is monotonic in its exponent, this problem is equivalent to

$$\max_{\omega} \omega'(\bar{R} - R_f e) - \frac{1}{2}b_r \omega' V \omega \qquad (51)$$

- The n first-order conditions are

$$\bar{R} - R_f e - b_r V \omega = 0 \qquad (52)$$

- Solving for ω, we obtain

$$\omega^* = \frac{1}{b_r} V^{-1}(\bar{R} - R_f e) \qquad (53)$$

Example with Negative Exponential Utility cont'd

- Comparing (53) to (39), note that

$$\frac{1}{b_r} = \lambda \equiv \frac{\overline{R}_p - R_f}{\left(\bar{R} - R_f e\right)' V^{-1} \left(\bar{R} - R_f e\right)} \qquad (54)$$

so that the greater is b_r, the smaller is $\overline{R}_p$ and the proportion of wealth invested in risky assets.

- Multiplying both sides of (53) by W_0, we see that the absolute amount of wealth invested in the risky assets is

$$W_0 \omega^* = \frac{1}{b} V^{-1} (\bar{R} - R_f e) \qquad (55)$$

implying that with constant absolute risk aversion the amount invested in the risky assets is independent of initial wealth.

Cross-hedging (Anderson & Danthine, 1981)

- Consider a one-period model of an individual required to trade a commodity in the future and wants to hedge the risk using futures contracts.

- Assume that at date 0 she is committed to buy (sell) $y > 0$ ($y < 0$) units of a risky commodity at date 1 at the spot price p_1. As of date 0, y is deterministic, while p_1 is stochastic.

- There are n financial securities (futures contracts) where the date 0 price of the ith financial security as p_{i0}^s. Its risky date 1 price is p_{i1}^s.

- Let s_i denote the amount of the ith security purchased at date 0, where $s_i < 0$ indicates a short position.

Cross-hedging (Anderson & Danthine, 1981) cont'd

- Define $n \times 1$ quantity and price vectors $s \equiv [s_1 \ldots s_n]'$, $p_0^s \equiv [p_{10}^s \ldots p_{n0}^s]'$, and $p_1^s \equiv [p_{11}^s \ldots p_{n1}^s]'$. Also define $p^s \equiv p_1^s - p_0^s$ as the $n \times 1$ vector of security price changes.

- Thus, the date 1 profit from securities trading is $p^{s\prime} s$

- Define the moments $E[p_1] = \bar{p}_1$, $Var[p_1] = \sigma_{00}$, $E[p_1^s] = \bar{p}_1^s$, $E[p^s] = \bar{p}^s$, $Cov[p_{i1}^s, p_{j1}^s] = \sigma_{ij}$, $Cov[p_1, p_{i1}^s] = \sigma_{0i}$, and the $(n+1) \times (n+1)$ covariance matrix of the spot commodity and financial securities is

$$\Sigma = \left[\begin{array}{cc} \sigma_{00} & \Sigma_{01} \\ \Sigma_{01}' & \Sigma_{11} \end{array} \right] \qquad (56)$$

where Σ_{11} is an $n \times n$ matrix whose i, j^{th} element is σ_{ij}, and Σ_{01} is a $1 \times n$ vector whose i^{th} element is σ_{0i}.

Cross-hedging (Anderson & Danthine, 1981) cont'd

- The end-of-period profit (wealth) of the financial operator, W, is

$$W = p^{s\prime}s - p_1 y \qquad (57)$$

- Assuming constant absolute risk aversion (CARA) utility, the problem is to choose s in order to maximize:

$$\max_s E[W] - \frac{1}{2}\alpha Var[W] \qquad (58)$$

- Substituting in for the operator's expected profit and variance:

$$\max_s \bar{p}^{s\prime}s - \bar{p}_1 y - \frac{1}{2}\alpha \left[y^2\sigma_{00} + s'\Sigma_{11}s - 2y\Sigma_{01}s \right] \qquad (59)$$

- The first-order conditions are

$$\bar{p}^s - \alpha \left[\Sigma_{11}s - y\Sigma_{01}' \right] = 0 \qquad (60)$$

Cross-hedging (Anderson & Danthine, 1981) cont'd

- Solving for s, the optimal financial security positions are

$$
\begin{aligned}
s &= \frac{1}{\alpha} \Sigma_{11}^{-1} \bar{p}^s + y \Sigma_{11}^{-1} \Sigma_{01}' \qquad (61) \\
&= \frac{1}{\alpha} \Sigma_{11}^{-1} \left(\bar{p}_1^s - p_0^s \right) + y \Sigma_{11}^{-1} \Sigma_{01}'
\end{aligned}
$$

- First consider $y = 0$. This can be viewed as a trader who has no requirement to hedge.
- If $n = 1$ and $\bar{p}_1^s > p_0^s$ ($\bar{p}_1^s < p_0^s$), the speculator buys (sells) the security. The size of the position is adjusted by the volatility of the security ($\Sigma_{11}^{-1} = 1/\sigma_{11}$), and the level of risk aversion α.
- For the general case of $n > 1$, expectations are not enough to decide to buy/sell. All of the elements in Σ_{11}^{-1} need to be considered to maximize diversification.

Cross-hedging (Anderson & Danthine, 1981) cont'd

- For the general case $y \neq 0$, the situation faced by a *hedger*, the demand for financial securities is similar to that of a pure speculator in that it also depends on price expectations.

- In addition, there are hedging demands, call them s^h :

$$s^h \equiv y\Sigma_{11}^{-1}\Sigma_{01}' \qquad (62)$$

- This is the solution to the variance-minimization problem, yet in general expected returns matter for hedgers.

- From (62), note that when $n = 1$ the pure hedging demand per unit of the commodity purchased, s^h/y, is

$$\frac{s^h}{y} = \frac{Cov(p_1, p_1^s)}{Var(p_1^s)} \qquad (63)$$

Cross-hedging (Anderson & Danthine, 1981) cont'd

- For the general case, $n > 1$, the elements of the vector $\Sigma_{11}^{-1}\Sigma_{01}'$ equal the coefficients $\beta_1, ..., \beta_n$ in the multiple regression model:

$$\Delta p_1 = \beta_0 + \beta_1 \Delta p_1^s + \beta_2 \Delta p_2^s + ... + \beta_n \Delta p_n^s + \varepsilon \qquad (64)$$

where $\Delta p_1 \equiv p_1 - p_0$, $\Delta p_i^s \equiv p_{1i}^s - p_{0i}^s$, and ε is a mean-zero error term.

- An implication of (64) is that an operator might estimate the *hedge ratios*, s^h/y, by performing a statistical regression using a historical time series of the $n \times 1$ vector of security price changes. In fact, this is a standard way that practitioners calculate hedge ratios.

Summary

- Normal distribution of individual asset returns is sufficient for mean-variance optimization to be valid.

- Two efficient portfolios are enough to span the entire mean-variance efficient frontier.

- When a riskless asset exists, only one efficient portfolio (tangency portfolio) and the riskless asset is required to span the frontier.

- Hedging can be expressed as an application of mean-variance optimization.