FIN 591: Homework #2

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Problem 1

a. Since there is only one risky asset, consumption at date 1 C_1 is represented as $C_1 = y_1 + (W_0 + y_0 - C_0)(R_f + W(R - R_f))$. By applying first order condition, the following equations should hold.

$$\begin{aligned} &U'(C_0) - \delta \mathrm{E}[U'(C_1)(R_f + W(R - R_f))] \\ &= be^{-bC_0} - \delta \mathrm{E}[be^{-bC_1}(R_f + W(R - R_f))] = 0 \\ &\Rightarrow e^{-bC_0} - \delta \mathrm{E}[e^{-bC_1}(R_f + W(R - R_f))] = 0 \end{aligned} \tag{1}$$

$$\begin{aligned} &\delta \mathrm{E}[U'(C_1)(W_0 + y_0 - C_0)(R - R_f)] \\ &= \delta \mathrm{E}[be^{-bC_1}(W_0 + y_0 - C_0)(R - R_f)] = 0 \\ &\Rightarrow \delta \mathrm{E}[e^{-bC_1}(W_0 + y_0 - C_0)(R - R_f)] = 0 \\ &\Rightarrow \mathrm{E}[e^{-bC_1}R] = R_f \mathrm{E}[e^{-bC_1}] \end{aligned} \tag{2}$$

Plugging (2) in (1), equation (3) is obtained. Therefore, the optimal choice should satisfy (3).

$$e^{-bC_0} = R_f \delta \mathrm{E}[e^{-bC_1}] \tag{3}$$

If we plug $C_1 = y_1 + (W_0 + y_0 - C_0)(R_f + W(R - R_f))$ into (3), the following equation should hold.

$$\mathrm{E}[e^{-by_1} \times e^{-b(W_0 + y_0 - C_0)(R_f + W(R - R_f))}] = \frac{e^{-bC_0}}{R_f \delta} \tag{4}$$

Since the RHS is nonrandom, the optimal portfolio weight W has negative relationship with the future wage income y_1.

b. By using the equation (3), optimal consumption at time 0 should satisfy the following equation.

$$C_0 = -b \log(R_f \delta \mathrm{E}[e^{-bC_1}]) \tag{5}$$

Problem 2

a. Under the optimal choice, $U_C(C_{T-1}, T-1) = \mathrm{E}_{T-1}[B_W(W_T, T)R_{T-1}]$ holds. If we plug given utility and bequest function to the equation, the following equation holds.

$$\begin{aligned} &\delta^{T-1} C_{T-1}^{\gamma-1} = \mathrm{E}_{T-1}[\delta^T W_T^{\gamma-1} R_{T-1}] \\ &\Rightarrow \delta^{T-1} C_{T-1}^{\gamma-1} = \delta^T S_{T-1}^{\gamma-1} \mathrm{E}[R_{T-1}^{\gamma}] \quad \text{where } W_T = S_{T-1}R_{T-1}, S_{T-1} = W_{T-1} - C_{T-1} \end{aligned} \tag{6}$$

Therefore, if we rearrange the equation (6), the optimal consumption at time $T-1$, C_{T-1}^* can be obtained as follows.

$$C_{T-1}^* = \frac{\delta^{\frac{1}{\gamma-1}} \mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}}}{1 - \delta^{\frac{1}{\gamma-1}} \mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}}} W_{T-1} \tag{7}$$

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Another condition under optimal choice is $\mathrm{E}_{T-1}[B_W(W_T,T)(R_{i,T-1}-R_f)]=0$ for $i=1,2,3,\ldots,n$. Therefore, the following equation holds.

$$\begin{aligned}
&\mathrm{E}_{T-1}[\delta^T W_T^{\gamma-1}R_{i,T-1}]=R_f\mathrm{E}_{T-1}[\delta^T W_T^{\gamma-1}]\\
&\Rightarrow \mathrm{E}_{T-1}[(S_{T-1}R_{T-1})^{\gamma-1}R_{i,T-1}]=R_f\mathrm{E}_{T-1}[\delta^T(S_{T-1}R_{T-1})^{\gamma-1}]\\
&\Rightarrow \mathrm{E}_{T-1}[R_{T-1}^{\gamma-1}R_{i,T-1}]=R_f\mathrm{E}_{T-1}[R_{T-1}^{\gamma-1}]
\end{aligned}\tag{8}$$

b. Let $\delta^{\frac{1}{\gamma-1}}\mathrm{E}_{T-1}[R_{T-1}^{\gamma}]^{\frac{1}{\gamma-1}}=a$. Then $C_{T-1}^*=\frac{a}{1+a}W_{T-1}$. Since $J(W_{T-1},T-1)=U(C_{T-1}^*,T-1)=\mathrm{E}_{T-1}[B(W_T,T)]$, $J(W_{T-1},T-1)$ can be represented as follows.

$$\begin{aligned}
J(W_{T-1},T-1)&=\frac{\delta^{T-1}C_{T-1}^{*\gamma}}{\gamma}+\mathrm{E}_{T-1}[\frac{\delta^T W_T^{\gamma}}{\gamma}]\\
&=\frac{\delta^{T-1}}{\gamma}(\frac{a}{1+a}W_{T-1})^{\gamma}+\frac{\delta^T}{\gamma}\mathrm{E}_{T-1}[((1-\frac{a}{1+a})W_{T-1}R_{T-1})^{\gamma}]\\
&=\frac{\delta^{T-1}}{\gamma}((\frac{a}{1+a})^{\gamma}W_{T-1}^{\gamma}+\delta(\frac{1}{1+a})^{\gamma}W_{T-1}^{\gamma}\mathrm{E}_{T-1}[R_{T-1}^{\gamma}])\\
&=\frac{\delta^{T-1}}{\gamma}(\frac{1}{1+a})^{\gamma}(a^{\gamma}+\delta\mathrm{E}_{T-1}[R_{T-1}^{\gamma}])W_{T-1}^{\gamma}
\end{aligned}\tag{9}$$

c. Let $\frac{1}{1+a}^{\gamma}(a^{\gamma}+\delta\mathrm{E}_{T-1}[R_{T-1}^{\gamma}])=b$. Then $J(W_{T-1},T-1)$ can be represented as $\frac{\delta^{T-1}}{\gamma}bW_{T-1}^{\gamma}$. Since under optimal choice, $U_C(C_{T-2},T-2)=\mathrm{E}_{T-2}[J_W(W_{T-1},T-1)R_{T-2}]$ holds, the following equation must hold.

$$\begin{aligned}
\delta^{T-2}C_{T-2}^{\gamma-1}&=\mathrm{E}_{T-2}[\delta^{T-1}bW_{T-1}^{\gamma-1}R_{T-2}]\\
&=\delta^{T-1}b\mathrm{E}_{T-2}[S_{T-2}^{\gamma-1}R_{T-2}^{\gamma}]\\
&=\delta^{T-1}b(W_{T-2}-C_{T-2})^{\gamma-1}\mathrm{E}_{T-2}[R_{T-2}^{\gamma}]\\
C_{T-2}&=\delta b(W_{T-2}-C_{T-2})\mathrm{E}_{T-2}[R_{T-2}]^{\frac{1}{\gamma-1}}
\end{aligned}\tag{10}$$

By rearranging the terms in equation (10), we can get an explicit form of C_{T-2}^* as follows.

$$\begin{aligned}
C_{T-2}^*&=\frac{\delta b\mathrm{E}_{T-2}[R_{T-2}^{\gamma}]^{\frac{1}{\gamma-1}}}{1+\delta b\mathrm{E}_{T-2}[R_{T-2}^{\gamma}]^{\frac{1}{\gamma-1}}}W_{T-2}\\
&=\frac{c}{1+c}W_{T-2}\quad\text{where } c=\delta b\mathrm{E}_{T-2}[R_{T-2}^{\gamma}]^{\frac{1}{\gamma-1}}
\end{aligned}\tag{11}$$

Another optimal condition is $\mathrm{E}_{T-2}[R_{i,T-2}J_W(W_{T-1},T-1)]=R_f\mathrm{E}_{T-2}[J_W(W_{T-1},T-1)]$. Therefore, the following equations hold.

$$\begin{aligned}
&\mathrm{E}_{T-2}[R_{i,T-2}\delta^{T-1}bW_{T-1}^{\gamma-1}]=R_f\mathrm{E}_{T-2}[\delta^{T-1}bW_{T-1}^{\gamma-1}]\\
&\Rightarrow \mathrm{E}_{T-2}[R_{i,T-2}W_{T-1}^{\gamma-1}]=R_f\mathrm{E}_{T-2}[W_{T-1}^{\gamma-1}]\\
&\Rightarrow \mathrm{E}_{T-2}[R_{i,T-2}(S_{T-2}R_{T-2})^{\gamma-1}]=R_f\mathrm{E}_{T-2}[(S_{T-2}R_{T-2})^{\gamma-1}]\\
&\Rightarrow \mathrm{E}_{T-2}[R_{i,T-2}R_{T-2}^{\gamma-1}]=R_f\mathrm{E}_{T-2}[R_{T-2}^{\gamma-1}]
\end{aligned}\tag{12}$$

d. Since $J(W_{T-2}, T-2) = U(C^*_{T-2}, T-2) + \mathrm{E}_{T-2}[J(W_{T-1}, T-1)]$, the following equation holds.

$$\begin{aligned} J(W_{T-2}, T-2) &= \frac{\delta^{T-2} C^{*\gamma}_{T-2}}{\gamma} + \mathrm{E}_{T-2}[\frac{\delta^{T-1}}{\gamma} b W^{\gamma}_{T-1}] \\ &= \frac{\delta^{T-2}}{\gamma} (\frac{c}{1+c})^{\gamma} W^{\gamma}_{T-2} + \mathrm{E}_{T-2}[\frac{\delta^{T-1}}{\gamma} b (W_{T-2} - C_{T-2})^{\gamma} R^{\gamma}_{T-2}] \\ &= \frac{\delta^{T-2}}{\gamma} (\frac{c}{1+c})^{\gamma} W^{\gamma}_{T-2} + \mathrm{E}_{T-2}[\frac{\delta^{T-1}}{\gamma} b (\frac{1}{1+c})^{\gamma} W^{\gamma}_{T-2} R^{\gamma}_{T-2}] \\ &= \frac{\delta^{T-2}}{\gamma} (\frac{1}{1+c})^{\gamma} ((c^{\gamma} + \delta b) W^{\gamma}_{T-2} \mathrm{E}_{T-2}[R^{\gamma}_{T-2}]) \\ &= \frac{\delta^{T-2}}{\gamma} d W^{\gamma}_{T-2} \quad \text{where } d = (\frac{1}{1+c})^{\gamma} (c^{\gamma} + \delta b) \mathrm{E}_{T-2}[R^{\gamma}_{T-2}] \end{aligned} \tag{13}$$

From the pattern, the optimal consumption at $T-t$ is kW_{T-t} for some constant k, and the optimal portfolio weight satisfies $\mathrm{E}_{T-t}[R_{i,T-t} R^{\gamma-1}_{T-t}] = R_f \mathrm{E}_{T-t}[R^{\gamma-1}_{T-t}]$.