Continuous-Time Consumption and Portfolio Choice

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Introduction

- Assuming that asset prices follow diffusion processes, we derive an individual's continuous consumption and portfolio choices.
- Asset demands reflect single-period mean-variance terms as well as components that hedge against changes in investment opportunities.
- Consumption and portfolio choices can be solved using stochastic dynamic programming or, when markets are complete, a martingale technique.

Model Assumptions

- Let x be a $k \times 1$ vector of state variables that affect the distribution of asset returns, where $r(x, t)$ is the date t instananeous-maturity risk-free rate and the date t price of the i^{th} risky asset, $S_i(t)$, follows the process

$$dS_i(t) / S_i(t) = \mu_i(x, t)\, dt + \sigma_i(x, t)\, dz_i \qquad (1)$$

where $i = 1, ..., n$ and $(\sigma_i\, dz_i)(\sigma_j\, dz_j) = \sigma_{ij}\, dt$. The process (1) assumed the reinvestment of dividends.

- The i^{th} state variable follow the process

$$dx_i = a_i(x, t)\, dt + b_i(x, t)\, d\zeta_i \qquad (2)$$

where $i = 1, ..., k$. $d\zeta_i$ is a Brownian motion with $(b_i\, d\zeta_i)(b_j\, d\zeta_j) = b_{ij}\, dt$ and $(\sigma_i\, dz_i)(b_j\, d\zeta_j) = \phi_{ij}\, dt$.

Model Assumptions cont'd

- Define C_t as the individual's date t rate of consumption per unit time.
- Also, let $\omega_{i,t}$ be the proportion of total wealth at date t, W_t, allocated to risky asset i, $i = 1, ..., n$, so that

$$\begin{aligned} dW &= \left[\sum_{i=1}^{n} \omega_i dS_i / S_i + \left(1 - \sum_{i=1}^{n} \omega_i \right) rdt \right] W - Cdt \qquad (3) \\ &= \sum_{i=1}^{n} \omega_i (\mu_i - r) W \, dt \ + \ (rW - C) \, dt \ + \ \sum_{i=1}^{n} \omega_i W \sigma_i \, dz_i \end{aligned}$$

- Subject to (3), the individual solves:

$$\max_{C_s, \{\omega_{i,s}\}, \forall s, i} E_t \left[\int_t^T U(C_s, s) \, ds \ + \ B(W_T, T) \right] \qquad (4)$$

Continuous-Time Dynamic Programming

- Consider a simplified version of the problem in conditions (3) to (4) with only one choice and one state variable:

$$\max_{\{c\}} E_t \left[\int_t^T U(c_s, x_s)\, ds \right] \tag{5}$$

subject to

$$dx = a(x, c)\, dt + b(x, c)\, dz \tag{6}$$

where c_t is a *control* (e.g. consumption) and x_t is a *state* (e.g. wealth). Define the indirect utility function, $J(x_t, t)$:

$$\begin{aligned} J(x_t, t) &= \max_{\{c\}} E_t \left[\int_t^T U(c_s, x_s)\, ds \right] \qquad (7) \\ &= \max_{\{c\}} E_t \left[\int_t^{t+\Delta t} U(c_s, x_s)\, ds + \int_{t+\Delta t}^T U(c_s, x_s)\, ds \right] \end{aligned}$$

Continuous-Time Dynamic Programming cont'd

- Apply Bellman's *Principle of Optimality*:

$$
\begin{aligned}
J(x_t,\ t) &= \max_{\{c\}} E_t\left[\int_t^{t+\Delta t} U(c_s, x_s)\, ds \ +\ \max_{\{c\}} E_{t+\Delta t}\left[\int_{t+\Delta t}^{T} U(c_s, x_s)\, ds\right]\right] \\
&= \max_{\{c\}} E_t\left[\int_t^{t+\Delta t} U(c_s, x_s)\, ds \ +\ J(x_{t+\Delta t},\ t+\Delta t)\right] \qquad (8)
\end{aligned}
$$

- For Δt small, approximate the first integral as $U(c_t, x_t)\,\Delta t$ and expand $J(x_{t+\Delta t},\ t+\Delta t)$ around x_t and t in a Taylor series:

$$
\begin{aligned}
J(x_t,\ t) &= \max_{\{c\}} E_t\left[U(c_t, x_t)\Delta t + J(x_t, t) + J_x \Delta x + J_t \Delta t \right. \qquad (9) \\
&\quad \left. + \frac{1}{2} J_{xx}(\Delta x)^2 + J_{xt}(\Delta x)(\Delta t) + \frac{1}{2} J_{tt}(\Delta t)^2 + o(\Delta t)\right]
\end{aligned}
$$

where $o\left(\Delta t\right)$ represents higher-order terms.

Continuous-Time Dynamic Programming cont'd

- The state variable's diffusion process (6) is approximated

$$\Delta x \approx a(x, c)\Delta t + b(x, c)\Delta z + o(\Delta t) \qquad (10)$$

where $\Delta z = \sqrt{\Delta t}\widetilde{\varepsilon}$ and $\widetilde{\varepsilon} \sim N(0, 1)$. Substituting (10) into (9), and subtracting $J(x_t, t)$ from both sides,

$$0 = \max_{\{c\}} E_t \left[U(c_t, x_t)\Delta t + \Delta J + o(\Delta t)\right] \qquad (11)$$

where

$$\Delta J = \left[J_t + J_x a + \frac{1}{2} J_{xx} b^2\right] \Delta t + J_x b \Delta z \qquad (12)$$

This is just a discrete-time version of Itô's lemma. In equation (11), $E_t \left[J_x b \Delta z\right] = 0$. Divide both sides of (11) by Δt.

Continuous-Time Dynamic Programming cont'd

- We can take the limit as $\Delta t \rightarrow 0$:

$$0 = \max_{\{c\}} \left[U(c_t, x_t) + J_t + J_x a + \frac{1}{2} J_{xx} b^2 \right] \qquad (13)$$

- Equation (13) is the stochastic, continuous-time Bellman equation and can be rewritten as

$$0 = \max_{\{c\}} \left[U(c_t, x_t) + L[J] \right] \qquad (14)$$

where $L[\cdot]$ is the *Dynkin operator*; that is, the "drift" term (expected change per unit of time) in $dJ(x,t)$ obtained from applying Itô's lemma to J.

Solving the *Real* Continuous-Time Problem

- Returning to the consumption - portfolio choice problem, define the indirect utility-of-wealth $J(W, x, t)$:

$$J(W, x, t) = \max_{C_s, \{\omega_{i,s}\}, \forall s, i} E_t \left[\int_t^T U(C_s, s)\, ds + B(W_T, T) \right] \tag{15}$$

- In this problem, consumption, C_t, and portfolio weights, $\{\omega_{i,t}\}$, $i = 1, ..., n$ are the control variables.
- Wealth, W_t, and the variables affecting the distribution of asset returns, $x_{i,t}$, $i = 1, ..., k$ are the state variables that evolve according to (1) and (2), respectively.

Solving the Continuous-Time Problem

- Thus, the Dynkin operator in terms of W and x is

$$
\begin{aligned}
L[J] \quad = \quad & \frac{\partial J}{\partial t} + \left[\sum_{i=1}^{n} \omega_i(\mu_i - r)W + (rW - C) \right] \frac{\partial J}{\partial W} + \sum_{i=1}^{k} a_i \frac{\partial J}{\partial x_i} \\
& + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}\omega_i\omega_j W^2 \frac{\partial^2 J}{\partial W^2} + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} b_{ij} \frac{\partial^2 J}{\partial x_i \, \partial x_j} \\
& + \sum_{j=1}^{k} \sum_{i=1}^{n} W \omega_i \phi_{ij} \frac{\partial^2 J}{\partial W \partial x_j} \qquad (16)
\end{aligned}
$$

- From equation (14) we have

$$
0 = \max_{C_t, \{\omega_{i,t}\}} \left[U(C_t, t) + L[J] \right] \qquad (17)
$$

- We obtain first-order conditions wrt C_t and $\omega_{i,t}$:

Solving the Continuous-Time Problem cont'd

$$0 = \frac{\partial U(C^*, t)}{\partial C} - \frac{\partial J(W, x, t)}{\partial W} \tag{18}$$

$$0 = W\frac{\partial J}{\partial W}(\mu_i - r) + W^2 \frac{\partial^2 J}{\partial W^2}\sum_{j=1}^{n}\sigma_{ij}\omega_j^* + W\sum_{j=1}^{k}\phi_{ij}\frac{\partial^2 J}{\partial x_j\,\partial W}, \tag{19}$$

where $i = 1,...,n$.

- Equation (18) is the envelope condition while equation (19) has the discrete-time analog

$$E_t\left[R_{i,t} J_W\left(W_{t+1}, t+1\right)\right] = R_{f,t} E_t\left[J_W\left(W_{t+1}, t+1\right)\right],\ \ i = 1, ..., n$$

Solving the Continuous-Time Problem cont'd

- Define the inverse marginal utility function $G = [\partial U/\partial C]^{-1}$ and let J_W be shorthand for $\partial J/\partial W$. Condition (18) becomes

$$C^* = G\left(J_W, t\right) \tag{20}$$

- Denote $\Omega \equiv [\sigma_{ij}]$ as the $n \times n$ instantaneous covariance matrix whose i, j^{th} element is σ_{ij}, and denote v_{ij} as the i, j^{th} element of $\Omega^{-1} \equiv [\nu_{ij}]$.
- Then the solution to (19) can be written as

$$\omega_i^* = -\frac{J_W}{J_{WW} W} \sum_{j=1}^{n} \nu_{ij}(\mu_j - r) - \sum_{m=1}^{k} \sum_{j=1}^{n} \frac{J_{Wx_m}}{J_{WW} W} \phi_{im} \nu_{ij}, \quad i = 1, \ldots, n \tag{21}$$

- ω_i^* in (21) depends on $-J_W / \left(J_{WW} W\right)$ which is the inverse of relative risk aversion for lifetime utility of wealth.

Solving the Continuous-Time Problem cont'd

- Assuming specific functions for U and the μ_i's, σ_{ij}'s, and ϕ_{ij}'s, equations (20) and (21) can be solved in terms of the state variables W, x, and J_W, J_{WW}, and J_{Wx_i}.
- Substituting C^* and the ω_i^* back into equation (17) leads to a nonlinear partial differential equation (PDE) for J that can be solved subject to $J(W_T, x_T, T) = B(W_T, T)$.
- In turn, solutions for C_t^* and the $\omega_{i,t}^*$ in terms of only W_t, and x_t then result from (20) and (21).
- If all of the μ_i's (including r) and σ_i's are constants, asset returns are lognormally distributed and there is a *constant investment opportunity set*.
- In this case the *only* state variable is W, and the optimal portfolio weights in (21) simplify to

Constant Investment Opportunities

$$\omega_i^* = -\frac{J_W}{J_{WW} W} \sum_{j=1}^{n} \nu_{ij}(\mu_j - r), \quad i = 1, \ldots, n \qquad (22)$$

- Plugging (20) and (22) back into the optimality equation (17), and using the fact that $[\nu_{ij}] \equiv \Omega^{-1}$, we have

$$\begin{aligned} 0 &= U(G,t) + J_t + \left[\sum_{i=1}^{n} \omega_i(\mu_i - r)W + rW - C\right] J_W + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}\omega_i\omega_j W^2 J_{WW} \\ &= U(G) + J_t + J_W(rW - G) - \frac{J_W^2}{J_{WW}} \sum_{i=1}^{n}\sum_{j=1}^{n} \nu_{ij}(\mu_i - r)(\mu_j - r) \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}\omega_i\omega_j W^2 \frac{\partial^2 J}{\partial W^2} \\ &= U(G) + J_t + J_W(rW - G) - \frac{J_W^2}{2J_{WW}} \sum_{i=1}^{n}\sum_{j=1}^{n} \nu_{ij}(\mu_i - r)(\mu_j - r) \qquad (23) \end{aligned}$$

Constant Investment Opportunities cont'd

- This equation can be solved for J and, in turn, C^* and ω_i^* after specifying U.
- In any case, since ν_{ij}, μ_j, and r are constants, the proportion of each risky asset that is optimally held will be proportional to $-J_W/(J_{WW}W)$ which is common across all assets.
- Consequently, the proportion of wealth in risky asset i to risky asset k is a constant:

$$\frac{\omega_i^*}{\omega_k^*} = \frac{\sum_{j=1}^{n} \nu_{ij}(\mu_j - r)}{\sum_{j=1}^{n} \nu_{kj}(\mu_j - r)} \tag{24}$$

Constant Investment Opportunities cont'd

- Therefore, the proportion of risky asset k to all risky assets is

$$\delta_k = \frac{\omega_k^*}{\sum_{i=1}^n \omega_i^*} = \frac{\displaystyle\sum_{j=1}^n \nu_{kj}(\mu_j - r)}{\displaystyle\sum_{i=1}^n \sum_{j=1}^n \nu_{ij}(\mu_j - r)} \tag{25}$$

- Since all individuals regardless of U will hold r and the constant-proportion portfolio of risky assets defined by δ_k, we obtain a two-fund separation result: all individuals' optimal portfolios consists of the risk-free asset paying rate of return r and a single risky asset portfolio having the following expected rate of return, μ, and variance, σ^2:

Two-Fund Separation

$$\begin{aligned} \mu &\equiv \sum_{i=1}^{n} \delta_i \mu_i \\ \sigma^2 &\equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i \delta_j \sigma_{ij} \end{aligned} \tag{26}$$

- Indeed, recalling the single-period mean-variance portfolio weights, the i^{th} element of (2.42) can be written as $w_i^* = \lambda \sum_{j=1}^{n} \nu_{ij} \left(\overline{R}_j - R_f \right)$, which equals (22) when $\lambda = -J_W / \left(J_{WW} W \right)$.
- Hence, we obtain mean-variance portfolio weights with lognormally-distributed asset returns since the asset return diffusions are *locally* normal.

HARA Utility and Constant Investment Opportunities

- Analytic solutions to the constant investment opportunity problem exist with Hyperbolic Absolute Risk Aversion utility:

$$U(C,\, t) \;=\; e^{-\rho t} \frac{1-\gamma}{\gamma} \left(\frac{\alpha C}{1-\gamma} + \beta \right)^{\gamma} \tag{27}$$

- Optimal consumption in equation (20) is

$$C^{*} \;=\; \frac{1-\gamma}{\alpha} \left[\frac{e^{\rho t} J_W}{\alpha} \right]^{\frac{1}{\gamma-1}} \;-\; \frac{(1-\gamma)\beta}{\alpha} \tag{28}$$

and using (22) and (26), the risky-asset portfolio weights are

$$\omega^{*} \;=\; -\frac{J_W}{J_{WW} W} \frac{\mu - r}{\sigma^2} \tag{29}$$

HARA Utility and Constant Investment Opportunities

- Simplify equation (23) to obtain

$$\begin{aligned} 0 &= \frac{(1-\gamma)^2}{\gamma} e^{-\rho t} \left[\frac{e^{\rho t} J_W}{\alpha} \right]^{\frac{\gamma}{\gamma-1}} + J_t \\ &+ \left(\frac{(1-\gamma)\beta}{\alpha} + rW \right) J_W - \frac{J_W^2}{J_{WW}} \frac{(\mu - r)^2}{2\sigma^2} \end{aligned} \tag{30}$$

- Merton (1971) solves this PDE subject to $J(W,T) = B(W_T, T) = 0$, and shows (28) and (29) then take the form

$$C_t^* = aW_t + b \tag{31}$$

and

$$\omega_t^* = g + \frac{h}{W_t} \tag{32}$$

CRRA and Constant Investment Opportunities

- Here a, b, g, and h are, at most, functions of time.
- For the special case of constant relative risk aversion where $U(C,t) = e^{-\rho t} C^{\gamma}/\gamma$, the solution is

$$J(W,t) = e^{-\rho t}\left[\frac{1-e^{-\varkappa(T-t)}}{\varkappa}\right]^{1-\gamma} W^{\gamma}/\gamma \qquad (33)$$

$$C_t^* = \frac{\varkappa}{1-e^{-\varkappa(T-t)}} W_t \qquad (34)$$

and

$$\omega^* = \frac{\mu - r}{(1-\gamma)\sigma^2} \qquad (35)$$

where $\varkappa \equiv \frac{\gamma}{1-\gamma}\left[\frac{\rho}{\gamma} - r - \frac{(\mu-r)^2}{2(1-\gamma)\sigma^2}\right]$.

Implications of Continuous-Time Decisions

- When the individual's planning horizon is infinite, $T \to \infty$, a solution exists only if $\varkappa > 0$.
- In this case with $T \to \infty$, $C_t^* = \varkappa W_t$.
- Although we obtain the Markowitz result in continuous time, it is not the *same* result as in discrete time.
- For example, a CRRA individual facing normally distributed returns and discrete-time portfolio rebalancing will choose to put all wealth in the risk-free asset.
- In contrast, this individual facing lognormally-distributed returns and continuous portfolio rebalancing chooses $\omega^* = (\mu - r) / \left[(1-\gamma)\sigma^2\right]$, which is independent of the time horizon.

Changing Investment Opporunities

- Consider the effects of changing investment opportunities by simply assuming a single state variable so that $k = 1$ and x is a scalar that follows the process

$$dx = a\left(x,t\right)dt + b\left(x,t\right)d\zeta \tag{36}$$

where $b\,d\zeta\sigma_i\,dz_i = \phi_i\,dt$.

- The optimal portfolio weights in (21) are

$$\omega_i^* = -\frac{J_W}{WJ_{WW}}\sum_{j=1}^{n}\upsilon_{ij}\left(\mu_j - r\right) - \frac{J_{Wx}}{WJ_{WW}}\sum_{j=1}^{n}\upsilon_{ij}\phi_j,\ \ i = 1,\ldots,n \tag{37}$$

Portfolio Weights with Changing Investment Opportunities

- Written in matrix form, equation (37) is

$$\boldsymbol{\omega}^* = \frac{A}{W}\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{e}) + \frac{H}{W}\Omega^{-1}\boldsymbol{\phi} \tag{38}$$

  where $\boldsymbol{\omega}^* = (\omega_1^* \ldots \omega_n^*)'$ is the $n \times 1$ vector of portfolio weights for the n risky assets; $\boldsymbol{\mu} = (\mu_1 \ldots \mu_n)'$ is the $n \times 1$ vector of these assets' expected rates of return; $\mathbf{e}$ is an n-dimensional vector of ones, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)'$, $A = -\frac{J_W}{J_{WW}}$, and $H = -\frac{J_{Wx}}{J_{WW}}$.

- A and H will, in general, differ from one individual to another, depending on the form of the particular individual's utility function and level of wealth.

Three Fund Theorem

- Thus, unlike in the constant investment opportunity set case (where $J_{Wx} = H = 0$), ω_i^* / ω_j^* is not the same for all investors.
- A *two mutual fund theorem* does *not* hold, but with one state variable, x, a *three fund theorem* does hold.
- Investors will be satisfied choosing between
  1. A fund holding the risk-free asset.
  2. A mean-variance efficient fund with weights $\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{e})$.
  3. A fund with weights $\Omega^{-1}\phi$ that best hedges against changing investment opportunities.

Portfolio Demands

- Recall $J_W = U_C$, which allows us to write $J_{WW} = U_{CC}\partial C/\partial W$.
- Therefore, A can be rewritten as

$$A = -\frac{U_C}{U_{CC}\left(\partial C/\partial W\right)} > 0 \tag{39}$$

by the concavity of U. Also, since $J_{Wx} = U_{CC}\partial C/\partial x$,

$$H = -\frac{\partial C/\partial x}{\partial C/\partial W} \gtreqless 0 \tag{40}$$

- A is proportional to the reciprocal of the individual's absolute risk aversion, so the smaller is A, the smaller in magnitude is the individual's demand for any risky asset.

Hedging Demand

- An unfavorable shift in investment opportunities is defined as a change in x such that consumption falls, that is, an increase in x if $\partial C/\partial x < 0$ and a decrease in x if $\partial C/\partial x > 0$.
- For example, suppose Ω is a diagonal matrix, so that $\upsilon_{ij} = 0$ for $i \neq j$ and $\upsilon_{ii} = 1/\sigma_{ii} > 0$, and also assume that $\phi_i \neq 0$. In this case, the hedging demand for risky asset i in (38) is

$$H\upsilon_{ii}\phi_i = -\frac{\partial C/\partial x}{\partial C/\partial W}\upsilon_{ii}\phi_i > 0 \text{ iff } \frac{\partial C}{\partial x}\phi_i < 0 \qquad (41)$$

- Thus, if $\partial C/\partial x < 0$) and if x and asset i are positively correlated ($\phi_i > 0$), then there is a positive hedging demand for asset i; that is, $H\upsilon_{ii}\phi_i > 0$ and asset i is held in greater amounts than what would be predicted based on a simple single-period mean-variance analysis.

Changing Interest Rate Example

- Let $r = x$ and $\boldsymbol{\mu} = r\mathbf{e} + \mathbf{p} = x\mathbf{e} + \mathbf{p}$ where $\mathbf{p}$ is a vector of risk premia for the risky assets.
- Thus, an increase in the risk-free rate r indicates an improvement in investment opportunities.
- Recall that in a simple certainty model with constant relative-risk-aversion utility, the elasticity of intertemporal substitution is given by $\epsilon = 1/\left(1-\gamma\right)$.
- When $\epsilon < 1$, implying that $\gamma < 0$, an increase in the risk-free rate leads to greater current consumption consistent with equation (34) where, for the infinite horizon case $C_t = \frac{\gamma}{1-\gamma}\left[\frac{\rho}{\gamma} - r - \frac{(\mu-r)^2}{2(1-\gamma)\sigma^2}\right] W_t = \frac{\gamma}{1-\gamma}\left[\frac{\rho}{\gamma} - r - \frac{\mathbf{p}^2}{2(1-\gamma)\sigma^2}\right] W_t$, so that $\partial C_t / \partial r = -\gamma W_t / \left(1-\gamma\right)$.

Asset Allocation Puzzle

- Given empirical evidence that risk aversion is greater than log ($\gamma < 0$), the intuition from these simple models would be that $\partial C_t / \partial r > 0$ and is increasing in risk aversion.
- From equation (41) we have

$$H\upsilon_{ii}\phi_i = -\frac{\partial C/\partial r}{\partial C/\partial W}\upsilon_{ii}\phi_i > 0 \text{ iff } \frac{\partial C}{\partial r}\phi_i < 0 \qquad (42)$$

- Thus, there is a positive hedging demand for an asset that is negatively correlated with changes in the interest rate, r.
- An obvious candidate asset is a long-maturity bond.
- This insight can explain why financial planners recommend both greater cash and a greater bonds-to-stocks mix for more risk-averse investors (the Asset Allocation Puzzle of Canner, Mankiw, and Weil AER 1997).

Log Utility

- Logarithmic utility is one of the few cases in which analytical solutions are possible for consumption and portfolio choices when investment opportunities are changing.
- Suppose $U(C_s, s) = e^{-\rho s} \ln(C_s)$ and $B(W_T, T) = e^{-\rho T} \ln(W_T)$.
- Consider a trial solution to (17) for the indirect utility function of the form $J(W, x, t) = d(t)\, U(W_t, t) + F(x, t)$ $= d(t)\, e^{-\rho t} \ln(W_t) + F(x, t)$.
- If so, then (20) is

$$C_t^* = \frac{W_t}{d(t)} \tag{43}$$

and (37) simplifies to

$$\omega_i^* = \sum_{j=1}^{n} \upsilon_{ij} \left(\mu_j - r \right) \tag{44}$$

Log Utility

- Substituting C_t^* and ω_i^* into the Bellman equation (17):

$$\begin{aligned} 0 &= U\left(C_t^*, t\right) + J_t + J_W\left[rW_t - C_t^*\right] + a\left(x,t\right)J_x \\ &\quad + \frac{1}{2}b\left(x,t\right)^2 J_{xx} - \frac{J_W^2}{2J_{WW}}\sum_{i=1}^{n}\sum_{j=1}^{n}\upsilon_{ij}\left(\mu_j - r\right)\left(\mu_i - r\right) \\ &= e^{-\rho t}\ln\left[\frac{W_t}{d\left(t\right)}\right] + e^{-\rho t}\left[\frac{\partial d\left(t\right)}{\partial t} - \rho d\left(t\right)\right]\ln\left[W_t\right] + F_t \\ &\quad + e^{-\rho t}d\left(t\right)r - e^{-\rho t} + a\left(x,t\right)F_x + \frac{1}{2}b\left(x,t\right)^2 F_{xx} \\ &\quad + \frac{d\left(t\right)e^{-\rho t}}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\upsilon_{ij}\left(\mu_j - r\right)\left(\mu_i - r\right) \end{aligned} \tag{45}$$

Log Utility

- Simplifying, the equation becomes

$$
\begin{aligned}
0 \quad = \quad & -\ln\left[d\left(t\right)\right]+\left[1+\frac{\partial d\left(t\right)}{\partial t}-\rho d\left(t\right)\right]\ln\left[W_t\right]+e^{\rho t}F_t \\
& +d\left(t\right)r-1+a\left(x,t\right)e^{\rho t}F_x+\frac{1}{2}b\left(x,t\right)^2 e^{\rho t}F_{xx} \\
& +\frac{d\left(t\right)}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\upsilon_{ij}\left(\mu_j-r\right)\left(\mu_i-r\right)
\end{aligned}
\tag{46}
$$

Log Utility

- Since a solution must hold for all values of wealth, we must have

$$\frac{\partial d\left(t\right)}{\partial t}-\rho d\left(t\right)+1=0 \tag{47}$$

subject to the boundary condition $d\left(T\right)=1$.

- The solution to this first-order ordinary differential equation is

$$d\left(t\right)=\frac{1}{\rho}\left[1-\left(1-\rho\right)e^{-\rho\left(T-t\right)}\right] \tag{48}$$

Log Utility

- The complete solution to (46) is then to solve

$$
\begin{aligned}
0 &= -\ln\left[d(t)\right] + e^{\rho t}F_t + d(t)\,r - 1 + a(x,t)\,e^{\rho t}F_x \qquad (49)\\
&\quad +\frac{1}{2}b(x,t)^2\,e^{\rho t}F_{xx} + \frac{d(t)}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\upsilon_{ij}\left(\mu_j - r\right)\left(\mu_i - r\right)
\end{aligned}
$$

  subject to the boundary condition $F(x,T) = 0$.
- The solution depends on how r, the μ_i's, and Ω are assumed to depend on the state variable x.
- However, these relationships influence only the level of indirect utility via $F(x,t)$ and do not affect C_t^* and ω_i^*.

Log Utility

- Substituting (48) into (43), consumption is

$$C_t = \frac{\rho}{1-(1-\rho)\,e^{-\rho(T-t)}} W_t \tag{50}$$

which is comparable to our earlier discrete-time problem.

- The log utility investor behaves myopically by having no desire to hedge against changes in investment opportunities, though the portfolio weights $\omega_i^* = \sum_{j=1}^{n} \upsilon_{ij} \left(\mu_j - r\right)$ will change over time as υ_{ij}, μ_j, and$r$ change.

The Martingale Approach

- Modify process (1) to write the return on risky i as

$$dS_i/S_i = \mu_i dt + \Sigma_i \mathbf{dZ}, \quad i = 1, ..., n \tag{51}$$

where $\Sigma_i = (\sigma_{i1}...\sigma_{in})$ is a $1 \times n$ vector of volatility terms and $\mathbf{dZ} = (dz_1...dz_n)'$ is an $n \times 1$ vector of independent Brownian motions.

- μ_i, Σ_i, and $r(t)$ may be functions of state variables driven by the Brownian motion elements of $\mathbf{dZ}$.
- If Σ is the $n \times n$ matrix whose i^{th} row equals Σ_i, then the covariance matrix of the assets' returns is $\Omega \equiv \Sigma\Sigma'$.

Complete Market Assumptions

- Importantly, we now assume that uncertain changes in the means and covariances of the asset return processes in (51) are driven only by the vector $\mathbf{dZ}$.
- Equivalently, each state variable, say x_i as represented in (2), has a Brownian motion process, $d\zeta_i$, that is a linear function of $\mathbf{dZ}$.
- Thus, changes in investment opportunities can be *perfectly* hedged by the n assets so that markets are dynamically complete.

Pricing Kernel

- Using a Black-Scholes hedging argument and the absence of arbitrage, we showed that a stochastic discount factor exists and follows the process

$$dM/M = -rdt - \Theta(t)' \mathbf{dZ} \tag{52}$$

where $\Theta = (\theta_1 ... \theta_n)'$ is an $n \times 1$ vector of market prices of risks associated with each Brownian motion and

$$\mu_i - r = \Sigma_i \Theta, \quad i = 1, ..., n \tag{53}$$

Optimal Consumption Plan

- Note that the individual's wealth equals the expected discounted value of the dividends (consumption) that it pays over the individual's planning horizon plus discounted terminal wealth

$$W_t = E_t \left[\int_t^T \frac{M_s}{M_t} C_s ds + \frac{M_T}{M_t} W_T \right] \tag{54}$$

- Equation (54) can be interpreted as an intertemporal budget constraint.

Static Optimization Problem

- The choice of consumption and terminal wealth can be transformed into a static optimization problem by the following Lagrange multiplier problem:

$$\max_{C_s \forall s \in [t,T], W_T} E_t \left[\int_t^T U(C_s, s)\, ds + B(W_T, T) \right] + \lambda \left(M_t W_t - E_t \left[\int_t^T M_s C_s ds + M_T W_T \right] \right) \tag{55}$$

- Later, we address the portfolio choice problem that would implement the consumption plan.

First Order Conditions

- Treating the integrals in (55) as summations over infinite points in time, the first-order conditions for optimal consumption at each date and for terminal wealth are

$$\frac{\partial U\left(C_s, s\right)}{\partial C_s} = \lambda M_s, \quad \forall s \in [t, T] \tag{56}$$

$$\frac{\partial B\left(W_T, T\right)}{\partial W_T} = \lambda M_T \tag{57}$$

- Define the inverse functions $G = [\partial U / \partial C]^{-1}$ and $G_B = [\partial B / \partial W]^{-1}$:

$$C_s^* = G\left(\lambda M_s, s\right), \quad \forall s \in [t, T] \tag{58}$$

$$W_T^* = G_B\left(\lambda M_T, T\right) \tag{59}$$

Determining the Lagrange multiplier

- Substitute (58) and (59) into (54) to obtain

$$W_t = E_t \left[\int_t^T \frac{M_s}{M_t} G\left(\lambda M_s, s\right) ds + \frac{M_T}{M_t} G_B\left(\lambda M_T, T\right) \right] \quad (60)$$

- Given the initial wealth, W_t, the distribution of M_s from (52), and the forms of the utility and bequest functions (which determine G and G_B), the expectation in equation (60) can be calculated to determine λ as a function of W_t, M_t, and any date t state variables.

Alternative Solution for the Multiplier

- Since W_t represents a contingent claim that pays a dividend equal to consumption, it must satisfy a particular Black-Scholes-Merton partial differential equation (PDE).
- For example, assume that μ_i, Σ_i, and $r(t)$ are functions of a single state variable, say, x_t, that follows the process

$$dx = a(x,t)\,dt + \mathbf{B}(x,t)'\,\mathbf{dZ} \tag{61}$$

  where $\mathbf{B}(x,t) = (B_1 ... B_n)'$ is an $n \times 1$ vector of volatilities multiplying the Brownian motion components of $\mathbf{dZ}$.
- Based on (60) and the Markov nature of M_t in (52) and x_t in (61), the date t value of optimally invested wealth is a function of M_t and x_t and the individual's time horizon, $W(M_t, x_t, t)$.

Wealth Process

- By Itô's lemma, $W(M_t, x_t, t)$ follows the process

$$\begin{aligned} dW &= W_M dM + W_x dx + \frac{\partial W}{\partial t} dt + \frac{1}{2} W_{MM} (dM)^2 \\ &\quad + W_{Mx} (dM)(dx) + \frac{1}{2} W_{xx} (dx)^2 \\ &= \mu_W dt + \Sigma_W' \mathbf{dZ} \end{aligned} \tag{62}$$

where

$$\begin{aligned} \mu_W &\equiv -rMW_M + aW_x + \frac{\partial W}{\partial t} + \frac{1}{2}\Theta'\Theta M^2 W_{MM} \\ &\quad -\Theta'\mathbf{B} MW_{Mx} + \frac{1}{2}\mathbf{B}'\mathbf{B} W_{xx} \end{aligned} \tag{63}$$

$$\Sigma_W \equiv -W_M M\Theta + W_x \mathbf{B} \tag{64}$$

No Arbitrage Condition for Wealth

- The expected return on wealth must earn the instantaneous risk-free rate plus its risk premium:

$$\mu_W + G(\lambda M_t, t) = rW_t + \Sigma_W' \Theta \tag{65}$$

- Substituting in for μ_W and Σ_W' leads to the PDE

$$\begin{aligned} 0 \;=\; & \Theta'\Theta M^2 \frac{W_{MM}}{2} - \Theta' \mathbf{B} M W_{Mx} + \mathbf{B}'\mathbf{B} \frac{W_{xx}}{2} + \left(\Theta'\Theta - r\right) M W_M \\ & + \left(a - \mathbf{B}'\Theta\right) W_x + \frac{\partial W}{\partial t} + G(\lambda M_t, t) - rW \end{aligned} \tag{66}$$

which is solved subject to the boundary condition $W(M_T, x_T, T) = G_B(\lambda M_T, T)$.

Solution for Consumption

- Either equation (60) or (66) leads to the solution $W(M_t, x_t, t; \lambda) = W_t$ that determines λ as a function of W_t , M_t, and x_t.
- The solution for λ is then be substituted into (58) and (59) to obtain $C_s^*(M_s)$ and $W_T^*(M_T)$.
- When the individual follows this optimal policy, it is time consistent in the sense that should the individual resolve the optimal consumption problem at some future date, say, $s > t$, the computed value of λ will be the same as that derived at date t.

Portfolio Allocation

- Market completeness permits replication of the individual's optimal process for wealth and its consumption dividend.
- The individual's wealth follows the process

$$dW = \boldsymbol{\omega}' (\boldsymbol{\mu} - r\mathbf{e}) W\,dt + (rW - C_t)\,dt + W\boldsymbol{\omega}'\Sigma\mathbf{dZ} \quad (67)$$

  where $\boldsymbol{\omega} = (\omega_1 ... \omega_n)'$ are portfolio weights and $\boldsymbol{\mu} = (\mu_1 ... \mu_n)'$ are assets' expected rates of return.
- Equating the coefficients of wealth's Brownian motions in (67) and (62) implies $W\boldsymbol{\omega}'\Sigma = \Sigma_W'$.
- Substituting in (64) for Σ_W and rearranging:

$$\boldsymbol{\omega} = -\frac{MW_M}{W}\Sigma'^{-1}\Theta + \frac{W_x}{W}\Sigma'^{-1}\mathbf{B} \quad (68)$$

Optimal Portfolio Weights

- The no-arbitrage condition (53) in matrix form is

$$\boldsymbol{\mu} - r\mathbf{e} = \Sigma\Theta \tag{69}$$

- Using (69) to substitute for Θ, equation (68) is

$$\begin{aligned} \boldsymbol{\omega} &= -\frac{MW_M}{W}\Sigma^{-1}\Sigma'^{-1}\left(\boldsymbol{\mu} - r\mathbf{e}\right) + \frac{W_x}{W}\Sigma'^{-1}\mathbf{B} \\ &= -\frac{MW_M}{W}\Omega^{-1}\left(\boldsymbol{\mu} - r\mathbf{e}\right) + \frac{W_x}{W}\Sigma'^{-1}\mathbf{B} \end{aligned} \tag{70}$$

- A comparison to (38) for the case of perfect correlation between assets and state variables shows that $MW_M = J_W / J_{WW}$ and $W_x = -J_{Wx}/J_{WW}$.
- Given $W(M, x, t)$ in (60) or (66), the solution is complete.

Example of Wachter JFQA (2002)

- Let there be a risk-free asset with contant rate of return $r > 0$, and a single risky asset with price process

$$dS/S = \mu(t)\,dt + \sigma dz \tag{71}$$

- Volatility, σ, is constant but the market price of risk, $\theta(t) = [\mu(t) - r]/\sigma$, satisfies the Ornstein-Uhlenbeck process

$$d\theta = a\left(\overline{\theta} - \theta\right) dt - bdz \tag{72}$$

  where a, $\overline{\theta}$, and b are positive constants.

- Since $\mu(t) = r + \theta(t)\sigma$ so that $d\mu = \sigma d\theta$, the expected rate of return is lower (*higher*) after its realized return has been high (*low*).

Individual's Expected Utility

- With CRRA and a zero bequest, (55) is

$$\max_{C_s \forall s \in [t,T]} E_t \left[\int_t^T e^{-\rho s} \frac{C^\gamma}{\gamma} \, ds \right] + \lambda \left(M_t W_t - E_t \left[\int_t^T M_s C_s ds \right] \right) \tag{73}$$

- The first-order condition (58) is

$$C_s^* = e^{-\frac{\rho s}{1-\gamma}} \left(\lambda M_s \right)^{-\frac{1}{1-\gamma}}, \quad \forall s \in [t, T] \tag{74}$$

so that (60) is

$$\begin{aligned} W_t &= E_t \left[\int_t^T \frac{M_s}{M_t} e^{-\frac{\rho s}{1-\gamma}} \left(\lambda M_s \right)^{-\frac{1}{1-\gamma}} ds \right] \qquad (75) \\ &= \lambda^{-\frac{1}{1-\gamma}} M_t^{-1} \int_t^T e^{-\frac{\rho s}{1-\gamma}} E_t \left[M_s^{-\frac{\gamma}{1-\gamma}} \right] ds \end{aligned}$$

Wealth and the Pricing Kernel

- $E_t\left[M_s^{-\frac{\gamma}{1-\gamma}}\right]$ could be computed by noting that $dM/M = -rdt - \theta dz$ and θ follows the process in (72).
- Alternatively, W_t can be solved using PDE (66):

$$\begin{aligned} 0 &= \frac{1}{2}\theta^2 M^2 W_{MM} + \theta b M W_{M\theta} + \frac{1}{2} b^2 W_{\theta\theta} + \left(\theta^2 - r\right) M W_M \\ &\quad + \left[a\left(\overline{\theta} - \theta\right) + b\theta\right] W_\theta + \frac{\partial W}{\partial t} + e^{-\frac{\rho t}{1-\gamma}} \left(\lambda M_t\right)^{-\frac{1}{1-\gamma}} - rW \end{aligned} \tag{76}$$

subject to boundary condition $W\left(M_T, \theta_T, T\right) = 0$.

Solution

- When $\gamma < 0$, so the individual is more risk averse than log utility, the solution to (76) is

$$W_t = (\lambda M_t)^{-\frac{1}{1-\gamma}} e^{-\frac{\rho t}{1-\gamma}} \int_0^{T-t} H(\theta_t, \tau)\, d\tau \tag{77}$$

where $H(\theta_t, \tau)$ is the exponential of a quadratic function of θ_t given by

$$H(\theta_t, \tau) \equiv e^{\frac{1}{1-\gamma}\left[A_1(\tau)\frac{\theta_t^2}{2} + A_2(\tau)\theta_t + A_3(\tau)\right]} \tag{78}$$

Solution – continued

and where

$$
\begin{aligned}
A_1(\tau) &\equiv \frac{2c_1\left(1-e^{-c_3\tau}\right)}{2c_3-(c_2+c_3)\left(1-e^{-c_3\tau}\right)} \\
A_2(\tau) &\equiv \frac{4c_1 a\bar{\theta}\left(1-e^{-c_3\tau/2}\right)^2}{c_3\left[2c_3-(c_2+c_3)\left(1-e^{-c_3\tau}\right)\right]} \\
A_3(\tau) &\equiv \int_0^\tau \left[\frac{b^2A_2^2(s)}{2(1-\gamma)}+\frac{b^2A_1(s)}{2}+a\bar{\theta}A_2(s)+\gamma r-\rho\right]ds
\end{aligned}
$$

with $c_1 \equiv \gamma/(1-\gamma)$, $c_2 \equiv -2(a+c_1 b)$, and $c_3 \equiv \sqrt{c_2^2-4c_1 b^2/(1-\gamma)}$.

Optimal Consumption

- Equation (77) can be inverted to solve for λ, but since from (74) $(\lambda M_t)^{-\frac{1}{1-\gamma}} e^{-\frac{\rho t}{1-\gamma}} = C_t^*$, (77) can be rewritten

$$C_t^* = \frac{W_t}{\int_0^{T-t} H(\theta_t, \tau)\, d\tau} \tag{79}$$

- Note that wealth equals the value of consumption from now until $T - t$ periods into the future.
- Therefore, since $\int_0^{T-t} H(\theta_t, \tau)\, d\tau = W_t / C_t^*$, the function $H(\theta_t, \tau)$ equals the value of consumption τ periods in the future scaled by current consumption.

Consumption Implications

- When $\gamma < 0$ and $\theta_t > 0$, so that $\mu(t) - r > 0$, then $\partial\left(C_t^*/W_t\right)/\partial\theta_t > 0$; that is, the individual consumes a greater proportion of wealth the larger is the risky asset's excess rate of return.

- This is what one expects given our earlier analysis showing that the "income" effect dominates the "substitution" effect when risk aversion is greater than that of log utility.

Portfolio Choice

- The weight (70) for a single risky asset is

$$\omega = -\frac{MW_M}{W}\frac{\mu(t) - r}{\sigma^2} - \frac{W_\theta}{W}\frac{b}{\sigma} \tag{80}$$

- Using (77), $-MW_M/W = 1/(1-\gamma)$ and W_θ can be computed. Substituting these two derivatives into (80) gives

$$\begin{aligned} \omega &= \frac{\mu(t) - r}{(1-\gamma)\sigma^2} - \frac{b\int_0^{T-t} H(\theta_t, \tau)\left[A_1(\tau)\theta_t + A_2(\tau)\right]d\tau}{(1-\gamma)\sigma\int_0^{T-t} H(\theta_t, \tau)\,d\tau} \\ &= \frac{\mu(t) - r}{(1-\gamma)\sigma^2} \\ &\quad -\frac{b}{(1-\gamma)\sigma}\int_0^{T-t}\frac{H(\theta_t, \tau)}{\int_0^{T-t} H(\theta_t, \tau')\,d\tau'}\left[A_1(\tau)\theta_t + A_2(\tau)\right]d\tau \end{aligned} \tag{81}$$

Portfolio Implications

- The first term of (81) is the mean-variance efficient portfolio.
- The second term is the hedging demand.
- $A_1(\tau)$ and $A_2(\tau)$ are negative when $\gamma < 0$, so that if $\theta_t > 0$, the term $[A_1(\tau)\theta_t + A_2(\tau)]$ is unambiguously negative and, therefore, the hedging demand is positive.
- Hence, individuals more risk averse than log invest more wealth in the risky asset than if investment opportunities were constant.
- Because of negative correlation between risky-asset returns and future investment opportunities, overweighting in the risky asset means that unexpectedly good returns today hedge against returns that are expected to be poorer tomorrow.

Summary

- We considered an individual's continuous-time consumption and portfolio choice problem when asset returns followed diffusion processes.
- With constant investment opportunities, asset returns are lognormally distributed and optimal portfolio weights are similar to those of the single-period mean-variance model.
- With changing investment opportunities, optimal portfolio weights reflect demand components that seek to hedge against changing investment opportunities.
- The Martingale Approach to solving for an individual's optimal consumption and portfolio choices is applicable to a complete markets setting where asset returns can perfectly hedge against changes in investment opportunities.