Consumption-Savings Decisions and State Pricing

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Introduction

- We now consider a consumption-savings decision along with the previous portfolio choice decision.
- These decisions imply a *stochastic discount factor* (SDF) based on marginal utilities of consumption at different dates.
- This SDF can value all traded assets and can bound assets' expected returns and volatilities.
- The SDF can also be derived by assuming *market completeness* and no arbitrage.
- We can modify the SDF to value assets using *risk-neutral probabilities*.

Consumption and Porftolio Choices

- Let W_0 and C_0 be an individual's initial date 0 wealth and consumption, respectively. At date 1, the individual consumes all of his wealth C_1.
- The individual's utility function is:

$$U(C_0) + \delta E\left[U\left(\widetilde{C}_1\right)\right] \tag{1}$$

where $\delta = \frac{1}{1+\rho}$ is a subjective discount factor. A rate of time preference $\rho > 0$ reflects impatience for consuming early.

- There are n assets where P_i is the date 0 price per share and X_i is the date 1 random payoff of asset i, $i = 1, ..., n$. Hence $R_i \equiv X_i / P_i$ is asset i's random return.

Consumption and Porftolio Choices cont'd

- The individual receives labor income of y_0 at date 0 and random labor income of y_1 at date 1.
- Let ω_i be the proportion of date 0 savings invested in asset i. Then the individual's intertemporal budget constraint is

$$C_1 = y_1 + (W_0 + y_0 - C_0) \sum_{i=1}^{n} \omega_i R_i \tag{2}$$

where $(W_0 + y_0 - C_0)$ is date 0 savings. The individual's maximization problem is

$$\max_{C_0, \{\omega_i\}} U(C_0) + \delta E[U(C_1)] \tag{3}$$

subject to $\sum_{i=1}^{n} \omega_i = 1$. Substituting in (2), the first-order conditions wrt C_0 and ω_i, $i = 1, ..., n$ are

Consumption and Porftolio Choices cont'd

$$U'\left(C_0\right) - \delta E\left[U'\left(C_1\right)\textstyle\sum_{i=1}^{n}\omega_i R_i\right] = 0 \tag{4}$$

$$\delta E\left[U'\left(C_1\right)R_i\right] - \lambda = 0, \quad i = 1, ..., n \tag{5}$$

where $\lambda \equiv \lambda' / \left(W_0 + y_0 - C_0\right)$ and λ' is the Lagrange multiplier for the constraint $\sum_{i=1}^{n}\omega_i = 1$.

- From (5), for any two assets i and j:

$$E\left[U'\left(C_1\right)R_i\right] = E\left[U'\left(C_1\right)R_j\right] \tag{6}$$

- Equation (6) implies that the investor's optimal portfolio choices are such that the expected marginal utility-weighted returns of any two assets are equal.

Consumption and Porftolio Choices cont'd

- To examine the optimal intertemporal allocation of resources, substitute (5) into (4)

$$\begin{aligned} U'(C_0) &= \delta E\left[U'(C_1)\textstyle\sum_{i=1}^n \omega_i R_i\right] = \textstyle\sum_{i=1}^n \omega_i \delta E\left[U'(C_1) R_i\right] \\ &= \textstyle\sum_{i=1}^n \omega_i \lambda = \lambda \end{aligned} \tag{7}$$

- Then substituting $\lambda = U'(C_0)$ into (5) gives

$$\delta E\left[U'(C_1) R_i\right] = U'(C_0), \quad i = 1, ..., n \tag{8}$$

or, since $R_i = X_i / P_i$,

$$P_i U'(C_0) = \delta E\left[U'(C_1) X_i\right], \quad i = 1, ..., n \tag{9}$$

Consumption and Porftolio Choices cont'd

- Equation (9) implies the individual invests in asset i until the marginal utility of giving up P_i dollars at date 0 just equals the expected marginal utility of receiving X_i at date 1.
- Equation (9) for a risk-free asset that pays R_f (gross return) is

$$U'\left(C_0\right) = R_f \delta E\left[U'\left(C_1\right)\right] \tag{10}$$

- With CRRA utility $U\left(C\right) = C^{\gamma}/\gamma$, for $\gamma < 1$, equation (10) is

$$\frac{1}{R_f} = \delta E\left[\left(\frac{C_0}{C_1}\right)^{1-\gamma}\right] \tag{11}$$

  implying that when the interest rate is high, so is the expected growth in consumption.

Consumption and Porftolio Choices cont'd

- If there is only a risk-free asset and nonrandom labor income, so that C_1 is nonstochastic, equation (11) is

$$R_f = \frac{1}{\delta}\left(\frac{C_1}{C_0}\right)^{1-\gamma} \tag{12}$$

- Note that

$$\begin{aligned} \frac{\partial R_f}{\partial \frac{C_1}{C_0}} &= \frac{1-\gamma}{\delta}\left(\frac{C_1}{C_0}\right)^{-\gamma} \\ &= (1-\gamma)\frac{R_f}{\frac{C_1}{C_0}} \end{aligned} \tag{13}$$

Intertemporal Elasticity

- So that the intertemporal elasticity of substitution is

$$\epsilon \equiv \frac{R_f}{\frac{C_1}{C_0}} \frac{\partial \frac{C_1}{C_0}}{\partial R_f} = \frac{\partial \ln \left(C_1 / C_0 \right)}{\partial \ln \left(R_f \right)} = \frac{1}{1-\gamma} \tag{14}$$

- Thus for CRRA utility, ϵ is the reciprocal of the coefficient of relative risk aversion. When $0 < \gamma < 1$, ϵ exceeds unity and a higher interest rate raises second-period consumption more than one-for-one.
- Conversely, when $\gamma < 0$, then $\epsilon < 1$ and a higher interest rate raises second-period consumption less than one-for-one, implying a decrease in initial savings.

Intertemporal Elasticity cont'd

- The individual's response reflects two effects from an increase in interest rates.
  1. A *substitution effect* raises the return from transforming current consumption into future consumption, providing an incentive to save more.
  2. An *income effect* from the higher return on a given amount of savings makes the individual better off and, ceteris paribus, would raise consumption in both periods.
- For $\epsilon > 1$, the substitution effect outweighs the income effect, while the reverse occurs when $\epsilon < 1$. When $\epsilon = 1$, the income and substitution effects exactly offset each other.

Equilibrium Asset Pricing Implications

- The individual's consumption - portfolio choice has asset pricing implications. Rewrite equation (9):

$$\begin{aligned} P_i &= E\left[\frac{\delta U'(C_1)}{U'(C_0)} X_i\right] \\ &= E\left[m_{01} X_i\right] \end{aligned} \tag{15}$$

  where $m_{01} \equiv \delta U'(C_1)/U'(C_0)$ is the *stochastic discount factor* or *state price deflator* for valuing asset returns.

- In states of nature where C_1 is high (due to high portfolio returns or high labor income), marginal utility, $U'(C_1)$, is low and an asset's payoffs are not highly valued.
- Conversely, in states where C_1 is low, marginal utility is high and an asset's payoffs are much desired.

Stochastic Discount Factor

- The SDF or "*pricing kernel*" may differ across investors due to differences in random labor income that causes the distribution of C_1, and hence $\delta U'(C_1)/U'(C_0)$, to differ.
- Nonetheless, $E[m_{01}X_i] = E[\delta U'(C_1)X_i/U'(C_0)]$ is the same for all investors who can trade in asset i since individuals adjust their portfolios to hedge individual-specific risks, and differences in $\delta U'(C_1)/U'(C_0)$ reflect only risks uncorrelated with asset returns.
- Utility depends on real consumption, C_1. If P_i^N and X_i^N are the initial price and end-of-period payoff measured in currency units (nominal terms), they need to be deflated by a price index to convert them to real quantities.

Real Pricing Kernel

- Let CPI_t be the consumer price index at date t. Equation (15) becomes

$$\frac{P_i^N}{CPI_0} = E\left[\frac{\delta U'(C_1)}{U'(C_0)}\frac{X_i^N}{CPI_1}\right] \tag{16}$$

- If we define $I_{ts} = CPI_s / CPI_t$ as 1 plus the inflation rate between dates t and s, equation (16) is

$$\begin{aligned} P_i^N &= E\left[\frac{1}{I_{01}}\frac{\delta U'(C_1)}{U'(C_0)}X_i^N\right] \\ &= E\left[M_{01}X_i^N\right] \end{aligned} \tag{17}$$

  where $M_{01} \equiv (\delta / I_{01})\, U'(C_1) / U'(C_0)$ is the SDF for nominal returns, equal to the real pricing kernel, m_{01}, discounted at the (random) rate of inflation between dates 0 and 1.

Risk Premia and Marginal Utility of Consumption

- Equation (15) can be rewritten to shed light on an asset's risk premium. Divide each side of (15) by P_i:

$$\begin{aligned} 1 &= E\left[m_{01} R_i\right] && (18) \\ &= E\left[m_{01}\right] E\left[R_i\right] + Cov\left[m_{01}, R_i\right] \\ &= E\left[m_{01}\right] \left(E\left[R_i\right] + \frac{Cov\left[m_{01}, R_i\right]}{E\left[m_{01}\right]} \right) \end{aligned}$$

- Recall from (10) that for the case of a risk-free asset, $E\left[\delta U'\left(C_1\right)/U'\left(C_0\right)\right] = E\left[m_{01}\right] = 1/R_f$. Then (18) can be rewritten

$$R_f = E\left[R_i\right] + \frac{Cov\left[m_{01}, R_i\right]}{E\left[m_{01}\right]} \tag{19}$$

or

Risk Premia and Marginal Utility of Consumption cont'd

$$
\begin{aligned}
E\left[R_i\right] &= R_f - \frac{Cov\left[m_{01}, R_i\right]}{E\left[m_{01}\right]} \\
&= R_f - \frac{Cov\left[U'\left(C_1\right), R_i\right]}{E\left[U'\left(C_1\right)\right]}
\end{aligned}
\tag{20}
$$

- An asset that tends to pay high returns when consumption is high (*low*) has $Cov\left[U'\left(C_1\right), R_i\right] < 0$ ($Cov\left[U'\left(C_1\right), R_i\right] > 0$) and will have an expected return greater (*less*) than the risk-free rate.
- Investors are satisfied with negative risk premia when assets hedge against low consumption states of the world.

Relationship to the CAPM

- Suppose there is a portfolio with a random return of $\widetilde{R}_m$ that is perfectly negatively correlated with the marginal utility of date 1 consumption, $U'\left(\widetilde{C}_1\right)$, so that it is also perfectly negatively correlated with m_{01}:

$$U'(\tilde{C}_1) = -\kappa \widetilde{R}_m, \quad \kappa > 0 \tag{21}$$

- Then this implies

$$Cov[U'(C_1),\, R_m] = -\kappa\, Cov[R_m, R_m] = -\,\kappa\, Var[R_m] \tag{22}$$

and

$$Cov[U'(C_1),\, R_i] = -\kappa\, Cov[R_m,\, R_i] \tag{23}$$

Relationship to the CAPM cont'd

- From (20), the risk premium on this portfolio is

$$E[R_m] = R_f - \frac{Cov[U'(C_1),\, R_m]}{E[U'(C_1)]} = R_f + \frac{\kappa\, Var[R_m]}{E[U'(C_1)]} \quad (24)$$

- Using (20) and (24) to substitute for $E[U'(C_1)]$, and using (23), we obtain

$$\frac{E[R_m] - R_f}{E[R_i] - R_f} = \frac{\kappa\, Var[R_m]}{\kappa Cov[R_m,\, R_i]} \quad (25)$$

and rearranging:

$$E[R_i] - R_f = \frac{Cov[R_m, R_i]}{Var[R_m]}\left(E[R_m] - R_f\right) \quad (26)$$

Relationship to the CAPM cont'd

- Equation (26) is the CAPM relation

$$E[R_i] = R_f + \beta_i \left(E[R_m] - R_f \right) \tag{27}$$

- Note that under CAPM assumptions the market portfolio is perfectly negatively correlated with consumption:
  1. There is no wage income, so end of period consumption derives only from asset portfolio returns.
  2. With a risk-free asset and normally distributed asset returns, everyone holds the same risky asset (market) portfolio.
- Hence, the only risk to C_1 is the return on the market portfolio.

Bounds on Risk Premia

- $m_{01} \equiv \delta U'(C_1)/U'(C_0)$ places a bound on the Sharpe ratio of all assets. Rewrite equation (20) as

$$E[R_i] = R_f - \rho_{m_{01},R_i} \frac{\sigma_{m_{01}} \sigma_{R_i}}{E[m_{01}]} \tag{28}$$

  where $\sigma_{m_{01}}$, σ_{R_i}, and ρ_{m_{01},R_i} are the standard deviation of the discount factor, the standard deviation of the return on asset i, and the correlation between the discount factor and the return on asset i, respectively.

- Rearranging (28) leads to

$$\frac{E[R_i] - R_f}{\sigma_{R_i}} = -\rho_{m_{01},R_i} \frac{\sigma_{m_{01}}}{E[m_{01}]} \tag{29}$$

Hansen-Jagannathan Bounds

- Since $-1 \leq \rho_{m_{01},R_i} \leq 1$, we know that

$$\left| \frac{E\left[R_i\right] - R_f}{\sigma_{R_i}} \right| \leq \frac{\sigma_{m_{01}}}{E\left[m_{01}\right]} = \sigma_{m_{01}} R_f \tag{30}$$

- Equation (30) was derived by Robert Shiller (1982) and generalized by Hansen and Jagannathan (1991).
- If there exists a portfolio of assets whose return is perfectly negatively correlated with m_{01}, then (30) holds with equality. The CAPM implies such a situation, so that the slope of the capital market line, $S_e \equiv \frac{E[R_m]-R_f}{\sigma_{R_m}}$, equals $\sigma_{m_{01}} R_f$.

Ex: Bounds with Power Utility

- If $U(C) = C^{\gamma}/\gamma$ so $m_{01} \equiv \delta \left(C_1/C_0 \right)^{\gamma - 1} = \delta e^{(\gamma-1)\ln(C_1/C_0)}$ and C_1/C_0 is lognormal with parameters μ_c and σ_c, then

$$
\begin{aligned}
\frac{\sigma_{m_{01}}}{E[m_{01}]} &= \frac{\sqrt{Var\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]}}{E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]} \\
&= \frac{\sqrt{E\left[e^{2(\gamma-1)\ln(C_1/C_0)}\right] - E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]^2}}{E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]} \\
&= \sqrt{E\left[e^{2(\gamma-1)\ln(C_1/C_0)}\right] / E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]^2 - 1} \\
&= \sqrt{e^{2(\gamma-1)\mu_c + 2(\gamma-1)^2\sigma_c^2} / e^{2(\gamma-1)\mu_c + (\gamma-1)^2\sigma_c^2} - 1} = \sqrt{e^{(\gamma-1)^2\sigma_c^2} - 1} \\
&\approx \pm(\gamma-1)\sigma_c = (1-\gamma)\sigma_c
\end{aligned} \tag{31}
$$

The fourth line evaluates expectations assuming C_1 log-normality, $E(X) = e^{\mu + \frac{1}{2}\sigma^2}$.The fifth line takes a two-term approximation of the series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, which is reasonable for small positive x. The $(+)$ solution is negative for $\gamma < 1$.

Ex: Bounds with Power Utility

- Hence, with power utility and lognormal consumption:

$$\left| \frac{E\left[R_i\right] - R_f}{\sigma_{R_i}} \right| \leq (1 - \gamma)\, \sigma_c \tag{32}$$

- For the S&P500 over the last 75 years, $E\left[R_i\right] - R_f = 8.3\%$ and $\sigma_{R_i} = .17$, implying a Sharpe ratio of $\frac{E[R_i] - R_f}{\sigma_{R_i}} = 0.49$.
- U.S. per capita consumption data implies estimates of σ_c between 0.01 and 0.0386.
- Assuming (32) holds with equality for the S&P500, $\gamma = 1 - \left(\frac{E[R_i] - R_f}{\sigma_{R_i}}\right) / \sigma_c$ is between -11.7 and -48.
- Other empirical estimates of γ are -1 to -5. The inconsistency of theory and empirical evidence is what Mehra and Prescott (1985) termed the *equity premium puzzle*.

Ex. Bounds on R_f

- Even if high risk aversion is accepted, it implies an unreasonable value for the risk-free return, R_f. Note that

$$\begin{aligned} \frac{1}{R_f} &= E\left[m_{01}\right] \quad (33) \\ &= \delta E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right] \\ &= \delta e^{(\gamma-1)\mu_c+\frac{1}{2}(\gamma-1)^2\sigma_c^2} \end{aligned}$$

and therefore

$$\ln\left(R_f\right) = -\ln\left(\delta\right) + \left(1-\gamma\right)\mu_c - \frac{1}{2}\left(1-\gamma\right)^2\sigma_c^2 \quad (34)$$

- If we set $\delta = 0.99$, and $\mu_c = 0.018$, the historical average real growth of U.S. per capita consumption, then with $\gamma = -11$ and $\sigma_c = 0.036$ we obtain:

Ex. Bounds on R_f cont'd

$$\begin{aligned} \ln(R_f) &= -\ln(\delta) + (1-\gamma)\mu_c - \frac{1}{2}(1-\gamma)^2 \sigma_c^2 \\ &= 0.01 + 0.216 - 0.093 = 0.133 \end{aligned} \tag{35}$$

which is a real risk-free interest rate of 13.3 percent.

- Since short-term real interest rates have averaged about 1 percent in the U.S., we end up with a *risk-free rate puzzle*: the high γ results in an unreasonable R_f.
- So a SDF derived from the marginal utility of consumption doesn't fit the data. However, we can derive a SDF of the form $P_i = E_0[m_{01}X_i]$ using another approach.

Complete Markets Assumptions

- An alternative SDF derivation is based on the assumptions of a complete market and the absence of arbitrage.
- Suppose that an individual can freely trade in n assets and assume that there is a finite number, k, of end-of-period states of nature, with state s having probability π_s.
- Let X_{si} be the cashflow returned by one share (unit) of asset i in state s. Asset i's cashflows can be written as:

$$X_i = \begin{bmatrix} X_{1i} \\ \vdots \\ X_{ki} \end{bmatrix} \tag{36}$$

Complete Markets Assumptions cont'd

- Thus, the per-share cashflows of the universe of all assets can be represented by the $k \times n$ matrix

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{k1} & \cdots & X_{kn} \end{bmatrix} \tag{37}$$

- We will assume that $n = k$ and that X is of full rank, implying that the n assets *span* the k states of nature and the market is *complete*.
- An implication is that an individual can purchase amounts of the k assets that return target levels of end-of-period wealth in each of the states.

Complete Markets Assumptions cont'd

- To show this complete markets result, let W be an arbitrary $k \times 1$ vector of end-of-period levels of wealth:

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_k \end{bmatrix} \tag{38}$$

  where W_s is the level of wealth in state s.

- To obtain W, at the initial date the individual purchases shares in the k assets. Let the vector $N = [N_1 \ \ldots N_k]'$ be the number of shares purchased of each of the k assets. Hence, N must satisfy

$$XN = W \tag{39}$$

Complete Markets Assumptions cont'd

- Since X is a nonsingular, its inverse exists so that

$$N = X^{-1}W \tag{40}$$

is the unique solution.

- Denoting $P = [P_1 \ \ldots \ P_k]'$ as the $k \times 1$ vector of beginning-of-period, per-share prices of the k assets, then the initial wealth required to produce the target level of wealth given in (38) is $P'N$.
- The absence of arbitrage implies that the price of a new, redundant security or contingent claim that pays W is determined from the prices of the original k securities, and this claim's price must be $P'N$.

Arbitrage and State Prices

- Consider the case of a *primitive*, *elementary*, or *Arrow-Debreu* security which has a payoff of 1 in state s and 0 in all other states:

$$e_s = \begin{bmatrix} W_1 \\ \vdots \\ W_{s-1} \\ W_s \\ W_{s+1} \\ \vdots \\ W_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{41}$$

Arbitrage and State Prices

- Then p_s, the price of elementary security s, is

$$p_s = P'X^{-1}e_s, \quad s = 1, ..., k \tag{42}$$

so a unique set of state prices exists in a complete market.

- These elementary state prices should each be positive, since wealth received in any state will have positive value when individuals are nonsatiated. Hence (42) and $p_s > 0 \ \forall s$ restrict the payoffs, X, and the prices, P, of the original k securities.
- Note that the portfolio composed of the sum of all elementary securities gives a cashflow of 1 unit with certainty and determines the risk-free return, R_f:

Arbitrage and State Prices cont'd

$$\sum_{s=1}^{k} p_s = \frac{1}{R_f} \tag{43}$$

- For a general multicashflow asset, a, whose cashflow in state s is X_{sa}, absence of arbitrage ensures its price, P_a, is

$$P_a = \sum_{s=1}^{k} p_s X_{sa} \tag{44}$$

- Consider the connection to state probabilities, π_s, by defining $m_s \equiv p_s/\pi_s$. Since $p_s > 0\ \forall s$, then $m_s > 0\ \forall s$ when $\pi_s > 0$.

Arbitrage and State Prices cont'd

- Then equation (44) can be written as

$$\begin{aligned} P_a &= \textstyle\sum_{s=1}^{k} \pi_s \frac{p_s}{\pi_s} X_{sa} \\ &= \textstyle\sum_{s=1}^{k} \pi_s m_s X_{sa} \\ &= E\left[m X_a\right] \end{aligned} \tag{45}$$

  where m denotes a stochastic discount factor whose expected value is $E[m] = \sum_{s=1}^{k} \pi_s m_s = \sum_{s=1}^{k} p_s = 1/R_f$, and X_a is the random cashflow of the multicashflow asset a.

- In terms of the consumption-based model, $m_s = \delta U'(C_{1s}) / U'(C_0)$ where C_{1s} is consumption at date 1 in state s, p_s is greater when C_{1s} is low.

Risk-Neutral Probabilities

- Define $\widehat{\pi}_s \equiv p_s R_f$. Then

$$\begin{aligned} P_a &= \textstyle\sum_{s=1}^{k} p_s X_{sa} \\ &= \frac{1}{R_f} \textstyle\sum_{s=1}^{k} p_s R_f X_{sa} \\ &= \frac{1}{R_f} \textstyle\sum_{s=1}^{k} \widehat{\pi}_s X_{sa} \end{aligned} \tag{46}$$

- Now $\widehat{\pi}_s$, $s = 1, ..., k$, have the characteristics of probabilities because they are positive, $\widehat{\pi}_s = p_s / \sum_{s=1}^{k} p_s > 0$, and they sum to 1, $\sum_{s=1}^{k} \widehat{\pi}_s = R_f \sum_{s=1}^{k} p_s = R_f / R_f = 1$.

Risk-Neutral Probabilities cont'd

- Using this insight, equation (46) can be written

$$\begin{aligned} P_a &= \frac{1}{R_f} \sum_{s=1}^{k} \widehat{\pi}_s X_{sa} \\ &= \frac{1}{R_f} \widehat{E}\left[X_a\right] \end{aligned} \tag{47}$$

  where $\widehat{E}[\cdot]$ denotes the expectation operator using the "pseudo" probabilities $\widehat{\pi}_s$ rather than the true probabilities π_s.

- Since the expectation in (47) is discounted by the risk-free return, we can recognize $\widehat{E}[X_a]$ as the certainty equivalent expectation of the cashflow X_a.

Risk-Neutral Probabilities cont'd

- Since $m_s \equiv p_s/\pi_s$ and $R_f = 1/E[m]$, $\widehat{\pi}_s$ can be written as

$$\begin{aligned} \widehat{\pi}_s &= R_f p_s = R_f m_s \pi_s \\ &= \frac{m_s}{E[m]} \pi_s \end{aligned} \qquad (48)$$

- In states where the SDF m_s is greater than its average, $E[m]$, the pseudo probability exceeds the true probability.
- Note if $m_s = \frac{1}{R_f} = E[m]$ then $P_a = E[mX_a] = E[X_a]/R_f$ so the price equals the expected payoff discounted at the risk-free rate, as if investors were risk-neutral.

Risk-Neutral Probabilities cont'd

- Hence, $\widehat{\pi}_s$ is referred to as the *risk-neutral* probability.
- $\widehat{E}\left[\cdot\right]$, also often denoted as $E^Q\left[\cdot\right]$, is referred to as the risk-neutral expectations operator.
- In comparison, the true probabilities, π_s, are frequently called the *physical*, or *statistical*, probabilities.

State Pricing Extensions

- This complete markets pricing, also known as *State Preference Theory,* can be generalized to an infinite number of states and elementary securities.
- Suppose states are indexed by all possible points on the real line between 0 and 1; that is, the state $s \in (0, 1)$.
- Also let $p(s)$ be the price (density) of a primitive security that pays 1 unit in state s, 0 otherwise.

State Pricing Extensions cont'd

- Further, define $X_a(s)$ as the cashflow paid by security a in state s.
- Then, analogous to (43),

$$\int_0^1 p(s)\, ds = \frac{1}{R_f} \tag{49}$$

and the price of security a is

$$P_a = \int_0^1 p(s)\, X_a(s)\, ds \tag{50}$$

State Pricing Extensions cont'd

- In *Time State Preference Theory,* assets pay cashflows at different dates in the future and markets are complete.
- For example, an asset may pay cashflows at both date 1 and date 2 in the future: let s_1 be a state at date 1 and let s_2 be a state at date 2. States at date 2 can depend on which states were reached at date 1.
- Suppose there are two events at each date, economic recession (r) or economic expansion (boom) (b). Then define $s_1 \in \{r_1, b_1\}$ and $s_2 \in \{r_1 r_2, r_1 b_2, b_1 r_2, b_1 b_2\}$.
- By assigning suitable probabilities and primitive security state prices for assets that pay cashflows of 1 unit in each of these six states, we can sum (or integrate) over both time and states at a given date to obtain prices of complex securities.

Summary

- An optimal portfolio is one where assets' expected marginal utility-weighted returns are equalized, and the individual's optimal savings trades off expected marginal utility of current and future consumption.
- Assets can be priced using a SDF that is the marginal rate of substitution between current and future consumption.
- A SDF can also be derived based on assumptions of market completeness and no arbitrage.
- A risk-neutral pricing formula transforms physical probabilities to account for risk.